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Product Innovation Investment under Bankruptcy Threat*

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Abstract

We analyze the effect of external financing and associated bankruptcy threat on the speed of product innovation in a market characterized by technological and demand uncertainty. In a dynamic market setting we characterize the optimal R&D investment strategy of a monopolistic incumbent firm that can invest to develop a new product with uncertain demand. The size of the R&D investment flow determines the distribution of the stochastic innovation time and at the same time influences the dynamic evolution of firm's liquidity. If liquidity is negative the firm faces bankruptcy risk. We show that the optimal investment is a U-shaped function of liquidity and characterize under which circumstances it is optimal for the firm to go into debt in order to speed up innovation. Furthermore, we show that, due to the existence of financial frictions, the relationship between the incumbent's profit on the existing market and the expected innovation time for the new product is non-monotone and follows a titled-z shape. We empirically verify the theoretically derived prediction of a U-shaped relationship between liquidity and investment using a dataset consisting of a sample of 400.000 Italian manufacturing companies.

Keywords: Product Innovation, Bankruptcy Threat, Optimal Investment, Uncertainty, Dynamic Optimization

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1 Introduction

Product Innovation is a crucial strategic activity for many firms. When firms are able to offer distinctive and innovative products, they not only extend their existing product lines, but also exert advantage against competitors. According to a widely circulated McKinsey survey, 84% of executives believe that innovation is critical for their business.[1] An innovation process takes time (Gee, 1978) and requires continuous financial investment from the firm. Furthermore, product innovation is associated with different types of uncertainties, in particular technological uncertainty and market uncertainty. Technological uncertainty implies it is difficult to predict the time and effort required for the successful innovation, and consequently, the firm has incomplete information about investment costs. Market uncertainty refers to the uncertain demand for the new product in case of a successful innovation, and leads to uncertain profitability. These two uncertainties affect the return to firms' product innovation investments.

Due to their risky nature, access to external financing for innovation projects is for many firms problematic (Brown et al., 2009) and therefore a large fraction of such projects have to be financed internally. This implies that innovation investment decisions of firms are often influenced by financial frictions. According to the data from CIS survey in Germany 2012-2014 (Behrens et al. (2017)), 18.5% of innovative firms have sacrificed innovation projects due to lack of finance. Moreover, 23% of all firms and 48% of firms in R&D-intensive industries would like to increase their innovation expenditures in case of an exogenous positive shock to their cash flow. The combination of uncertainties, with respective to the success of product innovation and to the future profitability, might jeopardize firms' financial standing. Consider Kodak in 1996. Then CEO George Fisher knew that the company's core business might be invaded, or even replaced by digital photography. Kodak was so worried about the threat posed by the new technology that they invested more than $2 billion in R&D for digital imaging. However, in spite of these investments Kodak failed to develop a new product that was successful on the market. As a result of this, they could not save their position in the traditional market but also failed to find a profitable new niche in the market. Due to this, Kodak had to file for bankruptcy in 2012. This example illustrates that product innovation investments, which might be necessary for a firm to ensure future market success, also exposes a firm to the risk of facing large expenses before a positive return is generated. In case a firm relies heavily on external financing for its R&D investments, cost overruns of the innovation project can affect a firm's ability to roll over future debt, and R&D investment hence increases the risk of going bankrupt before a new successful product can be developed (Buddelmeyer et al. (2010)).

[1] See https://hbr.org/2019/12/real-innovation-requires-more-than-an-rd-budget

The main agenda of this paper is to study how the need for external financing and the induced risk of bankruptcy influence optimal investment strategies for product innovation by incumbent firms in a market. In particular, we analyze, both theoretically and empirically, how optimal investment depends on the firms' financial standing, formally expressed as the firm's liquidity, as well as on its strength in the established market, formally expressed as the size of firm's profits on that market. Furthermore, we explore the role of frictions in access to external financing, expressed as the firm's bankruptcy risk when being unable to roll-over existing debt. Our focus on incumbent firms is motivated by empirical evidence that a large fraction of product innovations is developed by incumbents rather than by new market entrants (Chandy and Tellis (2000)). The main insights from our analysis are, first, that the relationship between the firm's liquidity and its product innovation investment is characterized by a U-shape, and, second, that the relationship between a firm's profitability on the established market and its product innovation time is non-monotone and (under certain conditions) resembles a tilted-z. The existence of a bankruptcy threat is crucial for both of these relationships.

Intuitively, when considering the effect of a firm's financial standing and strength on the established market on product innovation incentives, several countervailing effects come into play. First, with respect to the firm's available liquidity, a larger stock induces a lower demand for external financing, which reduces the risk to go bankrupt before the returns to investment have been gained. This reasoning implies that larger firm liquidity has a positive impact on product innovation investments. Conversely, for a firm with substantial debt (i.e. negative liquidity), the high risk of bankruptcy implies a reduction in expected costs of investment due to limited liability in case bankruptcy actually occurs. Second, with respect to the firm's strength on the established market, at least in case the new product is a partial substitute for the firm's existing product, cannibalization arguments induce a negative relationship between the incumbent's strength on the established market and product innovation incentives. However, higher profits of the incumbent on the established market, also increases the firm's liquidity and reduces the need for external financing. As discussed above, this tends to increase incentives for product innovation investment.

In this paper we disentangle these effects and shed light on their interplay in the framework of a dynamic market model, which incorporates technological and market uncertainty as well as bankruptcy risk. This risk is induced by the accumulation of debt due to the inability to internally finance innovation investments. By this approach, we bring together an industrial organization perspective that focuses on firms' innovation incentives, and a corporate finance perspective that emphasizes the impact of financial frictions. Building on our characterization of firms' optimal investment strategies we can also address how the interplay of between profitability of established markets and access to external financing influences the speed of innovation and bankruptcy risk in an industry.

We consider a monopolistic firm offering an established product on a mature market with a constant demand function. The firm receives a continuous profit stream from its sales on this market and at the same time can invest to develop a new product, which is a partial substitute to the established product. The completion time of the new product development is stochastic and the innovation rate depends on the firm's R&D investment. Once the product development is completed the firm puts the new product on the market. The demand for the new product evolves stochastically, starting from a low level and approaching a long-term market size, which is higher than that of the established product. The dynamics of the firm's liquidity is driven by the difference between market profits and the sum of dividends and innovation investment. It is assumed that the firm has access to external financing, such that liquidity can become negative. A firm with negative liquidity runs the risk of going bankrupt, and the risk increases with the size of negative liquidity. This formulation captures in reduced form that firms in debt might loose access to credit when trying to roll-over debt and that the probability for this to happen increases with the firm's leverage (see e.g. Sapienza (2002) for empirical evidence in this respect). The firm determines its R&D investment in order to maximize the expected discounted future dividend stream.

Our analysis shows that the optimal R&D strategy, as a function of the firm's liquidity, has a U-shape. Investments are highest when liquidity is either very high, which implies that the firm essentially faces no bankruptcy threat, or when the firm is already heavily indebted, in which case a fast product innovation is the only chance to avoid future bankruptcy. For firms with positive initial liquidity two different scenarios might arise, depending on the firm's profitability on the established market and the frictions in the firm's access to credit: a *debt scenario* or a *no-debt scenario*. In the former case it is optimal for a firm with positive initial liquidity to invest so heavily in R&D that it might eventually accumulate debt (if the innovation does not arrive sufficiently quickly) and faces a positive bankruptcy probability, whereas in the latter case the firm never goes into debt if the initial liquidity is non-negative. In the no-debt scenario the firm also over time eliminates potential initial debt, if it is not too large. Our analysis also characterizes under which circumstances the firm's optimal investment strategy induces a no-debt scenario in which the liquidity level of zero is a stable fixed point of the liquidity dynamics. In such a case it is optimal for the firm to eliminate any initial positive liquidity or debt and then to invest all incoming profits in R&D till the innovation is successful without relying on any external financing. From a theory perspective we show that the optimal R&D investment strategy might exhibit jumps and in particular is discontinuous at the liquidity level of zero in such a no-debt scenario. Combining analytical results with an extensive numerical analysis we fully characterize how the occurrence of the different scenarios depends on the interplay of the key model parameters. Furthermore, we highlight that for small initial firm liquidity a highly non-monotone relationship between the firm's profitability on the established market and the speed of innovation arises. A higher profitability on the established market induces lower R&D investment and slower innovation if the firm is in a debt scenario (arising for low levels of profitability). In a no-debt scenario whether an increase of profitability leads to faster or slower innovation depends on whether the steady state liquidity under optimal investment is zero (arising for intermediate levels of profitability) or strictly positive (arising for high levels of profitability). We explain this non-monotone relationship as the result of the interplay of three different effects of an increases in the firm profitability: the cannibalization effect, the bankruptcy loss effect and the financing effect, with the first two inducing a negative relationship between profitability and speed of innovation and the last one a positive relationship.

In the final part of the paper we empirically test the main qualitative results of our theoretical analysis using firm level data from Italian manufacturing companies from 2015-2018. We classify the firms with respect to their bankruptcy risk using a standard assessment system and estimate two types of multivariate regressions models relating firm investments to their risk classifications. Both formulations confirm the U-shaped relationship predicted by our model. Furthermore, we find a positive relationship between firms' market share and their investment, which we interpret as an indication that on average the financing effect is dominant for the considered firm population.

The remainder of the paper is organized as follows. In Section 2 we discuss how our paper contributes to different streams of related literature and in Section 3 we introduce our model. Analytical results characterizing the optimal investment strategy and the resultant liquidity dynamics are presented in Section 4. In Section 5 we extend these findings with an extensive numerical analysis illustrating the optimal investment strategies and the corresponding expected innovation times and bankruptcy probabilities for different parameter constellations. In Section 6 we empirically verify the predictions of our model using Italian firm level data. Concluding remarks are given in Section 7. All proofs can be found in Appendix B, Appendix C contains additional numerical results and Appendix D a detailed description of our numerical method.

2 Related Literature

Our research contributes to the following streams of related literature. First, we extend theoretical analysis of incumbent firms' product introduction and innovation incentives in the absence of financial frictions. A new product not only generates new demand for an incumbent firm but could also come at the expense of (cannibalizes) the demand of its old product (Van Heerde et al., 2010; Druehl and Schmidt, 2008). In case the new product is already initially available to be introduced onto the market, Moorthy and Png (1992) were among the first to discuss the product cannibalization effect. They considered under market segmentation the sequential and simultaneous introduction of two differentiated products, high-end and low-end, and suggested that the sequential introduction (first high-end and then low-end) is better, because the cannibalizing low-end product is made unavailable. Gezer (2019) investigated and compared new production introduction strategies such as immediate, delayed and abstained introduction. In case of delayed introduction, the incumbent can reduce the capacity of the old product to reduce the cannibalization effect. When a new product is not immediately available, Dawid et al. (2013) considered in a duopoly setting where firms' interactions takes place in three stages, e.g., innovation stage, capacity investment stage and pricing stage. They showed that a firm with a larger capacity on the old market has lower incentives to introduce the new product, and the firm with a larger capacity for the old product can prevent its competitor from innovating. Compared to the previous literature, this paper considers also an R&D investment to develop a new product, but in a continuous time setting. In particular, we obtain also the cannibalization effect, i.e., a stronger incumbent on the established market has less incentive to invest and develop a new product. Furthermore, we identify two additional effects, the bankruptcy loss effect and the financing effect, and both are related positively to firm's strength on the established market. The bankruptcy loss effect decreases firm's innovation incentive, similar as the cannibalization effect. The financing effect, on the other hand, increases the innovation incentive.

Our second contribution is to the stream of literature that features product innovation in a dynamic setting. Part of the literature treats production innovation as to increase the products differentiation, see e.g., Lin and Saggi (2002). The other strand of literature takes product innovation as to develop a new product and analyzes the relation between product innovation and the process innovation that decreases the marginal cost of production. For instance, Lambertini and Mantovani (2009) showed there exists substitutability between process and product innovations. Li and Ni (2016) took into account the effect of learning-by-doing and showed that a larger rate intensifies firm's knowledge accumulation and reduces the optimal investment in both the process and the product innovations. In this stream of literature, Dawid et al. (2015) were among the first to analyze how a firm's product innovation depends on the its initial product capacity and knowledge stock. In particular, they showed it could be optimal for a relatively small firm on the established market to innovate and eventually give up once it has built up a sufficiently large capacity for the old product. Later Dawid et al. (2023) extended the monopolistic incumbent's product innovation to a competitive setting and revealed that when sufficiently large, the knowledge leader's investment in R&D could be so small that its innovation rate is lower than the knowledge laggard's. It should be duly noted that all the above mentioned literature is under the assumption that the R&D investment, either in process innovation or in product innovation, can be financed internally by firms' equities. The present paper incorporates the possibility that such investments need to be financed through external sources. The main new contribution of this research lies in analyzing the influence of financial frictions, i.e., the inability to roll over debts and thus being confronted with real bankruptcy threat, on an incumbent firm's innovation investments. More specifically, our results support that it is possible for an initially indebted incumbent to eliminate its debt and break even on its liquidity under the optimal R&D investment in equilibrium.

This research further contributes to the literature regarding dynamic investment under financial frictions. Milne and Robertson (1996) analyzed firm's optimal investment in capital stock under uncertain cash flows and financial constraints, which is taken as a consequence of information asymmetries between providers of finance and the firm management. The purpose is to characterize a threshold of internal cash so as to balance the desire to pay dividends and the need to retain cash against liquidation threat. A following research by Holt (2003) studied how a firm's investment and dividend payout decisions interact over time under investment irreversibility and financial constraints. The obtained result shows there is a capital stock threshold such that investment is optimal below this threshold and dividend payout is optimal above this threshold. Note that most of the research in this stream of literature focuses on solving an optimal stopping problem and applies real options analysis. For instance, Boyle and Guthrie (2003); Lyandres and Zhdanov (2010) showed that the financial constraint and the default possibility accelerate investment, because they reduce the value of the firm's timing options. Shibata and Nishihara (2012) examined timings for the investment and the default of a firm with debt financing capacity constraint, i.e., part of the investment is financed through debts. They showed that the financing constraints may encourage over-investment. Similarly, Lin (2022) considered the firm's optimal investment rate and found that over-investment is more likely for a financial-constrained firm, which has to terminate its R&D due to limited access to financing.

Furthermore, Lin showed that a financially constrained firm could win the R&D race against an unconstrained opponent when the innovation rate is positively related to R&D investment. Compared with this stream of literature, we investigate an optimal control problem for a firm's R&D innovation investment under bankruptcy threat. This allows us to characterize the firm's optimal control path in the Markov Perfect Equilibrium, i.e., how its R&D investment changes with liquidity dynamics. In particular, we show that the optimal investment has a U-shaped fashion with respect the firm's initial liquidity. Thus, for a very negative liquidity the firm invests a lot because the successful innovation is the only way to be safe from bankruptcy. For a liquidity level around 0, the firm invests less because of the potential bankruptcy risk and an incentive to avoid getting (deeper) into debt.

Our findings are also related to the long-lasting debate on the relationship between cash-flow and investment under financial frictions. For example, Tanrısever et al. (2012) explored the tradeoff between investing in process innovation to reduce unit cost and conserving cash to reduce the bankruptcy likelihood for an indebted firm. A conservative (aggressive) investment is accompanied by less (more) production than the monopoly level to increase survival chances against bankruptcy. Empirical results in that respect are mixed. Many paper starting with Fazzari et al. (1988, 2000) find that financially constrained firms have a stronger positive relationship between cash flow and investment compared to where financial constraints do not play a role. This view has been challenged on both theoretical and empirical grounds by Kaplan and Zingales (1997, 2000). Gomes (2001) and Alti (2003) have put forward models showing a positive relationship between cash-flow and investment in settings with perfect financial markets. In this realm of literature dealing with generic investments, typically in physical capital, several papers have also studied the relationship between R&D investment and cash flow, and in how far it is related to financial frictions. The general conclusion from this body of literature is that there is evidence that firms face constraints for financing R&D investments and this gives rise to high sensitivity of innovative firm's investment to cash flows (see Hall and Lerner (2010) for a survey of this literature). Our paper complements this mainly empirical literature from a theoretical perspective and provides several innovative aspects. First, while these papers consider general R&D investment, we specifically focus on product innovation investments of incumbents by taking into account that, the firm's strength on the established market influences both the firm's revenues and its incentive to extend its product range. Second, our analysis characterizes the optimal investment strategy of the firm as a function of its current financial state, thereby capturing how the innovation investments of the firm evolves over time as its liquidity changes. This perspective allows us to show that, even within the same market environment the sign of the relationship between cash flow (i.e. change in firm liquidity) and R&D investment might change according to the firm's financial standing. Third, we characterize under which circumstances a rational and far-sighted incumbent should risk bankruptcy and jeopardize its position on the established market in order to pursue product innovation.

3 The Model

We consider a monopoly firm which produces an established product o and at the same time invests in the development of a new product n, which is a partial substitute to the established product. At the stochastic innovation time τ the firm introduces the new product and afterwards offers both products. The inverse demand is assumed to be linear and of the form

$$\begin{aligned} p_o(t) &= \alpha_o - q_o(t) - \eta q_n(t), \\ p_n(t) &= \bar{\alpha}_n + \alpha_n - q_n(t) - \eta q_o(t). \end{aligned}$$

Here $p_i(t)$, $i \in \{o, n\}$, denotes the price for product i and $q_i(t)$ the output of product $i \in \{o, n\}$ at time t. The parameter $\eta \in [0, 1)$ indicates the degree of horizontal differentiation between the two products. The consumers' maximal willingness to pay for the established product, α_o, is assumed to be constant. The maximal willingness to pay for the new product is denoted by $\bar{\alpha}_n + \alpha_n$, where α_n evolves stochastically from the moment of successful innovation τ and follows a mean-reverting stochastic process:

$$d\alpha_n = \delta\left(\tilde{\alpha}_n - \alpha_n\right) dt + \sigma \alpha_n dW(t), \quad \delta, \sigma > 0, \tag{1}$$

with $\alpha_n(\tau) = 0$ and $W(t)$ a Wiener process. We assume $\tilde{\alpha}_n + \bar{\alpha}_n > \alpha_o$. Our formulation captures that the evolution of the demand for the new product can not be perfectly predicted and that the long-run market potential of the new product is larger than that of the established product. We assume that $\bar{\alpha}_n > 0$ is sufficiently large to guarantee that the equilibrium output for the new product stays non-negative even for $\alpha_n = 0$.

The number of products, that the firm offers, is captured by $m(t) \in \{m_0, m_1, m_2\}$, which we denote as the *mode* of the problem. In mode m_0 the firm has exited the market (see below) and therefore $q_o = q_n = 0$, in mode m_1 the firm is active only on the established market, such that $q_o > 0, q_n = 0$ and in mode m_2 the firm has both products on the market, i.e. $q_o, q_n \geq 0$. At time $t = 0$, we have $m(0) = \{m_1\}$ indicating that initially the firm is active on the established market, and has not innovated yet.

At each time t the monopoly firm chooses the optimal output quantities taking into accout the current mode. For reasons of simplicity we normalize the unit costs of production to zero. For $m(t) = m_1$ standard calculations (see Appendix A) show that

$$q_o(m_1) = \frac{\alpha_o}{2}.$$

Similarly, in mode m_2 the optimal output quantities are given by

$$\begin{aligned} q_o(\alpha_n, m_2) &= \frac{\alpha_o - \eta(\bar{\alpha}_n + \alpha_n)}{2(1-\eta^2)}, \\ q_n(\alpha_n, m_2) &= \frac{(\bar{\alpha}_n + \alpha_n) - \eta\alpha_o}{2(1-\eta^2)}, \end{aligned}$$

whenever both expressions are non-negative. This is ensured by assuming that $\bar{\alpha}_n > \eta\alpha_o$ and that $\alpha_o - \eta(\bar{\alpha}_n + \tilde{\alpha}_n)$ is sufficiently large such that the probability that $\alpha_o < \eta(\bar{\alpha}_n + \alpha_n)$ is negligible. This generates the following market profits $\pi(\alpha_n, m) = q_o p_o + q_n p_n$ in the different modes:

$$\begin{aligned} \pi(\alpha_n, m_0) &= 0, \\ \pi(\alpha_n, m_1) &= \frac{\alpha_o^2}{4}, \\ \pi(\alpha_n, m_2) &= \frac{(\bar{\alpha}_n + \alpha_n)^2 + \alpha_o^2 - 2\eta(\bar{\alpha}_n + \alpha_n)\alpha_o}{4 - 4\eta^2}. \end{aligned}$$

It is easy to verify that $\pi(\alpha_n, m_2) \geq \pi(\alpha_n, m_1)$ with strict inequality whenever $q_n(\alpha_n, m_2) > 0$.

The transition from mode m_1 to m_2 corresponds to a successful innovation. The arrival process of the innovation is assumed to be memoryless and the innovation rate, is given by $\lambda^{12} = \gamma_I I(t)$, where $I(t)$ denotes the R&D investment by the monopoly firm and the *innovation parameter* $\gamma_I > 0$ captures the efficiency of the firm's R&D activities.[2] R&D investment is associated with quadratic costs of the form $\frac{\xi}{2} I^2$, $\xi > 0$.

