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Dynamically Consistent Intertemporal Dual-Self Expected Utility

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Abstract

Experimental evidence on intertemporal choice has documented a preference for consumption smoothing that cannot be explained by discounted utility. We study a general class of dynamically consistent intertemporal dual-self preferences that accommodate a preference for consumption smoothing. We show that these general preferences have a simple and tractable structure. They are characterized by a gain-loss asymmetry where gains with respect to future utility are discounted differently than losses. As applications, first, we show that under the stationarity axiom, these preferences are convex or concave. Second, we show that dynamically consistent intertemporal Choquet expected utility coincides with discounted expected utility.

1 Introduction

The most used model of intertemporal choice is the exponential discounted utility that was proposed by Samuelson (1937). This evaluates a consumption stream $(x_1, x_2, \dots)$ with the recursive formulation

$$V_t(x_t, x_{t+1}, \dots) = u(x_t) + \delta\Big(V_{t+1}(x_{t+1}, x_{t+2}, \dots) - u(x_t)\Big) = (1-\delta)\sum_{t'=t}^{\infty} \delta^{t'-t} u(x_{t'})$$

where u is the utility function and $\delta \in (0,1)$ is the discount factor. This model is analytically and axiomatically tractable (Koopmans, 1960) and has become the standard model for intertemporal choice.

Violations of the exponential discounted utility are well-documented. One of the characterizing features of this model is the use of the same discount factor δ in each period

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regardless of the distribution of utility. Loewenstein (1987) offered evidence for preference of spread with the following example

$$(G,B,B) \prec (B,G,B) \text{ and } (G,B,G) \succ (B,G,G)$$

where G stands for "good" consumption and B for "bad" consumption. This violates the discounted utility's additive separability since changing the common consumption B into G changes the preferences.

This violation follows from the decision maker valuing costs and benefits differently at different times. We study a dynamically consistent decision maker who values costs and benefits relative to a smooth consumption differently at different times. We show that this decision maker is characterized by a gain-loss asymmetry where gains with respect to future utility are discounted differently from losses with respect to future utility. Formally a consumption stream $(x_1, x_2, \dots)$ is evaluated with the recursive formulation

$$\begin{aligned} &V_t(x_t, x_{t+1}, \dots) \qquad (1)\\ &= u(x_t) + \delta_t^+ \min\left\{V_{t+1}(x_{t+1}, x_{t+2}, \dots) - u(x_t), 0\right\} + \delta_t^- \max\left\{V_{t+1}(x_{t+1}, x_{t+2}, \dots) - u(x_t), 0\right\}\end{aligned}$$

where u is the utility function, $\delta_t^+ \in (0,1)$ is the discount factor for a gain with respect to future utility and $\delta_t^- \in (0,1)$ is the discount factor for a loss with respect to future utility. This shows that our general dynamically consistent preferences have a simple and tractable structure.

Formally, we consider a decision maker who has preferences over risky consumption streams $(x_1, x_2, \dots)$ where each x_i is a lottery over consumption. We make two main assumptions. First, we assume that the decision maker evaluates the consumption streams relative to a smooth consumption and linearity in rescaling the variance of the consumption by assuming that for all consumption streams $(x_1, x_2, \dots), (y_1, y_2, \dots)$, a smooth consumption $(c, c, \dots)$, and $\alpha \in (0,1)$, we have[1]

$$(x_1, x_2, \dots) \succsim (y_1, y_2, \dots) \Longleftrightarrow \alpha(x_1, x_2, \dots) + (1-\alpha)(c, c, \dots) \succsim \alpha(y_1, y_2, \dots) + (1-\alpha)(c, c, \dots)$$

where the mixtures of lotteries are done periodwise.

[1] Formally, this is the C-independence axiom from Gilboa and Schmeidler (1989).

