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Double Marginalization due to External Financing: Capacity Investment under Uncertainty⋆

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Abstract

This paper considers a firm's investment decision determining the timing and capacity level in a dynamic setting with demand uncertainty. Its investment is financed by borrowing from a lender that has market power, generating a capital market inefficiency. We show that the firm's investment is subject to double marginalization in the sense that the need for external financing results in a considerably smaller investment and thus a reduction in welfare. In addition, we find that the presence of the bankruptcy option mitigates the double marginalization effect unless the bankruptcy cost is small. The firm's investment size is increasing in bankruptcy costs albeit at the expense of an investment delay. Based on this, an increase of bankruptcy costs raises social welfare.

Keywords: Double Marginalization; Uncertainty; Debt; Bankruptcy; Capacity Investment; Real Options

1 Introduction

Limited financing access is one of the main constraints for firms' investment. According to a survey by Behrens et al. (2017), 76% of German firms reported that they had to rely, at least partly, on cash flows to finance innovation in the period of 2011-2013. In terms of external financing of investments, 36% of German firms used long-term bank loans and 30% short-term oriented overdraft facilities. Especially young firms have to rely on external financing and often face lenders that have a certain degree of market power (Ryan et al. (2014); Schwert (2020)). Despite of this, the effect of market power of lenders on the investment behavior of firms is underinvestigated.

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The agenda of this paper is to analyze how market power of lenders influences investment decisions of firms, which rely on external financing and at the same time face uncertainty about the future evolution of demand on the product market. More precisely, we consider the problem of a monopolistic firm that holds an option to invest in the build-up of production capacity in a stochastically growing market. To finance the investment it needs to enter the lending market, and it is assumed that the suppliers of debt capital, i.e. the lender firms, have market power. In particular, the lender offers a coupon scheme to the firm. Once the firm accepts the deal, it determines the size of its investment (and hence the size of the loan) and starts repaying coupons to the lender. The coupon rate depends on the market size at the time of investment. Employing a continuous-time real options approach, we analyze this problem with and without bankruptcy option for the firm. Due to the coupon payments being fixed and the demand being stochastic, situations can arise where it is optimal for the firm to default. When this happens, the lender takes over the firm where the value of the firm is reduced due to bankruptcy costs.

In this setup we address the following main research questions. How does an optimally chosen coupon scheme by the lender affect size and timing of the firm's investment? How does the option for the firm to declare bankruptcy and terminate the coupon payments influence the optimal coupon scheme and the induced investment pattern? What is the role of the costs that the lender faces when taking over the firm's business upon bankruptcy? How does the interplay of lender market power and the risk of bankruptcy influence the evaluation of the investment size and timing from a welfare perspective?

Our main results are as follows. Above all, we find that market power on the lending market has a *double marginalization effect* on the firm's investment decision. In general, double marginalization arises when there are vertically separated firms in the supply chain, and the upstream firm sells an input product to the downstream firm with a profit margin, i.e., the wholesale price exceeds the cost of the input. The downstream firm needs to earn back this wholesale price, which then results in a higher output price, and thus a lower quantity, lower industry profit and lower welfare, compared to the integrated monopoly case. In the context of our setting the double marginalization effect arises from the lender setting a coupon rate above the cost of capital (which corresponds to the risk-free interest rate) and the firm reacting strategically to this behavior of the lender. To the best of our knowledge, this paper is the first to connect the double marginalization characteristic with financial markets.

In our investment decision problem, double marginalization is affected by two dimensions: next to quantity there is also timing. In the non-bankruptcy option case, our setting with a separate lender and firm results in the same timing but a lower investment size, compared to the integrated monopoly case.[1] The firm chooses a lower investment size because the lender's coupon rate, charged in return for providing debt funds, exceeds the risk-free interest rate, so that the firm's marginal cost of investment is higher. The implication is the usual double marginalization effect with higher output price, and lower welfare.

In the bankruptcy option case, we find that in addition to investment size, also timing is affected by the lender's market power. Therefore, we evaluate the effect of double marginalization from the perspective of welfare. In that light, we find that the double marginalization effect is mitigated because the firm invests more compared to the non-bankruptcy case, albeit at the expense of an investment delay.[2] The reason that the firm invests more follows from the fact that the downside potential of demand uncertainty is being limited by the availability of the bankruptcy option.

Considering the implications of an increase in bankruptcy costs we arrive at the surprising result that an

[1] Huisman and Kort (2015) analyze this problem for the case where the firm is assumed to be able to finance the investment internally at the risk-free rate. Hence, in the context of our work, their model can be interpreted as the integrated monopoly case and their results are used as the benchmark in our analysis.

[2] This result does not hold if bankruptcy costs are close to zero and demand uncertainty is not too high.

increase in such costs yields larger investment size and a gain in welfare. The reason for this finding is that the incentive for the lender to avoid bankruptcy of the firm increases and therefore, for a given market size, the coupon rate set by the lender decreases with increasing bankruptcy costs.

Our paper relates to different streams of literature, which we discuss below. First, we provide evidence that especially start-up firms are in need of *external financing*. Second, we review contributions dealing with *market power on the lending market*. Third, we provide a short overview of the *double marginalization* literature. Fourth, we list *real options* papers that take debt financing into account, and we state what we add to this literature. Finally, we consider real options contributions considering *vertically related firms*.

First, we consider the role of *external financing* for firms' investment. When a firm is already operating on the market, it happens frequently that investments are financed internally (Chandy and Tellis, 2000). Such internally financed investment could help firms to, for instance, expand their current production capacity (Willems and Zwart, 2018). However, especially for a start-up firm, investments usually have to be financed externally via bank loans or through the capital market. Recent empirical studies show that small and young firms heavily rely on debt financing (Robb and Robinson (2014), Hochberg et al. (2018), Davis et al. (2018)). Start-ups usually lack stable cash flows or collaterals but rely greatly on intangible resources (Hall, 2002). In some countries there exist loan guarantee programs to ease the access to financial resources (Minniti, 2008), like, e.g., the Italian Start-up Act (Giraudo et al., 2019). Wong and Yu (2022) analyze the effect of protecting the lender with the use of credit default swaps (CDS). They find that with this setup firms can raise more debt, but it also increases default rates and thereby investments are postponed. Venture capital is another relevant financing resource for capital investment (see e.g., Kortum and Lerner (2000)). The difference between these two financing sources is that venture capital appreciates the high-risk projects with high returns and is typically relevant for expanding established firms, whereas the bank lender appreciates start-ups with a steady and foreseeable growth path (Giraudo et al., 2019). A report on access to business finance by the OECD reveals, however, that in the EU only 0.8% of start-ups is financed by venture capital, whereas loans form the second most popular source (18.1%), after financial assistance from family and friends (26.6%) (OECD/EC, 2014).[3] Yet another way of external financing is to issue new equity. We refer to Sethi and Taksar (2002), which consider the problem of finding an optimal mix of retained earnings and external equity to maximize the value of a corporation in a stochastic environment.

An important aspect of our work is that we explicitly model *market power on the lending market*. The latter characteristic has been well recognized in the literature. Ryan et al. (2014) construct the bank Lerner indices for a sample of 20 European countries and obtain that the Lerner Index ranges from 0.27 to 0.63. Rajan (1992) finds that the bank can "hold up" the borrower, i.e., if a borrower seeks to switch banks, it may be deemed as a "lemon" regardless of its true financial condition. In this way, the bank is able to charge higher interest rates. Schenone (2009) supports that information asymmetry grants the lending banks an information monopoly compared to prospective lenders. Hale and Santos (2009) and Santos and Winton (2008) provide empirical evidence that the bank lends at lower interest rates when firms have access to the public bond market. Schwert (2020) concludes from an empirical investigation that banks earn a large premium relative to the bond-implied credit spread, and questions the nature of competition in the loan market. Petersen and Rajan (1995) obtain that, although banks charge higher rates when they have monopoly power, they also extend loans to riskier young firms because their future rents on the survivors make up for the additional failures. Our paper extends this literature by designing a *dynamic theoretical framework* to analyze the effects of the lender's market power and bankruptcy risk on the investment decision of the borrower.

[3] In their theoretical analysis, Kumar et al. (2020) show how crowdfunding might induce efficiency gains and characterize how the design of crowdfunding contracts depends on financial constraints.

As stated above, we believe that the present paper is the first to connect the *double marginalization* characteristic with financial markets. There is abundant literature on double marginalization. Spengler (1950) states that if the downstream firm in a successive monopoly setup adds its own markup to the markup of the upstream firm, the final price will be higher compared to where one monopolist sells directly to consumers. Tirole (1988) refers to such a "vertical chain monopoly" or a "successive monopoly" as a double marginalization problem. This has a direct link with our work in that in our model the bank as upstream supplier owns capital at the cost of the risk-free interest rate and lends capital at a larger coupon rate/lending rate to the downstream producer. Thus, we can characterize capital as the intermediate input. Among the many recent contributions on double marginalization, Qu and Raff (2021) discuss how the interplay of uncertainty and double marginalization affect the occurrence of the bullwhip effect in supply chains. There are also studies on double marginalization in a dynamic framework, see. e.g. Roy et al. (2019) and Desai et al. (2010). They consider a two-stage setting, where in each stage the upstream firm produces and sells to the retailer, and then the retailer sells to the customers. Anand et al. (2008) argue that, as the number of periods increases, the qualitative results from the two-period model still hold. A dynamic treatment of double marginalization in a continuous time setting is for example provided by El Ouardighi et al. (2016), again in a supply chain context. In spite of the extensive literature on the double marginalization phenomenon in fields like industrial organization and anti-trust (see Linnemer (2022) for a recent discussion), to our knowledge, this occurrence has not been addressed in the framework of a finance problem. Our paper attempts to fill this gap, in a stochastic dynamic setting.

Next, we like to mention what we add to the *real options* literature. The real options framework is designed to study investment decisions under uncertainty (Dixit and Pindyck (1994), Trigeorgis (1996)), where the standard approach is to establish the optimal time to invest in a project of a given size. However, as also recognized in the present work, in general, an investment decision is not only about timing but also about size. Pindyck (1988) is an early reference developing a real options model with capacity choice. By extending the monopoly setup of Bar-Ilan and Strange (1999) and Dangl (1999), Huisman and Kort (2015) consider the optimal investment timing and capacity choice in a monopoly as well as a duopoly framework, albeit within a perfect capital market, as mentioned above.

Several contributions consider an imperfect capital market by introducing debt financing into the real options framework (see, e.g., Mello and Parsons (1992), Hennessy and Whited (2005), Sundaresan and Wang (2007), Pawlina (2010), and Ritchken and Wu (2021)). A general finding is that, when the investment decision is just about the timing, risky debt provision makes firms endogenously more impatient and accelerate the investment timing (Boyle and Guthrie (2003), Mauer and Sarkar (2005), Lyandres and Zhdanov (2010)). In case that a firm has the flexible technology, i.e., to suspend (resume) production when prices drop below (rise above) the marginal cost, it invests earlier and uses less debt financing (Ritchken and Wu, 2021). However, if also the investment size is endogenous, Sarkar (2011) obtains that debt financing results in a delayed but larger investment. Similarly, Lukas and Thiergart (2019) find that levered firms invest more than unlevered firms, and that the corresponding investment threshold may be higher than that of their unlevered counterpart. Furthermore, for a firm that is in early stages of its life cycle, Bolton et al. (2019) shows that the risky debt also leads to over-investment because of the incentive to build up the cash-flow generating capacity. Lin (2022) on the other hand finds that such over-investment could be due to the incentive to increase the success of a firm's R&D project and thus to increase the chance of retaining the project value.

Compared to this literature, the new feature of our work is that we consider a lender who explicitly pursues its own objective regarding the investment decision, i.e., timing and size, and uses its market power to influence the firm's investment decision. For instance, if the lender prefers the firm not to invest, it can charge a large coupon rate so that acquiring capital is too expensive for the firm. On the other hand, when the lender prefers early investment it could charge less for lending, but then it is still possible that the firm prefers to wait with investing. The main results of our work, listed above, make clear that the lender's market power has a substantial effect on the firm's investment decision.

Finally, the present paper extends the real options contributions that consider *vertically related firms*. Pennings (2017) studies ex-post (investment) bargaining between a firm and its (downstream) buyer, whose activities enable the firm to sell on the market. It is shown that, the lower the bargaining power of the firm, the longer it waits to invest. Due to a positive value of waiting, the firm does not necessarily under-invest. A more related contribution to this research work is de Villemeur et al. (2014), where an upstream supplier of an input used in the production process of the downstream firm exercises market power, resulting in a price exceeding the cost of the input. Then the downstream firm decides on investment timing. This setup has been extended in Zormpas (2020) by assuming asymmetric information concerning the upstream supplier's production cost of capital, and in Zormpas and Agliardi (2021) by considering a stochastic production cost of capital. A common result in these contributions is that upstream market power delays investment. In our case, besides timing, the investment decision also involves size. We find that, in the absence of the bankruptcy option, upstream market power reduces investment size but leaves the timing of the investment unaffected. Furthermore, adding the bankruptcy option mitigates the size effect at the expense of a later investment, given a sufficient presence of bankruptcy cost.

The paper is organized as follows. Section 2 presents the model and thus formulates the problem for the firm and the lender. Section 3 derives the optimal decisions for both the firm and the lender, where it separately considers the influence of the bankruptcy option and market uncertainty. Section 4 conducts a further robustness analysis and focuses on the effect of the bankruptcy costs. Section 5 concludes. The proofs for all propositions can be found in Appendix.

2 Model

Consider the situation of a risk-neutral value-maximizing monopolist that has the option to enter a new market through undertaking investment and a (private) lender that has the opportunity to provide external financing to the firm. Assuming that the firm has no equity, it fully relies on debt to finance the investment. The debt structure considered in this paper takes the form of coupon payments by the firm to the lender in exchange for a lump-sum amount upon investment that covers the cost of investment. Coupon payments are incurred after investment.

When making the investment decision, the firm has to decide when and how much to invest, where the latter determines the production capacity. The lender decides on the coupon rate to be charged. For a stipulatory coupon rate, the firm has to repay the coupon to the lender, until the firm defaults. In case the firm defaults, the lender takes over the firm and incurs bankruptcy costs corresponding to a certain proportion of the firm value at the time of default.

Market Environment Denote by $I \geq 0$ the scale of investment by the firm measured in units of production capacity and denote by $Q(t)$ the firm's output at each time $t \geq 0$. The market is characterized by the inverse demand function that reads

$$p(t) = x(t)(1 - \eta Q(t)), \quad \text{with} \quad \mathrm{d}x(t) = \mu x(t)\mathrm{d}t + \sigma x(t)\mathrm{d}\omega(t).$$

Here, $p(t)$ denotes the market-clearing price, $\eta > 0$ denotes the price sensitivity parameter, and $x(t)$ is an exogenous shock process, which we label as the *level of demand*. The process $(x(t))_{t\geq 0}$ is governed by a geometric Brownian motion (gBm) with trend μ and volatility parameter σ. The term $\mathrm{d}\omega(t)$ represents the increment of a Wiener process. Further, denote by X the initial value of the demand level, i.e., $X = x(0) \geq 0$.

The investment cost for the lender of the funds provided to the firm is linear in the scale of investment where δ is the unit investment cost, i.e. the total cost of investment equals δI. Denote by ρ the coupon rate so that instantaneous profits for the firm, after investment has been undertaken, are given by

$$\pi(x(t), I; \rho) = \sup_{0 \leq Q \leq I} x(t)(1 - \eta Q)Q - \rho\delta I,$$

for $t \geq 0$.

The quantity maximizing the revenue part is given by $Q = \frac{1}{2\eta}$. It follows that the optimal investment size satisfies $I \leq \frac{1}{2\eta}$, so that it is always optimal for the firm to produce $Q(t) = I$. Therefore, hereafter, we will replace $Q(t)$ by I. Discounting is done under fixed risk-free rate r, where we make the usual assumption that $r > \mu$.

Strategies and Equilibrium Concept Both the lender and the firm use Markovian feedback strategies, thereby conditioning their actions on the current demand level $x(t)$, and the lender chooses its strategy first and the firm reacts to that. At $t = 0$, the lender offers a scheme $\rho(X)$ determining the coupon rate if the firm invests at a demand level $X \geq 0$. The lender stays committed to this scheme throughout the game, making it the Stackelberg leader. The firm takes this scheme into account when subsequently deciding on the timing and scale of investment. In particular, the firm determines a set $S \subseteq \mathbb{R}_+$ in which it invests, given the coupon scheme $\rho(\cdot)$ offered by the lender, and it determines an investment quantity $I(x, \rho(x))$ for each $x \in S$. Once the firm has invested at some demand level $X \in S$, and thereby the size of investment $\tilde{I} = I(X, \rho(X))$ and the coupon rate $\tilde{\rho} = \rho(X)$ have been fixed, it has the option to go bankrupt. This involves determining for each combination of $\tilde{I}$ and $\tilde{\rho}$, a bankruptcy threshold X_B such that the firm defaults once demand falls below this level. In our analysis we consider a Markov Perfect Equilibrium for our Stackelberg framework and adopt a backward induction approach to obtain it. To do so, we first characterize the bankruptcy threshold of the firm as a function of $\tilde{I}$ and $\tilde{\rho}$, then determine the firm's investment decision given a coupon scheme $\rho(\cdot)$, and finally derive the lender's optimal coupon scheme.