The financial situation of the firm is expressed in terms of its liquidity $e(t)$, which evolves according to

$$\dot{e} = \pi(\alpha_n, m) - \frac{\xi}{2} I^2 - D + re, \tag{2}$$

where $r > 0$ is the interest rate, and $D(t)$ denotes the dividends paid out at time t to the shareholders. We assume that while the firm is active, i.e. $m \in \{m_1, m_2\}$, the dividend policy reads $D(e, m) = \nu_m \max\{0, e\}$ with $\nu_m \in (0, 1)$. According to this policy the firm pays dividends

[2]Formally, the innovation rate is defined as $\lambda^{12} = \lim_{\epsilon \to \infty} \frac{1}{\epsilon} \mathbb{P}[m(t+\epsilon) = 2 | m(t) = 1]$.

as a fixed proportion of the positive liquidity reserve and does not pay any dividends if it has negative liquidity. Since our focus in this paper is on innovation investment under bankruptcy threat rather than on the optimal dividend policy, we keep the treatment of the dividend policy very simple and assume that ν_m is constant, rather than a dynamic control variable of the firm. This assumption is made for reasons of tractability, since the derivation of the optimal dividend policy in this framework with demand uncertainty and endogenous investment is highly complex and hardly feasible.[3]

We assume that the firm finances all investments internally as long as liquidity is positive, but if liquidity becomes negative, the firm takes on debt. To capture the risk for a firm associated with external financing of its investment, it is assumed that there is a positive probability that an indebted firm is not able to extend its debt contracts, and has to exit the market. This probability is assumed to be increasing with the amount of debt.[4] More precisely, the transition rate (bankruptcy rate) from mode $m = \{m_1, m_2\}$ to $m = m_0$ is given by

$$\lambda^{k0} := \gamma_B \max[0, -e], \; k = 1, 2. \tag{3}$$

This formulation can also be interpreted as a reduced form representation of a situation where the firm repeatedly has to refinance parts of its debt and the probability to be able to find a lender willing to provide a loan decreases with the firms' debt volume (see e.g. Meijers et al. (2019) for a similar formulation in discrete time).[5] The *bankruptcy parameter* γ_B determines how strong the financial frictions are. The case $\gamma_B = 0$ indicates unlimited access to external financing without any bankruptcy risk, and $\gamma_B \to \infty$ implies that the firm has no access to credit. Note that in our formulation the (expected) costs of external financing are completely captured by the dependence of the bankruptcy risk from liquidity, whereas the interest rate in the liquidity dynamics (2) does not depend on the level of firm liquidity. Assuming that the interest the firm pays for loans also depends positively on its level of debt would certainly be a reasonable assumption, but we abstain from such a formulation here in order to keep the model as simple as possible. Our aim here is to study the optimal investment behavior of a

[3]Characterizations of intertemporally optimal dividend policies in settings with stochastic cash flows and bankruptcy risk can be found e.g. in Moreno-Bromberg and Rochet (2014); Reppen et al. (2020). However, the settings considered in these models do not incorporate firm investment decisions and (stochastic) effects of R&D.

[4]Empirical evidence that higher leverage and higher R&D investment increase the risk of loosing a banking relationship and having to exit the market are provided e.g. in Sapienza (2002) and Buddelmeyer et al. (2010).

[5]Formally, one could consider a situation where the firm holds a portfolio of debt contracts with varying maturity and the probability that the maturity of one of the contracts ends in the interval $[t, t+\epsilon]$ for sufficiently small ϵ is given by $\zeta\epsilon$. Assuming that the probability that the firm is not able to find a lender for refinancing is proportional to its level of debt, i.e. is given by $\omega|e|$ for $e < 0$, yields a bankruptcy rate λ^{k0} as given in (3) with $\gamma_B = \zeta\omega$. It should be noted that even if in debt the firm has positive market profits and the perspective on even larger profits after innovation, such that providing credit to the firm is potentially profitable for the lender, but the incentives to provide credit decrease as the firm's level of debt increases.

firm which is aware that its expenditures for innovation activities might jeopardize its existence if these activities remain unsuccessful for too long. This effect is captured in the form of the bankruptcy rate. Once the firm declares bankruptcy, it stops operating and also stops paying dividends, i.e. $D(e, m_0) = 0$ for all $e \in \mathbb{R}$.

The firm's investment does not generate any expected revenue after a successful innovation in mode m_2, which trivially implies that $I(t) = 0$ for all t with $m(t) = m_2$. Therefore the overall dynamics of the liquidity is given by the state dynamics

$$\dot{e} = \begin{cases} 0 & m = m_0, \\ \pi(0, m_1) - \frac{\xi}{2} I^2 - \nu_1 \max\{0, e\} + re & m = m_1, \\ \pi(\alpha_n, m_2) - \nu_2 \max\{0, e\} + re & m = m_2, \end{cases} \tag{4}$$

and the Markov process m in $\{m_0, m_1, m_2\}$ with transition states

$$\lambda^{ij} = \begin{cases} \gamma_I I & (i,j) = (1,2), \\ \gamma_B \max[0, -e] & (i,j) \in \{(2,0),(1,0)\}, \\ 0 & \text{else}. \end{cases} \tag{5}$$

Before the successful innovation in mode m_1, the firm is constantly balancing the two effects from the innovation investment. One effect is that the investment decreases the firm's liquidity reserves and brings about a possible bankruptcy once the liquidity becomes negative. The other effect is the likelihood of transition into the more profitable mode m_2 that is boosted by investment. Note however, that a negative liquidity also implies a positive bankruptcy probability in mode m_2. Since the firm's profit flow in mode m_2 is always non-negative, such negative liquidity has to be due to negative liquidity at the moment of transition from mode m_1, i.e. at the innovation time. In other words, there is a certain risk of bankruptcy even after successful innovation if the debt accumulated during the innovation phase is too high, but the level of debt decreases over time once the firm has innovated.

The firm's objective is to maximize the expected dividend stream received by the firm's shareholders. Formally, this problem is given by

$$\max_{I(\cdot)} J = \mathbb{E}\left[\int_0^\infty e^{-rt} D(e, m) \mathrm{d}t\right]. \tag{6}$$

subject to the state equation (4), the Markov process $m(t)$ characterized by the transition rates (5) and the initial conditions $e(0) = e^{ini}$, $m(0) = m_1$. Since $I(t) = 0$ in mode m_2, in what follows we focus entirely on the characterization of optimal firm investment in mode m_1. Due to the time autonomous structure of the problem and the infinite time horizon, optimal investment depends only on the current state, but is independent of time. Hence, we express the optimal investment in mode m_1 as a function of the state e and denote the optimal investment function by $\phi(e)$.

4 Optimal R&D Investment

In order to solve the firm's investment problem we use a Dynamic Programming approach and as a first step specify the Hamilton-Jacobi-Bellman (HJB) equations which characterize the value functions V_k in modes $m_k, k = 1, 2$. Mode m_0 is absorbing and no more dividends are paid once the firm is bankrupt, which means that the value function is given by $V_0(e, \alpha_n) = 0$ for all values of the state (e, α_n). In mode m_2, no more investments are made and hence there is no control for the decision maker to choose. Standard arguments (see e.g. Chapter 8 in Dockner et al. (2000)) show that under appropriate smoothness assumptions a function $V_2(e, \alpha_n)$, solving the HJB equation

$$
\begin{aligned}
rV_2(e,\alpha_n) \quad = \quad & \nu_2 \max\{0, e\} + \delta(\tilde{\alpha}_n - \alpha_n)\frac{\partial V_2(e,\alpha_n)}{\partial \alpha_n} + \frac{\sigma^2\alpha_n^2}{2}\frac{\partial^2 V_2(e,\alpha_n)}{\partial \alpha_n^2} \\
& + \frac{\partial V_2(e,\alpha_n)}{\partial e}\dot{e} + \gamma_B \max\{0, -e\}(V_0(e,\alpha_o) - V_2(e,\alpha_n)) \qquad (7)
\end{aligned}
$$

is the value function for the monopoly firm. The first term on the right hand side (RHS) is the dividend received by the share holders. The next three terms indicate the expected change in the value function due to the dynamics of market demand and the liquidity. The last term states the expected change in the value resulting from the possibility of bankruptcy in case of negative liquidity.

Considering the investment problem in mode m_1, we drop the second argument of $V_1(e, \alpha_n)$ in light of the fact that $\alpha_n(\tau) = 0$ at the innovation time τ, and write the value function only as a function of the firm's liquidity e. The corresponding HJB equation for a given dividend rate ν_1 in this mode can be written as

$$
rV_1(e) = \max_I \left[\nu_1 \max\{e, 0\} + \frac{\mathrm{d}V_1(e)}{\mathrm{d}e}\dot{e} + \gamma_I I\left(V_2(e,0) - V_1(e)\right) + \gamma_B \max\{0, -e\}\left(V_0(e) - V_1(e)\right)\right]. \quad (8)
$$

On the RHS, the expected change of the value results from the changes of firm's liquidity and the possibility of transition into modes m_2 and m_0. Using the Bellman equation for m_1, i.e., equation (8), we obtain the following characterization of optimal investment.

Lemma 1. *For all levels of liquidity e at which $V_1(e)$ is differentiable the optimal investment before innovation, i.e., in mode m_1, is given by*

$$\phi(e) = \frac{\gamma_I}{\xi} \frac{(V_2(e,0) - V_1(e))}{\mathrm{d}V_1(e)/\mathrm{d}e} > 0. \tag{9}$$

Lemma 1 can be easily derived by taking the first order derivative of the RHS of equation (8) with respect to I. It shows there are several factors that influence the firm's optimal innovation investment. R&D investment increases with respect to the jump in the value $(V_2(e,0) - V_1(e))$ at the moment of successful innovation, but decreases with respect to $\xi \frac{\mathrm{d}V_1(e)}{\mathrm{d}e}$. To interpret this expression it should be noted that in light of (2) marginally increasing investment reduces the firm's liquidity by ξI and this decrease in liquidity is associated with a decrease in the firm owner's value of $\xi I \frac{\mathrm{d}V_1(e)}{\mathrm{d}e}$. Thus, optimal investment can be interpreted as the ratio of marginal (expected) returns to investment and the associated marginal cost coefficient. The firm owner always profits from additional liquidity, which implies that we have $\mathrm{d}V_1(e)/\mathrm{d}e > 0$. Note that no general statement about effect of the innovation parameter γ_I, which determines the marginal effect of an increase of R&D investment on the innovation rate, on the optimal investment level can be made, because this parameter has a direct positive effect, but also an indirect negative effect by positively influencing $V_1(e)$.

4.1 Optimal Investment Unaffected by Bankruptcy Threat

In order to better understand the effect of the bankruptcy threat for the optimal investment, we first study, as a benchmark, the special scenario without bankruptcy risk. In particular, we consider a situation where the initial liquidity is sufficiently large such that the firm never faces a positive bankruptcy probability even if it chooses its unconstrained optimal investment level. For such a scenario we can explicitly derive the value functions in both modes. Moreover, the firm's optimal innovation investment can also be calculated. These results are summarized in the following proposition.

Proposition 1. *Assume that $e^{ini} > \tilde{e} = \max\left[\frac{\xi(I^{nc})^2 - \alpha_o^2/2}{2(r-\nu_1)}, 0\right]$ with*

$$I^{nc} = \sqrt{\frac{r^2}{\gamma_I^2} + \frac{2rc}{\xi} - \frac{\alpha_o^2}{2\xi}} - \frac{r}{\gamma_I} > 0 \tag{10}$$

and either $r > \nu_1$ or $\tilde{e} > 0$. Then the optimal investment in mode m_1 is constant over time *with $I(t) = I^{nc}$ for all $t \in [0, \tau]$. The value function in mode m_1 for all $e \geq \tilde{e}$ is given by*

$$V_1(e) \quad = \quad e + c + \frac{1}{\gamma_I^2}\left(r\xi - \sqrt{r^2\xi^2 + 2cr\xi\gamma_I^2 - \frac{\xi\gamma_I^2\alpha_o^2}{2}}\right) \tag{11}$$

with

$$c = \frac{\delta^2\tilde{\alpha}_n^2 + \delta\tilde{\alpha}_n(\bar{\alpha}_n - \alpha_o\eta)(r + 2\delta - \sigma^2)}{2r(r+\delta)(1-\eta^2)(r+2\delta-\sigma^2)} + \frac{\bar{\alpha}_n^2 + \alpha_o^2 - 2\eta\alpha_o\bar{\alpha}_n}{4r(1-\eta^2)}. \tag{12}$$

In mode m_2 the value function for all $e \geq 0$ is given by

$$V_2(e, \alpha_n) \quad = \quad \frac{\delta\tilde{\alpha}_n + (\bar{\alpha}_n - \alpha_o\eta)(r + 2\delta - \sigma^2)}{2(r+\delta)(1-\eta^2)(r+2\delta-\sigma^2)}\alpha_n + \frac{\alpha_n^2}{4(1-\eta^2)(r+2\delta-\sigma^2)} + e + c. \tag{13}$$

Proposition 1 covers two scenarios. First, if $r > \nu_1$ and $e^{ini} \geq \tilde{e}$ then liquidity grows throughout mode m_1 even if the firm chooses the unconstrained optimal investment level and therefore liquidity never becomes negative. If $r < \nu_1$ then $\tilde{e} > 0$ only holds if $\alpha_o^2/4 > \xi(I^{nc})^2/2$, which means that the market profit in mode m_1 is sufficiently large to cover the investment costs under the unconstrained investment I^{nc}. In this case the sum of the market profit, net of investments, and earned interest is exactly equal to the firm's dividend payout at the liquidity level $\tilde{e}$. For any initial level of liquidity $e^{ini} > \tilde{e}$, the liquidity of a firm investing I^{nc} converges from above to $\tilde{e}$ in mode m_1 and therefore never becomes negative. Since the firm no longer invests in mode m_2 liquidity stays non-negative throughout mode m_2 if it is non-negative at the time of the innovation. As a corollary of Proposition 1 we can clarify under which conditions for a non-negative initial liquidity the firm's optimal investment coincides with the unconstrained optimum it would choose in the absence of any financial frictions.

Corollary 1. *If either $2rc \leq \alpha_o^2$ or the conditions $2rc > \alpha_o^2$ as well as $\gamma_I \leq \underline{\gamma}_I$ with*

$$\underline{\gamma}_I = \frac{r\alpha_o\sqrt{2\xi}}{2rc - \alpha_o^2} \tag{14}$$

and c given by (12) *hold, then the optimal investment reads $\phi(e) = I^{nc}$ for all $e \geq 0$. For all $e^{ini} \geq 0$ liquidity $e(t)$ stays non-negative for all $t \geq 0$ and either converges monotonously towards the steady state $\tilde{e} \geq 0$ (for $r < \nu_1$) or diverges towards infinity while the firm is in mode m_1 (for $r \geq \nu_1$).*

The conditions given in Corollary 1 are very intuitive. The firm is not concerned about bankruptcy risk if either the expected size of the new market is not sufficiently large compared to the size of the established market (this is captured by the condition $c \leq \alpha_o^2/2r$), or alternatively, the efficiency of the firm's R&D activities is relatively low ($\gamma_I \leq \underline{\gamma}_I$). In both cases, even without considering financial frictions, the firm would choose an R&D investment level, which is so small that it can be fully covered by the profits made on the established market.

Given that the firm's investment for the scenarios covered in Proposition 1 in mode m_1 is constant the expected innovation time can be easily calculated as

$$\mathbb{E}[\tau] = \int_0^\infty t\gamma_I I^{nc} \exp(-\gamma_I I^{nc} t) \mathrm{d}t = \frac{1}{\gamma_I I^{nc}}. \tag{15}$$

The value functions in both modes can be interpreted as a summation of the instantaneous liquidity reserve e and the discounted future profits. Since the interest rate is equal to the discount rate and the firm lives eternally, moving the payout of liquidity across time does not influence the value of the discounted dividend stream of the firm owner as long as it is guaranteed that liquidity never becomes negative. Furthermore, as long as the firm does not face any (future) bankruptcy risk, the innovation investment is determined by the relationship between marginal costs and marginal future returns, but is independent of the liquidity and also of the dividend rate, see (10). From this equation also the following very intuitive effects of the key parameters on the unconstrained optimal investment level can be directly derived.

Corollary 2. *The unconstrained optimal R&D investment level, I^{nc}, increases with the efficiency of R&D (γ_I) but decreases with respect to investment costs (ξ) and the size of the established market (α_o). If $r + 2\delta > \sigma^2$, then optimal investment increases with the market potential of the new product ($\tilde{\alpha}_n$).*

For the following analysis in particular the negative dependence of optimal R&D investment on the size of the established market is important. Intuitively, this dependence is due to a standard cannibalization effect. The introduction of the new product leads to a reduction in the price of the old product. Hence, the monopolist's incentive to introduce the new product is smaller the more profitable it is on the established market. If the monopolist has sufficiently large liquidity such that it can always internally finance its optimal R&D investments, cannibalization is the only effect induced by an increase of α_o. However, if the firm is financially constrained an increase of α_o also reduces the demand for external financing and hence the relationship between the optimal investment and the size of the established market is less clear cut. We now turn to analyzing this scenario in which the firm also has to take into account a potential bankruptcy risk.

4.2 Optimal Investment Affected by Bankruptcy Threat

If the unconstrained level of investment cannot be internally financed through profits on the established market, the monopolist, even if it initially does not have any debt, might face bankruptcy risk if it does not adjust the investment size. Building on Proposition 1, the following proposition shows that the efficiency of R&D activities, as well as the size of the established market, play a key role in determining whether the bankruptcy threat is relevant for the firm.

Proposition 2. *If $2rc > \alpha_o^2$ and $\gamma_I > \underline{\gamma}_I$, with $\underline{\gamma}_I$ given in (14), then either $\phi(e) < I^{nc}$ for some liquidity $e \geq 0$ or for some $e^{ini} \geq 0$ there is a positive probability for bankruptcy under the investment strategy $\phi(.)$, or both.*

In light of our agenda to study the effects of financial frictions on product innovation investment, from here onward we will focus on the case where financial frictions might influence such investment. Taking into account Corollary 1 and Proposition 2 we make the following formal assumption.

Assumption 1. *Throughout the following analysis it is assumed that $2rc > \alpha_o^2$.*

The existence of bankruptcy risk makes the characterization of the optimal investment strategy much more challenging compared to the case without such risk. Formally, this is due to the fact that the last terms on the right hand side of the HJB equations (7) and (8), which disappear if only positive values of e are considered, prevent us from obtaining closed form solutions for the value functions in modes m_1 and m_2.

Before analyzing the firm's optimal R&D investment in mode m_1 we first need to consider the firm's continuation value after a successful innovation, i.e. the firm's value function in the post innovation mode m_2. Since market profit in mode m_2 is non-negative and the firm makes no investments, the value function $V_2(e, \alpha_n)$ can be explicitly calculated for a non-negative initial liquidity $e \geq 0$. If $\nu_2 > r$ liquidity in the long run oscillates around the positive steady state $e_2^* = \pi(\tilde{\alpha}, m_2)/(\nu_2 - r) > 0$. For $r \leq \nu_2$, liquidity would diverge to positive infinity, but it is clear that such a dividend policy would be sub-optimal. Because liquidity never decreases in mode m_2, the bankruptcy rate is zero for all $t \geq \tau$ if $e(\tau) \geq 0$, and the value function $V_2(e, \alpha_n)$ has the same expression as equation (13) for $e \geq 0$. Since for negative liquidity no dividends are paid, these considerations and the fact that (13) does not depend on ν_2 show that also in the presence of bankruptcy risk, the value functions V_1 and V_2 as well as the optimal R&D investment strategy in mode m_1 do not depend on the value of $\nu_2 > r$. The non-linear form of the HJB equation for $e < 0$ does not allow us to obtain a closed form solution for $V_2(e, \alpha_n)$ on this part of the state space. Therefore, in Section 5 we will resort to numerical calculations to determine the value function on the half-plane with $e < 0$.

We now turn to the characterization of optimal R&D investment strategies in the case where the firm would face a threat of bankruptcy if choosing the unconstrained optimal investment size. Taking into account Proposition 2, in general it is not clear whether for $\gamma_I > \underline{\gamma}_I$ the liquidity stays non-negative under the optimal investment strategy even if it starts evolving from a non-negative initial level. Since the problem in mode m_1 is an optimal control problem with one-dimensional state-space the liquidity trajectory under the optimal control has to be monotonous (see Hartl (1987)). Therefore, the analysis of the locations of the steady states of the problem provide clear insights on whether liquidity might become negative, if the monopolist invests optimally. The following lemma provides a characterization of steady state candidates e^* under the assumption of differentiability of the value function at e^*.