Second, we assume that the decision maker is dynamically consistent and satisfies history independence of consumption: for all consumption streams a, b, x, y and period t,

$$(a_1, \dots, a_{t-1}, x_t, x_{t+1}, \dots) \succsim (a_1, \dots, a_{t-1}, y_t, y_{t+1}, \dots)$$
$$\iff (b_1, \dots, b_{t-1}, x_t, x_{t+1}, \dots) \succsim (b_1, \dots, b_{t-1}, y_t, y_{t+1}, \dots).$$

Under standard monotonicity and continuity assumptions, we show that in each period t, the decision maker has gain-loss discount factors $\delta_t^+, \delta_t^- \in (0, 1)$ that satisfy (1) and the recursive solution V_1 represents the preferences.

We offer two applications for our results. First, we consider strengthening the history independence axiom with the stationarity axiom. This assumes that for all consumption streams x and y and consumption c

$$(x_1, x_2, \dots) \succsim (y_1, y_2, \dots) \iff (c, x_1, x_2, \dots) \succsim (c, y_1, y_2, \dots).$$

Under this additional assumption, the decision maker will have convex or concave preferences. That is, there exists uncertainty about the discount factor and the decision maker is uncertainty averse or uncertainty loving towards the uncertainty. Formally, there exist discount factors $\delta^1 \leq \delta^2$ such that for all t

$$V_t(x) = \min_{\delta_t \in [\delta^1, \delta^2]} (1 - \delta_t) u(x_t) + \delta_t V_{t+1}(x)$$

or for all t

$$V_t(x) = \max_{\delta_t \in [\delta^1, \delta^2]} (1 - \delta_t) u(x_t) + \delta_t V_{t+1}(x)$$

and the recursive solution V_1 represents the preferences.

Our second application shows that the dynamically consistent intertemporal version of the Choquet expected utility (Schmeidler, 1989) is equivalent to the discounted expected utility. The intertemporal Choquet expected utility captures the idea that we can smooth out the consumption in x by considering a lottery between x and y where the consumption stream y has good consumption in the periods where x has bad consumption. However, we show that if we add the previous dynamic consistency axiom to the intertemporal Choquet expected utility, then we recover the discounted expected utility.

Our work contributes to the literature on intertemporal consumption. Wakai (2008) studied intertemporal consumption with a preference for consumption smoothing under convex preferences and when preferences conditional on each period are observed. This representation is characterized recursively for each period t by $\delta_t^+, \delta_t^- \in (0, 1)$ with $\delta_t^+ \leq \delta_t^-$ and a utility u that satisfy (1). We show that by generalizing to any non-convex preferences, we relax the requirement $\delta_t^+ \leq \delta_t^-$ and so maintain the tractability of the model while generalizing it axiomatically substantially. Additionally, we simplify the setting by only considering preferences at time 1. We show in the appendix that our history independence axiom is equivalent to the dynamic consistency axiom in Epstein and Schneider (2003) and Wakai (2008).

Our model is based on the dual-self expected utility (Chandrasekher et al., 2022). We show that this model simplifies remarkably under the history independence axiom to only gain-loss asymmetry. Formally, the gain-loss asymmetry is similar to Fehr and Schmidt's (1999) other regarding preferences that was axiomatized in Rohde (2010) but in our model, the reference utility is the future utility. Relatedly, Beissner et al. (2020) studied dynamically consistent α-maxmin in continuous time and its convergence to the continuous time limit.

Our application on dynamically consistent intertemporal Choquet expected utility extends previous results by Sarin and Wakker (1998) and Delbaen (2021). Sarin and Wakker (1998) show in a 2-stage dynamic choice situation that under dynamic consistency, consequentialism, and sequential consistency with a rank-dependent utility function, the decision maker uses expected utility at the first stage. Delbaen (2021) shows in a 2-period model that a time-consistent, comonotonic, and convex risk measure corresponds to expectation in the first period. We show that under infinitely many periods, dynamically consistent Choquet expected utility is equivalent to discounted utility.

Additionally, Mononen (2024) applies the dynamically consistent intertemporal dual-self expected utility to intergenerational welfare aggregation to provide a general characterization for dynamically consistent intergenerational welfare aggregation.