Problem of the Firm Once investment is undertaken, the firm is assumed to operate until it defaults. In line with the literature (see, e.g., Leland (1994)), bankruptcy is modeled as an optimal stopping problem given by

$$\tau_B(X, \tilde{I}, \tilde{\rho}) = \arg\sup_{\tau \geq 0} \mathbb{E}_{\tau_F} \Big\{ \int_{\tau_F}^{\tau_F + \tau} \pi(x(t), \tilde{I}; \tilde{\rho}) e^{-rt} \mathrm{d}t \;\Big|\; x(\tau_F) = X \Big\},$$

where τ_F denotes the firm's investment time and X the demand level where the firm undertakes investment. Here, τ_B denotes the (stochastic) bankruptcy time with investment size $\tilde{I}$ and coupon rate $\tilde{\rho}$, both fixed at the time of investment. Hence, τ_B is given by the first hitting time $\tau_B(X, \tilde{I}, \tilde{\rho}) = \inf\{t \mid x(t) \leq X_B(\tilde{I}, \tilde{\rho}),\ x(0) = X\}$.[4] Note that τ_B can be interpreted as the time elapsing between investment and bankruptcy. Figure 1 depicts the timeline of our problem.

[4] Following, e.g., Dixit and Pindyck (1994), the optimal stopping problem can be written in terms of the state process. Then, as we prove later in Proposition 2, under the optimal behavior the state space can be divided into two regions: for $X > X_B(\tilde{I}, \tilde{\rho})$, for some bankruptcy threshold $X_B(\tilde{I}, \tilde{\rho})$, the firm remains active in the market and for $X \leq X_B(\tilde{I}, \tilde{\rho})$ the firm defaults.

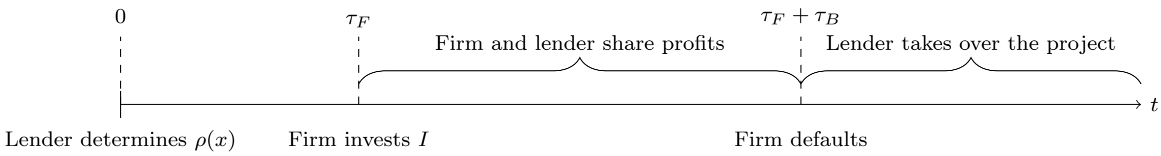


Figure 1: Illustration of the timeline for our model.

Consequently, at the moment of investment the firm's net present value is given by

$$\tilde{J}_F(X,\rho(\cdot),I) = \mathbb{E}_0\Big\{ \int_0^{\tau_B(X,I,\rho(X))} e^{-rt}\pi(x(t),I;\rho(X))\mathrm{d}t \;\Big|\; x(0)=X \Big\}. \tag{1}$$

Given a scheme $\rho(\cdot)$, the firm's optimal capacity size follows from

$$\arg\sup_{I\geq 0} \tilde{J}_F(X,\rho(X),I).$$

The scale of investment that follows from the solution to the optimization problem is denoted by $I^*(X,\rho(X))$ and the corresponding value function of the firm reads

$$J_F(X,\rho(X)) = \tilde{J}_F(X,\rho(X),I^*(X,\rho(X))).$$

Finally, the firm has to determine the timing of its investment for a given coupon scheme $\rho(\cdot)$ set by the lender. Similar to the bankruptcy option problem, the investment timing problem can be written as an optimal stopping problem distinguishing a region with respect to the state variable where immediate investment is optimal, the *stopping region*, and a region where it is optimal to delay investment, the *continuation region*. Following the real options literature, we assume that timing is determined by an investment threshold X_F and proceed in our analysis by considering a stopping region of the form $[X_F,\infty)$.[5] We will later verify numerically that in our framework this assumption is true for the optimal coupon scheme $\rho^*(\cdot)$ of the lender. Hence, for $X < X_F$ it is optimal for the firm to delay investment and for $X \geq X_F$ it is optimal to immediately undertake investment. Define function $\tilde{\tau}(\tilde{X};X) = \inf\Big\{t\geq 0 \mid x(t)\geq \tilde{X}; x(0)=X\Big\}$, i.e., the first time that gBm process $x(t)$ hits $\tilde{X}$ from below. Then, assuming that $x(0)$ is sufficiently small such that it is not optimal for the firm to invest immediately, the optimal investment threshold X_F^* solves the problem

$$\sup_{X_F\geq 0} \mathbb{E}_0\left\{ e^{-r\tilde{\tau}(X_F;x(0))} J_F(X_F,\rho(X_F),I^*(X_F,\rho(X_F)) \right\}. \tag{2}$$

It is well known that the solution to this problem does not depend on $x(0)$. Put differently, X_F^* is the minimal level of demand at which investment takes place. Clearly the optimal investment trigger depends on $\rho(\cdot)$, which can be formally expressed by writing $X_F^*(\rho(\cdot))$. Unlike standard real option problems, different investment thresholds lead to different unit costs of investment because the coupon rate depends on the state at the time of investment. It should be noted that the lender might prevent the firm from investing in parts of the state space by setting a sufficiently high coupon rate, e.g. $\rho(X)=\infty$. For further reference we define

[5] Without making any additional assumptions on the shape of the function $\rho(\cdot)$, strictly speaking it is not evident that the stopping region of the problem is given by an interval, although this is typically the case for real option problems like ours. Dixit and Pindyck (1994), e.g., show that for standard real options problems the state space can be split up into two consecutive regions giving the stopping region and continuation region in this fashion. For models where capacity choice is explicitly modeled, their result is extended by Huberts et al. (2019). Using a verification theorem based on, e.g., Gozzi and Russo (2006), optimality can be shown. This result however does not cover the case of a state-dependent coupon rate.

the firm's value function as

$$V_F(X,\rho(\cdot)) = \begin{cases} \mathbb{E}_0\left\{e^{-r\tilde{\tau}(X_F^*;x(0))} J_F(X_F^*,\rho(X_F^*)) \mid x(0)=X\right\} & \text{for all } X < X_F^*(\rho(\cdot)), \\ J_F(X,\rho(X)) & \text{otherwise,} \end{cases}$$

Problem of the Lender In our model it is assumed that the capital market is imperfect, that is, the lender has market power. As such the lender sets a coupon scheme $\rho(\cdot)$ as to maximize its net present value. In order to formulate the lender's net present value we first define

$$\begin{aligned} J_D(X,\tilde{\rho}) = \ & \mathbb{E}_0\Big\{ \int_0^{\tau_B(X,I^*(X,\tilde{\rho}),\tilde{\rho})} \tilde{\rho}\delta I^*(X,\tilde{\rho})e^{-rt}\mathrm{d}t \\ & +(1-\alpha)\int_{\tau_B(X,I^*(X,\tilde{\rho}),\tilde{\rho})}^{\infty} \pi(x(t),I^*(X,\tilde{\rho});0)e^{-rt}\mathrm{d}t - \delta I^*(X,\tilde{\rho}) \ \Big| \ x(0)=X\Big\} \end{aligned} \tag{3}$$

as the expected net present value for the lender if the firm invests at $x(t)=X$ with a coupon rate set to $\tilde{\rho}$. The first integral term represents the coupon payments from the firm. The second integral term captures the project value taken over by the lender after the firm defaults. Upon bankruptcy the assets are transferred to the lender, who will not terminate production. Following, e.g., Miao (2005), Nishihara and Shibata (2021), and Wong and Yu (2022), we assume that after bankruptcy the lender receives a proportion $1-\alpha$ of the firm value, i.e., the project is supposed to lose a proportion $\alpha \in [0,1]$ of its value and the net value is transferred to the lender. In what follows we refer to α as the bankruptcy cost parameter. The final term δI is the investment outlay that the firm borrows from the lender. The lender's problem then is to choose the coupon scheme $\rho(\cdot)$ in order to solve

$$\sup_{\rho(\cdot)} V_D(X,\rho(\cdot)), \tag{4}$$

with

$$V_D(X,\rho(\cdot)) = \begin{cases} \mathbb{E}_0\left\{e^{-r\tilde{\tau}(X_F^*(\rho(\cdot));X)} J_D(X_F^*(\rho(\cdot)),\rho(X_F^*(\rho(\cdot)))) \mid x(0)=X\right\} & X < X_F^*(\rho(\cdot)), \\ J_D(X,\rho(X)) & X \geq X_F^*(\rho(\cdot)). \end{cases}$$

The lender's optimal coupon scheme, solving (4) is denoted by $\rho^*(\cdot)$.

Welfare Denoting by $w(x(t),I) = x(t)I(1-\eta I/2)$ the welfare flow after investment,[6] the expected discounted total welfare can be written as

$$W(X;X_F,I(\cdot)) = \mathbb{E}_0\left\{\int_{\tau'}^{\infty} e^{-rt}w(x(t),I(X_F))\mathrm{d}t - e^{-r\tau'}\delta I(X_F)\right\},$$

where τ' is the (stochastic) first-hitting time of the the investment threshold X_F from X and $I(\cdot)$ the investment schedule. Direct calculations give

$$W(X;X_F,I(\cdot)) = \begin{cases} \left(\frac{X}{r-\mu}\left(1-\frac{\eta}{2}I(X)\right)-\delta\right)I(X) & \text{for all } X \geq X_F, \\ \left(\frac{X}{X_F}\right)^{\beta_1}\left(\frac{X_F}{r-\mu}\left(1-\frac{\eta}{2}I(X_F)\right)-\delta\right)I(X_F) & \text{for all } X < X_F. \end{cases} \tag{5}$$

Note that welfare only depends on $I(\cdot)$ and X_F, but not directly on $\rho(\cdot)$. However $\rho(\cdot)$ affects welfare indirectly through its impact on the investment threshold and size.

6 For any t after investment, when the firm is still active, the expression for $w(x(t),I)$ can be derived from summing consumer surplus and firm profit using standard calculations. Because the bankruptcy event does not lead to a shift in the demand curve, nor does it change the marginal cost of production, welfare is not directly affected by bankruptcy. Hence, even after firm bankruptcy the welfare flow is given by the same expression. This implies that α and X_B^* have an indirect impact on W only.

3 Model Analysis

In this section we characterize the optimal decisions of the firm and the lender. In order to gain additional intuition for the key mechanisms at work, we first consider a simplified version of the model where the firm commits to not to default. From this we can analyze the equilibrium investment scale and timing in isolation, without the effect of bankruptcy playing a part. We then consider the scenario where the firm holds the bankruptcy option after investment.

3.1 Firm Commits not to Default

We first consider the case where the firm is able to commit to paying coupons perpetually after investment. Such commitment can be due to legal environments where firm owners are personally liable for firm losses or can be generated by provision of sufficient collateral by the firm. In what follows we refer to this case as the no-bankruptcy option (NBO) scenario. Proposition 1 characterizes the optimal coupon scheme and the corresponding investment threshold for this scenario.

Proposition 1 *Assume that there is no bankruptcy option. Then, the lender's optimal strategy is given by*

$$\rho^{NBO}(X) = \begin{cases} \frac{r(X+\delta(r-\mu))}{2\delta(r-\mu)} & \text{for all } X \geq \frac{\beta_1+1}{\beta_1-1}\delta(r-\mu), \\ \infty & \text{otherwise.} \end{cases} \tag{6}$$

There is an investment trigger X_F^* such that for $X < X_F^*$ the firm waits until the demand level reaches X_F^* to install capacity $I^{opt} = I^{NBO}(X_F^*, \rho^{NBO}(X_F^*))$. From that moment on the firm will pay coupons at rate $\rho^{opt} = \rho^{NBO}(X_F^*)$. The firm's investment threshold, the associated investment size and the coupon rate are given by

$$X_F^* = \frac{\beta_1+1}{\beta_1-1}\delta(r-\mu), \quad I^{opt} = \frac{1}{2\eta(\beta_1+1)}, \quad \rho^{opt} = r\frac{\beta_1}{\beta_1-1} > r,$$

where $\beta_1 > 1$ is the larger root of $\frac{1}{2}\sigma^2\beta^2 + (\mu - \frac{1}{2}\sigma^2)\beta - r = 0$. For $X \geq X_F^*$ the firm invests immediately and installs capacity $I^{NBO}(X, \rho^{NBO}(X))$ with coupon rate $\rho^{NBO}(X)$. The optimal investment size is then given by

$$I^{NBO}(X, \rho^{NBO}(X)) = \frac{1}{4\eta}\left(1 - \frac{\delta(r-\mu)}{X}\right).$$

	External Financing	Internal Financing	Social Optimum
Threshold (X_F^*)	$\frac{\beta_1+1}{\beta_1-1}\delta(r-\mu)$	$\frac{\beta_1+1}{\beta_1-1}\delta(r-\mu)$	$\frac{\beta_1+1}{\beta_1-1}\delta(r-\mu)$
Size (I^{opt})	$\frac{1}{2\eta(\beta_1+1)}$	$\frac{1}{\eta(\beta_1+1)}$	$\frac{2}{\eta(\beta_1+1)}$
Welfare (W)	$\frac{7\delta}{8\eta(\beta_1^2-1)}$	$\frac{3\delta}{2\eta(\beta_1^2-1)}$	$\frac{2\delta}{\eta(\beta_1^2-1)}$

Table 1: Investment decisions and their corresponding welfare under different scenarios of financing for sufficiently small initial demand. The last two columns are taken from Huisman and Kort (2015).

Proposition 1 provides several important insights about the effects of the interplay between a lender and a firm, where both have market power. To interpret these insights, it is useful to compare the outcome of this strategic interaction with the scenario where the firm can finance investments internally, i.e. the integrated monopoly case, and hence faces unit finance costs of $r\delta$, where again r is the risk-free interest rate. This comparison is demonstrated in Table 1. Interestingly, the threshold X_F^* at which the firm invests, is identical in both settings. However, the size of the investment I^{opt} is only half in our framework with an endogenous coupon rate, compared to the investment size under internal financing. This indicates the same qualitative result as in the standard static *double marginalization* literature. The reason why the firm is investing less under external financing is that the coupon rate requested by the lender exceeds the risk-free rate. Hence, our result can be interpreted as an instance of the phenomenon of double marginalization, in the sense that the exploitation of market power on two subsequent stages of the value chain leads to distortions that are more pronounced than those resulting under an integrated monopoly. Although the double marginalization phenomenon occurs in numerous supply chain studies, to our knowledge, so far, double marginalization has not been identified as an important factor in the framework of optimal investment under external financing.

Compared to the case of internal financing, the endogenous choice of coupon rate in our model gives rise to two qualitative effects influencing the firm's investment timing. First, the equilibrium coupon rate ρ^{opt} is larger than the risk-free interest rate r and, second, by choosing the investment threshold X_F^* the firm can influence the size of the coupon rate, which is an increasing function of X.[7] Concerning the first effect it can easily be derived that the optimal investment threshold under a fixed coupon rate is increasing in ρ, $\rho > r$. The second effect, driven by the market power of the lender, however, gives the firm an incentive to accelerate the investment in order to keep the cost of investment low. Overall, in our framework the two countervailing effects exactly cancel such that the timing of investment under external financing is identical to that under internal financing.

Considering the effect of demand uncertainty on the coupon rate and the equilibrium investment pattern, we observe that for a given level of $x(t)$, at the time of investment the coupon rate does not depend on σ (see (6)). This is intuitive since the income stream of the lender is fixed and equal to $\rho\delta I$, which does not depend on the evolution of market demand once the firm has invested. Nevertheless, increased uncertainty induces a larger coupon rate in equilibrium. This is due to the fact that the coupon rate is an increasing function of the value of $x(t)$ at the time of investment, and a larger σ triggers a higher investment threshold, as is standard in real option models of this type, see, e.g., Dixit and Pindyck (1994). In other words, the wedge is increasing in the level of uncertainty.

In general, the phenomenon of double marginalization leads to a negative externality on social welfare relative to an integrated monopoly benchmark case (see, e.g., Spengler (1950)). This externality follows directly from a higher output price and a lower quantity on the market. In our stochastic dynamic model, as shown before, we find that the timing is not affected, but the size is reduced by double marginalization as well. This reduction has clear negative welfare implications since already under internal financing the socially optimal investment level is twice as high as that chosen by the firm, whereas the investment timing is the same for both (Huisman and Kort, 2015). Proposition 1 shows that double marginalization results in the investment size being brought down further, which means that welfare is reduced even more. This is confirmed in Table 1, which compares the optimal investment and the corresponding welfare under external financing to that under internal financing and the social optimum.

3.2 Firm Has the Option to Default

We now consider the problem where the firm does not commit to paying the coupon rate forever, i.e. it might declare bankruptcy at some point, as described in Section 2. This case is referred as the bankruptcy option

[7] It follows from (6) that the coupon rate in the stopping region is increasing in X, i.e., a higher willingness-to-pay by consumers (a shift in the demand curve) allows the lender to extract more rent from the market by charging the firm a higher coupon rate.

(BO) scenario. We first treat the problem of the firm choosing the investment threshold, an investment schedule $I(\cdot)$, and a bankruptcy threshold $X_B(\cdot)$ in order to maximize its expected payoff, given in (1), for a given coupon scheme $\rho(\cdot)$. Then we determine the optimal coupon scheme to be offered by the lender.