Lemma 2. *Assume that e^* is a steady state of the liquidity dynamics under the optimal investment strategy $\phi(e)$ in mode m_1 and that the associated value function $V_1(e)$ is differentiable at e^*. Then the following conditions have to be satisfied:*

$$\frac{\alpha_o}{2} - \frac{\xi\Phi^2(e^*)}{2} + re^* - \nu_1 \max\{0, e^*\} = 0, \tag{16}$$

$$\xi\phi(e^*)\frac{\mathrm{d}V_1(e^*)}{\mathrm{d}e} - \gamma_I\big(V_2(e^*, 0) - V_1(e^*)\big) = 0, \tag{17}$$

$$rV_1(e^*) = \max\{0, \nu_1 e^*\} + \gamma_I\phi(e^*)\big(V_2(e^*, 0) - V_1(e^*)\big) - \gamma_B \max\{0, -e^*\}V_1(e^*), \tag{18}$$

$$\begin{aligned}\gamma_I\phi(e^*)\left(\frac{\partial V_2(e^*, 0)}{\partial e} - \frac{\mathrm{d}V_1(e^*)}{\mathrm{d}e}\right) + \nu_1 1\!\!1_{[e^*\geq 0]} - r\frac{\mathrm{d}V_1(e^*)}{\mathrm{d}e} \\ + \gamma_B\left(1\!\!1_{[e^*\leq 0]}V_1(e^*) - \max\{0, -e^*\}\frac{\mathrm{d}V_1(e^*)}{\mathrm{d}e}\right) = 0.\end{aligned} \tag{19}$$

This system of necessary conditions is derived by taking into account the steady state condition $\dot{e} = 0$ (16), the first order condition for investment (17), the HJB equation at the steady state (18) and the state derivative of the HJB equation at the steady state (19). Assuming that the problem in mode m_2 is solved and $V_2(e, \alpha_n)$ is known, then there are four unknowns in the above equations, e^*, $\Phi(e^*)$, $V_1(e^*)$, and $\mathrm{d}V_1(e^*)/\mathrm{d}e$. Though closed form solutions to this system of equations in general cannot be obtained, Lemma 2 provides the basis for identifying via numerical analysis all candidates for steady states with local differentiability of the value function.

Before applying this lemma in the numerical analysis we first derive conditions under which zero liquidity is a steady state. From Corollary 2 (and its proof) we already know that for$\gamma_I < \underline{\gamma}_I$ we have $\dot{e} > 0$ at $e = 0$, such that a steady state with zero liquidity of the firm can only exist if $\gamma_I \geq \underline{\gamma}_I$. Due to the kink in the dividend policy and the bankruptcy rate at $e = 0$ we must expect that in general the value function V_1 is not differentiable at $e = 0$. Hence, Lemma 2 cannot be directly applied and we must resort to a viscosity solution of the HJB equation when determining the value function of the problem (see e.g. Bardi and Capuzzo-Dolcetta (2008)). Based on this we can characterize the conditions under which $e^* = 0$ is a steady state under optimal investment.

Proposition 3. *The liquidity $e^* = 0$ is a stable steady state under the optimal investment strategy $\phi(e)$ in mode m_1 if*

$$\gamma_I \in \left[\underline{\gamma}_I, \bar{\gamma}_I\right] \tag{20}$$

with $\underline{\gamma}_I$ given by (14) *and*

$$\bar{\gamma}_I = \frac{(r + \gamma_B c)\, \alpha_o \sqrt{2\xi}}{2rc - \alpha_o^2}, \tag{21}$$

where c is given by (12)*. Optimal R&D investment in the steady state is then given by*

$$\phi(0) = \frac{\alpha_o}{\sqrt{2\xi}} \tag{22}$$

and for $\gamma_I \in \left(\underline{\gamma}_I, \bar{\gamma}_I\right)$ optimal investment is discontinuous at $e = 0$ such that

$$\lim_{\epsilon \to 0+} \phi(-\epsilon) < \phi(0) < \lim_{\epsilon \to 0+} \phi(\epsilon).$$

Proposition 3 gives the upper and lower bounds for the efficiency of R&D activities, γ_I, such that at $e = 0$ investing an amount that equals exactly the market profits is optimal. The proposition also implies that if $e = 0$ is a steady state then the optimal investment strategy $\phi(e)$ exhibits a jump at this value of the liquidity. Clearly, this jump is due to fact that, as soon as liquidity becomes negative, an increase of investment increases the bankruptcy risk and therefore the incentive to invest is lower compared to a situation where no such effect on bankruptcy risk exists. For γ_I in the interval (20) the optimal investment without considering the effect on bankruptcy risk is larger than the profit on the established market, whereas the optimal investment taking into account the effect on bankruptcy risk is below market profit. In such a scenario, for positive initial liquidity the firm invests above the profit on the established market until liquidity has been depleted to zero and then reduces investment such that it equals the current profit. If γ_I is sufficiently large, then for small negative liquidity the optimal investment, even under the consideration of its effect on bankruptcy risk, is larger than what can be internally financed by the profits on the established market. In this scenario the optimal investment strategy induces that the firm goes into debt even if it starts with a non-negative liquidity. This intuition is formalized in the following corollary, which follows directly from Proposition 3 together with Corollary 2.

Corollary 3. *For $\gamma_I < \underline{\gamma}_I$ the optimal R&D investment of the firm induces $\dot{e} > 0$ for $e = 0$, whereas for $\gamma_I > \bar{\gamma}_I$, at $e = 0$ we have $\dot{e} < 0$.*

Note that the lower bound $\underline{\gamma}_I$ does not depend on the bankruptcy parameter γ_B, the upper bound, given by (20), is an increasing function of γ_B. The reason for this is that, whereas $\underline{\gamma}_I$ is determined by the condition that liquidity decreases under $I = I^{nc}$ for small positive values of e, the upper bound $\bar{\gamma}_I$ is determined by the condition that $\dot{e}$ is positive for negative values of liquidity close to zero. The bankruptcy parameter only becomes relevant if the firm's liquidity is negative, and therefore only $\bar{\gamma}_I$ depends on this parameter. Taking into account the expression for $\bar{\gamma}_I$ given in (21), it follows that for any given value of the R&D efficiency parameter γ_I we have $\gamma_I < \bar{\gamma}_I$ if the bankruptcy parameter γ_B is sufficiently large. Furthermore, if there is no bankruptcy risk, in the sense that $\gamma_B = 0$, then $\underline{\gamma}_I = \bar{\gamma}_I$ and $e = 0$ is a steady state under optimal R&D investment only if $\gamma_I = \underline{\gamma}_I$. Based on these arguments we can formulate the following corollary of Proposition 3, which we will use in the following section to distinguish between scenarios where a firm with positive initial liquidity either eventually accumulates debt or keeps a non-negative liquidity.

Corollary 4. *For $\gamma_I > \underline{\gamma}_I$ there exists a unique threshold $\bar{\gamma}_B > 0$ such that $e^* = 0$ is a stable steady state if and only if $\gamma_B \geq \bar{\gamma}_B$.*

Before numerically exploring in the next section in more detail the properties of the optimal investment policy and the resultant innovation rate and liquidity dynamics, we conclude this analytical section by briefly discussing the implications of a variation of the dividend rate ν_1 in mode m_1. In particular, we show in the following proposition that if for non-negative initial liquidity the firm never enters the negative liquidity domain in mode m_1, then it is optimal to delay all dividend payments till after the successful innovation.

Proposition 4. *Denote by $\tilde{\phi}(e)$ the optimal solution to the problem* (6) *under the dividend rate $\nu_1 = 0$ in mode m_1. If $\dot{e} \geq 0$ at $e = 0$, i.e. $\alpha_o^2 \geq 2\xi\tilde{\phi}(0)^2$, value function of the firm owner under $\nu_1 = 0$ (weakly) dominates the value function under any $\nu_1 > 0$.*

The intuition for this result is that while investing in R&D the firm should keep as high a liquidity as possible in order to avoid the bankruptcy risk associated with debt. Paying out dividends during mode m_1 could either make the firm go into debt or restrict its future innovation investments. Both of these effects are associated with costs for the firm and hence reduce the expected total dividend stream. Since the firm does not pay dividends if it has negative liquidity, the value of ν_1 matters only in the domain where $e \geq 0$. If the firm for $\nu_1 = 0$ optimally avoids to go into debt once it has reached a liquidity $e(t) \geq 0$, there is no bankruptcy risk and therefore no costs associated with delaying the payout of dividends to mode m_2. As discussed above, this is due to the fact that interest and discount rate coincide and that the firm has a positive income stream in mode m_2. The condition that liquidity stays non-negative under the optimal investment is crucial for the claim of Proposition 4. If initial liquidity is positive but at some point becomes negative under the optimal investment strategy, it can no longer be claimed that in general $\nu_1 = 0$ is optimal. In such a scenario it might be profitable for the owner to receive dividends before a potential bankruptcy, which would stop all dividend flows. As mentioned above, studying the optimal (liquidity dependent) dividend policy for this more complex case is not the focus of our analysis. Proposition 4 however provides some foundation for assuming in the following numerical analysis that $\nu_1 = 0$.

5 Economic Analysis

The main aim of this section is to study the influence of the bankruptcy parameter γ_B and the firm's strength on the established market α_o on the firm's investment, the expected innovation time and the bankruptcy probabilities. The form of the HJB equations does not correspond to that for a linear-quadratic problem and does not have a polynomial (exact) solution in the domain $e < 0$. This makes it challenging to get closed form solutions for the value function and the investment strategy when $\gamma_B > 0$. In order to analyze the effect of the bankruptcy threat on optimal investment, we need to numerically determine the value function of $V_1(e)$, which requires to approximate $V_2(e, \alpha_n)$ first. To achieve this goal, we resort to numerical methods. More specifically, we rely on a collocation method to calculate the approximate solution for $V_2(e, \alpha_n)$ for $e < 0$ and for $V_1(e)$ on the entire state space. Details of our numerical approach, built on Vedenov and Miranda (2001) and Dawid et al. (2015), are provided in Appendix C. A more extensive and detailed discussion of the numerical treatment of the problem at hand is provided in Banas et al. (2022). In particular, it is demonstrated there that the collocation method used in this paper yields qualitatively equivalent results in comparison with alternative methods that rely on finite difference or finite element approaches.

5.1 Parameter Calibration

Our numerical analysis is based on a standard parameter setting shown in the following table. Based on this standard parameter setting we will analyze the effects of variations of several of these parameters, in particular γ_B, and α_o.

$\nu_1 = 0$	pre-innovation dividend rate	$\delta = 1.55$	adjustment speed for α_n to reach $\tilde{\alpha}_n$
$\nu_2 = 0.2$	post-innovation dividend rate	$\sigma = 0.1$	uncertainty in new market dynamics
$\bar{\alpha}_n = 0.6$	base size of the new market	$r = 0.02$	interest rate
$\tilde{\alpha}_n = 0.8$	expansion of new market	$\gamma_I = 0.1$	efficiency of innovation
$\eta = 0.5$	horizontal differentiation	$\xi = 0.025$	invesment costs
$\lambda = 0.5$	parameter for state-space transformation		
Default values of parameter to be varied			
$\gamma_B = 0.05$	bankruptcy parameter	$\alpha_o = 0.8$	size of the old market

Table 1: Parameter values

Although this parameter setting is not based on a systematic empirical calibration for a specific industry, they have been chosen with clear theoretical and empirical foundations in mind. As mentioned above, our choice of $\nu_1 = 0$ is based on Proposition 4, and we will further discuss in Section 5.2 the choice of ν_2 does not affect any of our results.

The choice of the parameter values for α_o, $\bar{\alpha}_n$, $\tilde{\alpha}_n$ and η is guided by the aim to generate meaningful values for the resulting demand elasticity. Empirical evidence indicates that the unitary elasticity is reasonable for many established consumption goods (Anderson et al., 1997). For the established market without the influence of the new product, the chosen parameter values would yield a price elasticity before innovation as

$$-\left(\frac{\mathrm{d}p_o}{\mathrm{d}q_o}\right)^{-1}\left(\frac{p_o}{q_o}\right)\Bigg|_{q_o(m_1)} = 1,$$

and the price elasticity for the new product, in the long-run when $\alpha_n = \tilde{\alpha}_n$, is equal to

$$-\left(\frac{\partial p_n}{\partial q_n}\right)^{-1}\left(\frac{p_n}{q_n}\right)\Bigg|_{q_o(\tilde{\alpha}_n, m_2);\ q_n(q_o(\tilde{\alpha}_n, m_2))} = 1.05.$$

The parameter values for σ and δ are chosen in a way that the expected duration in mode m_2 until the new product price reaches its peak $\bar{\alpha}_n + \tilde{\alpha}_n$ is approximately 2.5 years, which is consistent with empirical observations about the time till full development of the demand for a new product in industries like the car industry (Volpato and Stocchetti, 2008).

Parameters γ_B, γ_I and ξ are calibrated such that for the default set of parameter values, the average innovation time is 2 to 2.5 years, which is consistent with empirical data about the average length of innovation projects (Behrens et al., 2017).

In the following analysis, we first calculate the value function $V_2(e, \alpha_n)$ for mode m_2. Using the estimated values $V_2(e, 0)$, we then numerically determine the (approximate) value function $V_1(e)$ in mode m_1. This allows us then to analyze the influence of γ_B and α_o on the optimal investment, the liquidity dynamics, expected innovation time and bankruptcy probability.

5.2 Post-innovation

After the firm has successfully innovated, additional investment in R&D has no value for the firm and therefore it is assumed to be zero. Hence, in mode m_2 no control has to be chosen by the firm. The value function $V_2(e, \alpha_n)$ in mode m_2 is shown in Figure 1. $V_2(e, \alpha_n)$ increases with both the market demand for new product α_n, and the liquidity e. On the half-space with $e \geq 0$, $V_2(e, \alpha_n)$ grows in a linear way with liquidity e regardless of the value of the bankruptcy risk parameter γ_B (see (13)). For $e < 0$ the bankruptcy risk has a crucial influence on the value in mode m_2 as can be seen in Figure 1. Note that without bankruptcy risk (i.e. $\gamma_B = 0$) the value function $V_2(e, \alpha_n)$ is linear in e with a slope of 1, and has the same functional form for both positive and negative liquidity reserves. However, for $\gamma_B = 0.05$, which is the case depicted in Figure 1, $V_2(e, \alpha_n)$ in the negative domain is convex-convace with respect to e and clearly below the value that would emerge for $\gamma_B = 0$. This highlights that the bankruptcy risk decreases the value, when the liquidity reserves are negative, and that the size of the negative effect of the bankruptcy risk depends in a non-linear way on the liquidity.

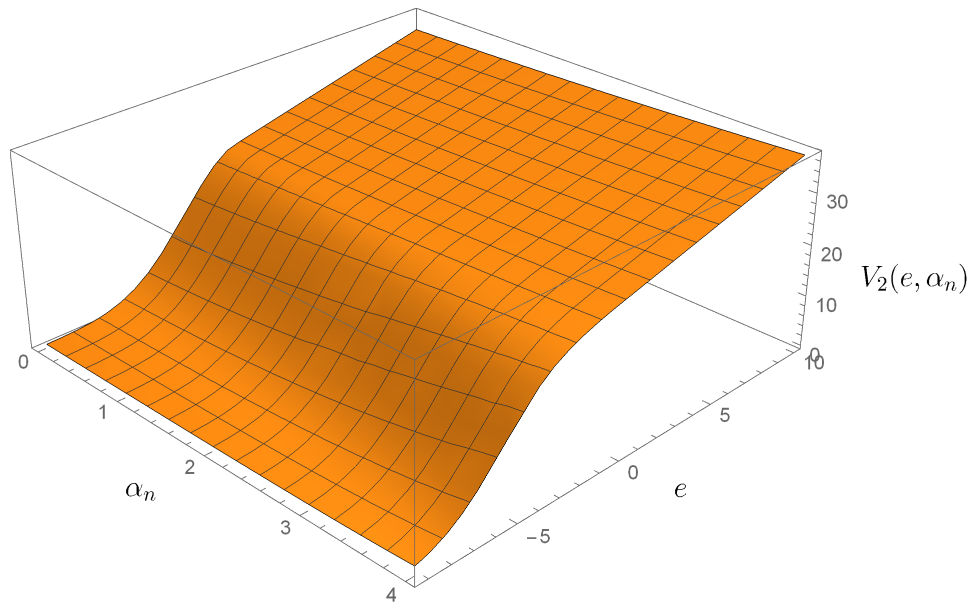


Figure 1: Value function $V_2(e, \alpha_n)$.

5.3 Pre-innovation

We now turn to the analysis of optimal investment during the innovation phase in mode m_1. First, we note our default scenario has the property that $\tilde{e} > 0$ and $\gamma_I > \underline{\gamma}_I$.[6] It's worth mentioning that Assumption 1 is satisfied for our parameter setting. Corollary 4 then implies that it depends on the value of γ_B whether $e = 0$ is a fixed point under the optimal investment strategy. If $e = 0$ is a fixed point a firm with non-negative initial liquidity never goes into debt under the optimal investment and we therefore refer to these cases as *no debt scenarios*. On the contrary we label situations where optimal investment implies that the firm shall enter the negative domain of the liquidity as *debt scenarios*.

5.3.1 Debt vs. No Debt Scenarios

Corollary 4 implies that if all other parameters are given according to their default values we have a debt scenario for $\gamma_B < \bar{\gamma}_B = 0.0069$, whereas a no debt scenario arises for $\gamma_B \geq 0.0069$. With respect to our second key parameter, α_o, the effect of a parameter variation on the occurrence of the no debt scenario is less clear cut, since both boundaries $\underline{\gamma}_I$ and $\bar{\gamma}_I$ in Proposition 3 depend in a highly non-linear way on α_o. In order to gain insights regarding how increasing the size of the established market affects the occurrence of the no debt scenario and how this effect depends on the value of the bankruptcy parameter, we show in Figure 2 the influence of γ_B and α_o on the occurrence of the no debt scenario. Specifically, the shaded area shows the combination of γ_B and α_o such that $e^* = 0$ is a steady state for the default parameter setting. In our analysis, we assume that $0.7 \leq \alpha_o \leq 2.8$ to make sure that the output quantities for both the old and the new products are non-negative after innovation. The shaded area is bounded from above by $\alpha_o = 0.992$ and below by $\alpha_o = 0.7$.[7] It can be clearly seen that for sufficiently large values of γ_B it is never optimal for the firm to go into debt, however for values of the bankruptcy parameter below approximately $\gamma_B = 0.012$ the firm avoids to go into debt only if the size of the established market is sufficiently large. Two effects, both pointing in the same direction, drive this result. First, due to the cannibalization effect the additional profit from a successful innovation, and accordingly also investment incentives, become lower as α_o grows, and, second, the profit on the established market increases with α_o and therefore the firm is able to internally finance larger investments. We summarize this discussion as our first numerical result.

Result 1. *The no debt scenario arises if the bankruptcy parameter γ_B is above a threshold $\bar{\gamma}_B$, which decreases as the profitability of the established market (α_o) increases.*

[6]Under the default parameter setting we have $I^{nc} = 4.93, \tilde{e} = 7.2$ and $\underline{\gamma}_I = 0.01 < 0.1 = \gamma_I$.

[7]For $\alpha_o \geq 0.992$ liquidity dynamics is positive at $e = 0$ even under the unconstrained investment level I^{nc}, such that $e = 0$ is no steady state regardless of the value of γ_B.

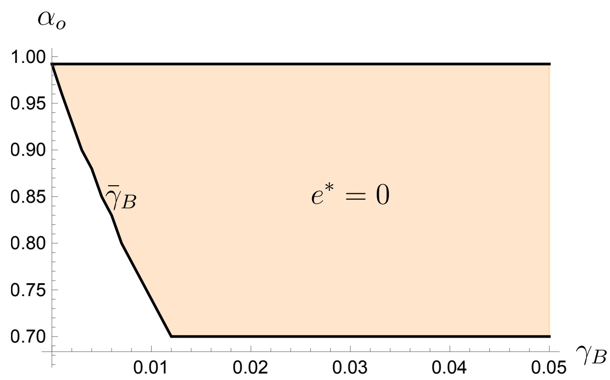


Figure 2: Combinations of the bankruptcy risk parameter γ_B and the firm's strength in the old market α_o for which a liquidity of zero is a steady state, i.e., $e^* = 0$, and therefore a no-debt scenario arises.

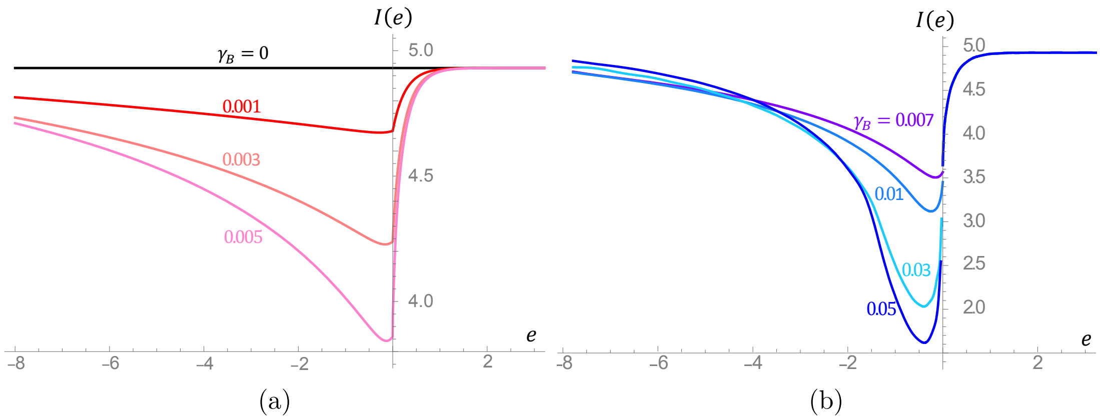


Figure 3: Effect of the bankruptcy risk parameter γ_B on the optimal investment strategy $\phi(e)$ for debt scenarios (a) and no debt scenarios (b).