The remainder of the paper proceeds as follows: Section 2.1 axiomatizes the dynamically consistent dual-self expected utility. Section 2.2 sketches the proof for the dynamically consistent dual-self expected utility. Section 3.1 studies the stationary dual-self expected utility. Section 3.2 shows the equivalency between the dynamically consistent Choquet expected utility and discounted expected utility. The appendix connects our history independence axiom to the dynamic consistency and contains the proofs for all the results.

2 Model

We adapt the Anscombe-Aumann (1963) framework to temporal interpretation. Time is discrete over an infinite time-horizon.[2] The consumption set is X and $\Delta(X)$ is the set of (simple) lotteries on X. Choice alternatives are streams of lotteries over consumption, $f : \mathbb{N} \to \Delta(X)$. f_t denotes the consumption of the stream f at time $t \in \mathbb{N}$. When $x \in \Delta(X)$ is a lottery, we often abuse notation and use x to denote the constant stream $(x, x, \dots)$.

Our first primitive is a binary relation $\succsim_c$ on lotteries, $\Delta(X)$. The set of bounded consumption streams is denoted $H = \{f : \mathbb{N} \to \Delta(X) | \, \exists\, x, y \in \Delta(X) \, \forall\, t \in \mathbb{N}, x \succsim_c f_t \succsim_c y\}$. Our main primitive is a binary relation $\succsim$ on H that extends $\succsim_c$.[3] As usual, $\succ$ and $\sim$ denote the asymmetric and symmetric parts of $\succsim$ respectively.

For $f, g \in H, t \in \mathbb{N}$, $(f_1, \dots, f_{t-1}, g_t, g_{t+1}, \dots)$ denotes the consumption stream where the consumption in the periods $t' < t$ is $f_{t'}$ and in the periods $t' \geq t$ is $g_{t'}$. Mixtures of consumption streams are defined periodwise: for $f, g \in H, \alpha \in [0, 1], t \in \mathbb{N}$, define $(\alpha f + (1-\alpha)g)_t = \alpha f_t + (1-\alpha) g_t$.

Remark The typical assumption in the literature has been to assume that there exists the best and the worst consumption streams such as in Koopmans (1960) and Wakai (2008). Our approach generalizes this setting to any bounded consumption streams that are crucial for some applications such as in intergenerational welfare in Mononen (2024).

2.1 Recursive Dual-Self Expected Utility

Our first axioms are standard axioms that $\succsim$ is a weak order and satisfies mixture continuity.

Axiom 1 (Weak Order) $\succsim$ is complete and transitive.

Axiom 2 (Mixture Continuity) For any $f, g, h \in H$ such that $f \succ g$, there exist $\alpha, \beta \in (0, 1)$ such that

$$\alpha f + (1-\alpha) h \succ g \text{ and } f \succ \beta g + (1-\beta) h.$$

[2] The results for the finite time-horizon follow symmetrically.

[3] Formally, for all $x, y \in \Delta(X)$, $x \succsim_c y \iff (x, x, \dots) \succsim (y, y, \dots)$.

Our next axiom is the standard strict monotonicity on consumption for all the periods.

Axiom 3 (Strict Monotonicity) For any $f, g \in H$, if for all $t \in \mathbb{N}$, $f_t \succsim g_t$, then $f \succsim g$. Additionally, if for some $t' \in \mathbb{N}$, $f_{t'} \succ g_{t'}$, then $f \succ g$.

We relax the independence axiom to allow for the spread of the consumption stream to affect utility by assuming only independence to mixing with a smooth consumption. This allows for the possibility that the same discounting is applied only to a subset of utility sequences that share a similar pattern of utility changes. This axiom captures that the decision maker evaluates the consumption streams relative to a smooth consumption and assumes linearity in rescaling the variance of the consumption

Axiom 4 (C-Independence) For all $f, g \in H, c \in \Delta(X)$, and $\alpha \in (0, 1)$,

$$f \succsim g \iff \alpha f + (1-\alpha)c \succsim \alpha g + (1-\alpha)c.$$

Next, we assume the history independence axiom from Bommier et al. (2017). We show in the appendix that this is the dynamic consistency axiom from Epstein and Schneider (2003) and Wakai (2008) when applied to our setting.