3.2.1 Firm's bankruptcy decision

Before solving the firm's investment problem, we consider the firm's exit option. After investment, market demand evolves stochastically, and $x(t)$ reaches $X_B^*(I,\rho)$ for the first time at time $\tau_B(X,I;\rho)$. The value of the firm after $X_B^*(I,\rho)$ is reached, is zero and it no longer pays coupons to the lender. The firm exercises the bankruptcy option at the threshold X_B^*, characterized as follows.

Proposition 2 *Let $\beta_2 < 0$ be the smaller root of $\frac{1}{2}\sigma^2\beta^2 + (\mu - \frac{1}{2}\sigma^2)\beta - r = 0$. There exists a threshold $X_B^* > 0$ such that it is optimal for the firm to file for bankruptcy for all $X \leq X_B^*$. The bankruptcy threshold for a given coupon rate $\tilde{\rho} > 0$ and capacity size $\tilde{I} \in (0, \frac{1}{\eta})$ is given by*

$$X_B^*(\tilde{I},\tilde{\rho}) = \frac{\beta_2}{\beta_2 - 1}\frac{\tilde{\rho}\delta}{r}\frac{r-\mu}{1-\eta\tilde{I}}. \tag{7}$$

The bankruptcy threshold in Proposition 2 implies that, for a given investment size $\tilde{I}$, $X_B^*(\tilde{I},\tilde{\rho})$ increases with $\tilde{\rho}$. Because $x(t)$ reaches this trigger from above after investment, an increased $X_B^*(\tilde{I},\tilde{\rho})$ is reached sooner. So a larger coupon rate makes it more likely for the firm to default up to a given point in time, which follows directly from the firm's reduction in instantaneous profits. Similarly, for a given coupon rate $\tilde{\rho}$, an increase in the capacity size also leads to a higher exit trigger. This result can be explained by noting that the coupon payments increase proportionally with $\tilde{\rho}$ and $\tilde{I}$, whereas market revenue is independent of $\tilde{\rho}$ and marginal revenue is decreasing in $\tilde{I}$. Hence, a larger value of $\tilde{\rho}$ respectively $\tilde{I}$ implies that a larger value of $x(t)$ is needed to compensate the higher coupon payments. Requiring a larger value of $x(t)$ translates into a higher bankruptcy threshold. For a given $\tilde{\rho}$ and $\tilde{I}$, the volatility parameter σ influences X_B^* negatively through β_2. More uncertainty implies a higher bankruptcy option value. The firm therefore wants to keep the bankruptcy option alive for a longer time. This implies that the firm delays bankruptcy, which is reflected by a smaller X_B^*.

3.2.2 Firm's investment decision

For a given $X \in S$ and coupon rate $\tilde{\rho} = \rho(X)$, the firm's investment size follows from $\sup_{I\geq 0} \tilde{J}_F(X,\tilde{\rho},I)$, with

$$\begin{aligned}\tilde{J}_F(X,\tilde{\rho},I) &= \mathbb{E}_0 \int_0^{\tau_B(X,I;\tilde{\rho})} e^{-rt}\Big(x(t)(1-\eta I)I - \tilde{\rho}\delta I\Big)\mathrm{d}t \\ &= \tfrac{X}{r-\mu}I(1-\eta I) - \tfrac{\tilde{\rho}}{r}\delta I + \tfrac{\tilde{\rho}\delta I}{r(1-\beta_2)}\left(\tfrac{X}{X_B^*(I,\tilde{\rho})}\right)^{\beta_2}.\end{aligned} \tag{8}$$

The last expression in (8) consists of three terms. The first term captures the firm's total cash inflows and the second term represents the sum of the firm's coupon payments. The third term corrects for the fact that these cash flows only take place until the firm is bankrupt. Proposition 3 shows that the solution to (8) is unique and depends in a monotone way on X and the coupon rate $\tilde{\rho}$.

Proposition 3 *Let $\rho : [0,\infty) \to [r,\infty)$. Assume $\rho(X)$ is continuously differentiable almost everywhere. Then the firm's optimization problem in (8) has a unique solution $I^*(X,\rho(X))$ for every $X > 0$. Moreover,*

(i) $I^*(X,\rho(X))$ is monotonically increasing in X if and only if $\rho(X)/X$ is a decreasing function of X, and

(ii) $I^*(X, \tilde{\rho})$ is monotonically decreasing in $\tilde{\rho}$, for a given X.

Proposition 3 confirms the intuition that an increase in the coupon rate results in the firm setting a lower quantity and that an increase in X, i.e. a shift in the demand curve leading to a higher willingness-to-pay, incentivizes the firm to set a higher quantity, as long as the coupon schedule is not increasing too steeply in X. Intuitively, if the coupon schedule is steep, then the marginal revenue is outweighed by the marginal cost of finance, in which case increasing the investment size is not optimal for the firm as X goes up.

The next step is to determine when the firm will invest. To that end, Proposition 4 shows that for a given coupon scheme $\rho(\cdot)$ there exists a threshold below which it is not optimal for the firm to undertake investment.

Proposition 4 *Assume that the lender offers $\rho(X)$ for all $X \geq 0$, $\rho(\cdot)$ is C^1, and assume that $\rho(X)/X$ is a monotonically decreasing function of X. There exists a $X_F(\rho(\cdot)) > 0$ such that for $X < X_F(\rho(\cdot))$ the firm optimally delays investment till the threshold $X_F(\rho(\cdot))$ is reached and then invests $I^*(X_F(\rho(\cdot)), \rho(X_F(\rho(\cdot))))$.*

To determine the firm's actual investment decision we need to know the coupon scheme which is chosen by the lender. This scheme is characterized in the following section.

3.2.3 Lender's coupon scheme and investment timing in equilibrium

Since the characterization of the lender's optimal coupon scheme requires an extensive and rather technical analysis, in this section we provide only the main steps, results and economic insights. The full exposition of the analysis with all technical details is provided in Appendix A. A key role in characterizing the optimal behavior of the lender is played by the coupon scheme which for each X maximizes the value of the lender under the assumption that the firm invests immediately. This coupon scheme, which we denote as $\rho^{imm}(X)$, solves the optimization problem

$$\begin{aligned} \rho^{imm}(X) &= \arg\sup_{\rho\geq 0} J_D(X,\rho) \\ &= \arg\sup_{\rho\geq 0} \left\{ \tfrac{\rho-r}{r}\delta I^*(X,\rho) - \tfrac{\rho\delta I^*(X,\rho)}{r}\tfrac{1-\alpha\beta_2}{1-\beta_2}\left(\tfrac{X}{X_B^*(I^*(X,\rho),\rho)}\right)^{\beta_2}\right\}, \end{aligned} \tag{9}$$

where $I^*(X,\rho)$ satisfies (8). It follows directly that for all values of X, where it is optimal for the firm to invest immediately under $\rho(X) = \rho^{imm}(X)$, the optimal coupon rate set by the lender has to coincide with $\rho^{imm}(X)$. In Appendix A we show that there exists a threshold $\tilde{X}_D$ with the property that, under the assumption that the lender chooses the coupon scheme $\rho^{imm}(\cdot)$, and for all X the firm invests immediately under that scheme, it is optimal for the lender to offer financing if and only if $X \geq \tilde{X}_D$. However, generally speaking it is not optimal for the firm to invest immediately for all values of X if the lender offers the coupon scheme $\rho^{imm}(\cdot)$. Rather, there exists a threshold $\tilde{X}_F$ such that it is optimal for the firm to invest immediately under the scheme $\rho^{imm}(\cdot)$ if and only if $X \geq \tilde{X}_F$.

If $\tilde{X}_F \leq \tilde{X}_D$, the firm is willing to invest immediately under coupon scheme $\rho^{imm}(\cdot)$ for any value of X where the lender is willing to provide financing, i.e. for all $X \geq \tilde{X}_D$. In such a scenario it is optimal for the lender to offer financing with a coupon rate $\rho^{imm}(X)$ for any $X \geq \tilde{X}_D$ and to refuse financing as long as X is below $\tilde{X}_D$ (formally the lender offers a coupon rate of infinity for these values). If $\tilde{X}_D < \tilde{X}_F$, then for $X \in [\tilde{X}_D, \tilde{X}_F)$ the firm is not willing to invest immediately under a coupon rate $\rho^{imm}(X)$. In this case there is a value $\hat{X}_D < \tilde{X}_F$ such that it is optimal for the lender to offer on the interval $[\hat{X}_D, \tilde{X}_F]$ a coupon rate $\rho^{ind}(X) \leq \rho^{imm}(X)$. The scheme $\rho^{ind}(\cdot)$ has the property that $\rho^{ind}(\tilde{X}_F) = \rho^{imm}(\tilde{X}_F)$ and for any $X \in [\hat{X}_D, \tilde{X}_F)$ it makes the firm indifferent between investing immediately and delaying investment, which implies that it is the highest coupon rate for which the firm is willing to invest immediately. We refer to such a coupon scheme as an *indifference coupon scheme*.[8] Overall, we obtain the following characterization of the optimal coupon scheme of the lender and the resulting investment behavior of the firm.

Proposition 5 *There exists a threshold $\hat{X}_D$ and an indifference coupon scheme $\rho^{ind}(X)$ for $X \in [\hat{X}_D, \tilde{X}_F]$ such that the optimal coupon scheme has the form*

$$\rho^*(X) = \begin{cases} \rho^{imm}(X) & \text{for all } X \in [\max[\tilde{X}_F, \tilde{X}_D], \infty), \\ \rho^{ind}(X) & \text{for all } X \in [\hat{X}_D, \tilde{X}_F), \\ \infty & \text{for all } X \in [0, \hat{X}_D). \end{cases} \tag{10}$$

If $\tilde{X}_D < \tilde{X}_F$ then $\hat{X}_D < \tilde{X}_F$, whereas $\hat{X}_D = \tilde{X}_D$ holds for $\tilde{X}_D \geq \tilde{X}_F$. Under the optimal coupon scheme it is optimal for the firm to invest immediately at any X with $\rho^*(X) < \infty$. The optimal coupon scheme of the lender satisfies $\rho^*(X) > r$ for all $X > 0$.

It should be noted that for $\tilde{X}_D \geq \tilde{X}_F$ the interval $[\hat{X}_D, \tilde{X}_F)$ is empty, which means that in this case the lender never offers an indifference coupon scheme.

Denoting by X_F^* the smallest value of market demand at which the firm is willing to invest under the optimal coupon scheme $\rho^*(\cdot)$, we obtain from Proposition 5 that

$$X_F^* = X_F(\rho^*(\cdot)) = \begin{cases} \tilde{X}_D & \text{if } \tilde{X}_D \geq \tilde{X}_F, \\ \hat{X}_D & \text{otherwise.} \end{cases} \tag{11}$$

We denote the equilibrium coupon rate when investment is undertaken at X_F^* as $\rho^{opt} = \rho^*(X_F^*)$ and the corresponding optimal investment size by $I^{opt} = I^*(X_F^*, \rho^{opt})$.

An important implication of Proposition 5 is that, like in the case without bankruptcy option, the coupon rate at which the firm invests is always strictly larger than the risk free rate r. With respect to investment timing, there is however a qualitative difference between the cases with and without the bankruptcy option. Recall that for the scenario where the firm holds no bankruptcy option the need for external financing has no impact on the investment timing, compared to the scenario of an "integrated" monopoly and the social optimum. The firm sets a lower capacity, which directly negatively impacts welfare. When the firm holds an option to default, in general the need for external financing has an impact on the timing of investment. Moreover, it is not clear how the bankruptcy option changes the firm's optimal investment size. This also raises the question how the double marginalization phenomenon's impact on welfare is affected by the presence of the bankruptcy option and the size of the bankruptcy cost α. The next section aims to answer these questions.

4 Managerial and Economic Implications

The characterization of the Markov Perfect Equilibrium summarized in Proposition 5 unfortunately does not allow for a closed form representation of the firm's equilibrium investment strategy and the lender's coupon scheme. Therefore, in this section we resort to numerical analysis to gain insights about the effect of key parameters on the optimal strategies of the lender and the firm. We first focus on optimal behavior under different values of the bankruptcy cost parameter α and then analyze implications of different degrees of

[8] See Appendix A for a formal definition of an indifference coupon scheme.

demand uncertainty, σ.[9] In particular, we analyze the outcomes of the model with bankruptcy option and contrast them with the case where the firm commits not to default.

4.1 Effect of Bankruptcy Costs

To structure our discussion we distinguish between two scenarios. The *Large Initial Demand* scenario arises if the market demand X at the point in time when the firm considers to enter the market is sufficiently large such that it invests immediately if the lender offers the coupon rate $\rho^{imm}(X)$. Conversely, the *Small Initial Demand* scenario arises if initial demand is so low that the firm does not investment immediately.

4.1.1 Large initial demand

In the Large Initial Demand scenario the firm invests immediately and therefore timing of investment is no issue. Hence, our analysis in this section focuses on the determinants of the investment size and the coupon rate set by the lender.

We start by considering the implications of a change in the bankruptcy cost parameter α on the optimal choice of the coupon rate. Figure 2a illustrates the effect of the bankruptcy costs on the optimal coupon rate. Higher bankruptcy costs incentivize the lender to prolong the period until bankruptcy. By lowering the coupon rate, the bankruptcy trigger is lowered, as illustrated by Figure 2b, so that the expected time till bankruptcy goes up. A larger value of α thus leads to a smaller coupon rate, which reduces the firm's financing costs. As illustrated in Figure 2c, this implies that the firm's optimal investment size increases with the size of the bankruptcy costs. We summarize this discussion in our first main finding.

Result 1 *Assume that initial demand is sufficiently large. Then the lender should react to an increase in the bankruptcy costs with a decrease of the coupon rate, thereby triggering a larger investment by the firm.*

In Figure 2c we observe that, apart from very small values of α, the firm investments are higher in the presence of the bankruptcy option compared to a scenario where it has committed not to default. The reason is that it can avoid the losses in case of a negative development of demand, while still benefiting from the upward potential. However, the lender now covers this downside risk. This generates an incentive for the lender to offer a higher coupon rate, thereby reducing the firm's investment size. This effect is highlighted in Figure 2c, where we depict, apart from the firm's optimal investment size in the bankruptcy (BO) and non-bankruptcy (NBO) cases, also the investment size the firm would choose in the BO case if the lender offers the same coupon rate as in the NBO case (red line). Furthermore, in the BO scenario the choice of the coupon rate also affects the bankruptcy trigger for the firm, directly and indirectly through the firm's optimal investment size, as is illustrated by Figure 2b. Due to the latter effect there emerges an incentive for the lender to delay bankruptcy by lowering the coupon rate and this incentive is higher, the larger is the bankruptcy loss parameter α. This explains why for a sufficiently large value of α the lender's optimal coupon rate is lower than the rate in the NBO scenario.

Since the lender incurs the bankruptcy cost, the firm is only indirectly affected by α, namely through the lender's choice of the coupon rate. A higher coupon rate induces a smaller investment size so that under high bankruptcy cost ($\alpha = 1$) the firm's optimal size in the BO case is higher than in the NBO case but the opposite holds under low bankruptcy cost ($\alpha = 0$).

[9] Since changing δ, r, η, or μ does not lead to qualitatively different results, as can be expected, we omit a full analysis of these parameters. Section 5 provides a robustness check for all parameters and shows that our results remain valid for other parametrizations.

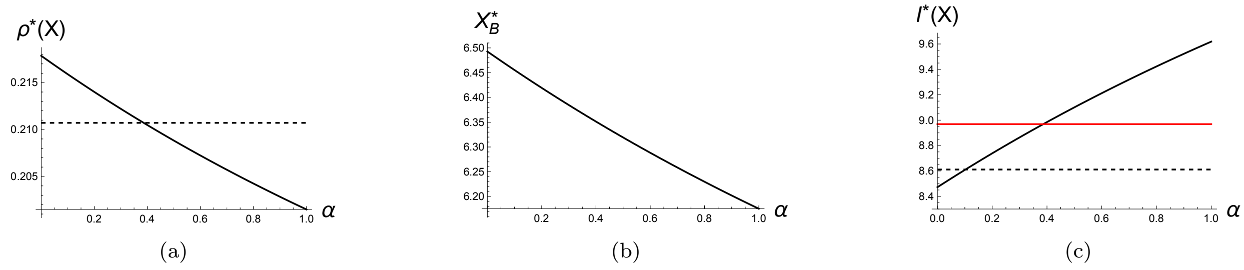


Figure 2: The effect of α on the optimal coupon scheme $\rho^*(X)$ (panel a), the bankruptcy trigger $X_B^*(I^*(X), \rho^*(X))$ (b), and the optimal investment size $I^*(X)$ (c) for large initial demand ($X = 9$) in the BO (solid) and the NBO scenario (dashed). In panel (c) the red line depicts the firm's optimal investment size in the BO case if the lender would offer the same coupon rate as in the NBO case.

$\mu = 0.03,\ r = 0.1,\ \sigma = 0.1,\ \delta = 40, \text{ and } \eta = 0.02.$

The fact that the lender, at least for small values of α, charges a higher coupon rate if the firm has the option to default induces larger financing costs for the firm and, as a result the firm value is smaller than in the NBO case. The larger investment size of the firm in the BO case has a positive impact on the lender's value. In case bankruptcy costs are sufficiently small this effect dominates and the lender's value is larger in the BO than in the NBO scenario. These observations are illustrated in Figure 3. To disentangle the direct effect of the firm's option to default from the effect of the adjustment of the coupon rate and the investment size, we depict in this figure also the lender's and firm's value if the coupon were set as in the NBO case but the firm nevertheless has the option to default (red line). Comparing this to the NBO case clearly demonstrates the positive direct effect of the option to default on the firm value. The strategic reaction of the lender to this option makes the firm worse off if it has the option to default. As illustrated by the figure, for small bankruptcy costs the lender already profits directly from the increased investment incentive of the firm (red line) and can even further increase its gains by optimally adapting the coupon rate.[10] This leads to our second main result.