5.3.2 The Effect of the Bankruptcy Risk

We are now in a position to characterize the shape of the optimal investment strategy $\phi(e)$ and to explore how this optimal strategy changes if the bankruptcy risk parameter grows. The optimal strategies depicted in this and the following sections have all been calculated based on the numerically determined value functions, as described in Appendix C.

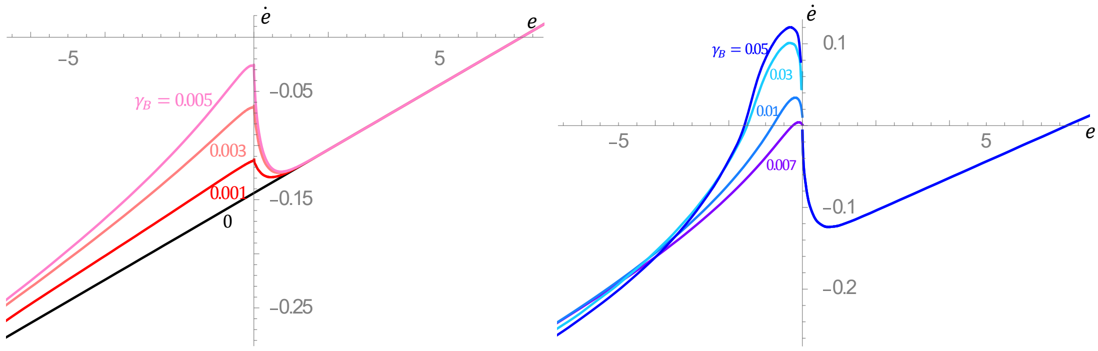
In Figure 3 we show the optimal investment strategies for our default parameter setting and different values of γ_B. As noted above, for our standard parameter setting $e^* = 0$ is a steady state whenever $\gamma_B \geq \bar{\gamma}_B = 0.0069$. In panel (a) we show the optimal investment strategy for values of γ_B below this threshold, i.e. debt scenarios, whereas in panel (b) the optimal investment strategy in no debt scenarios are depicted. The value functions $V_1(e)$ corresponding to these optimal investment strategies can be found in Appendix C.

Figure 3 illustrates our theoretical result that the optimal investment strategy exhibits a downward jump at zero liquidity in the no debt scenario. Furthermore, it shows that investment is continuous at $e = 0$ in the debt scenario. Intuitively, one might expect that even in the debt scenario investment changes discontinuously when the bankruptcy risk kicks in at $e = 0$, but under the optimal strategy the firm at a time t, when $e(t)$ is still positive, already foresees that liquidity will turn negative in the future and therefore already takes into account that current investment will influence future bankruptcy risk. Since $\tilde{e} > 0$ we have that liquidity decreases under optimal investment for all $e \in (0, \tilde{e})$, see also Figure 4.

A main insight from Figure 3 is that the optimal investment strategy $\phi(e)$ is U-shaped when $\gamma_B > 0$ for both the debt and the no debt scenarios. When liquidity is positive and large, the optimal investment is not influenced by the bankruptcy threat and the optimal investment is equal to that with no bankruptcy risks, i.e., I^{nc}, as given in equation (10). If liquidity is positive but close to 0, optimal investment is an increasing function of liquidity. For the debt scenario the firm has an incentive to delay the point in time when liquidity becomes negative and thus the bankruptcy threat arises, and the firm does this by reducing its investment as liquidity gets closer to zero. Moreover, Figure 3a shows that the larger γ_B is, the steeper is the decrease of $\phi(e)$ as liquidity approaches zero. For the no debt scenario $\phi(e)$ decreases as e approaches zero because the firm anticipates the downward jump of investment once $e = 0$ is reached, and in light of the convex investment costs smoothes this investment path by reducing investment already before the zero liquidity steady state is reached. Since in the no debt scenario liquidity never becomes negative for $e^{ini} \geq 0$, it is evident that the branch of $\phi(e)$ for $e \geq 0$ is identical across different values of γ_B. Furthermore, considering the significantly negative liquidity levels, the firm invests more the larger the negative liquidity is. The intuition for this behavior is that if the firm is deeply in debt, then there is a large probability that the firm will go bankrupt if it does not innovate quickly and thereby can generate higher profits. The amount of debt the firm holds at the time of bankruptcy does not influence owners' value (which is zero due to limited liability of owners) and therefore it is optimal to invest heavily in order to try to speed up innovation. Summarizing this discussion we get our next main result.

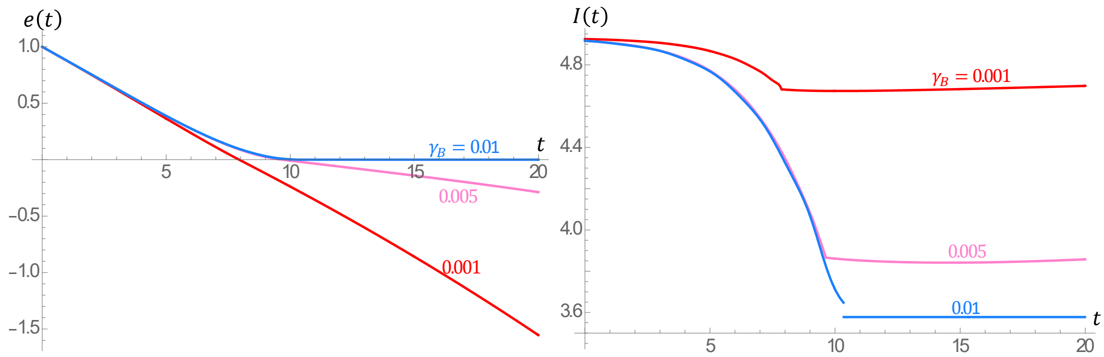
Result 2. *There is a U-shaped relationship between a firm's liquidity and its optimal investment.*

We now consider the liquidity dynamics under the optimal investment. Figure 4, which depicts $\dot{e}$ for the debt and no debt scenarios, shows that there is always a positive steady state, $e = \tilde{e}$, which is unstable. In the debt scenario this is the only steady state and the liquidity decreases for any $e < \tilde{e}$ (see Figure 4a). Hence, the liquidity diverges to $-\infty$ in the long run as long as the firm is in mode m_1, i.e. has neither innovated nor gone bankrupt. For larger values of the bankruptcy parameter γ_B, i.e. for the no debt scenario, two additional steady states emerge (see 4b). The locally stable steady state at $e^* = 0$ and an unstable negative steady



(a) (b)

Figure 4: Effect of bankruptcy risk parameter γ_B on the liquidity dynamics $\dot{e}$ for debt scenarios (a) and no debt scenarios (b).

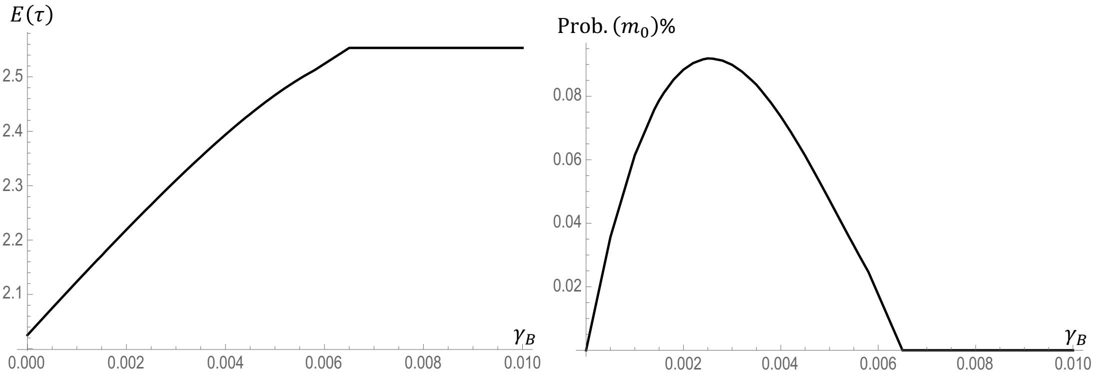


(a) Liquidity dynamics (b) Dynamics of optimal investment

Figure 5: Effect of the bankruptcy risk parameter γ_B on the dynamics of liquidity and optimal investment for an initial liquidity of $e(0) = 1$.

state constituting the lower boundary of the basin of attraction of $e^* = 0$. Hence, if the initial liquidity of the firm is negative, but the amount of debt is small, then it is optimal for the firm to choose a sufficiently small R&D investment such that its debt is reduced to zero over time.

Figure 5 illustrates these findings by showing the dynamics of liquidity and the optimal investment for an initial liquidity of $e(0) = 1$. The figure highlights that even in the debt scenario (i.e., for $\gamma_B = 0.001, 0.005$) the firm accumulates debt rather slowly, and once entering the negative liquidity domain the firm chooses an invesmtent level that is almost constant over time and substantially below the unconstrained optimal level $I^{nc} = 4.93$. For the case where the bankruptcy risk parameter is sufficiently large to induce the no debt scenario (i.e., $\gamma_B = 0.01$), the downward jump in investment, once liquidity hits zero, can be clearly seen in Figure 5b. As is illustrated in panel (a) of the figure, this downward jump indeed implies that liquidity stays constant at the steady state level of $e^* = 0$. Overall, figure 5b also illustrates that an increase of γ_B has a negative impact on the firm's level of investment throughout time, where this effect becomes more pronounced as liquidity gets close to zero.



(a) Expected innovation time $E(\tau)$ (b) Bankruptcy probability (percentage)

Figure 6: Effect of the bankruptcy risk parameter γ_B on the expected innovation time $E(\tau)$ and the bankruptcy probability for a given the initial liquidity $e(0) = 0.1$.

To conclude our analysis for the effect of an increase in the bankruptcy risk parameter we now consider the impact of γ_B on the expected innovation time and the actual ex-ante expected probability for the firm to go bankrupt. Restricting attention to scenarios with a non-negative initial firm liquidity, it follows directly from our previous analysis that, if $\gamma_B \geq \bar{\gamma}_B$, then we are in a no debt scenario, where the bankruptcy probability is zero and the exepcted innovation time does not depend on the actual value of γ_B. The latter observation is due to the fact that in the no debt scenario the level of investment for non-negative liquidity is not influenced by γ_B. This in confirmed in Figure 6a, which also shows that as long as we remain in the debt scenario the firm's expected innovation time increases with γ_B, due to the negative effect of this parameter on investment.

With respect to the bankruptcy probability an inverse U-shaped relationship with γ_B emerges (see Figure 6b). As long as γ_B is small, the direct effect of an increase of this parameter dominates, thereby leading to a higher bankruptcy probability. However, as discussed above, such an increase induces a reduction of firm investment and therefore a slower build-up of debt, which reduces the bankruptcy probability. As γ_B grows this latter effect starts to dominate and the bankruptcy probability decreases with γ_B. As γ_B crosses the threshold $\bar{\gamma}_B$, and we enter the no debt scenario, the negative effect of γ_B on investment is so strong that the firm never accumulates any debt and hence the bankruptcy probability is zero.

Result 3. *For a given initial liquidity a firm's expected innovation time increases with the bankruptcy parameter γ_B. Furthermore, there is an inverse U-shaped relationship between γ_B and the firm's bankruptcy probability.*

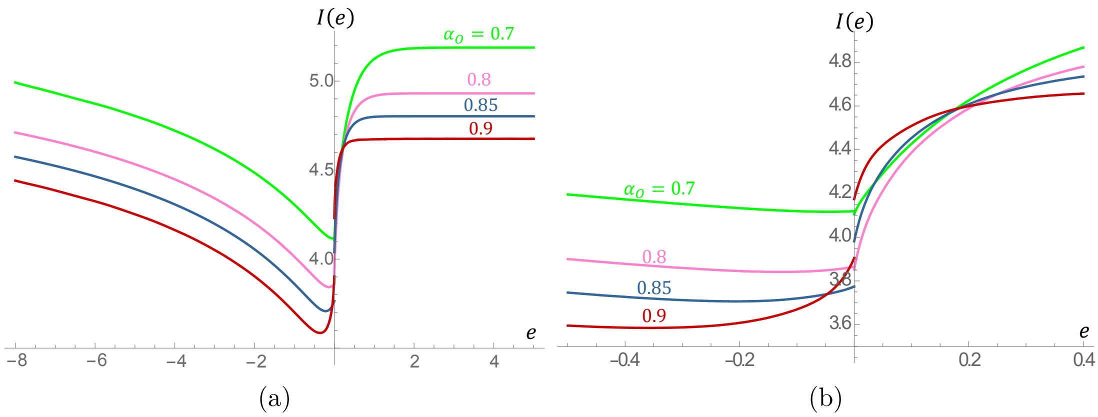


Figure 7: Effect of the established market size α_o on the optimal investment $I(e)$ for the bankruptcy risk parameter $\gamma_B = 0.005$. Panel (a) shows the entire relevant part of the state space whereas panel (b) zooms into the neighborhood of $e = 0$.

5.4 The Effect of the Profitability of the Established Market

The profitability of the established market in our model is proxied by the market size parameter α_o, which determines the quantity sold by the firm on the established market. In particular, this parameter therefore influences the firm's ability to finance innovation expenditures internally. Understanding how optimal innovation investments depend on α_o allows us to gain insight on the question under which circumstances larger sales and higher profits on the established market lead to higher R&D investments and faster innovation.

Figure 7, shows the optimal investment strategy as a function of liquidity for $\gamma_B = 0.005$ and different values of α_o. Different from panel (a) where the entire relevant part of the state space is shown, in panel (b) we zoom in to liquidity values close to zero. First, it should be noted that for the default value $\alpha_o = 0.8$ we are in a debt scenario because $\gamma_B = 0.005 < 0.0069 = \bar{\gamma}_B$. However, increasing the market size of the established market to $\alpha_o = 0.85$ lowers the threshold to $\bar{\gamma}_B = 0.0048$ such that a no debt scenario arises for $\gamma_B = 0.005$. Hence, the investment strategy is continuous at zero liquidity for $\alpha_o = 0.7, 0.8$, but exhibits a jump for $\alpha_o = 0.85, 0.9$.

Concerning the effect of α_o on the level of investment, it becomes clear that if liquidity is strongly positive or strongly negative the optimal R&D investment is smaller the larger the established market is. For large liquidity, where the bankruptcy threat hardly influences investment, this is due to a standard cannibalization effect. The larger quantity of the established product that the firm sells, the stronger negative implication the drop in the price of the established product has, which is triggered by product innovation. Hence, large sales on the established market reduce the incentive to invest in the development of the new product. This result is consistent with Dawid et al. (2015), which shows that, if investment is fully financed internally, a larger production capacity on the established market induces lower investment in

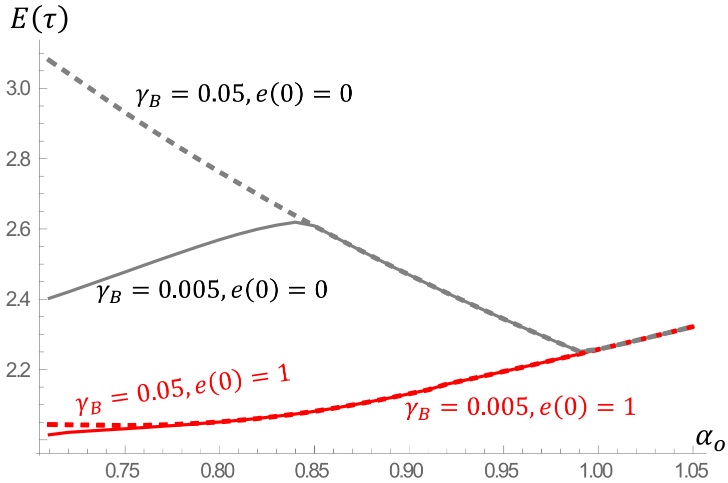


Figure 8: Effect of the profitability of the established market α_o on the expected innovation time $E[\tau]$ under bankruptcy risk parameters $\gamma_B = 0.05$ (dashed) and $\gamma_B = 0.005$ (solid line) in combination with initial liquidity $e(0) = 0$ (black) and $e(0) = 1$ (red).

new product development. The cannibalization effect is also present in the case of negative liquidity, however here it is complemented by a second effect. If the firm has negative liquidity then an increase in investment instantaneously increases the firm's bankruptcy rate. The larger the established market is the larger is the loss in expected future dividends induced by bankruptcy. Hence, an increase in α_o has a negative effect on R&D expenditures. We refer to this effect as the *bankruptcy loss effect*.

A third effect of an increase of α_o is that it pushes up the limit of the firm's expenditure that can be financed internally and therefore reduces the amount of debt needed for a certain investment size. This effect, which we label as the *financing effect*, increases the optimal size of R&D investment. A close look at the optimal investment $I(e)$ around $e = 0$ reveals that, this effect may dominate canibalization in this part of the state space. In particular, in the no debt scenario, where $e^* = 0$ is a stable steady state ($\alpha_o = 0.85, 0.9$ in Figure 7b), a larger value of α_o induces higher R&D investments. This is quite obvious in the steady state $e^* = 0$, where investment is given by $\alpha_o^2/4$, and it also holds in an interval around zero liquidity. However, in a debt scenario ($\alpha_o = 0.7, 0.8$ in Figure 7b), the cannibalization and bankruptcy loss effects dominate, and a larger size of the established market induces lower product innovation investments also around $e = 0$. Intuitively, the main difference to the no debt scenario is that the bankruptcy loss effect is present here, whereas in the no debt scenario this effect is absent for any non-negative liquidity and negligible for slightly negative liquidity, because under the optimal investment the negative liquidity quickly disappears.

The interplay of these three effects determines how an increase in the size of the established market α_o influences the firm's expected innovation time. Figure 8 depicts the expected innovation time as a function of α_o under four cases with different values of the bankruptcy risk parameter γ_B and different initial liquidity $e(0)$. Focusing first on the case where the firm's initial liquidity is zero, it follows directly from our analysis above that for $\gamma_B = 0.05$ the state $e^* = 0$ is a stable steady state for all values of $\alpha_o \in [0.7, 0.992]$, where for $\alpha_o > 0.992$ we have $\bar{\gamma}_I > \gamma_I = 0.1$. Hence, for $\alpha_o \in [0.7, 0.992]$ R&D investment is constant over time and equal to $\phi(e(t)) = \phi(0) = \alpha_o/\sqrt{2\xi}\ \forall t \geq 0$, see (22). The financing effect dominates here and optimal investment increases with α_o. Therefore, the expected innovation time decreases as α_o becomes larger, which can be seen in the dashed grey line in Figure 8. For $\alpha_o > 0.992$ the state $e = 0$ is no longer a steady state, but starting from $e = 0$ liquidity grows over time in mode m_1, where investment is constant at $\phi(e(t)) = I^{nc}\ \forall t \geq 0$. Due to the cannibalization effect I^{nc} decreases with α_o, so on this interval the expected innovation time grows when the size of the established market becomes larger. If we assume a lower value of the bankruptcy risk parameter ($\gamma_B = 0.005$), then we get qualitatively the same picture as above as long as $\alpha_o \geq 0.845$, which is the threshold where $\bar{\gamma}_I = \gamma_I$ for this value of γ_B. For $\alpha_o \in [0.845, 0.992]$ zero liquidity is a stable steady state, whereas for $\alpha_o \geq 0.992$ liquidity grows and investment is the unconstrained optimum. However, for $\alpha_o < 0.845$ we are in the debt scenario and the firm accumulates debt over time. Consistent with the intuition developed above, in this interval the expected innovation time increases with α_o since the combination of the cannibalization and bankruptcy loss effects reduces overall R&D investment. Hence, for low values of the bankruptcy risk parameter the relationship between the size of the established market and expected innovation time is characterized by a highly non-monotone tilted z-shaped pattern (the solid grey line in Figure 8). If firm's initial liquidity is sufficiently large ($e(0) = 1$), then the probability that the firm innovates before liquidity gets close to zero is so large that for most parts of the considered range of α_o values it does not matter how large the bankruptcy risk parameter is (compare the dashed and solid red lines in Figure 8). Therefore, the cannibalization effect dominates and the expected innovation time grows with α_o. Only for very low values of α_o around 0.7 the financing effect starts to have a sizeable impact. In this region the expected innovation time is clearly larger under a higher bankruptcy risk parameter and also slightly decreasing with respect to α_o. The intuition for this observation is that in light of such a small size of the established market, a large fraction of the firm's investment has to be financed from the existing stock of liquidity rather than from instantaneous profit and therefore liquidity decreases fast. Hence, the effects driving incentives around $e = 0$ become relevant with a higher probability and also with a lower associated discount factor. We summarize this discussion in our final numerical result.