Axiom 5 (History Independence) For all $a, b, f, g \in H$ and $t \in \mathbb{N}$,

$$(a_1, \ldots, a_{t-1}, f_t, f_{t+1}, \ldots) \succsim (a_1, \ldots, a_{t-1}, g_t, g_{t+1}, \ldots)$$
$$\iff (b_1, \ldots, b_{t-1}, f_t, f_{t+1}, \ldots) \succsim (b_1, \ldots, b_{t-1}, g_t, g_{t+1}, \ldots).$$

The idea of this axiom is that the decision maker has preferences $\succsim_t$ at time period t for consumption streams starting at period t. Assume that $(f_t, f_{t+1}, \ldots) \succsim_t (g_t, g_{t+1}, \ldots)$. When evaluating the consumption streams $(a_1, \ldots, a_{t-1}, f_t, f_{t+1}, \ldots)$ and $(a_1, \ldots, a_{t-1}, g_t, g_{t+1}, \ldots)$, then the former consumption stream is weakly better at time period t than the latter one and until time period t, the consumption streams are equal and the choice between them does not matter. Hence, under dynamic consistency, the former consumption stream is weakly better than the latter one also at time period 1.

Finally, we assume the monotone continuity axiom from Villegas (1964), Arrow (1966), and Chateauneuf et al. (2005) adapted to consumption streams. This axiom states that the limit of a consumption stream is not given a positive weight. Relaxing this axiom has been studied in Drugeon and Ha Huy (2022).

Axiom 6 (Monotone Continuity) For all $f, g, h \in H$, if $f \succ g$, then there exists $t \in \mathbb{N}$ such that

$$(f_1, \ldots, f_{t-1}, h_t, \ldots) \succ g \text{ and } f \succ (g_1, \ldots, g_{t-1}, h_t, \ldots).$$

These six axioms characterize the dynamically consistent intertemporal dual-self expected utility representation.

Theorem 1 (Dynamically Consistent Dual-Self) $\succsim$ satisfies Axioms 1-6 iff. there exist an affine $u : \Delta(X) \to \mathbb{R}$ and for each $t \in \mathbb{N}$, $\delta_t^+, \delta_t^- \in (0, 1)$ such that $\prod_{t=1}^{\infty} \max\{\delta_t^+, \delta_t^-\} = 0$, for each $t \in \mathbb{N}$ and $f \in H$, we have a recursive function

$$\begin{aligned} &V_t(f_t, f_{t+1}, \ldots) \\ &= u(f_t) + \delta_t^+ \min\Big\{V_{t+1}(f_{t+1}, f_{t+2}, \ldots) - u(f_t), 0\Big\} + \delta_t^- \max\Big\{V_{t+1}(f_{t+1}, f_{t+2}, \ldots) - u(f_t), 0\Big\} \end{aligned}$$

with $\limsup_{t\to\infty} |V_t(f_t, f_{t+1}, \ldots)| < \infty$ and the recursive solution V_1 represents $\succsim$.

Additionally, if $\succsim$ is nontrivial, then u is unique up to a positive affine transformation and $(\delta_t^+, \delta_t^-)_{t\in\mathbb{N}}$ are unique.

Here, the restrictions $\prod_{t=1}^{\infty} \max\{\delta_t^+, \delta_t^-\} = 0$ and $\limsup_{t\to\infty} |V_t(f_t, f_{t+1}, \ldots)| < \infty$ capture Axiom 6 and that the recursive formulation has a convergent solution.

This representation can be alternatively written as

$$V_t(f) = \begin{cases} \min\limits_{\delta_t \in [\delta_t^+, \delta_t^-]} (1 - \delta_t) u(f_t) + \delta_t V_{t+1}(f), & \text{if } \delta_t^+ \leq \delta_t^- \\ \max\limits_{\delta_t \in [\delta_t^-, \delta_t^+]} (1 - \delta_t) u(f_t) + \delta_t V_{t+1}(f), & \text{if } \delta_t^+ > \delta_t^-. \end{cases} \tag{2}$$

Here, at each time period t, the decision maker considers discount factors between δ_t^+ and δ_t^- as possible and is either optimistic or pessimistic about the uncertainty. This result shows that the general dual-self expected utility (Chandrasekher et al., 2022) simplifies remarkably under history independence to only periodwise preference or dispreference for uncertainty on discount factors.