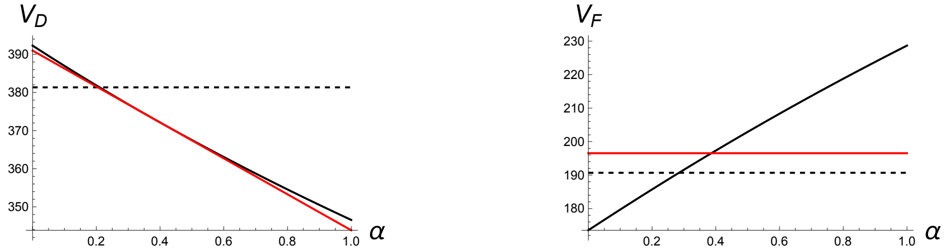
Result 2 *Assume that initial demand is sufficiently large. Then the option to default reduces the firm's value if the bankruptcy costs for the lender are small. For sufficiently small bankruptcy costs the lender gains value from the firm's bankruptcy option in spite of covering the downside risk of negative demand development.*

Figure 3 shows how the values of the lender and the firm change with α. Not surprisingly, the value of the lender decreases with α (see Figure 3a), whereas the value of the firm increases with α (Figure 3b). The first of these observations is directly driven by the increased bankruptcy costs the lender faces, and the second effect is due to the reduction in the coupon rate, which is induced by a larger α.

4.1.2 Small initial demand

We now consider the scenario where initial demand $x(0)$ is so small that immediate investment is not optimal. Hence, in addition to the size of the investment, also investment timing needs to be decided on. First, we

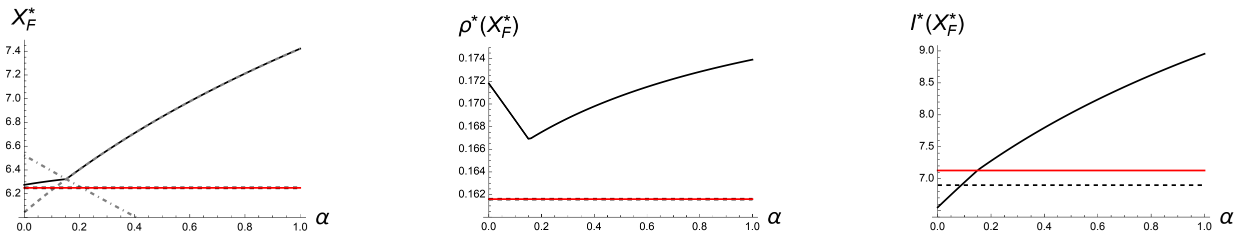
[10] The figure also illustrates that for $\alpha = 0.465$ the optimal coupon rate ρ^* coincides with the coupon rate in the absence of the firm's bankruptcy option. Hence, also the investment size and the values for firm and lender coincide between the BO and the NBO case for this value of α.



(a) Value of lender $V_D(X = 9, \rho(\cdot))$. (b) Value of the firm $V_F(X = 9, \rho(\cdot))$

Figure 3: Effect of α on the value of the lender and the firm for the scenarios with (solid line) and without (dashed line) bankruptcy option and large initial demand. The red lines depict the lender's and firm's value in the BO case if the lender would offer the same coupon rate as in the NBO case.

$\mu = 0.03,\ r = 0.1,\ \sigma = 0.1,\ \delta = 40,$ and $\eta = 0.02.$



(a) (b) (c)

Figure 4: Effect of α on the equilibrium investment threshold X_F^* (a), the coupon rate $\rho^*(X_F^*)$ (b) and the investment size I^{opt} for the scenarios with (solid line) and without (dashed line) bankruptcy option. In panel (a) the dotted line shows $\tilde{X}_F$ and the dashed-dotted line $\tilde{X}_D$. The red lines are generated assuming the lender commits to the NBO coupon scheme $\rho^{NBO}(\cdot)$ and the firm reacts optimally to that.

$\mu = 0.03,\ r = 0.1,\ \sigma = 0.1,\ \delta = 40,$ and $\eta = 0.02.$

investigate (investment) thresholds $\tilde{X}_F$, $\tilde{X}_D$, and X_F^*. As can be seen in Figure 4a, the threshold $\tilde{X}_D$ is increasing in α. Intuitively, an increase in bankruptcy cost lowers the lender's net present value of the project, and therefore "financing" is delayed. We find the opposite for the firm: the threshold $\tilde{X}_F$ is decreasing in α. The bankruptcy cost parameter α only has an indirect effect on the firm's investment problem in the sense that the coupon scheme $\rho^*(\cdot)$ decreases with α (see Figure 2a). Lower coupon payments lower the firm's cost of investment and make it more attractive for the firm to invest, which implies earlier financing and therefore $\tilde{X}_F$ decreases with α.

The interplay between the opposite monotonicities of $\tilde{X}_D$ and $\tilde{X}_F$ implies that if bankruptcy costs are large, we have $\tilde{X}_D > \tilde{X}_F$. In this scenario, i.e., when the lender's threshold (under the assumption that the firm invests immediately) is not smaller than the firm's threshold (under the assumption that the lender offers credit at any value of X), (11) implies that $X_F^* = \tilde{X}_D$. Intuitively, for large α the willingness of the lender to provide the credit forms the bottleneck and we have $X_F^* = \tilde{X}_D$. On the contrary, if the bankruptcy costs are low, we have $\tilde{X}_F > \tilde{X}_D$, in which case the lender offers an indifference coupon scheme and the firm invests at a threshold $\hat{X}_D \in (\tilde{X}_D, \tilde{X}_F)$. Low bankruptcy costs make the project more attractive for the lender and therefore the investment timing depends on the willingness of the firm to carry out the investment. The lender is willing to provide financing for values of X where the firm is not willing to invest at a coupon rate $\rho^{imm}(X)$. This motivates the lender to offer a lower coupon rate to induce the firm to still undertake the investment. In this way we obtain our next main result.

Result 3 *Assume that initial demand is small. Then, if bankruptcy costs are small, the lender should induce the firm to invest already at a level of demand X_F^* where the firm would not have invested under the coupon scheme ρ^{imm}. Consequently, it is optimal for the lender to offer the indifference coupon scheme ρ^{ind} with $\rho^{ind}(X) < \rho^{imm}(X)$ for all $X \in [X_F^*, \tilde{X}_F)$.*

Focusing now on the scenario with sufficiently large bankruptcy costs, we observe from panels Figures 4b-c that the coupon rate as well as investment size goes up if α increases. The main driver of this observation is that for large bankruptcy costs the investment trigger is determined by the lender ($X_F^* = \tilde{X}_D$) and therefore increasing in α. When setting the coupon rate the lender has competing incentives, it wants to lower the coupon rate as a result of higher bankruptcy costs but investment taking place at a higher value of X outweighs that incentive.

The fact that both the investment trigger and the coupon rate increase with α generates opposing incentives for the firm with respect to determining the investment size. The increase in the coupon rate incentivizes the firm to invest in a lower quantity, whereas the higher investment trigger makes it more interesting to increase the scale of investment. The net result is positive, i.e., the effect of the increase in the coupon rate is dominated by the effect of the higher level of consumers' willingness-to-pay at the time of investment. Hence, for small initial demand we get the following main result characterizing the effects of an increase in bankruptcy costs on investment.

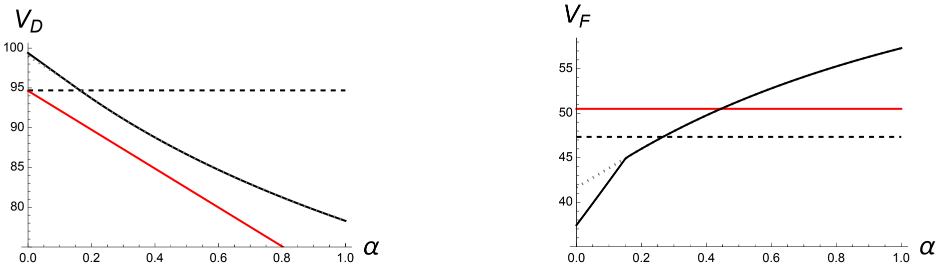
Result 4 *Assume that initial demand is small. Then, if bankruptcy costs are not too small, the lender should react to an increase in the bankruptcy cost by increasing the minimal level of market demand at which it is willing to offer financing. Consequently, the coupon rate and the investment size increase with the size of the bankruptcy costs.*

Comparing this main result to Result 1 highlights that the effect of an increase of bankruptcy costs on investment size is qualitatively similar regardless of whether initial demand is large or small. However, the underlying mechanisms are quite different. Whereas for large initial demand the increase in investment size is driven by the lower coupon rate charged by the lender, in the case of small initial demand the fact that the investment takes place at a higher trigger induces a larger investment size in spite of the higher coupon rate.

As long as α is not too small the dependence of the value of the lender and the firm on α is qualitatively similar to what we saw for the scenario with immediate investment: the lender's value is negatively affected by α whereas the firm's value is positively affected (see Figure 5). Furthermore, for small bankruptcy costs, similar to the case with large initial demand (see Result 2), the option to default reduces the firm's value, whereas the lender gains value from the firm's bankruptcy option in spite of covering the downside risk of negative demand development.

4.2 Welfare Effects and Double Marginalization

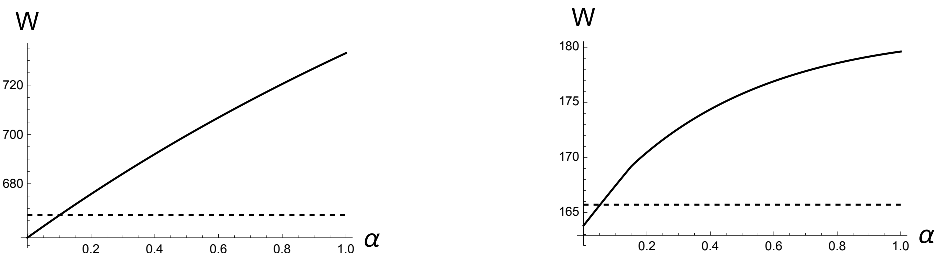
In Section 3 we have shown that in the case where the firm can commit not to default, double marginalization occurs in the sense that welfare is lower compared to the case with internal financing. This is due to $\rho^{NBO}(X) > r$. Furthermore, also in our main model with bankruptcy option we have shown that $\rho^*(X) > r$. Ceteris paribus this has a negative effect on investment size, but the bankruptcy option as such increases the investment incentive for the firm. Therefore, a detailed analysis is needed to determine whether also in



(a) (b)

Figure 5: Effect of α on (a) the value of the lender and (b) value of the firm for small initial demand ($X = 5$) and $\sigma = 0.1$ in scenarios with (solid line) and without (dashed line) bankruptcy option. The red lines are generated assuming the lender commits to the NBO coupon scheme $\rho^{NBO}(\cdot)$ and the firm reacts optimally to that.

$$\mu = 0.03, \ r = 0.1, \ \sigma = 0.1, \ \delta = 40, \text{ and } \eta = 0.02.$$



(a) (b)

Figure 6: Effect of α on social welfare for $\sigma = 0.1$ in scenarios with (solid line) and without (dashed line) bankruptcy option for (a) large initial demand ($X = 9$) and (b) small initial demand ($X = 5$).

$$\mu = 0.03, \ r = 0.1, \ \sigma = 0.1, \ \delta = 40, \text{ and } \eta = 0.02.$$

the scenario with bankruptcy option double marginalization, in the sense of a reduction of welfare due to external financing, occurs.

Based on our discussion in the previous section of the effects of the bankruptcy option on timing and size of investment we are now in a position to analyze the effect of this option and of the size of bankruptcy costs on the degree of double marginalization. In Figure 6 we compare welfare in equilibrium between the non-bankruptcy case and the case with bankruptcy option for different values of bankruptcy costs.[11] Both in the case of large and small initial demand welfare is below the welfare level corresponding to the internal financing case (for our parameter setting welfare under internal financing is $W = 1144$ for $X = 9$ and $W = 227$ for $X = 5$). This confirms that double marginalization occurs also in the scenario with bankruptcy option. However, Figure 6 clearly shows that, except for very small values of α, the welfare loss induced by external financing is smaller if there is a bankruptcy option for the firm. Together, these observations yield our next main result.

Result 5 *Regardless of whether the firm has the option to default, the need for external financing leads to*

[11] As discussed in Section 2, we assume that when the firm defaults, the lender takes over the assets, production is assumed to continue, and there is no direct effect on welfare. The bankruptcy option does however have an indirect impact by affecting $\rho^*(X)$, $I^*(X)$, and X_F^*, which all influence W.

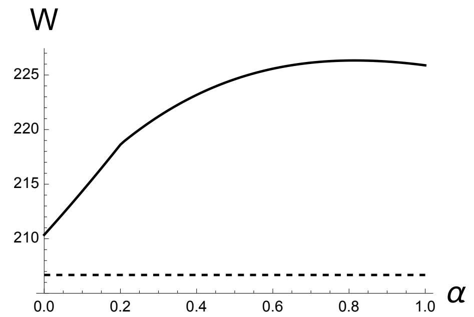


Figure 7: Effect of α on social welfare for $\sigma = 0.2$ in scenarios with (solid line) and without (dashed line) bankruptcy option.

$$\mu = 0.03,\ r = 0.1,\ \sigma = 0.05,\ \delta = 40,\ X = 5, \text{ and } \eta = 0.02.$$

the occurrence of double marginalization, resulting in a reduction of welfare. The firm's option to default mitigates the double marginalization effect if bankruptcy costs are not too small.

Connecting our main result 5 to our results 1 and 4 indicates that the lower welfare loss under the bankruptcy option is mainly driven by the fact that the investment size is larger in that case. In particular, it should be pointed out that for small initial demand the bankruptcy option mitigates double marginalization although it results in a higher coupon rate compared to the scenario without bankruptcy option.

Considering the effect of bankruptcy costs on welfare, the monotonic relationship between α and I depicted in Figures 2 and 4 suggests that welfare should be increasing in α. Figure 6 confirms that this holds true both for large and small initial demand. If initial demand is so small that it is not optimal to invest immediately, two opposing effects arise: welfare is decreased due to a longer delay in investment triggered by an increase in α, whereas welfare is increased due to an increase in investment size induced by higher α. For higher levels of α, the negative effect on the present value due to a delay in investment, relative to the positive effect due to an increase in output, becomes stronger. In our default setting the positive effect dominates such that welfare is increasing in α even for small initial demand (see Figure 6b). However, if volatility is sufficiently large, the negative effect might dominate for large values of α such that the relationship between α and welfare has an inverted U-shape. This can be observed in Figure 7. Overall, we obtain:

Result 6 *An increase in bankruptcy costs leads to higher welfare except for the scenario where initial demand is low and both bankruptcy costs and demand uncertainty are high.*

At first sight, Result 6 is surprising. An increase in bankruptcy costs, generates a friction between the lender and the firm in the sense that it widens the gap of the value of the firm's capital stock for the firm itself and the lender, which takes over the capital at the time of bankruptcy. In our setting this increase in friction is associated with an increase in welfare. The dominant force here is that an increase in bankruptcy costs incentivizes the lender to lower the coupon rate in order to reduce the risk of firm bankruptcy. This strategic reaction of the lender fosters investment size and thus mitigates the double marginalization effect.

4.3 Effect of Demand Uncertainty

To study the effect of demand uncertainty on the optimal choice of the coupon scheme and the resulting investment size, we first consider the scenario of high bankruptcy costs ($\alpha = 0.5$). Figures 8a-b illustrate the effect of uncertainty on the coupon rate and the investment size in a situation where initial demand is sufficiently high so that the firm invests immediately.