Result 4. *The interplay of the cannibalization, the financing and the bankruptcy loss effect de-*

termine the impact by the profitability of the established market on the expected innovation time. For small initial liquidity and a small value of the bankruptcy parameter the relationship between the profitability of the established market on the expected innovation time is non-monotone and of tilted-z fashion.

6 Empirical Verification

In this section we empirically test the predictions of our theoretical model. We mainly focus on the suggested U-shaped relation between investments and bankruptcy risk (see Result 2), since this result is the crucial driver of our other results. We also consider the effect of the firm's profits on the established market (proxied by market share), thereby relating to our analysis and discussion connected to Result 4. Unfortunately, the available data does not allow to include proxies for the size of the bankruptcy parameter γ_B in our empirical analysis, such that we cannot test Results 1 and 3. Our empirical analysis investigates investment in the Italian manufacturing industry in 2019, without potential influence of the pandemic. We extract financial information from Bureau Van Dijk's AIDA database (accessed in October 2022), and collect a sample of more than 80,000 manufacturing companies. The decision to analyze the Italian manufacturing industry relies on the available information and the specific market structure. We have the opportunity to collect both companies' financial information and their bankruptcy risks, and to deal with a pool of small and medium sized enterprises, for which trade credits have a potentially important role in funding their business and their investments, see Cosci et al. (2020). Although our analytical model assumes monopoly power for the considered firm, which is hardly realistic for the considered manufacturing companies, the actual amount of market power on the established market is not crucial for the qualitative findings of our theoretical analysis.

6.1 The Econometric Model

We look at firm i at time t and verify the proposed relation between investment and the bankruptcy risk by studying two OLS multivariate regression models of the following form:

$$\begin{aligned} INV_{i,t} = \beta_0 + \beta_1 MS_{i,t-1} + \beta_2 NTA_{i,t-1} + \beta_3 AGE_{i,t} + \textstyle\sum_{r=1}^{25} \alpha_r INDU_{r,i,t} \\ + \textstyle\sum_{q=1}^{5} h_q AREA_{q,i,t} + \sum_{z=1}^{8} \gamma_z BR_{i,z,t-1} + \mu_{i,t} \end{aligned} , \tag{23}$$

$$\begin{aligned} INV_{i,t} = \beta_0 + \beta_1 MS_{i,t-1} + \beta_2 NTA_{i,t-1} + \beta_3 AGE_{i,t} + \textstyle\sum_{r=1}^{25} \alpha_r INDU_{r,i,t} \\ + \textstyle\sum_{q=1}^{5} h_q AREA_{q,i,t} + \beta_4 BR_{i,z,t-1} + \beta_5 BR^2_{i,z,t-1} + \mu_{i,t} \end{aligned} . \tag{24}$$

In (23) and (24) we consider the firm's investment (INV), market share (MS), bankruptcy risk (BR), size (NTA), seniority (AGE), industrial sector ($INDU$), and geographical macro area in which it is located ($AREA$). The difference between (23) and (24) concerns the representation of the bankruptcy risk. In the first model represented in (23) we use a vector of dummy variables indicating the firm's bankruptcy risk class (i.e., 8 classes), whereas in the second model represented in (24) we introduce a linear and quadratic term of a count variable representing bankruptcy risk with values between 1 and 8. The formulation (24) represents a robustness test to confirm the collected evidence on the U-shaped relation between investments and bankruptcy risk. The selection of independent variables is based on the available financial information, as well as the current literature on investment-cash flow relationship (Carreira and Silva, 2010). Note that, the information about bankruptcy risk is at short run, i.e., this information represents companies' expected bankruptcy risk in 1 year. Moreover, we adopt one period lag variables to minimize the bias due to simultaneity in the dynamics under investigation (Bottazzi et al., 2014).

The dependent variable in these two models is a representative index of companies' investment relative to their corresponding industrial sector (i.e., INV). In particular, considering the j^{th} industrial sector, denoted by a three-digit NACE,[8] this proxy is equal to the difference in net tangible assets (NTA) of firm i between time t and $t-1$, divided by the total difference in net tangible assets of that j^{th} industrial sector between time t and $t-1$.[9] Hence, for firm i in an industrial sector with n companies classified in that sector, the investment degree is equal to:

$$INV_{i,t} = \frac{NTA_{i,t} - NTA_{i,t-1}}{\sum_{k=1}^{n} (NTA_{k,t} - NTA_{k,t-1})} \ .$$

The higher the index, the higher is the investment of firm i on the market. Note that, considering the Italian manufacturing industry, investments in net tangible assets (e.g., machinery and equipment) can be seen as a good proxy for the expected degree of innovation of companies, since they represent crucial drivers of innovative outputs (Pellegrino et al., 2012).[10]

[8]NACE represents the French term "nomenclature statistique des activités économiques dans la Communauté européenne", which is the industry standard classification system used in the European Union.

[9]In our sample there are 116 industrial sectors with NACE codes according to the 3 digits classification.

[10]Note that Pellegrino et al. identify four innovative inputs as proxies of investments in innovation: in-house expenditures in R&D (i), outsourced expenditures in R&D (ii), expenditures in equipment and machinery (iii), and expenditures in intangible assets (iv). However, Pellegrino et al. work with a selected sample of observations ($< 3{,}000$ firms) with key information extracted from a survey with firms' disclosure of their innovation strategies; while in this work we focus on the whole manufacturing industry, selecting the unique reliable information that can be extracted from their balance sheet (i.e., investments in tangible assets). Indeed, according to current accountability rules, expenditures in R&D can be identified if, and only if, they are capitalized and reported among intangible assets, and there is no regulation on this specific decision (i.e., firms do not have to justify their decision to capitalize R&D costs). Failing firms capitalize intangible assets more aggressively than the non-failed firms (Jones, 2011), making this proxy inconsistent to represent the investments in innovation.

Among the explanatory variables, we use firms' market share, in terms of sales, with respect to their industrial sector (i.e., MS) as a proxy for the profitability of the firm on the established market.

The expected bankruptcy risk of firm i at time $t-1$ (i.e., BR), is represented both as a vector of dummy variables (formulation (23)) and as a count variable (formulation (24)) according to the classes of risk. For every observation, this bankruptcy risk was computed using an artificial neural network, following the assessment system of Standard & Poor's and classifying companies into 8 classes according to their expected solvency at short term (i.e., credit rating score): AAA (i.e., very high capacity to repay debts), AA (i.e., high capacity to repay debts), A (i.e., sound capacity to repay debts, which might be affected by adverse circumstances), BBB (i.e., adequate capacity of repayment, which might worsen), BB (i.e., predominantly speculative debt), B (i.e., high default risk), CCC (i.e., very high default risk) and D (i.e., failed enterprise).[11] Correspondingly, the value for variable BR ranges between 1 (expected worst creditworthiness, i.e., class D) and 8 (expected excellent creditworthiness, i.e., class AAA).

According to the current literature (e.g.,Czarnitzki (2006); Czarnitzki and Hottenrott (2011a); Peters et al. (2017); Falavigna and Ippoliti (2022b)), this stratification of company's financial health is indicative of both its expected bankruptcy risk and the difficulties that it might encounter in accessing external financial resources.[12] Moreover, our methodological approaches, based on a vector of dummy variables and a count variable, are coherent with the literature that investigates the relation between credit ratings and investments in R&D and innovation (Czarnitzki and Hottenrott, 2011b; Falavigna and Ippoliti, 2022a).

Finally, some control variables are introduced in the model. These include a representative proxy for firm size (i.e., NTA), which is equal to the total net assets at time $t-1$, and a representative proxy for firm seniority (i.e., AGE), which is equal to the number of years elapsed between the establishment of the firm and the year under investigation. Lastly, $AREA$ is a vector of dummy variables that are equal to 1 if the company is located in the q^{th} geographical macro area (i.e., North West, North East, Center, South or Islands), and 0 otherwise; while $INDU$ is also a vector of dummy variables equal to 1 if the productivity sector belongs to the

[11] Following Falavigna (2012), the present study estimates these indexes by means of a neural networks algorithm on the basis of key balance sheet information: total receivables due from shareholders, total tangible assets, total current assets, total shareholders' funds, total provisions for risks and charges, total payables, total value of production, total production costs, and total financial charges.

[12] Note that this proxy is highly correlated with the available liquidity (an index equal to financial and operating activities divided by debts), and also to the dividend payout (see Table 5 in the Appendix). According to data, on average, a company classified as "AAA" has an expected liquidity equal to 5.87, while a company classified as "CCC" equals 1.03.

Table 2: Descriptive statistics: dependent and independent variables Italian manufacturing industry (2019)

	Variable	Obs.	Mean	Std. Dev.	Min	Max
Dependent variable (Investment)	$\log INV_t$	80,669	-9.492	2.302	-16.441	0
Explanatory variables (Bankruptcy risk)	AAA_{t-1}	80,669	0.103	0.304	0	1
	AA_{t-1}	80,669	0.299	0.458	0	1
	A_{t-1}	80,669	0.176	0.381	0	1
	BBB_{t-1}	80,669	0.198	0.399	0	1
	BB_{t-1}	80,669	0.115	0.319	0	1
	B_{t-1}	80,669	0.081	0.272	0	1
	CCC_{t-1}	80,669	0.013	0.114	0	1
	D_{t-1}	80,669	0.015	0.123	0	1
Alternative explanatory variable (Bankruptcy risk)	BR_{t-1}	80,669	5.694	1.673	1	8
Control variables (Firm characteristics)	$\log MS_{t-1}$	80,669	-8.960	1.883	-17.682	0
	$\log NTA_{t-1}$	80,669	5.704	2.135	0	16.289
	$\log AGE_{t-1}$	80,669	2.675	1	0	5.030
Control variables (Fixed effects)	North-West	80,669	0.375	0.484	0	1
	North-East	80,669	0.311	0.463	0	1
	Center	80,669	0.179	0.384	0	1
	South	80,669	0.107	0.309	0	1
	Islands	80,669	0.028	0.165	0	1

r^{th} industrial sector, corresponding to a two-digit NACE code, and 0 otherwise.[13] Note that, in order to satisfy the assumptions on normality distribution, a logarithmic transformation was applied for variables INV, MS and NTA.

Table 2 shows some descriptive statistics of the dependent variable (i.e., INV) and the two key explanatory variables (i.e., MS and BR), as well as the main control ones (i.e., AGE, NTA, and $AREA$).

Table 3 presents further information on the relation under investigation, showing levels of investments according to different market shares and bankruptcy risks. The table indicates that for a larger market share, we can observe higher average levels of investments in net tangible assets. Interpreting a larger market share as an indication that the established market is more profitable for the firm, this empirical evidence is consistent with our theoretical analysis for the cases where the financing effect is dominant (see Figure 8 and its discussion). Moreover, Table

[13]Note that we adopt the 3 digit classification to estimate INV and MS, collecting more precise indexes, while we adopt the 2 digit classification as control variable. This choice is due to the necessity of controlling the number of covariates and potential collinearity among them.

Table 3: Market share and investments in net tangible assets according to different bankruptcy risks for Italian manufacturing industry (2019)

Market Share (MS)	Bankruptcy Risk (BR)								
	D	CCC	B	BB	BBB	A	AA	AAA	Total
> 0.05%	1.99%	0.30%	0.31%	0.35%	0.41%	0.48%	0.50%	0.63%	4.49%
	53	78	664	1,231	3,322	3,659	6,482	1,991	17,480
> 0.025%	1.11%	0.19%	0.18%	0.20%	0.27%	0.34%	0.35%	0.45%	0.32%
	96	132	1,294	2,300	5,441	5,466	9,504	2,918	27,151
> 0.0125%	0.82%	0.12%	0.11%	0.13%	0.18%	0.25%	0.26%	0.34%	0.23%
	177	221	2,342	3,805	8,086	7,669	13,125	3,966	39,391

Within each category of market share, the first row represents the average (unconditional) investments level, while the second row denotes the number of observations characterized according to the MS and the relative BR class (i.e., credit rating scores).

3 gives an indication of the theoretically predicted U-shaped relation between bankruptcy risk and investment level for the different pooled sub-samples of firms generated by different lower bounds on market share. Observing the number of firms in the different sub-samples, we can also detect the market structure of Italian manufacturing industry, which is mainly composed by small and medium enterprises.

6.2 Results

Table 4 shows the results of our OLS multivariate regression models with robust standard errors, and with lagged variables (1 year). All the models are statistically significant according to the F-test, i.e., the coefficients are not jointly equal to zero. Moreover, the R-squared is extremely high (i.e., 0.86), and the coefficients are all statistically significant (p-values < 0.01). Finally, the pairwise correlations, the distribution of residuals and the Variance Inflation Factors (VIFs) for the independent variables specified in the linear regression model are tested, with good results in all cases. Based on the estimated coefficients shown in Table 4 and keeping the firm with a class D as reference, Figure 9 plots the (conditional) average levels of investments in net tangible assets according to the credit rating scores for an otherwise identical firm.

In accordance with the preliminary evidence of Table 3, the figure confirms the U-shaped relation between bankruptcy risk and investments as predicted in Result 2 of our theoretical analysis. Indeed, also after controlling for other factors, firms' investments are highest when financial health is either very high (i.e., credit rating class AAA, where the firm essentially faces no bankruptcy risk) or very low (i.e., credit rating class D, where a fast product innovation is the only chance to avoid future default).

Table 4: OLS regression models with robust standard error
Italian manufacturing industry (2019)

VARIABLES	Model (23) $\log INV_t$	Model (24) $\log INV_t$
CCC_{t-1}	-0.216***	
	(0.0423)	
B_{t-1}	-0.336***	
	(0.0323)	
BB_{t-1}	-0.284***	
	(0.0317)	
BBB_{t-1}	-0.273***	
	(0.0312)	
A_{t-1}	-0.243***	
	(0.0314)	
AA_{t-1}	-0.216***	
	(0.0311)	
AAA_{t-1}	-0.180***	
	(0.0321)	
$\log MS_{t-1}$	0.196***	0.195***
	(0.00370)	(0.00369)
$\log NTA_{t-1}$	0.856***	0.856***
	(0.00278)	(0.00278)
$\log AGE_{t-1}$	0.0894***	0.0914***
	(0.00359)	(0.00359)
BR_{t-1}		-0.0936***
		(0.0120)
BR^2_{t-1}		0.0104***
		(0.00112)
Constant	-12.76***	-12.84***
	(0.0605)	(0.0610)
NACE code 2 digits $(\text{FE})_t$	Yes	Yes
Macro area (FE)	Yes	Yes
VIF	2.84	3.39
Observations	80,669	80,669
R-squared	0.864	0.864

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Turning to the other coefficients, ceteris paribus, our results point to the fact that a 10% increase in total net tangible assets can drive up investments by 8.50%, while considering seniority a 10% increase can raise investments by 0.86%. These results are coherent with the current literature and previous empirical findings, which highlight that the investments of young and small firms are more sensitive to cash flow and the access to external financial resources (e.g., Hyytinen and Väänänen (2006); Ughetto (2008); Hadlock and Pierce (2010)). Lastly, concerning sales, ceteris paribus, a 10% market share increase is associated with an increase in investments by 1.88%. As discussed above, this is an indication that on average for the considered firms the financing effect is dominant.

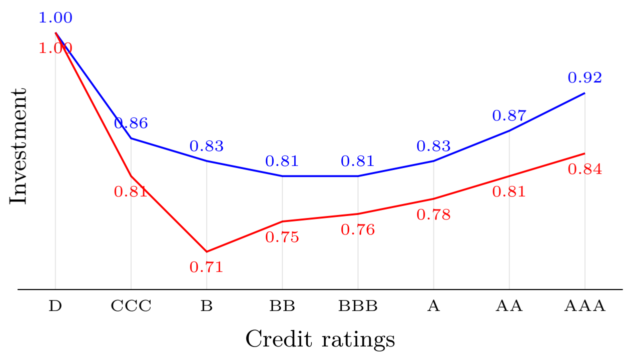


Figure 9: U-shaped relation between (conditional) average investments (at time t) and bankruptcy risk (at time $t-1$) of Italian manufacturing companies in 2019: model formulation (23) in red and (24) in blue.

7 Conclusions

This paper is one of the first to explicitly incorporate the bankruptcy risk associated with continuous investments in uncertain innovation projects in a dynamic market model. We analyze the optimal product innovation investment strategy of a monopolistic firm facing technological and demand uncertainty as well as financial frictions. The firm can finance investments externally, however faces a bankruptcy risk that grows with the size of the firm's debt. We analytically characterize scenarios in which it is optimal for the monopolist to refrain from the accumulation of any debt, thereby avoiding any bankruptcy risk, and scenarios where accumulating a positive amount of debt is optimal. Combining these insights with an extensive numerical analysis we show that the optimal investment strategy is U-shaped as a function of the firm's liquidity, such that investments are lowest around zero liquidity. We argue that this shape is driven by the interplay of three effects, the well-known cannibalization effect, the bankruptcy loss effect and the financing effect. Due to the induced adjustment of firm's investment strategy, an increase of the bankruptcy risk parameter has a non-monotone inverse U-shape effect on the actual bankruptcy probability of the firm. Finally, we show that there is a highly non-monotone relationship between the profitability of the established market for the firm and the expected innovation time under optimal investment. We empirically test our findings using Italian firm-level data and confirm the U-shaped relationship between the financial standing of a firm and its investment. Furthermore, we find a positive relationship between a firm's market share in the established market and its investment, which suggest that among the three effects we have theoretically identified the financing effect on average dominates in the considered firm population.

Our analysis has important implications for the design of the optimal product innovation strategy of firms, since it provides guidance on how to optimally account for firm's financial standing in light of financial frictions and technological uncertainty associated with product innovation. Apart from characterizing the non-monotone effect of the financial standing on optimal product innovation investment, our theoretical analysis also highlights the different qualitative effects whose interplay determines optimal investment. From the perspective of the speed of innovation, our results demonstrate that tightening firms' access to credit, e.g. due to stricter banking regulations, reduces the speed of innovation up to some level and is neutral beyond that. More importantly, our finding that the size of financial frictions might determine whether an increase in the firms' profit on the established market speeds up or slows down the introduction of new products provides a new perspective on the role of financial frictions on innovation incentives. Although we do not explicitly model competition, this insight also points out a role of the level of financial frictions with respect to the long lasting debate on the relationship between intensity of competition (which might determine the firm's profit on the established market) and innovation investment, see e.g. Aghion et al. (2005).

The framework developed in this paper can be extended in several directions, thereby allowing to address a number of important issues that were put aside in our analysis. First and foremost, we have considered a monopolistic firm and therefore have abstracted from the effect of strategic competition. On the one hand, competition should generate incentives to preempt the competitor and therefore increases the willingness of firms to take on debt. On the other hand, particularly in markets without strong patent protection, there exists risk that even after winning the innovation race the competitor might catch-up, and thereby eliminate pioneering profits. Such risk could make the accumulation of a large debt prior to innovation substantially more risky compared to the monopoly case. Addressing these issues in an oligopolistic framework of a multi-mode differential game is a natural extension to our work here. A second restriction of our analysis is that we have not fully characterized the combination of optimal investment and optimal dividend policy. Although we are confident that our qualitative insights about optimal R&D investment fully carry over to a setting where the dividend strategy is fully state-dependent, potentially singular and intertemporally optimal, determining such an optimal policy gives rise to a highly challenging control problem and it is unclear in how far clear cut results can be obtained. Finally, in this paper we have assumed that the firm has access to credit at a given interest rate even if it already has accumulated substantial debt. Alternatively, one could assume that the interest rate grows with the level of debt, that there is a maximal level of debt under which the firm still can get additional credit, or both. Whereas the addition of an upper bound for debt should hardly influence our results, endogenizing the interest rate, either assuming a competitive credit market or a potential debtor with some market power, would enrich the analysis and allow additional insights on the robustness of the U-shaped investment pattern identified here.

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A Calculation of optimal prices, quantities and profits

When the firm is in mode m_2 at time t and produces both the old product o and the new product n, the firm's profit equals[14]

$$p_o q_o + p_n q_n = (\alpha_o - q_o - \eta q_n)\, q_o + (\bar{\alpha}_n + \alpha_n - q_n - \eta q_o)\, q_n.$$

The first order conditions of the profit with respect to q_o and q_n are given by

$$\alpha_o - 2q_o - 2\eta q_n = 0 \quad \text{and} \quad \alpha_n + \bar{\alpha}_n - 2q_n - 2\eta q_o = 0.$$

So the firm's optimal quantities can be written as

$$q_o(\alpha_n, m_2) = \frac{\alpha_o - \eta(\bar{\alpha}_n + \alpha_n)}{2(1-\eta^2)} \quad \text{and} \quad q_n(\alpha_n, m_2) = \frac{(\bar{\alpha}_n + \alpha_n) - \eta\alpha_o}{2(1-\eta^2)}.$$

The corresponding prices for the old and the new products under the optimal outputs are $\alpha_o/2$ and $(\bar{\alpha}_n + \alpha_n)/2$, respectively. Thus, it can be derived that the market profits in mode 2 equals

$$\pi(\alpha_n, m_2) = \frac{(\bar{\alpha}_n + \alpha_n)^2 + \alpha_o^2 - 2\eta(\bar{\alpha}_n + \alpha_n)\alpha_o}{4 - 4\eta^2}.$$

Conducting similar calculations for the firm in mode m_1 with $q_n = 0$ yields the firm's optimal quantities as $q_o(m_1) = \alpha_o/2$. The corresponding market price is $\alpha_o/2$, and the market profits in m_1 are such that $\pi(\alpha_n, m_1) = \alpha_o^2/4$.