Here, the set of used sequences of discount factors is

$$\mathcal{D} = \left\{ \left((1 - \delta_t) \prod_{t'=1}^{t-1} \delta_{t'} \right)_{t\in\mathbb{N}} \middle| \forall\, t \in \mathbb{N}, \delta_t \in \{\delta_t^+, \delta_t^-\} \right\}.$$

Next, we connect this set of discount factors to the rectangularity of a set of priors as in Epstein and Schneider (2003) and Amarante and Siniscalchi (2019).

Definition For a finite partition $\Pi = \{E_i\}_{i=1}^n$ of $\mathbb{N}$, a set of probabilities $\mathcal{C} \subseteq \Delta(\mathbb{N})$ is Π-*rectangular* if for all $r, q_1, \dots, q_n \in \mathcal{C}$, the measure p on $\mathbb{N}$ defined by[4]

$$p(\cdot) = \sum_{i=1}^{n} r(E_i) q_i(\cdot | E_i)$$

is in $\mathcal{C}$.

Next, we show that the set $\mathcal{D}$ is rectangular with respect to the passage of time.

Proposition 2 (Rectangularity) For each $t \in \mathbb{N}$, denote the partition $\Pi_t = \Big\{\{i\}_{i=1}^{t-1}, \{t, t+1, \dots\}\Big\}$, then $\mathcal{D}$ is Π_t-rectangular.

Here, the partition Π_t captures the information at period t. Each of the previous periods $t' < t$ is fully resolved but the future $\{t, t+1, \dots\}$ is unknown.

This shows that the rectangularity of discount factors captures dynamic consistency also for non-convex preferences.

2.2 Proof Sketch

The proof for Theorem 1 follows in 2 steps. The first step is an observation that if there are only two time periods, then $\succsim$ that satisfies Axioms 1-4 is convex or concave[5]. By Ghirardato et al. (2004, Lemma 1), there exists an affine utility $u : \Delta(X) \to \mathbb{R}$ and $V : u(\Delta(X))^2 \to \mathbb{R}$ that is C-additive and positively homogeneous[6] that represent $\succsim$. If $(a, b), (c, d) \in u(\Delta(X))^2$, $a > b$ and $c > d$, then V is linear between (a, b) and (c, d) since

$$(c, d) = \frac{c-d}{a-b}(a, b) + \frac{c+d}{2} - \frac{c-d}{a-b}\frac{a+b}{2}$$

and so (c, d) can be achieved from (a, b) by a positive scaling and by adding a constant. Hence, the linearity follows from C-additivity and positive homogeneity of V. Similarly, if $(a, b), (c, d) \in u(\Delta(X))^2$, $a < b$ and $c < d$, then V is linear between (a, b) and (c, d). Thus, V is a piecewise linear function with two pieces consisting of half-spaces and so especially convex or concave.

[4] $q_i(\cdot|E_i)$ denotes the conditional probability measure of q_i conditional on the event E_i.

[5] $\succsim$ is *convex*(*concave*) if for all $f, g \in H$ and $\alpha \in (0, 1)$ with $f \succsim g$ we have $\alpha f + (1-\alpha) g \succsim g$ ($f \succsim \alpha f + (1-\alpha) g$).

[6] V is *C-additive*, if for all $\varphi \in u(\Delta(X))^2$ and $c \in \mathbb{R}$ such that $\varphi + (c, c) \in (\Delta(X))^2$, we have $V(\varphi + (c, c)) = V(\varphi) + c$. V is *positively homogeneous*, if for all $\varphi \in u(\Delta(X))^2$ and $a > 0$ such that $a\varphi \in (\Delta(X))^2$, we have $V(a\varphi) = aV(\varphi)$.