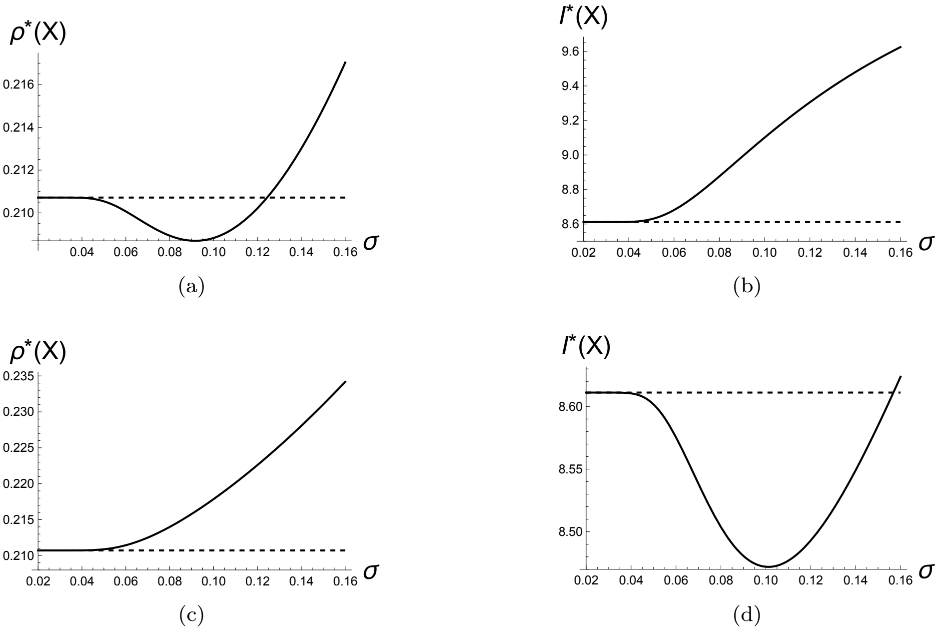


Figure 8: Effect of σ on the coupon rate and the investment size for $\alpha = 0.5$ (panels (a) and (b)) and for $\alpha = 0$ (panels (c) and (d)).

$$\mu = 0.03,\ r = 0.1,\ \delta = 40,\ X = 9, \text{ and } \eta = 0.02.$$

Considering panel (a) we should distinguish three regions with respect to σ. First, for high values of σ there is a positive relationship between σ and $\rho^*(X)$. Larger demand shocks raise the upside potential, so that the firm wants to invest with a large size I^*. The lender sets a high coupon rate to consequently reduce I^* and to benefit from the firm's incentive to invest a lot. Second, for values of σ very close to 0, bankruptcy becomes irrelevant. The process has a positive trend, so that the bankruptcy trigger is unlikely to be reached when σ is sufficiently small. For these values of σ the solid curve in Figure 8a becomes flat and the debt holder behaves as it would have in the case of no bankruptcy (dashed line). Finally, for intermediate values of σ, the bankruptcy risk becomes relevant again. The lender likes to avoid bankruptcy, because of the high bankruptcy cost in this scenario. As Figures 2a-b illustrate, the bankruptcy trigger increases with ρ. Therefore, it is optimal for the lender to let the coupon rate decrease with σ. For the lender, a downside is that a decrease of the coupon rate goes along with an increase of firm investment, since, as explained before, the presence of the bankruptcy option makes that the lender has to deal with the downside potential of the investment. However, the decrease of the coupon rate reducing the bankruptcy trigger is the dominating effect here.

In contrast to the coupon rate, the firm's optimal scale of investment is monotonic in σ. A higher value of σ increases the upward potential, incentivizing the firm to invest in a higher capacity. The downside risk is covered by the lender so that the firm does not lose the incentive to invest more when the level of uncertainty goes up.

The impact of uncertainty as described above, changes when α is small, as illustrated by Figures 8c-d for $\alpha = 0$, in which case the lender faces no bankruptcy cost. In such a scenario the lender is not afraid that bankruptcy occurs. Hence, as discussed in Section 4.1.1, the lender has no incentive to keep the coupon rate low. When the uncertainty parameter σ increases, then, as Figure 8c shows, the lender increases its coupon rate to lower the firm's incentive to undertake a large investment. This incentive arises because the firm benefits from the upside investment potential, which increases with uncertainty due to the larger demand shocks, while the downside is limited by the firm's bankruptcy option. Note that, as the lender takes over the project upon bankruptcy, it has to take care of the downside potential of the investment, which incentivizes the lender to try to limit the firm's investment size.

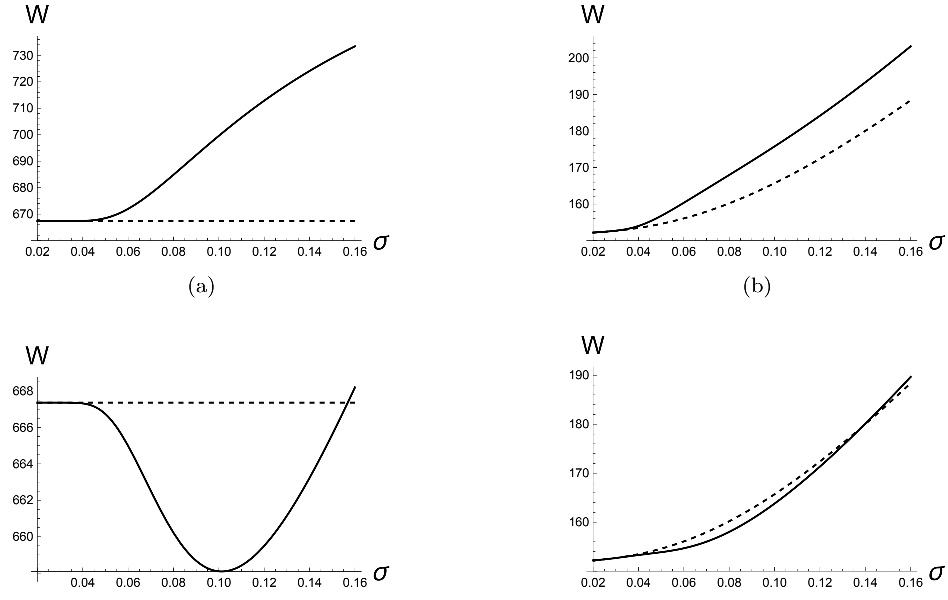
As we see in Figure 8d, for intermediate values of σ the negative effect of a higher coupon rate on the firm's investment size dominates the positive effect of an increased upside potential, due to which investment decreases with σ. For high uncertainty levels the upside potential effect dominates so then the firm's investment is increasing for higher uncertainty levels despite the higher coupon payments the firm incurs. Summarizing, we obtain:

Result 7 *Assume that initial demand is sufficiently large. An increase in market uncertainty has a non-monotonic effect on investment size provided that bankruptcy costs are sufficiently small. For large bankruptcy costs the lender reacts to an increase in market uncertainty by first decreasing and then increasing the coupon rate. As a result, investment size monotonically increases with market uncertainty if bankruptcy costs are high.*

Considering the effect of market uncertainty on welfare, we observe from Figures 9a and 9c that for large initial demand welfare follows the same pattern as the scale of investment. This is very intuitive, since investment timing is fixed for large initial demand (always immediate investment) and from a welfare perspective there is always under-investment in equilibrium, such that an increase in investment size is associated with a welfare increase. In particular, for large bankruptcy costs welfare increases with σ and is always higher than in the no-bankruptcy case, whereas for small bankruptcy costs welfare decreases with market uncertainty for intermediate levels of σ and for this range the firm's option to default reduces welfare compared to the no-bankruptcy case. The intuition for the non-monotonicity of welfare is analogous to that for the non-monotonicity of investment size. Also in case the double marginalization effect is more pronounced if the firm has the option to default, i.e. welfare is smaller with the bankruptcy option than without it, it follows directly from the observation that investment size is smaller with the bankruptcy option than without for this range of σ.

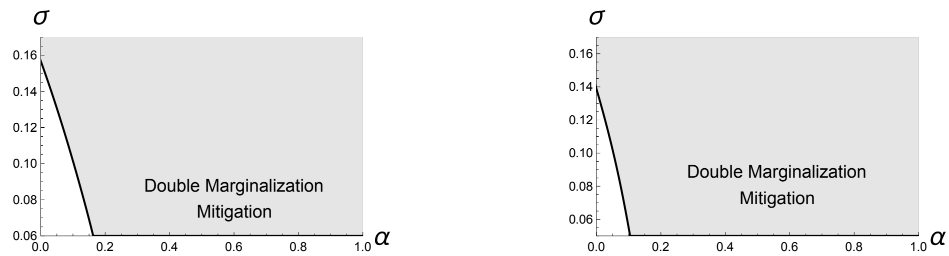
For small initial demand, uncertainty impacts welfare in two ways: through investment size and timing. Investment and therefore consumption is delayed as market uncertainty increases, but at the same time output is increased and thus (for a given level of $x(t)$) is offered at a lower price. Figures 9b and 9d illustrate that welfare benefits from both additional output at a lower price and an increase in the value of the investment option, and thus of the expected producer and consumer surplus, more than that it is harmed from foregone consumption in the short run. Hence, we have the unambiguous result that welfare is larger for a higher level of uncertainty both for small and large bankruptcy costs. Considering the effect of the bankruptcy option on welfare we obtain for small initial demand a qualitatively similar picture as for large initial demand. For large bankruptcy costs the option for the firm to default mitigates double marginalization and increases welfare regardless of the level of market uncertainty. However, if bankruptcy costs are small, then the option to default increases welfare only if demand uncertainty is not too small. For small α and σ, the lender's incentive to push up the coupon rate results directly in a loss in output, leading to a loss in welfare.

In Figure 10 we show for which combinations of bankruptcy costs and demand uncertainty the firm's option to default mitigates double marginalization in the sense that it increases welfare compared to the scenario without the bankruptcy option. The figure illustrates that the effects identified in our discussion above are robust. In particular, the figure shows that both for small and large initial demand the option



(a) (b) (c) (d)

Figure 9: Effect of σ on welfare for large initial demand (left panel) and small initial demand (right panel) for $\alpha = 0.5$ (panels (a) and (b)) and for $\alpha = 0$ (panels (c) and (d)).
$\mu = 0.03,\ r = 0.1,\ \delta = 40,\ X = 9$, and $\eta = 0.02$.



(a) Large initial demand $X = 9$ (stopping region).

(b) Small initial demand $X = 5$ (continuation region).

Figure 10: Combinations of α and σ that mitigate double marginalization.
$\mu = 0.03,\ r = 0.1,\ \delta = 40$, and $\eta = 0.02$.

to default mitigates double marginalization in all scenarios except for cases where bankruptcy costs and demand uncertainty are small. We summarize these insights in our last main result.

Result 8 *Regardless of whether initial demand is large or small, the firm's option to default mitigates double marginalization except under small bankruptcy costs and a limited market uncertainty.*

5 Robustness

In this section we first check the robustness of our main results against a wide range of tested parameter values. In addition, we study an extension where the firm has more than one investment option.

Results	σ	r	μ	η	δ
Baseline	0.1	0.1	0.03	0.02	40
Tested Interval	$[0.03, 0.3]$	$[0.04, 0.2]$	$[0, 0.09]$	$[0.01, 0.3]$	$[20, 60]$
Result 1	✓	✓	✓	✓	✓
Result 2	✓	✓	✓	✓	✓
Result 3	✓	✓	✓	✓	✓
Result 4	✓	✓	✓	✓	✓
Result 5	✓	✓	✓	✓	✓
Result 6[a]	✓	✓	$[0.03, 0.09]$	✓	✓
Result 7[b]	—	✓	✓	✓	✓
Result 8[c]	—	✓	✓	✓	✓

[a] We choose several values of α and check that, for a given parameter value in the interval, whether the induced welfare increases with α for both the stopping and continuation regions.

[b] We choose several values of σ and check that, for other given parameter value in the interval, how the investment size and the coupon rate in the stopping region change with σ for both $\alpha = 0$ (size decreases and then increases with σ) and $\alpha = 1$ (coupon decreases and then increases with σ, size increases with σ).

[c] We check for $\alpha = 0$ and small changes in each parameter, whether the social welfare with the default option is larger than that without the option for small-step changes in σ.

Table 2: Robustness check. A checkmark indicates this result is robust.

5.1 Changing the Parameter Values

Our eight main results presented in Section 4 were derived for our baseline parameter setting. In order to examine the robustness of these findings we vary each parameter of our model within a reasonable range around the baseline and check whether our results still hold. From Table 2 we conclude that all results apart from Result 6 are fully robust for all considered parameter ranges. Also Result 6, which states that (with minor exceptions) welfare increases if bankruptcy costs go up, holds on the entire range of considered variations of all parameters except for μ. If the trend parameter μ is too small this result does no longer hold. Intuitively, for μ close to zero the delay in investment caused by the increase of the investment threshold, triggered by a higher value of α, yields a decrease in the discounted welfare stream that outweighs the increase in investment size. Overall, this exercise shows that our qualitative findings are very robust with respect to changes in the parameter values.

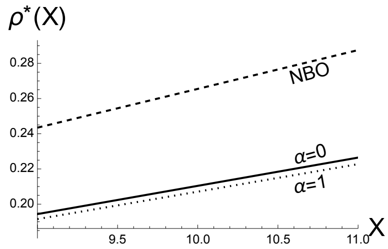
5.2 Multiple Investment Options

As usually assumed in the real options literature, we have considered a scenario in which the firm has only one option to invest during its lifetime. Here, we relax this assumption to investigate whether our main conclusions are still valid. To do so, we allow the firm to invest twice and check the robustness of our Result 8 that the bankruptcy option mitigates the double marginalization effect. In particular, we look at a scenario where the firm needs to borrow from the lender to make both the first and the second investment. As before, the coupon scheme is chosen by the lender at time $t = 0$ and the resulting coupon rate is set at the moment of the first investment and remains constant afterwards. Figure 11a illustrates the optimal coupon rate $\rho^*(X)$ in case initial market demand is sufficiently large that the firm undertakes the first investment immediately.

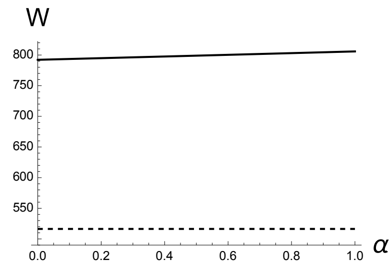
From this figure we derive that the presence of the bankruptcy option decreases the lender's optimal coupon rate. The event of bankruptcy results in bankruptcy costs incurred by the lender. For this reason the lender reduces the coupon rate to tempt the firm to delay bankruptcy occurrence.

However, in Figure 11a we observe that the coupon rate is also lower for $\alpha = 0$, which is different from the one-investment case (see Figure 2). The reason is that for the two-investment scenario there are implicit bankruptcy costs even when $\alpha = 0$, because an eventual firm bankruptcy after the first investment would kill the second (future) investment opportunity. As such, because of the presence of an implicit bankruptcy cost at any α, the scenario where $\alpha = 0$ is equivalent to the cases in our main model with a positive α where we find the same qualitative result.

Similar to the one-investment case, a lower coupon rate mitigates the double marginalization effect, resulting in higher welfare as shown in Figure 11b. We conclude that indeed we have robustness for this result.



(a) Optimal coupon scheme $\rho^*(X)$ in the stopping region for $\alpha = 0$ (solid), $\alpha = 1$ (dotted) and for the NBO scenario (dashed).



(b) Effect of α on social welfare in scenarios with (solid line) and without (dashed line) bankruptcy option for $X = 9$ (stopping region).

Figure 11: Optimal coupon rate $\rho^*(X)$ and the effect of α on welfare W when the firm has the option to invest twice.

$$\mu = 0.03,\ r = 0.1,\ \sigma = 0.1,\ \delta = 40, \text{ and } \eta = 0.02.$$

6 Concluding Remarks

This paper investigates the effect market power on the capital market has on the firm's optimal investment decision. In particular, the lender, which finances the firm's investment, can determine the coupon rate it will charge. Taking into account the coupon scheme offered by the lender, the firm decides at which point it accepts the loan offer in order to invest. Simultaneously the firm also determines the size of its investment.

Upon investing the firm is able to produce goods, which can be sold on a market facing demand uncertainty. This makes the firm's revenue uncertain, while at the same time the firm incurs a fixed coupon payment. When demand realizations are low, the firm makes losses and thus has an incentive to default. The paper investigates both the scenarios where the firm commits to not to file for bankruptcy and where it keeps this possibility open.

Focusing first on the non-bankruptcy case, we find that the lender offers the loan with a markup on the interest rate. Compared to a situation where the firm can finance the investment itself, the increased cost of finance leads the firm to invest less, implying a detrimental welfare effect. Hence, we identify double marginalization in the financial world: the markup by the lender causes higher investment costs for the firm, resulting in less investment and output, and thus a welfare loss.

If the firm keeps the option to declare bankruptcy open, the downside potential of the investment is limited for the firm. The upside potential generated by positive demand shocks is unchanged. Due to this asymmetry the firm has an incentive to invest in a larger scale. This contributes to a mitigation of double marginalization. Furthermore, the bankruptcy option influences also the size of the coupon rate, which affects the extend of double marginalization as well. Two opposing effects arise in this respect. First, the lender does not want bankruptcy to occur and hence lowers the coupon rate. By lowering the coupon rate it increases the firm's payoff of being active, thereby reducing its incentive to exercise the bankruptcy option. Second, the lender realizes that bankruptcy typically occurs at a moment of low demand realizations, and that it takes over the firm at this point. In this way, the lender is confronted with the downside demand potential. Because of this, it has an incentive to reduce the firm's investment which can be achieved by increasing the coupon rate. Among these two effects, the first mitigates double marginalization whereas the second strengthens it. Whereas the intensity of the second effect is not directly affected by the size of bankruptcy costs, the first effect becomes more pronounced the larger bankruptcy costs are. Overall, these consideration imply that the bankruptcy option mitigates double marginalization for all scenarios except for very low bankruptcy costs and that the mitigation of double marginalization is more pronounced the higher bankruptcy costs are.

All in all, this leads to the surprising result that introducing a significant amount of bankruptcy costs will increase welfare. In such a situation the lender fears bankruptcy, and therefore it reduces the markup of the interest rate of the loan. This stimulates the firm to increase investment even more, beyond the effect in the same direction that already exists due to the downside potential being limited by the bankruptcy option.