B Proofs

Proof of Lemma 1. Taking the derivative of both sides of HJB (8) with respect to I yields the following equation

$$\gamma_I(V_2(e,0) - V_1(e)) - \xi I \frac{\mathrm{d}V_1(e)}{\mathrm{d}e} = 0,$$

which leads to

$$I = \frac{\gamma_I}{\xi} \frac{V_2(e,0) - V_1(e)}{\mathrm{d}V_1(e)/\mathrm{d}e}.$$

Taking into account that $\frac{\mathrm{d}V_1(e)}{\mathrm{d}e} > 0$ for all e shows that also the second order optimality condition is satisfied. Furthermore, it follows from $\pi(\alpha_n, m_2) \geq \pi(0, m_1)\, \forall \alpha_n \geq 0$ with strict ineqaulity for some $\alpha_n > 0$ that $V_2(e,0) > V_1(e)$ for all e and therefore $\phi(e) > 0$. $\square$

[14]We dismiss the argument of time t when there can be no misunderstanding.

Proof of Proposition 1. Assuming that $e^{ini} \geq 0$ is sufficiently large such that $e(t) \geq 0$ for all t and the firm never faces a positive bankruptcy probability. In such a case the HJB (7) in mode 2 can be rewritten as

$$\begin{aligned} rV_2(e,\alpha_n) &= \nu_2 e + \delta(\tilde{\alpha}_n - \alpha_n)\frac{\partial V_2(e,\alpha_n)}{\partial \alpha_n} + \frac{\sigma^2\alpha_n^2}{2}\frac{\partial^2 V_2(e,\alpha_n)}{\partial \alpha_n^2} \\ &\quad + \frac{\partial V_2(e,\alpha_n)}{\partial e}\left(\frac{(\bar{\alpha}_n+\alpha_n)^2+\alpha_o^2-2\eta(\bar{\alpha}_n+\alpha_n)\alpha_o}{4(1-\eta^2)} - \nu_2 e + re\right). \end{aligned} \tag{25}$$

Assume $V_2(e,\alpha_n)$ takes the form of

$$V_2(e,\alpha_n) = a_2\alpha_n^2 + a_1\alpha_n + be + c, \tag{26}$$

with the unknown coefficients a_1, a_2, b and c that need to be determined. Substituting (26) into (25) and comparing the coefficients of $1, \alpha_n, \alpha_n^2$ and e on both sides of the equation yields the values of a_1, a_2, b and c and thus leads to the expression (13). Among the two solutions of this system of equations only the one with $V_1(0) < V_2(0,0)$ is relevant. A similar method can also be applied in mode m_1 to solve the HJB equation of (8), i.e.,

$$rV_1(e) = \nu_1 e + \frac{\mathrm{d}V_1(e)}{\mathrm{d}e}\left(\frac{\alpha_o^2}{4} + \frac{\gamma_I^2}{2\xi}\left(\frac{V_2(e,0)-V_1(e)}{\mathrm{d}V_1(e)/\mathrm{d}e}\right)^2 - \nu_1 e + re\right).$$

Assuming a value function of the form $V_1(e) = e + \tilde{c}$ yields, after the comparison of coefficients, the expression (11). Substituting equations (11) and (13) into (9) yields that the optimal investment without bankruptcy risk is equal to (10). Note that the constant term $\tilde{c}$ in the value function $V_1(e)$ is the smaller root of a quadratic equation. It follows from $V_1(e) < V_2(e,0)$ that the smaller root has to be considered.

As a last step we verify that for any $e > \tilde{e}$ indeed $e(t) > 0$ holds under the optimal investment strategy. Taking into account that liquidity is positive we have in mode m_2 that

$$\dot{e} = (r-\nu_2)e + \pi(\alpha_n, m_2).$$

Since $\pi(\alpha_n, m_2) > 0$ it follows that $\dot{e} > 0$ for sufficiently small positive values of e and therefore liquidity stays positive if it is positive at the time $t = \tau$ of the innovation. Considering m_1 we have under the optimal investment

$$\dot{e} = (r-\nu_1)e + \frac{\alpha_o^2}{4} - \frac{\xi}{2}(I^{nc})^2, \tag{27}$$

which is non-negative due to our assumptions that $e \geq \tilde{e}$. Hence, $e(t) > 0$ holds also in mode m_1. □

Proof of Corollary 1. For $2rc = \alpha_o$ we obtain

$$\xi (I^{nc})^2 = \frac{\alpha_o^2}{2} + 2\frac{\xi r}{\gamma_I}\left(\frac{r}{\gamma_I} - \sqrt{\frac{r^2}{\gamma_I^2} + \frac{\alpha_o^2}{2\xi}}\right) < \frac{\alpha_o^2}{2}$$

and therefore $\frac{\alpha_o^2}{2} > \xi (I^{nc})^2$ holds for all $\gamma_I > 0$. The unconstrained investment I^{nc} is an increasing function of c and therefore $\frac{\alpha_o^2}{2} > \xi (I^{nc})^2$ holds for all $\gamma_I \geq 0$ whenever $2rc \leq \alpha_o$. If $2rc > \alpha_o$ the value of of $\underline{\gamma}_I$ follows directly from inserting (10) into the inequality $\frac{\alpha_o^2}{2} \geq \xi \left(I^{nc}\right)^2$ and solving for γ_I. This implies that under the conditions given in the corollary we have $\tilde{e} = 0$ if $r - \nu_1 > 0$. Proposition 1 then implies that $\phi(e) = I^{nc}$ is optimal for all $e \geq 0$. Furthermore, according to (27) liquidity increases under this optimal investment for all $e \geq 0$.

For $r < \nu_1$ the equation $\dot{e} = 0$ has a unique positive solution $e^* = \frac{\alpha_o^2 - 2\xi(I^{nc})^2}{4(\nu_1 - r)}$. Considering again (27) we have $\dot{e} = \frac{\alpha_o^2}{4} - \frac{\xi (I^{nc})^2}{2} > 0$ for $e = 0$. Hence, $\dot{e} > 0$ for $e \in [0, e^*]$ and $\dot{e} < 0$ for $e > e^*$. □

Proof of Corollary 2. First it should be noted that $4rc - \alpha_o^2 > 0$, which can be verified by inserting (12) for c. Taking this into account, we have

$$\frac{\partial I^{nc}}{\partial \gamma_I} = -\frac{r}{\gamma_I^2}\underbrace{\left(\frac{2r/\gamma_I}{2\sqrt{\left(\frac{r}{\gamma_I}\right)^2 + \frac{4rc - \alpha_o^2}{2\xi}}} - 1\right)}_{<0} > 0.$$

The monotonicity of I^{nc} with respect to ξ follows directly from $4rc - \alpha_o^2 > 0$. Considering the effect of an increase of $\tilde{\alpha}_n$ it follows directly from (12) that, under the assumption $r + 2\delta > \sigma^2$ we have $\frac{\partial c}{\partial \tilde{\alpha}_n} > 0$, which implies that I^{nc} increases with $\tilde{\alpha}_n$. Finally, considering the effect of a change of α_o we obtain

$$\frac{\partial I^{nc}}{\partial \alpha_o} = \frac{1}{2\sqrt{\left(\frac{r}{\gamma_I}\right)^2 + \frac{4rc - \alpha_o^2}{2\xi}}} \frac{1}{2\xi}\left(4r\frac{\partial c}{\partial \alpha_o} - 2\alpha_o\right) < 0.$$

The sign of the term in the bracket is negative because

$$4r\frac{\partial c}{\partial \alpha_o} - 2\alpha_o \quad = \quad -\frac{4\eta\delta\tilde{\alpha}_n}{2(r+\delta)(1-\eta^2)} + \frac{2\alpha_o}{1-\eta^2} - \frac{2\eta\bar{\alpha}_n}{1-\eta^2} - 2\alpha_o$$

$$= \quad -\frac{4\eta\delta\tilde{\alpha}_n}{2(r+\delta)(1-\eta^2)} + \frac{2\eta}{1-\eta^2}\left(\eta\alpha_o - \bar{\alpha}_n\right) < 0,$$

due to our assumption that $\bar{\alpha}_n > \eta\alpha_o$. □

Proof of Proposition 2. Using the same arguments as applied in the proof of Corollary 1, it follows that for $2rc > \alpha_o$ and $\gamma_I > \underline{\gamma}_I$ we have $\frac{\alpha_o^2}{2} < \xi(I^{nc})^2$. Hence, in case the firm invests I^{nc}, we have $\dot{e}(0) = \frac{\alpha_o^2}{4} - \frac{\xi(I^{nc})^2}{2} < 0$ and the sign of $\dot{e}$ is negative for $e = 0$. Here and in what follows we use the notation $\dot{e}(e)$ to denote the value of $\dot{e}$ at liquidity level e. By continuity, if $\phi(e) = I^{nc}$ on some (small) interval $e \in (-\epsilon, \epsilon)$, then $\dot{e}(e) < 0$ holds on this entire interval. Therefore, either the firm chooses $\phi(e) < I^{nc}$ on some parts of this interval or we have $\dot{e} < 0$ on the entire interval $e \in (-\epsilon, \epsilon)$. In the latter case, for any $e^{ini} \in [0, \epsilon)$ there is a time $s > 0$ such that for any $t \geq s$ if $m(t) = m_1$ we have $e(t) < 0$. Put differently, if the firm has not innovated by time s, it has negative liquidity starting at $t = s$ till it either innovates or goes bankrupt. Since there is a positive probability that the realization of the innovation time satisfies $\tau > s$, and since $\lambda^{10} > 0$ whenever $e < 0$, this implies that there is a positive probability that the firm goes bankrupt before it moves to mode m_2. □

Proof of Proposition 3. Note that $e^* = 0$ being a stable steady state is equivalent to that, for any $\gamma_I \in [\underline{\gamma}_I, \bar{\gamma}_I]$, it holds that $\dot{e} < 0$ for a slightly positive liquidity and $\dot{e} > 0$ for a slightly negative liquidity, under the optimal investment. Since the proof is rather lengthy and technical we first provide a sketch of the main steps in our argument. In the first part of the proof we show that $\dot{e}(0)^- = \lim_{\epsilon\to 0+} \dot{e}(-\epsilon) > (=, <)\ 0$ if and only if $\gamma_I < (=, >)\ \bar{\gamma}_I$. To this end, we first show that $\dot{e}(0)^- > 0$ for all $\gamma_I < \underline{\gamma}_I$, that $\dot{e}(0)^- < 0$ for all $\gamma_I > \bar{\gamma}_I$ and that $\dot{e}(0)^- = 0$ can only hold if $\gamma_I = \bar{\gamma}_I$. We then use these insights to show that $\phi(0)^- > 0$ also has to hold on the interval $[\underline{\gamma}_I, \bar{\gamma}_I)$. It should be noted that in general we cannot be sure that $\phi(e)$ is a continuous function of γ_I, and hence this last step in not an obvious conclusion from the insights that $\dot{e}(0)^-$ is positive for $\gamma_I < \underline{\gamma}_I$ and that $\dot{e}(0)^- = 0$ can only hold if $\gamma_I = \bar{\gamma}_I$. After completion of the first part of the proof we argue that analogous arguments establish that $\dot{e}(0)^+ = \lim_{\epsilon\to 0+} \dot{e}(\epsilon) > (=, <)\ 0$ if and only if $\gamma_I < (=, >)\ \underline{\gamma}_I$. Finally, we show that for $\gamma_I \in [\underline{\gamma}_I, \bar{\gamma}_I]$ the investment level $\phi(0) = \alpha_o/\sqrt{2\xi}$ is a solution to the HJB equation at $e = 0$ in the viscosity sense. Together these steps establish the claim of the Proposition.

Turning now to the first part of the proof, i.e. showing that $\dot{e}(0)^- = \lim_{\epsilon\to 0+} \dot{e}(-\epsilon) > (=, <)\ 0$ if and only if $\gamma_I < (=, >)\ \bar{\gamma}_I$, we first note that it follows from Corollaries 1 and 2 that for $\gamma_I < \underline{\gamma}_I$ we have $I^{nc} < \frac{\alpha_o}{\sqrt{2\xi}}$ and since $\phi(e) \leq I^{nc}$ for all e, this implies $\dot{e}(0)^- > 0$.

Next we show that there is only a single value of γ_I for which under the optimal investment strategy we can have $\phi(0)^- = \phi(0) = \frac{\alpha_o}{\sqrt{2\xi}}$ and that this value is given by $\gamma_I = \bar{\gamma}_I$. Using (22), calculated from $\dot{e}(0) = 0$, and the fact the the value functions in both modes have to be continuous, we conclude from (9) that in such a scenario we must have

$$\frac{\gamma_I(V_2(0,0) - V_1(0))}{\xi \mathrm{d}V_1(0)/\mathrm{d}e^-} = \frac{\alpha_o}{\sqrt{2\xi}}. \tag{28}$$

In order to determine $\mathrm{d}V_1(0)/\mathrm{d}e^-$ we consider the HJB equation for $e < 0$ in mode m_1, which is given by

$$rV_1(e) = \frac{\mathrm{d}V_1(e)}{\mathrm{d}e}\left(re + \frac{\alpha_o^2}{4} - \frac{\xi}{2}\phi(e)^2\right) + \gamma_I\phi(e)\left(V_2(e,0) - V_1(e)\right) - \gamma_B e\left(V_0(e) - V_1(e)\right). \tag{29}$$

Taking the derivative on both sides with respect to e and considering the limit $e \to 0$, we obtain

$$\begin{aligned} 0 \;=\; & \gamma_I\frac{\alpha_o}{\sqrt{2\xi}}\left(\frac{\partial V_2(0,0)}{\partial e^-} - \frac{\mathrm{d}V_1(0)}{\mathrm{d}e^-}\right) + \gamma_B V_1(0) + \underbrace{\left(\frac{\alpha_o^2}{4} - \frac{\xi(\phi(0)^-)^2}{2}\right)}_{=0}\frac{\mathrm{d}^2V_1(0)}{\mathrm{d}e^2} \\ & + \frac{d\phi(0)}{de^-}\underbrace{\left(-\xi\frac{\mathrm{d}V_1(0)}{\mathrm{d}e^-}\phi(0)^- + \gamma_I\left(V_2(0,0) - V_1(0)\right)\right)}_{=0}, \end{aligned}$$

where the observation that the bracket in the second line is equal to zero follows from (28). Since $V_2(e,0)$ is smooth at $e = 0$[15], it holds that $\partial V_2(0,0)/\partial e = 1$ and we get

$$\frac{\mathrm{d}V_1(0)}{\mathrm{d}e^-} = \frac{\gamma_I\frac{\alpha_o}{\sqrt{2\xi}} + \gamma_B V_1(0)}{\gamma_I\alpha_o/\sqrt{2\xi}}. \tag{30}$$

Furthermore, (29) yields that the value function in mode m_1 at the steady state $e^* = 0$ equals to

$$V_1(0) = \frac{\gamma_I V_2(0,0)\alpha_o/\sqrt{2\xi}}{r + \gamma_I\alpha_o/\sqrt{2\xi}}. \tag{31}$$

Inserting this into (28) yields

$$\frac{r\gamma_I V_2(0,0)}{\xi\left(r + \gamma_I\alpha_o/\sqrt{2\xi}\right)\frac{\mathrm{d}V_1(0)}{\mathrm{d}e^-}} = \frac{\alpha_o}{\sqrt{2\xi}}$$

[15]The continuity of $\frac{\partial V_2(e,0)}{\partial e}$ for $e \to 0-$ can be seen by considering the HJB equation (7) in mode m_2, taking into account the continuity of $V_2(e,0)$ and the fact that in mode m_2 the term $\dot{e}(e,0)$ is continuous at $e = 0$, see (4).

and using (30) we obtain

$$\frac{r\gamma_I V_2(0,0)}{\xi\left(r+\gamma_I\alpha_o/\sqrt{2\xi}+\gamma_B V_2(0,0)\right)}=\frac{\alpha_o}{\sqrt{2\xi}}.$$

Solving for γ_I yields a unique solution, which, taking into account that $V_2(0,0)=c$, is given by (21). Hence, $\phi(0)^- = \phi(0) = \frac{\alpha_o}{\sqrt{2\xi}}$ can hold only if $\gamma_I = \bar{\gamma}_I$. This also implies that the only possible value of γ_I where $\phi(0)^- = \frac{\alpha_o}{\sqrt{2\xi}}$ can hold is $\gamma_I = \bar{\gamma}_I$.[16]

As a next step we now show that $\phi(0)^- > \frac{\alpha_o}{\sqrt{2\xi}}$ holds if $\gamma_I > \bar{\gamma}_I$. Consider an arbitrary fixed value of $\gamma_B = \tilde{\gamma}_B$ and arbitrary $\gamma_I > \bar{\gamma}_I(\tilde{\gamma}_B)$, where for expositional reasons in what follows we write $\bar{\gamma}_I$ explicitly as a function of γ_B. We show by contradiction that $\phi(0)^- > \frac{\alpha_o}{\sqrt{2\xi}}$ must hold. Since $\gamma_I > \bar{\gamma}_I(\tilde{\gamma}_B) > \underline{\gamma}_I$ and $I^{nc} > \frac{\alpha_o}{\sqrt{2\xi}}$ for all $\gamma_I > \underline{\gamma}_I$, the inequality $I^{nc} > \frac{\alpha_o}{\sqrt{2\xi}}$ holds for the considered parameter constellation. Note that both sides in this inequality are independent from γ_B. Assume now that $\phi(0)^- \leq \frac{\alpha_o}{\sqrt{2\xi}}$. Since $\gamma_I \neq \bar{\gamma}_I(\tilde{\gamma}_B)$ we know that this weak inequality cannot hold as equality and therefore we must have that $\phi(0)^- < \frac{\alpha_o}{\sqrt{2\xi}}$. Keeping γ_I fixed, it is straightforward to see that for any $e < 0$ we have $\lim_{\gamma_B \to 0} \phi(e) = I^{nc} > \frac{\alpha_o}{\sqrt{2\xi}}$, since the firm chooses unconstrained investment I^{nc} when there is no bankruptcy threat. Therefore, for sufficiently small $\gamma_B > 0$ we must have $\phi(0)^- > \frac{\alpha_o}{\sqrt{2\xi}}$, whereas by assumption we have $\phi(0)^- < \frac{\alpha_o}{\sqrt{2\xi}}$ for $\gamma_B = \tilde{\gamma}_B$. For a given level of liquidity $e < 0$ the optimal investment $\phi(e)$ changes continuously with γ_B. This follows from standard results about continuity of optimal investment with respect to the discount rate, since a change in γ_B is equivalent to a change in the discount rate, see (8). This implies that there must exist a value $\hat{\gamma}_B \in (0, \tilde{\gamma}_B)$ such that $\phi(0)^- = \frac{\alpha_o}{\sqrt{2\xi}}$ for $\gamma_B = \hat{\gamma}_B$. According to our arguments above, this can only hold if $\gamma_I = \bar{\gamma}(\hat{\gamma}_B)$. The threshold $\bar{\gamma}_I(\gamma_B)$ is an increasing function of γ_B (see (21)), and therefore we have

$$\gamma_I > \bar{\gamma}_I(\tilde{\gamma}_B) > \bar{\gamma}_I(\hat{\gamma}_B),$$

which contradicts $\gamma_I = \bar{\gamma}(\hat{\gamma}_B)$. Hence, we have shown that $\phi(0)^- > \frac{\alpha_o}{\sqrt{2\xi}}$ has to hold for all $\gamma_I > \bar{\gamma}_I$.