The second step of the proof is the recursive formulation for period t. First, there exists an affine utility $u : \Delta(X) \to \mathbb{R}$ that represents $\succsim$ on $\Delta(X)$. Second, we infer preferences at each time period t by

$$(f_t, f_{t+1}, \dots) \succsim_t (g_t, g_{t+1}, \dots) \Longleftrightarrow \forall\, h \in H, (h_1, \dots, h_{t-1}, f_t, f_{t+1}, \dots) \succsim (h_1, \dots, h_{t-1}, g_t, g_{t+1}, \dots).$$

By Axiom 5, $\succsim_t$ inherits Axioms 1-4. Third, define $\trianglerighteq$ on $\Delta(X)^2$ by

$$(a, b) \trianglerighteq (c, d) \iff (a, b, b, \dots) \succsim_t (c, d, d, \dots).$$

Also, $\trianglerighteq$ inherits Axioms 1-4. By the first observation, $\trianglerighteq$ is convex or concave. Assume, w.l.o.g. that $\trianglerighteq$ is convex. By Gilboa and Schmeidler's (1989) existence and uniqueness theorems and by strict monotonicity axiom, there exist $\delta_t^-, \delta_t^+ \in (0, 1)$ with $\delta_t^- \leq \delta_t^+$ such that for all $(a, b) \in \Delta(X)^2$

$$W_t(a, b) = \min_{\delta_t \in [\delta_t^-, \delta_t^+]} (1 - \delta_t) u(a) + \delta_t u(b)$$

and W_t represents $\trianglerighteq$.

Next, by Ghirardato et al. (2004, Lemma 1), there exists a function V_{t+1} that represents $\succsim_{t+1}$ and for all $x \in \Delta(X)$, $V_{t+1}(x, x, \dots) = u(x)$.

Finally, we have for all f, g when $c^f, c^g \in \Delta(X)$ are such that $c^f \sim_{t+1} f$ and $c^g \sim_{t+1} g$,

$$
\begin{aligned}
& (f_t, f_{t+1}, \dots) \succsim_t (g_t, g_{t+1}, \dots) \\
\iff & (f_t, c^f, c^f, \dots) \succsim_t (g_t, c^g, c^g, \dots) \iff (f_t, c^f) \trianglerighteq (g_t, c^g) \\
\iff & \min_{\delta_t \in [\delta_t^-, \delta_t^+]} (1 - \delta_t) u(f_t) + \delta_t u(c^f) \geq \min_{\delta_t \in [\delta_t^-, \delta_t^+]} (1 - \delta_t) u(g_t) + \delta_t u(c^g) \\
\iff & \min_{\delta_t \in [\delta_t^-, \delta_t^+]} (1 - \delta_t) u(f_t) + \delta_t V_{t+1}(c^f, c^f, \dots) \geq \min_{\delta_t \in [\delta_t^-, \delta_t^+]} (1 - \delta_t) u(g_t) + \delta_t V_{t+1}(c^g, c^g, \dots) \\
\iff & \min_{\delta_t \in [\delta_t^-, \delta_t^+]} (1 - \delta_t) u(f_t) + \delta_t V_{t+1}(f_{t+1}, f_{t+2}, \dots) \geq \min_{\delta_t \in [\delta_t^-, \delta_t^+]} (1 - \delta_t) u(g_t) + \delta_t V_{t+1}(f_{t+1}, f_{t+2}, \dots)
\end{aligned}
$$

where the first equivalency follows from the definition of $\sim_{t+1}$.

3 Applications

3.1 Stationarity Implies Convexity or Concavity

In our first application, we consider stationary preferences following Koopmans (1960).

Axiom 7 (Stationarity) Let $c \in \Delta(X)$ and $f, g \in H$. Then

$$(f_1, f_2, \dots) \succsim (g_1, g_2, \dots) \iff (c, f_1, f_2, \dots) \succsim (c, g_1, g_2, \dots).$$

This axiom captures that the passage of time does not affect the preferences and so the discounting is the same in each period. This is formalized in the next theorem.