It is interesting to relate this finding to the insight from the literature on hold-up problems (see e.g. Che and Sákovics (2008)) that in vertical production chains the more specific investment is to the vertical relationship the more are investment incentives distorted downwards and the lower is welfare. In our setting a high value of the bankruptcy cost parameter α, which determines how valuable the investment is for the lender after the break-up of the vertical relationship, can be interpreted as an indication of the specificity of the firm's investment. In this sense, our result can be interpreted that in a setting with a vertical relationship between lender and producer the more specific the investment is the larger is welfare. Hence, the effect of investment specificity on welfare is opposite to the one arising in a standard hold-up problem.

Our finding that having the bankruptcy option incentivizes the lender to choose a lower coupon rate and leads to larger investment provides an innovative addition to the literature, which typically highlights negative implications of young firms' inability to provide sufficient collateral. Our analysis shows that in a setting where the lender has market power, its strategic reaction to this inability, i.e. setting a lower coupon rate, is not only profitable for the firm but also leads to higher welfare.

Our findings also have clear implications for the design of regulatory schemes and policies on the credit market. In particular, policies and regulatory schemes which reduce the bankruptcy costs of lenders could have negative effects on investment and welfare if the lender has market power. Examples of such policies are schemes, e.g. by the European Investment Bank, insuring (parts of) loans by SMEs.[12] Also, institutional settings which deter firms from filing for bankruptcy, could lead to behavior of the lender which is detrimental for investment and welfare.

Our analysis is based on several assumptions fostering the tractability of our model, which might be relaxed in future work. First, we abstract from the firm's choice how to finance its investment, by assuming that the size of the loan equals the investment amount. In order to endogenize the firm's choice of debt/equity ratio, an extension of the model, which keeps track of the dynamics of firm savings, could be considered. Second, competition on the product and on the credit market could be analyzed. An inspiring paper combining

[12] See `https://www.eib.org/en/products/guarantees/sme-mid-cap-guarantees/index`.

financial frictions with strategic interactions is Doraszelski et al. (2022), where the focus is on the evolution of oligopolistic industries over time. Our aim will be to investigate the strategic interactions between firms and banks and the resulting decisions on investments and coupon rates.

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Appendix

A Details of the analysis of the lender's optimal coupon scheme

In this Appendix we provide the details of the analysis of the lender's optimal coupon scheme and the implications for firm investment, which is the basis for Proposition 5 in the main text. Using the definition of $\rho^{imm}(X)$, as given in (9) we first show some basic properties of this scheme.[13]

Lemma 1 *It holds for all $X > 0$ that $\rho^{imm}(X)$ is finite and $\rho^{imm}(X) > r$.*

We now proceed as follows in order to find the equilibrium coupon scheme and investment timing. First, we determine the firm's optimal investment timing under the assumption that the lender offers ρ^{imm} for all X under consideration. We derive the investment threshold $\tilde{X}_F$ of the firm, which is optimal under our assumption that the lender offers ρ^{imm} for all X. As a second step we determine a threshold $\tilde{X}_D$ with the property that under the assumption that the firm invests immediately for a coupon rate $\rho^{imm}(X)$ it is optimal for the lender to offer financing if and only if $X \geq \tilde{X}_D$. As a third step we distinguish between scenarios where $\tilde{X}_D \geq \tilde{X}_F$ and $\tilde{X}_D < \tilde{X}_F$ and characterize the optimal investment timing in equilibrium in both cases.

Turning first to the firm's investment timing under the assumption that the lender offers the coupon scheme $\rho^{imm}(X)$ we first note that it follows from Proposition 4 that under the condition that $\rho^{imm}(X)/X$ decreases with X the firm delays investment if X is sufficiently small. We will later verify numerically that in our framework this condition on $\rho^{imm}(X)/X$ is satisfied and proceed by defining $\tilde{X}_F = X_F^*(\rho^{imm})$ as the firm's investment threshold under the coupon scheme ρ^{imm}.

Second, we consider the lender's timing problem under the assumption that it offers a coupon rate $\rho^{imm}(\cdot)$ and the firm invests immediately. The lender then takes into account how the firm's optimal investment size depends on the coupon scheme. Analogous to the firm's problem, we assume that the lender is willing to offer financing for all values of X above a certain threshold $\tilde{X}_D$. This threshold is determined as the solution to the optimal stopping problem

$$\sup_{\tilde{X}_D \geq 0} \mathbb{E}_0 \left\{ e^{-r\tilde{\tau}(\tilde{X}_D;x(0))} J_D(\tilde{X}_D, \rho^{imm}(\tilde{X}_D), I^*(\tilde{X}_D, \rho^{imm}(\tilde{X}_D))) \right\},$$

for sufficiently small $x(0)$, where, as above, $\tilde{\tau}(\tilde{X}; x(0))$ denotes the first time that the gBm process $x(t)$ hits $\tilde{X}$ from below.

Proposition 6 shows that there indeed exists a positive optimal threshold $\tilde{X}_D$ for the lender such that it is only willing to offer financing under ρ^{imm} for values of X that are at least $\tilde{X}_D$.

Proposition 6 *There exists a $\tilde{X}_D > 0$ such that for $X < \tilde{X}_D$ it is not optimal for the lender to offer financing to the firm, i.e., it sets $\rho^* = \infty$.*

Propositions 4 and 6 show that there is a threshold $\tilde{X}_F$ respectively $\tilde{X}_D$ such that the firm respectively the lender wants to delay investment under the schedule ρ^{imm} for any X below the threshold. In the absence of the bankruptcy option standard results from real options theory establish that investment is undertaken immediately in the entire connected interval above this threshold (see Proposition 1). Due to the complexity added by the existence of the bankruptcy option, it seems infeasible to prove this property for the general

[13] The first order condition, which $\rho^{imm}(X)$ has to satisfy for all X, where the firm invests a positive amount under the coupon rate $\rho^{imm}(X)$, is provided in the proof of Lemma 1.

setting considered in this section. However, extensive numerical analyses (see also the results presented in Section 4) indicate that also in the setting with bankruptcy option, under the assumption that the coupon scheme is $\rho^{imm}(\cdot)$, the firm respectively the lender want immediate investment whenever X is larger or equal than the corresponding threshold. In our following analysis we assume that the solution to the optimal timing problem of the lender and the firm are indeed of this form.

Proposition 6 implies that the lender wants to delay investment by the firm if $X < \tilde{X}_D$. For $X \geq \tilde{X}_D$ the lender would like the firm to invest immediately at a coupon schedule $\rho^{imm}(\cdot)$, however it is not guaranteed that the firm is willing to invest at $\rho^{imm}(X)$. Hence, to determine the optimal investment timing in equilibrium, as the third step in our analysis, we distinguish between the cases where $\tilde{X}_D \geq \tilde{X}_F$ and $\tilde{X}_D < \tilde{X}_F$. In the first of these two cases the firm is willing to invest immediately under coupon scheme $\rho^{imm}(X)$ for any value of X where the lender is willing to provide financing, i.e. for all $X \geq \tilde{X}_D$. In such a scenario the lender's optimal coupon scheme is given by

$$\rho^*(X) = \begin{cases} \rho^{imm}(X) & \text{for all } X \in [\tilde{X}_D, \infty), \\ \infty & \text{for all } X \in [0, \tilde{X}_D). \end{cases} \tag{12}$$

If $\tilde{X}_D < \tilde{X}_F$, then for $X \in [\tilde{X}_D, \tilde{X}_F)$ the firm is not willing to invest immediately under a coupon rate $\rho^{imm}(X)$. Since $X \geq \tilde{X}_D$, it might be optimal for the lender to offer a lower coupon rate in order to still induce immediate investment.

In the following proposition we show that in this case there exists a region $[\hat{X}_D, \tilde{X}_F)$ with $\hat{X}_D > \tilde{X}_D$ such that it is optimal for the lender to offer financing at a coupon rate $\rho^{ind}(X) < \rho^{imm}(X)$ in this region. The coupon rate $\rho^{ind}(X)$ has the property that it makes the firm indifferent between investing immediately and delaying investment, which implies that it is the highest coupon rate for which the firm is willing to invest immediately. Formally, we use the following definition.

Definition 1 *A schedule $\rho^{ind}(X)$ for $X \in [\underline{X}, \tilde{X}_F]$ is called an indifference coupon scheme if for any $\breve{X} \in [\underline{X}, \tilde{X}_F)$ the following property holds. If the lender offers a coupon scheme of the form*

$$\rho(X) = \begin{cases} \infty & \textit{for all } X \in [0, \underline{X}), \\ \rho^{ind}(X) & \textit{for all } X \in [\underline{X}, \tilde{X}_F) \setminus \{\breve{X}\}, \\ \breve{\rho} & \textit{for } X = \breve{X}, \\ \rho^{imm}(X) & \textit{for all } X \in [\tilde{X}_F, \infty), \end{cases}$$

then it is optimal for the firm to invest immediately at $\breve{X}$ if and only if $\breve{\rho} \leq \rho^{ind}(\breve{X})$.

Clearly for $X = \tilde{X}_F$ we have $\rho^{ind}(\tilde{X}_F) = \rho^{imm}(\tilde{X}_F)$ and existence and uniqueness of the indifference coupon scheme can be easily obtained (see proof of Proposition 7). Using this definition we can characterize the optimal coupon scheme for the case where the investment trigger of the firm is above the trigger for the lender.

Proposition 7 *Assume that $\tilde{X}_D < \tilde{X}_F$. Then there exists a threshold $\hat{X}_D \in (0, \tilde{X}_F)$ and an indifference coupon scheme $\rho^{ind}(X) \in (r, \rho^{imm}(X))$ for $X \in [\hat{X}_D, \tilde{X}_F)$ such that $\rho^*(X) = \rho^{ind}(X)$ and the firm invests immediately under ρ^{ind} for $X \in [\hat{X}_D, \tilde{X}_F)$.*

Similar to Proposition 6 dealing with the lender's problem under the assumption that the firm invests immediately, also for the lender problem considered in Proposition 7 it is not possible to prove analytically that the continuation region is a connected interval if the lender chooses the coupon scheme ρ^{ind} below $\tilde{X}_F$.

However, also for this problem our extensive numerical exploration suggests that this intuitive property holds and that for all $X < \hat{X}_D$ it is optimal for the lender to delay firm investment. Assuming that this property holds the optimal coupon rate offered by the lender for $\tilde{X}_D < \tilde{X}_F$ is given by

$$\rho^*(X) = \begin{cases} \rho^{imm}(X) & \text{for all } X \in [\tilde{X}_F, \infty), \\ \rho^{ind}(X) & \text{for all } X \in [\hat{X}_D, \tilde{X}_F), \\ \infty & \text{for all } X \in [0, \hat{X}_D). \end{cases} \tag{13}$$

Note that, contrary to the scenario $\tilde{X}_D < \tilde{X}_F$, for the case where $\tilde{X}_D > \tilde{X}_F$, there is no need to consider a coupon scheme of the lender that leads to investment being undertaken for X below $\tilde{X}_D$. Under the scheme ρ^{imm}, the firm would be willing to invest immediately for any $X \in [\tilde{X}_F, \tilde{X}_D]$ whereas the lender prefers to delay investment. Therefore, because the lender value under any coupon rate different from $\rho^{imm}(X)$ is even lower, the lender has no incentive to offer a coupon rate that leads to earlier investment. Propositions 1 and 7 directly imply the following Corollary.

Corollary 1 *The optimal coupon scheme of the lender satisfies* $\rho^*(X) > r$ *for all* $X > 0$.

Proposition 5 in the main text summarizes the results derived in this Appendix.

B Auxiliary Results and Proofs

Proof of Proposition 1:
We proceed in three steps. First, we determine the firm's optimal investment size under the assumption that financing is offered for any X. Second, we will consider the problem of the lender. We determine the coupon rate for any X under the condition that the firm is willing to invest immediately. Third, we determine the firm's optimal investment timing, following from the standard value matching and smooth pasting conditions which provide necessary optimality conditions for the investment threshold for the firm. In order to be able to do that, we first assume that financing is offered for any X and show that there is a unique firm threshold $\tilde{X}_F$. Thus, the firm is willing to undertake investment immediately on $[\tilde{X}_F, \infty)$ for some $\tilde{X}_F > 0$. For the lender's problem, we show that there exists a unique threshold $\tilde{X}_D$ such that the lender is willing to offer financing on $[\tilde{X}_D, \infty)$ and that $\tilde{X}_D = \tilde{X}_F$. From this it follows that $X_F^* = \tilde{X}_F = \tilde{X}_D$ and, therefore, $S = [X_F^*, \infty)$.
(i) In order to determine the optimal investment size we start out by calculating the firm's net present value in the stopping region. This is given by

$$\tilde{J}_F(X, \rho(X), I) = \mathbb{E}_0 \int_0^\infty e^{-rt}\pi(x(t), I; \rho)\mathrm{d}t = \frac{X}{r-\mu}I(1-\eta I) - \frac{\rho(X)}{r}\delta I.$$

To find the optimal scale of investment, the first order condition gives

$$I^*(X, \rho(X)) = \frac{1}{2\eta}\left(1 - \frac{\delta(r-\mu)}{X}\frac{\rho(X)}{r}\right). \tag{14}$$

The second order condition confirms that (14) yields a (global) maximum. We will show later that values of X such that $I^* < 0$ are not considered, so that (14) gives a solution to the optimization problem. Inserting the optimal investment gives the value function for $X \in S$:

$$J_F^*(X) = \frac{(rX - (r-\mu)\delta\rho(X))^2}{4r^2(r-\mu)\eta X}. \tag{15}$$

(ii) Then turning to the lender's problem, let us first define $\rho^{imm}(X)$ as the optimal coupon rate for X, under the assumption that the firm undertakes investment immediately. The lender's net present value, for any X, is given by

$$\begin{aligned} J_D(X,\rho) &= \mathbb{E}_0 \int_0^\infty e^{-rt} \rho \delta I^*(X,\rho) \mathrm{d}t - \delta I^*(X,\rho) \\ &= \frac{\rho - r}{r} \delta I^*(X,\rho) \\ &= \frac{\rho - r}{2\eta r} \delta \left(1 - \frac{\delta(r-\mu)}{X} \frac{\rho}{r}\right). \end{aligned} \tag{16}$$

Taking the derivative with respect to ρ yields

$$\frac{\partial}{\partial \rho} J_D = \frac{1}{2\eta r} \delta \left(1 - \frac{\delta(r-\mu)}{X} \frac{\rho}{r}\right) - \frac{\rho - r}{2\eta r} \delta \frac{\delta(r-\mu)}{X} \frac{1}{r}.$$

From $\frac{\partial}{\partial \rho} J_D = 0$ we obtain

$$\rho^{imm}(X) = \frac{r(X + \delta(r-\mu))}{2\delta(r-\mu)} > 0. \tag{17}$$

Hence,

$$I^*(X, \rho^{imm}(X)) = \frac{X - \delta(r-\mu)}{4\eta X}. \tag{18}$$

(iii) Consider now the firm's optimal stopping problem. In particular we first treat the auxiliary problem where the lender offers differentiable coupon scheme $\rho^{imm}(\cdot)$ for all $X \in (0,\infty)$. Denote by $\mathcal{L}$ the infinitesimal generator, i.e.

$$\mathcal{L} = \mu X \frac{\partial}{\partial X} + \tfrac{1}{2}\sigma^2 X^2 \frac{\partial^2}{\partial X^2}.$$

Let C_F denote the continuation region, and let ∂C_F denote a (potential) boundary. As standard for these problems (see, e.g., Peskir and Shiryaev (2006)), the firm's value function is given by some function ϕ that solves a free boundary problem, so that then $V_F = \phi$. That is, in the stopping region it holds that $\phi = J_F^*$ (i.e., $S = \mathbb{R}_+ \backslash C_F$) and in the continuation region ϕ solves $\mathcal{L}\phi = r\phi$ with conditions $\frac{\partial}{\partial X}\phi(\tilde{X}) = \frac{\partial}{\partial X} J_F^*(\tilde{X})$ ("*smooth pasting*"), and $\phi(\tilde{X}) = J_F^*(\tilde{X})$ for all $\tilde{X} \in \partial C_F$ ("*value matching*").

The solution to $\mathcal{L}\phi = r\phi$, under the boundary condition $\phi(0) = 0$, (see, e.g., Dixit and Pindyck (1994)) is given by $\phi(X) = AX^{\beta_1}$ where A follows from the free boundary conditions and where β_1 is the positive root of the quadratic polynomial of

$$\frac{1}{2}\sigma^2\beta^2 + (\mu - \frac{1}{2}\sigma^2)\beta - r = 0.$$

Inserting $\phi(X) = AX^{\beta_1}$ into the value matching and smooth pasting conditions gives after some transformation the following equation to be satisfied for any X at the boundary ∂C,

$$\beta_1 \left(\frac{X}{r-\mu} I(1-\eta I) - \frac{\rho(X)}{r} \delta I \right) = \frac{X}{r-\mu} I(1-\eta I) - \frac{X}{r} \delta I \frac{\partial}{\partial X} \rho(X). \tag{19}$$

Solving (19) and (14) simultaneously, using (17), gives the unique solution

$$\begin{aligned} \tilde{X}_F &= \frac{\beta_1 + 1}{\beta_1 - 1} \delta(r-\mu), \\ I^*(\tilde{X}_F, \rho^{imm}(\tilde{X}_F)) &= \frac{1}{2\eta(\beta_1 + 1)}. \end{aligned}$$

Notice that $\rho^{imm} > r$ and $I^* > 0$ for all $X \geq \tilde{X}_F$, because $\beta_1 > 1$. Hence, under the coupon scheme where $\rho(X) = \rho^{imm}(X)$ for all $X > 0$, the stopping region under the firm's optimal investment strategy is given by $[\tilde{X}_F, \infty)$.