To complete the first part of the proof it remains to be shown that $\phi(0)^- < \frac{\alpha_o}{\sqrt{2\xi}}$ has to hold for all $\gamma_I \in [\underline{\gamma}_I, \bar{\gamma}_I)$. To this end, assume that $\phi(0; \tilde{\gamma}_I)^- > \frac{\alpha_o}{\sqrt{2\xi}}$ for some $\tilde{\gamma}_I \in [\underline{\gamma}_I, \bar{\gamma}_I)$, where for more clarity in this paragraph we write $\phi(e; \gamma_I)$ explicitly in dependence of parameter γ_I. Furthermore, we define $\tilde{\tilde{\phi}}(e; \gamma_I)$ and $\tilde{\tilde{V}}_1(e; \gamma_I)$ as the family of strategy and value functions

[16] This follows from the observation that $\phi(0)^- = \frac{\alpha_o}{\sqrt{2\xi}}$ implies $\phi(0) = \frac{\alpha_o}{\sqrt{2\xi}}$, which is an implication of the continuity of the value function.

which satisfy the optimality condition (9) with V_1 replaced by $\tilde{\tilde{V}}_1$ and the equation

$$r\tilde{\tilde{V}}_1(e;\gamma_I) = \nu_1 \max\{e,0\} + \frac{\mathrm{d}\tilde{\tilde{V}}_1}{\mathrm{d}e}\dot{e} + \gamma_I \tilde{\tilde{\phi}}(e;\gamma_I)\left(V_2(e,0) - \tilde{\tilde{V}}_1(e;\gamma_I)\right) + \gamma_B \max\{0,-e\}\left(V_0(e) - \tilde{\tilde{V}}_1(e;\gamma_I)\right) \tag{32}$$

for $e \le 0$ as well as the condition $\tilde{\tilde{\phi}}(e;\tilde{\gamma}_I) = \phi(0;\tilde{\gamma}_I)$ and $\tilde{\tilde{V}}_1(e;\tilde{\tilde{\gamma}}_I) = V_1(e;\tilde{\gamma}_I)$. Note that (32) corresponds to the HJB equation (8) as long as $\tilde{\tilde{\phi}}(e;\gamma_I)$ is the global maximizer of the right hand side of the HJB equation. Even if for $\gamma_I < \tilde{\tilde{\gamma}}_I$ the strategy $\tilde{\tilde{\phi}}(e,\gamma_I)$ is not (globally) optimal, the same arguments we have used in the first part of this proof to show that $\phi(0)^- \neq \frac{\alpha_o}{\sqrt{2\xi}}$ for $\gamma_I < \bar{\gamma}_I$ establish that $\tilde{\tilde{\phi}}(0;\gamma_I)^- \neq \frac{\alpha_o}{\sqrt{2\xi}}$ for all $\gamma_I < \tilde{\gamma}_I$. Since by construction $\tilde{\tilde{\phi}}(e;\gamma_I)$ is continuous with respect to γ_I, this implies $\tilde{\tilde{\phi}}(0;\gamma_I)^- > \frac{\alpha_o}{\sqrt{2\xi}}$ for all $\gamma_I < \tilde{\gamma}_I$. However, any investment satisfying the optimality condition (9) cannot exceed the unconstrained investment level, i.e. $\tilde{\tilde{\phi}}(e;\gamma_I) \le I^{nc}$ has to hold. This implies, we must have $\tilde{\tilde{\phi}}(0;\gamma_I)^- < \frac{\alpha_o}{\sqrt{2\xi}}$ for $\gamma_I < \underline{\gamma}_I$, which contradicts $\tilde{\tilde{\phi}}(0;\gamma_I)^- > \frac{\alpha_o}{\sqrt{2\xi}}$ for all $\gamma_I < \tilde{\gamma}_I$. This means that our assumption that there exists a $\tilde{\gamma}_I \in [\underline{\gamma}_I, \bar{\gamma}_I)$ with $\phi(0;\tilde{\gamma}_I)^- > \frac{\alpha_o}{\sqrt{2\xi}}$ is falsified and we have shown that also for all $\gamma_I \in [\underline{\gamma}_I, \bar{\gamma}_I)$ we must have $\phi(0;\gamma_I)^- < \frac{\alpha_o}{\sqrt{2\xi}}$.

Summarizing, we have shown that $\phi(0)^-$ has the following form:

$$\phi(0)^- \begin{cases} < \frac{\alpha_o}{\sqrt{2\xi}} & \gamma_I < \bar{\gamma}_I \\ = \frac{\alpha_o}{\sqrt{2\xi}} & \gamma_I = \bar{\gamma}_I \\ > \frac{\alpha_o}{\sqrt{2\xi}} & \gamma_I > \bar{\gamma}_I. \end{cases} \tag{33}$$

Using the notation $\phi(0)^+ = \lim_{\epsilon \to 0+} \phi(\epsilon)$ analogous arguments would show that

$$\phi(0)^+ \begin{cases} < \frac{\alpha_o}{\sqrt{2\xi}} & \gamma_I < \underline{\gamma}_I \\ = \frac{\alpha_o}{\sqrt{2\xi}} & \gamma_I = \underline{\gamma}_I \\ > \frac{\alpha_o}{\sqrt{2\xi}} & \gamma_I > \underline{\gamma}_I. \end{cases} \tag{34}$$

We are now in a position to show that for $\gamma_I \in [\underline{\gamma}_I, \bar{\gamma}_I]$ the optimal investment at $e = 0$ is given by $\phi(0) = \frac{\alpha_o}{\sqrt{2\xi}}$. Since the value function $V_1(e)$ in general can have a kink at $e = 0$, we are searching for a viscosity solution to the HJB equation, see Bardi and Capuzzo-Dolcetta (2008). Therefore, we have to show that for a continuous value function satisfying the HJB equation (8) on $e \neq 0$ and (31) for $e = 0$ the first order condition (17)

$$\phi(0) = \frac{\alpha_o}{\sqrt{2\xi}} = \frac{\gamma_I(V_2(0,0) - V_1(0))}{\xi\kappa} \tag{35}$$

holds for some $\kappa \in [\min[\mathrm{d}V_1(0)/\mathrm{d}e^+, \mathrm{d}V_1(0)/\mathrm{d}e^-], \max[\mathrm{d}V_1(0)/\mathrm{d}e^+, \mathrm{d}V_1(0)/\mathrm{d}e^-]]$ where $\mathrm{d}V_1(0)/\mathrm{d}e^-$ and $\mathrm{d}V_1(0)/\mathrm{d}e^+$ are again the one-sided derivatives. Taking into account

$$\phi(0)^- = \frac{\gamma_I(V_2(0,0) - V_1(0))}{\xi \mathrm{d}V_1(0)/\mathrm{d}e^-}, \quad \phi(0)^+ = \frac{\gamma_I(V_2(0,0) - V_1(0))}{\xi \mathrm{d}V_1(0)/\mathrm{d}e^+},$$

such a value of κ exists if and only if $\frac{\alpha_o}{\sqrt{2\xi}} \in [\min[\phi(0)^-, \phi(0)^+], \max[\phi(0)^-, \phi(0)^+]]$. Taking into account (33) and (34) this is true if and only if $\gamma_I \in [\underline{\gamma}_I, \bar{\gamma}_I]$. Stability of the steady state $e = 0$ and the claim about the jump of optimal investment for $\gamma_I \in (\underline{\gamma}_I, \bar{\gamma}_I)$ follows directly from (33) and (34). □

Proof of Proposition 4. We split the proof into two parts. First, we show our claim for the case $\gamma_I \leq \underline{\gamma}_I$, where under $\nu_1 = 0$ the firm can choose the unconstrained investment level I^{nc} without facing any bankruptcy risk as long as initial liquidity is non-negative. Second, we deal with the more complicated case of $\gamma_I > \underline{\gamma}_I$. Here, under our assumption that the optimal investment strategy $\tilde{\phi}$ for $\nu_1 = 0$ induces $\dot{e}(0) \geq 0$, we must have that $\tilde{\phi}(e) < I^{nc}$ at least for $e = 0$. Also, for this case, contrary to $\gamma_I \leq \underline{\gamma}_I$, we cannot give the value function in closed form. Hence, it is less obvious that it is never optimal to pay out some dividend already in mode m_1, thereby potentially risking to enter the zone of negative liquidity with the associated bankruptcy risk. We show our claim for this case by first establishing that under the assumptions made in the proposition for any value of ν_1 it is not optimal to enter the part of the state space with negative liquidity if initial liquidity is non-negative and then constructing for each possible investment path under $\nu_1 > 0$, an associated path under $\nu_1 = 0$, which generates at least the same value.

We first consider the case where $\gamma_I \leq \underline{\gamma}_I$. According to Corollary 1, for such small value of γ_I liquidity stays non-negative for any $e^{ini} \geq 0$ under the unconstrained optimal investment level I^{nc} and $\nu_1 = 0$. Hence, in this case $\tilde{\phi}(e) = I^{nc}$ and the firm does not have any bankruptcy risk. Accordingly the value function is equal to that of the problem without financial constraints (11). It follows from (11) that in the absence of bankruptcy risk the value function is constant with respect to the dividend rate ν_1. Since choosing a positive dividend rate ν_1 might lead to a positive bankruptcy probability this could only reduce the value and we conclude that for $\gamma_I \leq \underline{\gamma}_I$ we have $V_{1,0}(e) \geq V_{1,\nu_1}(e)$ for all $e \geq 0$. Here $V_{1,\nu_1}(e)$ denotes the value function of problem (6) under the dividend rate ν_1.

Considering now the case $\gamma_I > \underline{\gamma}_I$. In the remainder of the proof we denote by $\phi_{\nu_1}(e)$ the optimal investment strategy under dividend rate ν_1 and by $V_{1,\nu_1}(e)$ the corresponding value function. The condition stated in the proposition that $\dot{e} \geq 0$ under the optimal policy for $\nu_1 = 0$ then translates into $\phi_0(0) \leq \alpha_o/\sqrt{2\xi}$. We know from Proposition 3 and Corollary 3 that

$\phi_0(0) \geq \alpha_o/\sqrt{2\xi}$ must hold for $\gamma_I > \underline{\gamma}_I$ and therefore we must have $\phi_0(0) = \alpha_o/\sqrt{2\xi}$. Hence, we have $\dot{e}(0) = 0$ under this strategy. Now consider an arbitrary strictly positive dividend rate $\tilde{\nu}_1 > 0$. We first show that it must hold that $\phi_{\tilde{\nu}_1}(0) \leq \alpha_o/\sqrt{2\xi}$. Assume, on the contrary, that $\phi_{\nu_1}(0) > \alpha_o/\sqrt{2\xi}$. Then under strategy $\phi_{\tilde{\nu}_1}$ we have $\dot{e} < 0$ at $e = 0$. Therefore, under $\phi_{\tilde{\nu}_1}$ and $e^{ini} = 0$ liquidity would be negative for all $t > 0$ as long as we are in mode m_1. Since, no dividends are paid for $e < 0$, the expected value generated by this path would be independent from the choice of ν_1. Furthermore, since this is the optimal path under dividend rate $\tilde{\nu}_1$ from $e^{ini} = 0$, the value generated by this path must be larger than that generated by choosing a constant investment of $I = \alpha_o/\sqrt{2\xi}$ and staying at $e = 0$. However, also the value generated by this constant path is independent from ν_1 and therefore we get a contradiction to the optimality of $\phi_0(0) = \alpha_o/\sqrt{2\xi}$ for $\nu_1 = 0$. Hence, we must have $\phi_{\nu_1}(0) \leq \alpha_o/\sqrt{2\xi}$ for all $\nu_1 \geq 0$. We use this insight to show now that for any initial value e^{ini} the value under an optimal path for $\nu_1 > 0$ is (weakly) dominated by the value under an optimal path for $\nu_1 = 0$.

Consider first an arbitrary initial value $e^{ini} \geq 0$ and denote by $\tilde{e}(t)$ the liquidity trajectory under the optimal strategy, by $\tilde{D}(t) = \nu_1\tilde{e}(t) \geq 0$ the dividend stream, and by $\tilde{\Phi}(t) = \phi_{\nu_1}(e(t))$ the investment stream. Because of $\phi_{\nu_1}(0) \leq \alpha_o/\sqrt{2\xi}$, we have $\tilde{e}(t) \geq 0$ for all t. Now consider an alternative dividend and investment trajectory of the form $\hat{D}(t) = 0$ and $\hat{\Phi}(t) = \tilde{\Phi}(t)$ for all t. The corresponding expected values of the total dividend (in both modes) are denoted by $\tilde{J}$ and $\hat{J}$. Then showing that $\hat{J} \geq \tilde{J}$ proves the claim of Proposition 4.

To show that $\hat{J}(e^{ini}) \geq \tilde{J}(e^{ini})$ holds we first observe that both investment trajectories give rise to exactly the same trajectory of innovation rates, which implies that under both trajectories the distribution of the innovation time τ is identical. Hence, to establish $\hat{J}(e^{ini}) \geq \tilde{J}(e^{ini})$ it is sufficient to show that $\hat{J}_{\tilde{\tau}}(e^{ini}) \geq \tilde{J}_{\tilde{\tau}}(e^{ini})$ holds for any realisation $\tilde{\tau}$ of the stochastic innovation time. Here $\hat{J}_{\tilde{\tau}}(e^{ini})$ and $\tilde{J}_{\tilde{\tau}}(e^{ini})$ denote the values conditional on the innovation time, given by

$$\begin{aligned}\tilde{J}_{\tilde{\tau}}(e^{ini}) &= \int_0^{\tilde{\tau}} e^{-rt}\tilde{D}(t)dt + e^{-r\tilde{\tau}}V_2(\tilde{e}(\tilde{\tau}), 0),\\ \hat{J}_{\tilde{\tau}}(e^{ini}) &= e^{-r\tilde{\tau}}V_2(\hat{e}(\tilde{\tau}), 0).\end{aligned}$$

Furthermore, liquidity dynamics under the two trajectories read

$$\dot{\tilde{e}} = \frac{\alpha_o^2}{4} - \frac{\xi}{2}\tilde{\Phi}^2(t) - \tilde{D}(t) + r\tilde{e}, \text{ and } \tilde{e}(0) = e^{ini},$$

and

$$\dot{\hat{e}} = \frac{\alpha_o^2}{4} - \frac{\xi}{2}\tilde{\Phi}^2(t) + r\hat{e}, \text{ and } \hat{e}(0) = e^{ini}.$$

The difference between the two liquidity streams can be written as $\Delta e_t = \hat{e}_t - \tilde{e}_t$ and $\Delta e(0) = 0$. Then,

$$\dot{\Delta e} = \tilde{D} + r\Delta e.$$

and it follows that

$$\Delta e(t) = \int_0^t \exp\left(rt - r\rho\right)\tilde{D}(\rho)\mathrm{d}\rho.$$

Using this and noting that $V_2(e, 0)$ is linear with slope 1 in e for all $e \geq 0$ (see (13)), we obtain

$$\begin{aligned}
\hat{J}_{\tilde{\tau}}(e^{ini}) - \tilde{J}_{\tilde{\tau}}(e^{ini}) &= e^{-r\tilde{\tau}}(V_2(\hat{e}(\tilde{\tau}), 0) - V_2(\tilde{e}(\tilde{\tau}), 0)) - \int_0^{\tilde{\tau}} e^{-rt}\tilde{D}(t)dt \\
&= e^{-r\tilde{\tau}}\Delta e(\tilde{\tau}) - \int_0^{\tilde{\tau}} e^{-rt}\tilde{D}(t)dt \\
&= 0
\end{aligned}$$

Hence, $\hat{J}(e^{ini}) \geq \tilde{J}(e^{ini})$ and this completes the proof for $e^{ini} \geq 0$. Since by definition no dividends are paid as long as $e < 0$ and the value of ν_1 does not influence the liquidity dynamics for $e < 0$, this implies directly that $\hat{J}(e^{ini}) \geq \tilde{J}(e^{ini})$ also holds for any $e^{ini} < 0$. □

C Value Function in Mode m_1

Figure 10 shows the value function $V_1(e)$, calculated through our numerical procedure, for our default parameter setting and different values of γ_B.

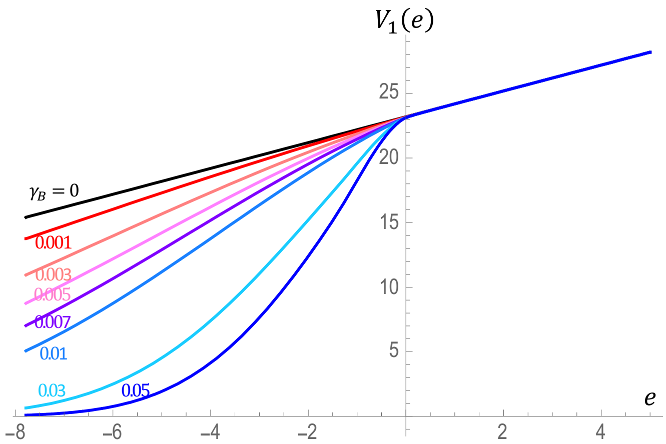

Figure 10: Effect of the bankruptcy risk parameter γ_B on the mode 1 value function $V_1(e)$

The figure shows that $V_1(e)$ increases with liquidity e, which is quite easy to understand because more liquidity induces higher dividends and a higher fraction of internally financed investments. Note that when $\gamma_B = 0$, the value function is linear with respect to e and has the functional form of (11). For a positive initial liquidity, the value function is not influenced by γ_B because there is no bankruptcy threat. Whereas for the negative liquidity, there is a positive possibility for the firm to go bankrupt, and a higher γ_B decreases the value. For $e \to -\infty$ the value function $V_1(e)$ goes to zero for any $\gamma_B > 0$ since expected time till bankruptcy goes to zero.

D Details of the Numerical Procedure

In order to numerically determine a Markov Perfect Equilibrium strategy profile for the entire game, we first need to calculate a solution for value function $V_2(\alpha_n, e)$ in mode $m = 2$ that solves the HJB equation (7). Based on the derived solution $\hat{V}_2(\alpha_n, e)$, we then numerically calculate the (approximate) value function $V_1(e)$ as the solution to the HJB equation (8) in mode m_1.

When applying the numerical method, we encounter two technical challenges. The first is that the collocation method operates on a finite state space, but in our model the state space for liquidity is infinite. The second challenge is that, the denominator term $\mathrm{d}V_1(e)/\mathrm{d}e$ in the optimal control (9) could be close to 0, especially when the initial liquidity is very negative and the bankruptcy probability is very large. This would make the optimal control I explode and the numerical calculations difficult. In order to solve these two technical problems, we propose a transformation from the state space of liquidity e to a state space of z according to $z(e) = (1 + \exp(-\lambda e))^{-1} \in (0, 1)$ with $0 < \lambda < 1$. The new state space of the problem is the interval $(0, 1)$, and therefore a bounded interval, which makes the application of the collocation method easier. Then $e(z) = \frac{-1}{\lambda} \ln\left(\frac{1}{z} - 1\right)$ and the calculations are carried out in the state space $(\alpha_n, z) \in [0, \alpha^u] \times [z_l, z_u]$ after innovation in mode m_2, and in the state space $z \in [z_l, z_u]$ before innovation in mode m_1, where $\alpha^u > \tilde{\alpha}$ is chosen sufficiently large, z_l is close to zero and z_u close to one. Note that after this transition, the dynamics read

$$\dot{z}(e) = \lambda z(e)(1 - z(e))\dot{e},$$

while $\dot{e}$ is given by (2). The denominator of the optimal investment is

$$\frac{\mathrm{d}V_1(e(z))}{\mathrm{d}z} = \frac{\mathrm{d}V_1(e(z))/\mathrm{d}e}{\lambda z(1-z)}.$$

For strongly negative liquidity e, which corresponds to $z(e)$ close to zero, both the nominator $\mathrm{d}V_1(e)/\mathrm{d}e$ and denominator $\lambda z(1-z)$ after the transition are close to 0. The technical problem associated with a small value of the derivative of the value function can thereby be alleviated. In the remainder of this section we provide a detailed description of the numerical procedure in both nodes.

D.1 Post-innovation Mode m_2

Note that for the state space of $e \geq 0$, the analytical solution of $V_2^+(\alpha_n, e)$ is (13), hence we only need to calculate $V_2^-(\alpha_n, e)$ for $e \leq 0$. Note that $V_2(\alpha_n, e)$ in our model is a continuous function in e, implying that $V_2^+(\alpha_n, 0) = V_2^-(\alpha_n, 0)$. Similar to the procedure sketched above for mode $m = 1$, we also use in m_2 a transformation from the state space of liquidity e to a state space of z according to the transformation rule $z(e) = (1 + \exp(-\lambda e))^{-1}$ with $\lambda \in (0, 1)$. The state space of $e \in (-\infty, +\infty)$ corresponds to $z \in (0, 1)$, and the negative liquidity corresponds to $z \in (0, 0.5]$. Thus, our numerical calculation is carried out in the state space $(\alpha_n, z) \in [0, \bar{\alpha}_n] \times (0, 0.5]$. From $e(z) = (\ln z - \ln(1 - z))/\lambda$, after the transition, it holds that

$$\frac{\partial V_2^-(\alpha_n, z)}{\partial z} = \frac{\partial V_2(\alpha_n, e)}{\partial e} \frac{\mathrm{d}e(z)}{\mathrm{d}z} = \frac{1}{\lambda z(1-z)} \frac{\partial V_2(\alpha_n, e)}{\partial e},$$

and the value function $V_2^-(\alpha_n, z)$ satisfies the revised HJB as

$$\begin{aligned} rV_2^-(\alpha_n, z) \;=\; & \delta(\tilde{\alpha}_n - \alpha_n) \frac{\partial V_2^-(\alpha_n, z)}{\partial \alpha_n} + \frac{\sigma^2 \alpha_n^2}{2} \frac{\partial^2 V_2^-(\alpha_n, z)}{\partial \alpha_n^2} + \frac{\gamma_B}{\lambda} V_2^-(\alpha_n, z) \ln\left(\frac{z}{1-z}\right) \\ & + \lambda z(1-z) \frac{\partial V_2^-(\alpha_n, z)}{\partial z} \left(\frac{(\bar{\alpha}_n + \alpha_n)^2 + \alpha_o^2 - 2\eta\alpha_o(\bar{\alpha}_n + \alpha_n)}{4(1-\eta^2)} + \frac{r}{\lambda} \ln\left(\frac{z}{1-z}\right) \right). \end{aligned} \tag{36}$$

In order to solve this nonlinear partial different equation, we resort to the numerical collocation method to calculate an approximate solution $\hat{V}_2^-(\alpha_n, z)$.