Now we consider for what values of X the lender is willing to offer ρ^{imm}. The value function for the lender for such X is given by inserting (17) into (16). Value matching and smooth pasting conditions using this value function imply that for any X on the boundary of the continuation region, $X \in \partial C_D$, we must have

$$\beta_1 \left(\frac{X}{\delta(r-\mu)} - 1 \right)^2 = \left(\frac{X}{\delta(r-\mu)} - 1 \right) \left(\frac{X}{\delta(r-\mu)} + 1 \right),$$

which has two solutions: $\tilde{X}_{D1} = \delta(r-\mu)$ and

$$\tilde{X}_{D2} = \frac{\beta_1 + 1}{\beta_1 - 1} \delta(r-\mu).$$

Under the first of these solutions $I^*(\tilde{X}_{D1}, \rho^{imm}(\tilde{X}_{D1})) = 0$, which implies that the only candidate for boundary ∂C_D is $\tilde{X}_D = \tilde{X}_{D2}$. Since $\tilde{X}_D = \tilde{X}_F$ under this solution we indeed have that $X_F^* = \tilde{X}_F = \tilde{X}_D$ and $S = \left[\frac{\beta_1+1}{\beta_1-1}\delta(r-\mu), \infty\right)$. At the threshold, it then holds that

$$\rho^{imm}(X_F^*) = r\frac{\beta_1}{\beta_1 - 1}.$$

Finally, notice that $I^*(X_F^*, \rho(X_F^*)) > 0$ so that $I^* > 0$ for all $X \in S$. Furthermore,

$$J_D(\tilde{X}_D, \rho^*(\tilde{X}_D)) = \frac{\delta}{2\eta(\beta_1^2 - 1)} > 0,$$

which shows that it is optimal for the lender to provide a coupon scheme. □

Proof of Proposition 2:

Because the default option is conditional on the firm being active in the market, we reset the present time to a point t' after investment ($t' \geq \tau_F$) and denote the corresponding current value of the geometric Brownian motion as $X = x(t')$. Then it holds that the firm's value equals

$$V_B(X; \tilde{I}, \tilde{\rho}) = \begin{cases} 0 & \text{for } X \in D_B, \\ \frac{X(1-\eta\tilde{I})\tilde{I}}{r-\mu} - \frac{\tilde{\rho}\delta\tilde{I}}{r} + A_B X^{\beta_2} & \text{for } X \notin D_B, \end{cases}$$

for some $D_B \subseteq \mathbb{R}_+$. Let

$$\chi_B(X) = -\beta_2 \left(\frac{X(1-\eta\tilde{I})\tilde{I}}{r-\mu} - \frac{\tilde{\rho}\delta\tilde{I}}{r} \right) + X\frac{\partial}{\partial X} \left(\frac{X(1-\eta\tilde{I})\tilde{I}}{r-\mu} - \frac{\tilde{\rho}\delta\tilde{I}}{r} \right).$$

According to standard real options theory (see, e.g., Dixit and Pindyck (1994)) it follows that for any X, if $\chi_B(X) > 0$ then the firm delays bankruptcy so that the firm exits when $x(t)$ enters a region with the property that $\chi_B(X) < 0$ for all interior points. Since $\tilde{I}$ and $\tilde{\rho}$ are fixed, χ_B is clearly linear in X and thus it is the case that D_B takes the shape of an interval so that $[0, \infty)$ can be split up by means of a threshold separating the stopping region and continuation region.

Note that $\chi_B(0) = \beta_2 \frac{\tilde{\rho}\delta\tilde{I}}{r} < 0$ and $\chi_B'(X) = -(\beta_2 - 1)\frac{(1-\eta\tilde{I})\tilde{I}}{r-\mu} > 0$, so that, due to the linearity of χ_B with respect to X, there exists a unique $X_B^* < \infty$ such that bankruptcy is optimal for all $X \in D_B = [0, X_B^*]$.

The value matching and smooth pasting conditions at the default threshold yield that

$$X_B^*(I, \rho) = \frac{\beta_2}{\beta_2 - 1} \frac{\rho\delta(r-\mu)}{r(1-\eta I)}.$$

□

The following lemma gives some properties of the firm's optimal choice of investment size, which will be useful in the following analysis.

Lemma 2 *Let* $\rho : [0,\infty) \to [r,\infty)$. *Assume* $\rho(X)$ *is continuously differentiable almost everywhere on* $[0,\infty)$. *Then,*

(i) The objective function in the firm's optimization problem in (8) *has at most one interior local maximum,*

(ii) $I^*(X,\rho(X)) < \frac{1}{2\eta}$ *and* $I^*(X,\rho(X))$ *is continuous almost everywhere on* $[0,\infty)$ *as a function of* X, *and*

(iii) $I^*(X,\tilde{\rho})$ *is continuous almost everywhere on* $[0,\infty)]$ *as a function of* $\tilde{\rho}$.

Proof of Lemma 2:
First notice that (8) is C^2 since we have $X_B^* > 0$ and $\rho > 0$. Taking the derivative of (8) with respect to I yields that $I^*(X,\tilde{\rho}) > 0$ satisfies

$$g(X,I,\tilde{\rho}) = \frac{\beta_2}{\beta_2 - 1}\left(\frac{X}{X_B^*(I,\tilde{\rho})}\right)\left(1 - \frac{\eta I}{1-\eta I}\right)\left(1 - \left(\frac{X}{X_B^*(I,\tilde{\rho})}\right)^{\beta_2 - 1}\right) - 1 + \left(\frac{X}{X_B^*(I,\tilde{\rho})}\right)^{\beta_2} = 0, \quad (20)$$

where it has to be kept in mind that X_B^* is a function of I and ρ and therefore also X. Furthermore, (8) increases in I if and only if $g(X,I) > 0$.
(i) Concerning $\frac{\partial g(X,I,\rho(X))}{\partial I}$, we have

$$\frac{\partial g(X,I,\rho(X))}{\partial I} = -\frac{\beta_2}{\beta_2 - 1}\left(\frac{X}{X_B^*(I,\rho(X))}\right)\frac{\eta}{(1-\eta I)^2}\left(1 - \left(\frac{X}{X_B^*(I,\rho(X))}\right)^{\beta_2 - 1}\right) + \frac{\partial g}{\partial\left(\frac{X}{X_B^*}\right)}\frac{\partial\left(\frac{X}{X_B^*(I,\rho(X))}\right)}{\partial I}.$$

The first term is clearly negative. To obtain a sign for $\partial\left(\frac{X}{X_B^*(I,\rho(X))}\right)/\partial I$ we observe that X is independent from I and it follows directly from (7) that X_B^* increases with I. Hence, $\partial\left(\frac{X}{X_B^*}\right)/\partial I < 0$ and taking into account $\partial g/\partial\left(\frac{X}{X_B^*}\right) > 0$ we obtain $\frac{\partial g(X,I,\rho(X))}{\partial I} < 0$. This implies that the equation (20) has at most one solution for $I \in \left[0, \frac{1}{2\eta}\right]$ and if it has a solution, then it is a local maximum of the firm's objective function.
(ii) To prove continuity of $I^*(X,\rho(X))$ with respect to X we first consider the case where $g(X,0,\rho(X)) > 0$. The optimal investment level $I^*(X,\rho(X))$ is then given by the unique interior root of $g(X,I,\rho(X))$ and since g depends continuously on X and I and strictly monotonously on I this root changes continuously with X. For all X with $g(X,0,\rho(X)) \leq 0$ the arguments above imply that $I^*(X,\rho(X)) = 0$.

Next, to show that $I < \frac{1}{2\eta}$, we consider two cases. For all X such that $X > X_B^*$, we have that for $I = \frac{1}{2\eta}$ it holds that $g(X,I,\rho(X)) = -1 + \left(\frac{X}{X_B^*}\right)^{\beta_2} < 0$. Thus, we have that the root lies to the left of $\frac{1}{2\eta}$. For any X with $X < X_B^*$, the firm goes bankruptcy immediately, and therefore optimal investment is zero. Hence, $I^*(X,\rho(X)) < \frac{1}{2\eta}$ for all $X > 0$.
(iii) To prove continuity of $I^*(X,\tilde{\rho})$ with respect to $\tilde{\rho}$ the same arguments as used in (ii) apply, taking into account that $g(X,I,\tilde{\rho})$ depends continuously on $\tilde{\rho}$. □

Proof of Proposition 3:
For the uniqueness of I^*, first, we observe that it follows from Lemma 2(ii) that the optimal value of I is in $\left[0, \frac{1}{2\eta}\right]$ and therefore finite. Furthermore, point (i) of the Lemma establishes that there is at most one value

in $\left[0, \frac{1}{2\eta}\right]$ satisfying the first order condition. Therefore, what remains to be shown is that if $I = 0$ is optimal then there is no strictly positive $\tilde{I}$ yielding the same value for the firm's objective function $\tilde{J}_F(X, \tilde{\rho}, I)$. Assume that such a $\tilde{I}$ exists, then $g(X, \tilde{I}, \tilde{\rho}) = 0$ with g given by (20). Furthermore, it has been shown in the proof of Lemma 2(ii) that $\frac{\partial g(X,I,\tilde{\rho})}{\partial I} < 0$. Hence, $g(X, I, \tilde{\rho}) > 0$ for all $I \in [0, \tilde{I}]$, which contradicts that $J_F(X, \tilde{\rho}, 0) = J_F(X, \tilde{\rho}, \tilde{I}) = 0$.

(i) We need to show that $\partial I^*/\partial X \geq 0$ if and only if $\rho(X)/X$ is decreasing in X. We focus on the case where the optimal value of I is strictly positive, since as long as optimal investment is zero it is constant in X. Taking into account that a strictly positive I^* has to satisfy (20), we obtain from total differentiation with respect to X

$$\frac{\partial I^*(X)}{\partial X} = -\frac{dg(X, I^*, \rho(X))}{dX} / \frac{\partial g(X, I^*, \rho(X))}{\partial I}.$$

Note that the differentiability of $g(X, I, \rho(X))$ with respect to X together with $\frac{\partial g(X,I,\rho(X))}{\partial I} < 0$ implies that $I^*(X, \rho(X))$ is differentiable. Let us write $\frac{dg(X,I^*,\rho(X))}{dX}$ as

$$\frac{dg(X, I^*, \rho(X))}{dX} = \frac{\partial g}{\partial\left(\frac{X}{X_B^*}\right)} \cdot \frac{d\left(\frac{X}{X_B^*}\right)}{dX}.$$

For the first of these terms we have

$$\frac{\partial g}{\partial\left(\frac{X}{X_B^*}\right)} = \frac{\beta_2}{\beta_2 - 1}\left(1 - \frac{\eta I^*}{1 - \eta I^*}\right)\left(1 - \beta_2\left(\frac{X}{X_B^*}\right)^{\beta_2 - 1}\right) + \beta_2\left(\frac{X}{X_B^*}\right)^{\beta_2 - 1}.$$

To show that this expression is positive, we first observe that $g(X, 0, \rho(X)) = 0$ has unique solution $X = X_B^*$. Because $I^* \searrow 0$ as $X \searrow X_B^*$ (see proof of Lemma 2(ii)), for X such that $X = X_B^*$ the expression for $\partial g/\partial\frac{X}{X_B^*}$ reduces to

$$\begin{aligned}\frac{\partial g}{\partial\left(\frac{X}{X_B^*}\right)} &= \frac{\beta_2}{\beta_2 - 1} - \frac{1}{\beta_2 - 1}\beta_2\left(\frac{X}{X_B^*}\right)^{\beta_2 - 1} \\ &\geq \frac{\beta_2}{\beta_2 - 1} - \beta_2\frac{1}{\beta_2 - 1} \\ &= 0,\end{aligned}$$

where the inequality follows from $\left(\frac{X}{X_B^*}\right) > 1$ and $\beta_2 < 0$. Furthermore, we have

$$\frac{\partial^2 g}{\partial\left(\frac{X}{X_B^*}\right)^2} = (\beta_2 - 1)\beta_2\left(\frac{X}{X_B^*}\right)^{\beta_2 - 2}\left(1 - \frac{\beta_2}{\beta_2 - 1}\left(1 - \frac{\eta I^*}{1 - \eta I^*}\right)\right) > 0,$$

since $\frac{\beta_2}{\beta_2 - 1}\left(1 - \frac{\eta I^*}{1 - \eta I^*}\right) < 1$. This implies that $\partial g/\partial\left(\frac{X}{X_B^*}\right) > 0$ for all $X \geq X_B^*$.

To finish the proof we now still have to show that $d\left(\frac{X}{X_B^*}\right)/dX > 0$ for a given value of I. Again using (7) we obtain

$$\frac{d\left(\frac{X}{X_B^*}\right)}{dX} = \frac{(\beta_2 - 1)r(1 - \eta I)}{\beta_2\delta(r - \mu)}\frac{d}{dX}\left(\frac{X}{\rho(X)}\right) > 0$$

if and only if $\rho(X)/X$ decreases with X.

(ii) We need to show that $\partial I^*/\partial\rho \leq 0$. Using g again, we obtain from total differentiation

$$\frac{\partial I^*(X, \rho)}{\partial\rho} = -\frac{\frac{\partial g}{\partial(X/X_B^*)}\frac{\partial(X/X_B^*)}{\partial\rho}}{\frac{\partial g}{\partial I^*} + \frac{\partial g}{\partial(X/X_B^*)}\frac{\partial(X/X_B^*)}{\partial I^*}}$$

Since $\frac{\partial g}{\partial(X/X_B^*)} \geq 0$ and $\frac{\partial(X/X_B^*)}{\partial\rho} < 0$, it is sufficient to show that $\frac{\partial g}{\partial I^*} + \frac{\partial g}{\partial(X/X_B^*)}\frac{\partial(X/X_B^*)}{\partial I^*} < 0$. For the denominator we have that

$$\begin{aligned}\frac{\partial g}{\partial I^*}+\frac{\partial g}{\partial(X/X_B^*)}\frac{\partial(X/X_B^*)}{\partial I^*} &= -\left(2-\left(\frac{X}{X_B^*}\right)^{\beta_2-1}\frac{2-(1+\beta_2)\eta I^*}{(1-\eta I^*)}\right)\frac{\eta\beta_2 X/X_B^*}{(\beta_2-1)(1-\eta I^*)}\\ &= -\left(\frac{2-2(1+\beta_2)\eta I^*}{2-(1+\beta_2)\eta I^*}-\left(\frac{X}{X_B^*}\right)^{\beta_2-1}\frac{1-(1+\beta_2)\eta I^*}{(1-\eta I^*)}\right)\frac{\eta\beta_2 X/X_B^*}{(\beta_2-1)(1-\eta I^*)}\frac{2-(1+\beta_2)\eta I^*}{1-(1+\beta_2)\eta I^*}.\end{aligned}$$

The inequality $\frac{2-2(1+\beta_2)\eta I^*}{2-(1+\beta_2)\eta I^*} > \frac{1-2\eta I^*}{1-\eta I^*}$ is equivalent to $(1-\beta_2)\eta I > 0$, which always holds. Using this inequality we have that

$$\begin{aligned}&\left(\frac{2-2(1+\beta_2)\eta I^*}{2-(1+\beta_2)\eta I^*}-\left(\frac{X}{X_B^*}\right)^{\beta_2-1}\frac{1-(1+\beta_2)\eta I^*}{(1-\eta I^*)}\right)\frac{\beta_2}{\beta_2-1}\\ &\quad > \left(\frac{1-2\eta I^*}{1-\eta I^*}-\left(\frac{X}{X_B^*}\right)^{\beta_2-1}\frac{1-(1+\beta_2)\eta I^*}{(1-\eta I^*)}\right)\frac{\beta_2}{\beta_2-1}\\ &\quad = \frac{\partial g(X,I^*)}{\partial(X/X_B^*)}\\ &\quad \geq 0.\end{aligned}$$

The equality follows from the proof of (ii). Therefore, $\dfrac{\partial g}{\partial I^*} + \dfrac{\partial g}{\partial X/X_B^*}\dfrac{\partial X/X_B^*}{\partial I^*} < 0$ and thus $\partial I^*/\partial\rho \leq 0$. □

Proof of Proposition 4:
Define $\chi_F(X;\rho) = -\beta_1 J_F(X,\rho) + X\frac{\partial}{\partial X}J_F(X,\rho)$, where

$$\begin{aligned}J_F(X,\rho) = &\frac{XI^*(X,\rho(X))(1-\eta I^*(X,\rho(X)))}{r-\mu}\\ &-\frac{\rho(X)}{r}\delta I^*(X,\rho(X))\left(1-\frac{1}{1-\beta_2}\left(\frac{X}{X_B^*(I^*(X,\rho(X)),\rho(X))}\right)^{\beta_2}\right),\end{aligned}$$

for all X such that $X \geq X_B^*(I^*(X,\rho(X)),\rho^*(X))$. The proof of Proposition 3 has shown that $I^*(X;\rho) \searrow 0$ as $X \searrow X_B^*$. Notice that the firm goes bankrupt instantly if it were to undertake investment immediately for all X such that $X \leq X_B^*(I^*(X,\rho(X)),\rho^*(X))$, leading to $J_F = 0$ and one can easily check that $\lim_{X\searrow X_B^*} J_F = 0$. Notice that χ_F and J_F are continuous functions which follows from ρ and I^* being C^1 on $[0,\infty)$. According to standard real options theory (see, e.g., Dixit and Pindyck (1994)) it follows that for any X, if $\chi_F(X;\rho) > 0$ then the firm delays investment so that investment is undertaken when X enters a region with the property that $\chi_F < 0$ for all interior points. Thus it is sufficient to show that the claim in Proposition 4 holds if $\inf\{X : \chi_F(X;\rho) < 0\} > 0$. For any X such that $X < X_B^*(I^*(X,\rho(X)),\rho^*(X))$, i.e. when the firm goes bankrupt instantly, it thus holds that $\chi_F = J_F = 0$. Moreover, since $J_F > 0$ for any X arbitrarily close to X_B^* with the condition that $X > X_B^*(I^*(X,\rho(X)),\rho^*(X))$ we must have that $\frac{\partial}{\partial X}J_F(X;\rho) > 0$ for X such that $X = X_B^*(I^*(X,\rho(X)),\rho^*(X))$ and thus $\chi_F > 0$ for X higher than, but arbitrarily close to, X_B^* from which we can conclude that the firm finds itself in the continuation region for sufficiently small X. Moreover, it is not difficult to see that the firm finds itself in the stopping region for $X = \infty$ so that indeed there exists a finite threshold $\tilde{X}_F(\rho) = \inf\{X > 0 : \chi_F(X;\rho) < 0\}$.

$\{\tilde{X}_F, I^*\}$ then solves $\chi_F(X;\rho) = 0$ with (20), i.e.

$$\frac{\beta_1(1-\eta I)}{r-\mu}\left(X - \left(\frac{X}{X_B(I,\rho(X))}\right)^{\beta_2} X_B(I,\rho(X))\right) - \frac{\beta_1\rho(X)}{r}\delta\left(1 - \left(\frac{X}{X_B(I,\rho(X))}\right)^{\beta_2}\right)$$
$$+ \frac{\delta}{r(\beta_2-1)}\left(\beta_2\rho(X) - X(\beta_2-1)\frac{\mathrm{d}\rho(X)}{\mathrm{d}X}\right)\left(\frac{rX(\beta_2-1)(1-\eta I)}{\beta_2\delta(r-\mu)\rho(X)}\right)^{\beta_2} - \frac{(1-\eta I)X}{r-\mu} + \frac{\delta X}{r}\frac{\mathrm{d}\rho(X)}{\mathrm{d}X} = 0 \quad (21)$$

and (20). □

Proof of Lemma 1:
Objective function J_D can alternatively be written as

$$J_D = \frac{\delta I^*(X,\rho)}{r}\left(\rho\underbrace{\left(1 - \frac{1-\alpha\beta_2}{1-\beta_2}\left(\frac{X}{X_B^*(I^*(X,\rho),\rho)}\right)^{\beta_2}\right)}_{\in[0,1)} - r\right). \quad (22)$$

First, since I^* is continuous in ρ (see Lemma 2) and since X_B^* is continuous in ρ (see (7)), it holds that J_D is continuous in ρ. Second, it follows directly from (22) that $J_D < 0$ if $\rho = r$. Third, rewriting the first order condition of (9) gives that ρ^{imm} satisfies

$$(1-\alpha\beta_2)\left(\frac{rX\big(1-\eta I^*(X,\tilde{\rho})\big)(\beta_2-1)}{\beta_2(r-\mu)\tilde{\rho}\delta}\right)^{\beta_2}\left(I^*(X,\tilde{\rho}) + \frac{\tilde{\rho}\big(1-\eta I^*(X,\tilde{\rho})(1+\beta_2)\big)}{(1-\beta_2)\big(1-\eta I^*(X,\tilde{\rho})\big)}\frac{\partial I^*}{\partial\rho}\right)$$
$$-I^*(X,\tilde{\rho}) - (\tilde{\rho}-r)\frac{\partial I^*}{\partial\rho} = 0. \quad (23)$$

One can easily check that $\rho = \infty$ is not optimal since for sufficiently high ρ the cost of investment is too high for the firm and investment is delayed indefinitely. In addition, any $\rho < r$ also cannot give a global optimum of J_D since for these values $J_D < 0$. Hence, the global maximum is attained at a coupon rate on (r,∞). All terms in (23) are continuous and finite and thus the root is well defined.

□

Proof of Proposition 6:
Let $\chi_D(X) = -\beta_1 J_D(X,\rho^{imm}(X)) + X\frac{\partial}{\partial X}J_D(X,\rho^{imm}(X))$. A similar argument as for the proof of Proposition 4 can be constructed with the help of $\chi_D(X)$. Since for sufficiently small X it holds that $J_D = \chi_D = 0$ and since the lender prefers the firm to invest for $X = \infty$ it must hold that $\inf\{X : \chi_D(X) < 0\} > 0$.

Then $\tilde{X}_D$ solves $\chi_D(X) = 0$, i.e.,

$$(1-\alpha\beta_2)\rho^{imm}(X)\left(\frac{rX(\beta_2-1)\left(1-\eta I^*(X,\rho^{imm}(X))\right)}{\beta_2\delta(r-\mu)\rho^{imm}(X)}\right)^{\beta_2}\left(\frac{\big(1-(\beta_2+1)\eta I^*(X,\rho^{imm}(X))\big)X}{(\beta_2-1)\big(1-\eta I^*(X,\rho^{imm}(X))\big)}\frac{\partial I^*}{\partial X}\right.$$
$$\left.+\frac{\beta_2-\beta_1}{\beta_2-1}I^*(X,\rho^{imm}(X))\right) - (\rho^{imm}(X)-r)\left(\beta_1 I^*(X,\rho^{imm}(X)) - X\frac{\partial I^*}{\partial X}\right) = 0, \quad (24)$$

with $\rho^{imm}(X)$ as given by (23).

Notice that in a scenario where $\tilde{X}_F < \tilde{X}_D$, the investment size $I^*(\tilde{X}_D, \rho^{imm}(\tilde{X}_D))$ is the solution of (20) with (23) and (24). □

Proof of Proposition 7:
We proceed in three steps. As a first step we show that there is ρ^{ind} which makes the firm indifferent. Consider

$X \in [0, \tilde{X}_F)$. Since $X < \tilde{X}_F$, for the firm, the value of waiting exceeds the value of immediate investment, i.e., it follows that $V_F(X, \rho^{imm}) > J_F(X, \rho^{imm}(X))$. Define a coupon scheme ρ^{ind} by $\rho^{ind}(\tilde{X}_F) = \rho^{imm}(\tilde{X}_F)$ and

$$\frac{\partial \rho^{ind}(X)}{\partial X} = \frac{\beta_1 J_F(X, \rho^{ind}(X)) - X \frac{\partial J_F(X, \rho^{ind}(X))}{\partial X}}{X \frac{\partial J_F(X, \rho^{ind}(X))}{\partial \rho}} \quad \forall X \in [0, \tilde{X}_F). \tag{25}$$

It follows from this definition that on the entire interval $[\tilde{X}_D, \tilde{X}_F]$ the following smooth pasting condition holds for the firm since in that case

$$\beta_1 J_F(X, \rho^{ind}(X)) = X \frac{\mathrm{d} J_F(X, \rho^{ind}(X))}{\mathrm{d} X}.$$

Since the right hand side of (25) is Lipschitz continuous on the compact interval $[\tilde{X}_D, \tilde{X}_F]$ there exists a continuous solution $\rho^{ind}(X)$ to this boundary problem. The smooth pasting condition implies that the firm is indifferent between investing immediately at $\rho = \rho^{ind}(X)$ and marginally delaying investment on the entire interval, and therefore the condition of Definition 1 is satisfied and ρ^{ind} is an indifference coupon scheme.

As a second step we show that $V_D(X, \rho^{ind}(\cdot)) > V_D(X, \rho^{imm}(\cdot))$ for $X \in [\tilde{X}_F - \epsilon, \tilde{X}_F)$. Since $V_D(\tilde{X}_F, \rho^{ind}(\cdot)) = V_D(\tilde{X}_F, \rho^{imm}(\cdot))$, it is sufficient to show that $\lim_{\xi \to 0+} \frac{\partial V_D(\tilde{X}_F - \xi, \rho^{ind}(\cdot))}{\partial X} < \lim_{\xi \to 0+} \frac{\partial V_D(\tilde{X}_F - \xi, \rho^{imm}(\cdot))}{\partial X}$. Since under ρ^{ind} the firm invests at any $X \in [\tilde{X}_F - \epsilon, \tilde{X}_F)$ we have $V_D(X, \rho^{ind}(\cdot)) = J_D(X, \rho^{ind}(X))$ for all $X \in [\tilde{X}_F - \epsilon, \tilde{X}_F)$. Therefore,

$$\frac{\partial V_D(X, \rho^{ind}(\cdot))}{\partial X} = \frac{\partial J_D(X, \rho^{ind}(X))}{\partial X} + \frac{\partial J_D(X, \rho^{ind}(X))}{\partial \rho} \frac{\partial \rho^{ind}(X)}{\partial X}.$$

Since $\rho^{ind}(\tilde{X}_F) = \rho^{imm}(\tilde{X}_F)$ and by definition of ρ^{imm} we have $\frac{\partial J_D(\tilde{X}_F, \rho^{imm}(\tilde{X}_F))}{\partial \rho} = 0$, we obtain

$$\lim_{\xi \to 0+} \frac{\partial V_D(\tilde{X}_F - \xi, \rho^{ind}(\cdot))}{\partial X} = \frac{\partial J_D(\tilde{X}_F, \rho^{imm}(\tilde{X}_F))}{\partial X}. \tag{26}$$

Furthermore, since under the schedule $\rho^{imm}(\cdot)$ the firm invests only at the trigger $\tilde{X}_F$ we have $V_D(X, \rho^{imm}(\cdot)) = \left(\frac{X}{\tilde{X}_F}\right)^{\beta_1} J_D(\tilde{X}_F, \rho^{imm}(\tilde{X}_F))$. Therefore,

$$\lim_{\xi \to 0+} \frac{\partial V_D(\tilde{X}_F - \xi, \rho^{imm}(\cdot))}{\partial X} = \frac{\beta_1}{\tilde{X}_F} J_D(\tilde{X}_F, \rho^{imm}(\tilde{X}_F)). \tag{27}$$

Since $\tilde{X}_F > \tilde{X}_D$ under the schedule ρ^{imm} the lender strictly prefers the firm to invest immediately at $\tilde{X}_F$ rather than delaying investment, and therefore

$$\frac{\beta_1}{\tilde{X}_F} J_D(\tilde{X}_F, \rho^{imm}(\tilde{X}_F)) > \frac{\partial J_D(\tilde{X}_F, \rho^{imm}(\tilde{X}_F))}{\partial X}. \tag{28}$$

Combining (28) with (26) and (27) we obtain

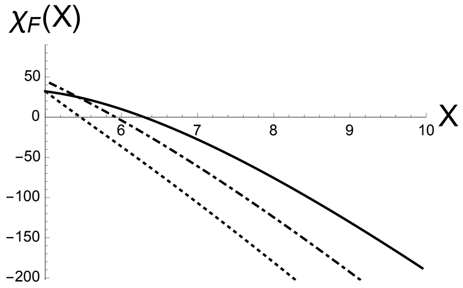
$$\lim_{\xi \to 0+} \frac{\partial V_D(\tilde{X}_F - \xi, \rho^{imm}(\cdot))}{\partial X} > \lim_{\xi \to 0+} \frac{\partial V_D(\tilde{X}_F - \xi, \rho^{ind}(\cdot))}{\partial X}.$$

Hence, there is an ϵ such that $V_D(X, \rho^{ind}(\cdot)) > V_D(X, \rho^{imm}(\cdot))$ for all $X \in [\tilde{X}_F - \epsilon, \tilde{X}_F)$. By continuity, there is a threshold $\hat{X}_D < \tilde{X}_F$ such that this inequality holds on the entire interval $[\hat{X}_D, \tilde{X}_F)$ and it is therefore optimal for the lender to offer the coupon scheme $\rho^{ind}(\cdot)$ on this interval.

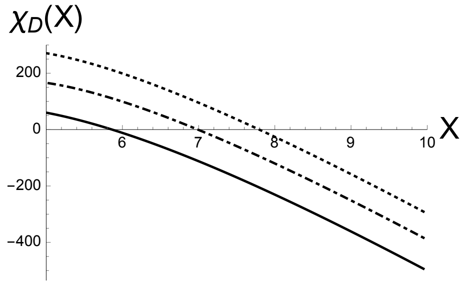
Finally, as a third step, we show that $\rho^{ind} > r$. Using analogous arguments to those in the proof of Lemma 1, we can infer that for $\rho^{ind} \le r$ we have $V_D(X, \rho^{ind}(\cdot)) = J_D(X, \rho^{ind}(X)) < 0$ with J_D given in equation (22). Hence, it can never be optimal for the lender to offer $\rho^{ind}(\cdot)$ for any X with $\rho^{ind}(X) \le r$. □

C Numerical Verification: S and D are intervals

Figure 12 shows that for our default parameter values, functions $\chi_F(X;\rho^{imm}(X))$ and $\chi_D(X)$ have a unique solution for $\tilde{X}_F$ and $\tilde{X}_D$. When plotting these two functions, we consider values of X such that the lender's value non-negative given the lender's coupon scheme $\rho^{imm}(X)$, and such that $\mathrm{d}\rho^{imm}(X)/\mathrm{d}X \geq 0$.



(a) $\chi_F(X;\rho^{imm}(X))$.



(b) $\chi_D(X)$.

Figure 12: $\chi_F(X;\rho^{imm}(X))$ and $\chi_D(X)$ as functions of X for $\alpha = 0$ (solid), $\alpha = 0.5$ (dot-dashed), and $\alpha = 1$ (dotted), conditional on $\hat{J}_D(X,\rho^{imm}(X)) \geq 0$ and $\hat{J}_F(X;\rho^{imm}(X)) \geq 0$ respectively.

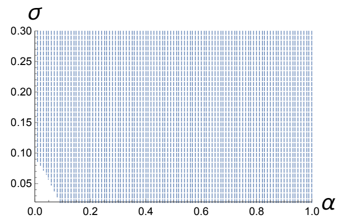
$\mu = 0.03,\ r = 0.1,\ \sigma = 0.1,\ \delta = 40$, and $\eta = 0.02$.

D Robustness Check (Small Initial Demand)

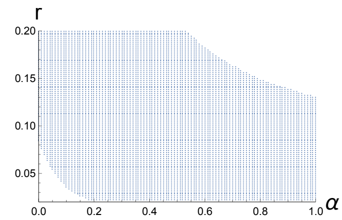
Figure 13 checks the robustness for our result that double marginalization can be mitigated in the case of small initial demand. The shaded areas represent all constellations of the corresponding parameter and the bankruptcy cost parameter α for which we obtain the mitigation result. The void areas represent all constellations for which we do not observe mitigation. This is because the effect of investment delay dominates the effect of the increased investment size, resulting in a smaller welfare compared with the NBO scenario. This typically happens for small values of α.

Panels (a), (d), and (e) show robustness. However, for large α, Figure 13b shows that large values of r may reside in the void area, implying no mitigation. This is similar for small μ in Figure 13c. This is because for a large r the opportunity cost for investment is larger, which delays investment for both the BO and the NBO scenarios. Correspondingly the lender sets a larger coupon rate and investment is decreased. Then, for a sufficiently large bankruptcy parameter α, the BO investment trigger increases faster than the NBO trigger, so the delay effect on welfare dominates and thus no mitigation.

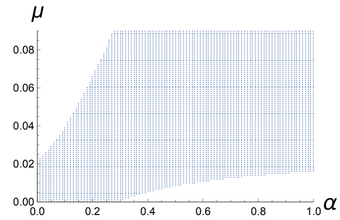
When μ is small, the bankruptcy option becomes very relevant. This in combination with the default event being expensive for the lender, i.e., when α is large, creates an equivalently large wedge between the BO and NBO investment thresholds so that the net result on welfare is negative. This trade-off was already visible in Figure 6b where we found that W is decreasing in α when α is sufficiently large. However, in Figure 6b, μ is sufficiently large so that mitigation happens for any α.



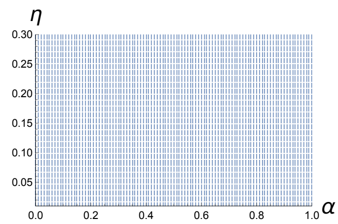
(a) Robustness check for $\sigma \in [0.02,\ 0.3]$.



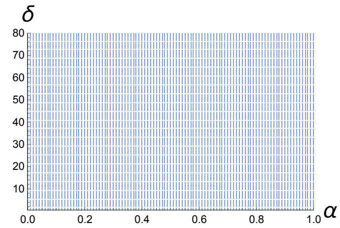
(b) Robustness check for $r \in [0.02,\ 0.2]$.



(c) Robustness check for $\mu \in [0,\ 0.09]$.



(d) Robustness check for $\eta \in [0.01,\ 0.3]$.



(e) Robustness check for $\delta \in [0.5,\ 80]$.

Figure 13: Robustness check that the bankruptcy cost α mitigates double marginalization for different parameters in case of a small initial demand.

$\mu = 0.02,\ r = 0.1,\ \sigma = 0.1,\ \delta = 40,\ X = 1$ and $\eta = 0.02$.