In a given state space $[\alpha_n^l, \alpha_n^u] \times [z^l, z^u]$ with l and u denoting the lower and the upper boundary for the corresponding interval, we first construct a sparse grid of collocation nodes $\mathcal{N} = \mathcal{N}_\alpha \times \mathcal{N}_z$, where $\mathcal{N}_\alpha = \{\alpha_n^i\}_{i=1,\ldots,n_\alpha}$ and $\mathcal{N}_z = \{z^j\}_{j=1,\ldots,n_z}$, and α_n^i and z^j are defined as

$$\alpha_n^i = \frac{\alpha_n^u + \alpha_n^l}{2} + \frac{\alpha_n^u - \alpha_n^l}{2} \cos\left(\frac{(n_\alpha - i + 0.5)\pi}{n_\alpha}\right), \tag{37}$$

$$z^j = \frac{z^u + z^l}{2} + \frac{z^u - z^l}{2} \cos\left(\frac{(n_z - j + 0.5)\pi}{n_z}\right). \tag{38}$$

Then we construct a set of basis functions $\{b_{k_\alpha, k_z}(\alpha_n, z)\}_{\{k_\alpha = 1,\ldots,n_\alpha\} \times \{k_z = 1,\ldots,n_z\}}$ corresponding to our Chebyshev sparse grid such that

$$b_{k_\alpha, k_z}(\alpha_n, z) = T_{k_\alpha - 1}\left(-1 + \frac{2\left(\alpha_n - \alpha_n^l\right)}{\alpha_n^u - \alpha_n^l}\right) \times T_{k_z - 1}\left(-1 + \frac{2\left(z - z^l\right)}{z^u - z^l}\right),$$

and function $T_k(x)$ is the Chebyshev polynomial of of degree k defined on the interval $[0,1]$. For the given state space of $[0,\bar{\alpha}_n]\times(0,0.5]$, our calculation is carried out in the space of $[0,\alpha_n^u]\times[z^l,0.5]$: $\alpha_n^l=0$ represents that $\alpha_n=0$ at the moment the mode jumps from $m=1$ to $m=2$, and $z^u=0.5$ corresponds to an upper boundary of $e=0$. In order to make sure the calculated value function is continuous at $e=0$, we specify further that

$$z^j \quad = \quad \begin{cases} \frac{0.5+z^l}{2}+\frac{0.5-z^l}{2}\cos\left(\frac{(n_z-j+0.5)\pi}{n_z}\right) & 1\le j\le n_z-1, \\ 0.5 & j=n_z. \end{cases} \tag{39}$$

The value function is assumed to take the form of

$$\hat{V}_2^-(\alpha_n,z)=\sum_{k_\alpha=1}^{n_\alpha}\sum_{k_z=1}^{n_z} c_{k_\alpha,k_z}\times b_{k_\alpha,k_z}(\alpha_n,z)=\vec{c}^\top\cdot\vec{b}(\alpha_n,z),$$

where $\vec{c}$ and $\vec{b}$ are column vectors with a length of $n_\alpha n_z$ such that $\vec{c}_k=c_{k_\alpha,k_z}$ and $\vec{b}_k(\alpha_n,z)=b_{k_\alpha,k_z}(\alpha_n,z)$ with $k=(k_z-1)n_z+k_\alpha$ for $k_\alpha\in\{1,...,n_\alpha\}$ and $k_z\in\{1,...,n_z\}$. $\vec{c}$ and $\vec{b}(\alpha,z)$ together can capture all the polynomial elements in the value function given a pair of $\{\alpha,z\}$. We aim to determine the weight vector of $\vec{c}$ such that the (approximate) value function $\hat{V}_2^-(\alpha_n,z)$ satisfies the HJB equation (36) on the collocation nodes $\{\alpha_n^i,z^j\}$ with $i\in\{1,...,n_\alpha\}$ and $j\in\{1,...,n_z-1\}$ in $\mathcal{N}$. For the other n_α nodes with $i\in\{1,...,n_\alpha\}$ and $j=n_z$ in $\mathcal{N}$, we have $\hat{V}_2^-(\alpha_n^i,z^j=0.5)=V_2^+(\alpha_n^i,e=0)$ to make $V_2(\alpha_n,e)$ continuous. In total there are $n_\alpha n_z$ number of nodes, implying $n_\alpha n_z$ number of equations.

Furthermore, for $i\in\{1,...,n_\alpha\}$ and $j\in\{1,...,n_z-1\}$ we introduce four $n_\alpha(n_z-1)\times n_\alpha n_z$ matrices $\mathbf{B}$, $\mathbf{B}^\alpha$, $\mathbb{B}^\alpha$, and $\mathbf{B}^z$ with entries

$$\mathbf{B}_{s,k}=b_k(\alpha_n^i,z^j),\quad \mathbf{B}^\alpha_{s,k}=\frac{\partial b_k(\alpha_n^i,z^j)}{\partial\alpha_n},\quad \mathbb{B}^\alpha_{s,k}=\frac{\partial^2 b_k(\alpha_n^i,z^j)}{\partial\alpha_n^2},\quad \mathbf{B}^z_{s,k}=\frac{\partial b_k(\alpha_n^i,z^j)}{\partial z},$$

where $s=(j-1)n_z+i$ denotes node s. These four matrices capture the values of all base functions and their partial derivatives at the nodes in $\mathcal{N}$ that are not on the boundary of $z^{n_z}=0.5$. For each node $\{\alpha_n^i,z^j\}$ with $i\in\{1,...,n_\alpha\}$ and $j\in\{1,...,n_z-1\}$, we define the following four column vectors in such a way that $\vec{\mathrm{g}}^{\dot{z}}$ captures the dynamics of the liquidity, $\vec{\mathrm{g}}^{\alpha^2}$ captures the quadratic of α_n, $\vec{\mathrm{g}}^{\alpha}$ captures α_n, and $\vec{\mathrm{g}}^{z}$ captures z. Specifically these four vectors read

$$\vec{\mathrm{g}}_s^{\alpha^2}=\left(\alpha_n^i\right)^2,\quad \vec{\mathrm{g}}_s^{\alpha}=\alpha_n^i,\quad \vec{\mathrm{g}}_s^{e}=\frac{1}{\lambda}\ln\left(\frac{z^j}{1-z^j}\right),$$

$$\vec{g}_s^{\dot{z}} = \lambda z^j \left(1 - z^j\right) \left(\frac{\left(\bar{\alpha}_n + \alpha_n^i\right)^2 + \alpha_o^2 - 2\eta(\bar{\alpha}_n + \alpha_n^i)\alpha_o}{4(1-\eta^2)} + \frac{r}{\lambda} \ln\left(\frac{z^j}{1-z^j}\right)\right),$$

and $s \in \{1, ..., n_\alpha(n_z - 1)\}$. Thus $\vec{c}$ has to be chosen to solve

$$r\mathbf{B} \cdot \vec{c} = \delta\tilde{\alpha}_n \mathbf{B}^\alpha \cdot \vec{c} - \delta\vec{g}^\alpha \cdot \mathbf{B}^\alpha \cdot \vec{c} + \frac{\sigma^2}{2}\vec{g}^{\alpha^2} \cdot \mathbb{B}^\alpha \cdot \vec{c} + \vec{g}^{\dot{z}} \cdot \mathbf{B}^z \cdot \vec{c} + \gamma_B \vec{g}^e \cdot \mathbf{B} \cdot \vec{c}\,. \tag{40}$$

and in addition

$$\vec{c}^\top \vec{b}(\alpha^i, 0.5) = V_2^+(\alpha^i, 0), \quad i \in \{1, ..., n_\alpha\}. \tag{41}$$

There are in total $n_\alpha n_z$ linear equations when combining (40) and (41), which can be solved using standard solvers. Note that there is no control in mode m_2, implying that solving these $n_\alpha n_z$ equations yields the solution $\vec{c}$. Thus, we can write the calculated value function in mode 2 as

$$\hat{V}_2(\alpha_n, e) = \begin{cases} \hat{V}_2^-(\alpha_n, z(e)) & e < 0\,, \\ V_2^+(\alpha_n, e) & e \geq 0\,. \end{cases}$$

D.2 Pre-innovation Mode m_1

There are several differences in mode m_1 compared with that in m_2: $V_1(\mathrm{e})$ is only defined on the domain of e, and the control is captured by

$$\phi(e) = \frac{\gamma_I}{\xi} \frac{V_2(0, e) - V_1(e)}{\mathrm{d}V_1(e)/\mathrm{d}e}.$$

In order to numerically calculate for $V_1(e)$, we carry out the same transformation as for mode m_2 from the state space of e to the state space of $z \in (0, 1)$ according to $e(z)$. After the transition, the optimal control can be rewritten as

$$\phi(e(z)) = \frac{\gamma_I}{\xi} \frac{\hat{V}_2(0, e(z)) - V_1(e(z))}{\lambda z(1-z)\mathrm{d}V_1(e(z))/\mathrm{d}z}.$$

We proceed with the same collocation method as in mode m_2, but on just one dimensional state space of z. Because the HJB in mode m_1 has different expressions for the positive and negative e, we need to calculate the value function separately for $V_1^+(z)$ with $z \in [0.5, 1)$, corresponding to $e \geq 0$, and for $V_1^-(z)$ with $z \in (0, 0.5]$, corresponding to $e \leq 0$. The value function $V_1(e(z))$ has to be continuous on the entire state space, however, there might exist a kink for $V_1(e(z))$ at $z = 0.5$ and a jump in the control function $\phi(e(z))$ because of different HJB expressions and the difference in $\mathrm{d}V_1(e(z))/\mathrm{d}e$ for positive and negative e. As has been shown in our analysis in Section4 such discontinuity can arises only if $e^* = 0$ is a stable steady state, which according to Proposition 3, happens if and only if $\gamma_I \in [\underline{\gamma}_I, \bar{\gamma}_I]$. If $\gamma_I \geq \underline{\gamma}_I]$ we are in the no debt scenario and the interval $z \in [0.5, 1]$ is invariant under the state dynamics under optimal investment. For $\gamma_I \geq \bar{\gamma}_I$ the value function V_1^+ of the problem is given in closed form by (11). For $\gamma_I \in [\underline{\gamma}_I, \bar{\gamma}_I)$. Finally, for $\gamma_I > \bar{\gamma}_I$ only the interval $z \in [0, 0.5]$ is invariant.

In any case the HJB equations on the positive domain, given by

$$\begin{aligned} rV_1^+(z) =& \nu_1 e(z) + \lambda z(1-z)\frac{\mathrm{d}V_1^+(z)}{\mathrm{d}z}\left(\frac{\alpha_o^2}{4} - \frac{\xi}{2}\phi^2(e(z)) + (r-\nu_1)e(z)\right) \\ &+ \gamma_I \phi(e(z))\left(V_2(0, e(z)) - V_1^+(z)\right), \quad z \in [0.5, 1]. \end{aligned} \tag{42}$$

and on the negative domain, given by

$$\begin{aligned} rV_1^-(z) =& \lambda z(1-z)\frac{\mathrm{d}V_1^-(z)}{\mathrm{d}z}\left(\frac{\alpha_o^2}{4} - \frac{\xi}{2}\phi^2(e(z)) + re(z)\right) \\ &+ \gamma_I \phi(e(z))\left(V_2(0, e(z)) - V_1^-(z)\right) + \gamma_B e(z) V_1^-(z), \quad z \in [0, 0.5] \end{aligned} \tag{43}$$

are solved separately in our numerical procedure.

With respect to the boundary conditions and the sequence of the numerical calculation of V_1^- and V_1^+ three cases have to be distinguished:

1. For $\gamma_I \in [\underline{\gamma}_I, \bar{\gamma}_I]$ both intervals $z \in [0, 0.5]$ and $z \in [0.5, 1]$ are invariant under the state dynamics under optimal investment and $e^* = 0$ is a stable steady state. Hence,

$$V_1^-(0.5) = V_1^+(0.5) = \int_0^\infty e^{-(r+\gamma_I\phi(e(0.5)))t}\gamma_I\phi(e(0.5))V_2^+(0,0)dt = \frac{\gamma_I\alpha_o V_2^+(0,0)}{r\sqrt{2\xi} + \gamma_I\alpha_o}\ . \tag{44}$$

   with $\phi(e(0.5)) = \phi(0) = \alpha_o/\sqrt{2\xi}$ has to hold both for V_1^- and V_1^+.

2. For $\gamma_I > \bar{\gamma}_I$ we have $\dot{e} < 0$ under optimal investment at $e = 0$. Hence, the interval $z \in [0, 0.5]$ is invariant under the state dynamics under optimal investment. Therefore, we first numerically find a function $\hat{V}_1^-$ (approximately) solving (43), where no explicit boundary conditions are imposed.[17] Then, as a second step, we numerically determine a solution of (42) with the boundary condition $V_1^+(0.5) = \hat{V}_1^-(0.5)$.

3. For $\gamma_I < \underline{\gamma}_I$ we have $\dot{e} > 0$ under optimal investment at $e = 0$. Hence, the interval $z \in [0.5, 1]$ is invariant under the state dynamics under optimal investment and the value function V_1^+ of the problem is given in closed form by (11). The value function on the

[17]Formally, we have the boundary condition $\lim_{z\to 0} V_1^-(z) = 0$ and we check in our numerical solution that $V_1^-(z^l)$ becomes small for a sufficiently small lower bound of the state interval considered in the numerical approximation of $\hat{V}_1^-$ (see below).

negative domain is determined as the solution of (43) with boundary condition $V_1^-(0.5) = V_1^+(0.5)$.

As has been explained in the main text, in this paper we only consider the first two of these three cases, since in case 3. financial constraints are irrelevant for $e(0) \geq 0$. In what follows we just describe our algorithm for the first case, the procedure in the second case is analogous.

In order to calculate an (approximate) value function $\hat{V}_1^+(z)$ that makes (42) hold on the interval $z \in [0.5, z^u)$, we first construct a set of collocation nodes $\mathcal{N}_z = \{z^j\}_{j=1,\dots,n_z}$. The idea is similar as to construct the grid in mode $m = 2$ except in mode $m = 1$ that $n_\alpha = 1$ and $\alpha^u = \alpha^l = 0$. Thus, the corresponding set of base functions is denoted by $\{b_{1,k_z}(0,z)\}_{k_z=1,\dots,n_z}$. In order to be able to incorporate the boundary condition (44) at $z = 0.5$, we further specify that

$$z^j = \begin{cases} 0.5 & j = 1, \\ \frac{z^u+0.5}{2} + \frac{z^u-0.5}{2}\cos\left(\frac{(n_z-j+0.5)\pi}{n_z}\right) & 1 < j \leq n_z. \end{cases}$$

Similarly to mode m_2 we consider an (approximate) value function of the form

$$\hat{V}_1^+(z) = \sum_{k_z=1}^{n_z} c_{k_z}^{pos} \times b_{1,k_z}(0,z) = \vec{c^{pos}}^\top \cdot \vec{b^{pos}}(z) \ ,$$

where $\vec{c^{pos}} = (c_k^{pos})_{k=1}^{n_z}$ and $\vec{b^{pos}}(z) = (b_k^{pos}(z))_{k=1}^{n_z}$ are column vectors with a length of n_z and $b_k^{pos}(z) = b_{1,k}(0,z)$. Finding the solution is equivalent to determine the vector $\vec{c^{pos}}$ such that $\hat{V}_1^+(z)$ satisfies the HJB (42) on the collocation nodes $z^j \in \mathcal{N}_z$ and $j \in \{2, ..., n_z\}$. Furthermore, $\hat{V}_1^+$ has to satsify (44) at node z^1. Thus, there are in total n_z equations and to be solved with n_z unknowns in vector $\vec{c^{pos}}$. It should be noted that, contrary to mode m_2, the right hand side of the HJB equations in this mode contain terms with the optimal control $\phi(z^j)$ at the considered node, where the optimal control function ϕ depends on the value function V_1 and its state derivative.

We use an iterative algorithm to solve this system of equations. In particular, we consider a sequence of vectors $\vec{c^{pos}}(it)$, with $it \in \{0, 1, ...\}$ is the indicator for the iterations. In iteration $it \geq 1$, we calculate for all nodes z^j in $\mathcal{N}_z$, i.e., $j \in \{1, ..., n_z\}$, the optimal control as

$$\vec{\phi}(it) = \frac{\gamma_I}{\xi}\text{Diag}\left(\left(V_2(0, \vec{g}^z) - \mathbf{B} \cdot \vec{c^{pos}}(it-1)\right) \cdot \left(\vec{g}^z \cdot \mathbf{I} \cdot \mathbf{B}^z \cdot \vec{c^{pos}}(it-1)\right)^{-1}\right), \qquad (45)$$

where $\text{Diag}(\mathbf{X})$ generates a column vector with elements on the diagonal line of $\mathbf{X}$, $\mathbf{I}$ is a $n_z \times n_z$

identity matrix, $\vec{g}^z$ is of length n_z with $\vec{g}^z_j = \lambda z^j(1-z^j)$ and $\mathbf{B}$ and $\mathbf{B}^z$ are such that

$$\mathbf{B}_{j,k} = b_k^{pos}(z^j), \quad \mathbf{B}^z_{j,k} = \frac{\mathrm{d}b_k^{pos}(z^j)}{\mathrm{d}z}, \quad j,k \in \{1,..,n_z\}.$$

Substitution of $\vec{\phi}(it)_{j\in\{2,...n_z\}}$ at node z^j into HJB (42) together with the boundary condition at node z_1 generates a system of n_z linear equations in $c^{\vec{pos}}(it)$, which can be solved by standard methods. This gives the value of $c^{\vec{pos}}(it)$ and updated optimal controls at each node under this new coefficient vector. The iteration is stoped once after inserting these updated controls into HJB equations the maximal absolute difference between the right and left hand side of (42) across nodes is sufficiently small. Overall, the numerical details can be summarized as follows:

(1) Choose n_z and calculate the nodes in $\mathcal{N}_z$. Choose the stopping criterion ϵ.
(2) Calculate $\mathbf{B}$, $\mathbf{B}^z$ and $\vec{g}^z$.
(3) Choose $c^{\vec{pos}}(0)$.
(4) Calculate the optimal control $\vec{\phi}(0)$.
(5) While the stopping criteria is not satisfied, iterate the following steps for $it = 1, 2, ...$
    (a) Calculate $c^{\vec{pos}}(it)$ by solving the combined n_z equations: (44) for node $z^1 = 0.5$ and (42) for node z^j using $\vec{\phi}_j(it-1)$, $j \in \{2, ..., n_z\}$.
    (b) Calculate the optimal control $\vec{\phi}(it)$.
    (c) Calculate the difference $\Delta_j(it)$ between left and right hand side of (44) for node z^1 and (42) for node z^j using $\vec{\phi}(it)$ and $c^{\vec{pos}}(it)$.
    (d) Checking the stopping criteria of $\max_{j\in\{1,...,n_z\}}\left[|\Delta_j(it)|/(\mathbf{B}\cdot c^{\vec{pos}}(it))_j\right] < \epsilon$.
(6) Set the value function $\hat{V}_1^+(z) = c^{\vec{pos}\top}(it)\cdot b^{\vec{pos}}(z)$ and calculate the optimal control $\phi(e(z))$ by $\hat{V}_1^+(z)$.

The numerical calculation of $\hat{V}_1^-(z)$ with $z \in (z^l, 0.5]$ is analogous to the numerical calculation for $\hat{V}_1^+(z)$ and we do not repeat the details here. The numerical approximation for value function $V_1(e)$ then reads

$$\hat{V}_1(e) = \begin{cases} \hat{V}_1^+(z(e)) & e \geq 0, \\ \hat{V}_1^-(z(e)) & e < 0. \end{cases}$$

E Empirical Analysis Tables

Table 5: Average level of dividend payout according to bankruptcy risk: Italian manufacturing industry (2015-2018)

Credit rating	D_{t-1}	CCC_{t-1}	B_{t-1}	BB_{t-1}	BBB_{t-1}	A_{t-1}	AA_{t-1}	AAA_{t-1}	Total
$\text{Dividends}_t = 0$	92%	97%	52%	28%	20%	17%	14%	9%	173085
$\text{Dividends}_t > 0$	8%	3%	48%	72%	80%	83%	86%	91%	366174
Total Firms	62833	13117	58498	69689	93634	76628	120687	44123	539259