Theorem 3 (Stationary Recursive Dual-Self) $\succsim$ satisfies Axioms 1-4 and 7 iff. there exist an affine $u : \Delta(X) \to \mathbb{R}$ and $\delta^1, \delta^2 \in (0, 1)$ with $\delta^1 \le \delta^2$ such that there exists a recursive function V defined by for each $f \in H$,

$$V(f_1, f_2, \dots) = \min_{\delta \in [\delta^1, \delta^2]} (1 - \delta) u(f_1) + \delta V(f_2, f_3, \dots)$$

or for each $f \in H$,

$$V(f_1, f_2, \dots) = \max_{\delta \in [\delta^1, \delta^2]} (1 - \delta) u(f_1) + \delta V(f_2, f_3, \dots)$$

with $\limsup_{t \to \infty} |V(f_t, f_{t+1}, \dots)| < \infty$ such that V represents $\succsim$.

Especially, this axiom shows that the stationarity axiom implies either convexity or concavity of the preferences. Additionally, the stationarity implies the monotone continuity axiom, Axiom 6.

This offers a simple and tractable characterization with only two parameters for the general stationary dual-self expected utility. This model has been studied from a programming perspective in Drugeon et al. (2019).

More generally, we can perform uncertainty attitude comparisons. This shows that the values of δ_t^+, δ_t^- capture the uncertainty attitude. Our condition captures dispreference for delays.

Theorem 4 Assume that there exist an affine $u : \Delta(X) \to \mathbb{R}$ and for each $t \in \mathbb{N}$, $\delta_t^+, \delta_t^- \in (0, 1)$ such that for each $t \in \mathbb{N}$ and $f \in H$, we have a recursive function

$$\begin{aligned} &V_t(f_t, f_{t+1}, \dots) \\ &= u(f_t) + \delta_t^+ \min\left\{V_{t+1}(f_{t+1}, f_{t+2}, \dots) - u(f_t), 0\right\} + \delta_t^- \max\left\{V_{t+1}(f_{t+1}, f_{t+2}, \dots) - u(f_t), 0\right\} \end{aligned}$$

with $\limsup_{t\to\infty} |V_t(f_t, f_{t+1}, \dots)| < \infty$ and the recursive solution V_1 represents $\succsim$. For all $t \in \mathbb{N}$, $\delta^+_{t+1} \geq \delta^+_t$ and $\delta^-_t \geq \delta^-_{t+1}$ iff for all $f \in H, x \in \Delta(X)$,

$$x \succsim (f_1, f_2, \dots) \Longrightarrow x \succsim (x, f_1, f_2, \dots).$$

Using the alternative recursive formulation (2), this result shows that the person can start as uncertainty loving and can at most once switch to uncertainty averse.

3.2 Dynamically Consistent Choquet EU Is Discounted EU

Our second application shows that the dynamically consistent intertemporal version of the Choquet expected utility is equivalent to the discounted expected utility. Here, we strengthen Axiom 4 to the comonotonic independence axiom. First, we define comonotonic consumption streams.

Definition f and g in H are *comonotonic* if there does not exist t and t' in $\mathbb{N}$ such that

$$f_t \succ g_{t'} \text{ and } f_{t'} \succ g_t.$$

The idea of comonotonic streams is that both of the streams give good consumption in the same time periods. In this case, creating a lottery between the two comonotonic streams does not smooth out the consumption. This is captured by the next comonotonic independence axiom that characterizes Choquet expected utility.

Axiom 8 (Comonotonic Independence) If f, g, and h are pairwise comonotonic, then for all $\alpha \in (0,1)$

$$f \succsim g \iff \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h.$$

However, we show next that any non-trivial version of Choquet expected utility is not compatible with dynamic consistency. But instead, the dynamically consistent Choquet expected utility is equivalent to the discounted expected utility.

Theorem 5 (Dynamically Consistent Choquet EU) $\succsim$ satisfies Axioms 1-3, 5, 6, and 8 iff. there exist an affine $u : \Delta(X) \to \mathbb{R}$ and for each $t \in \mathbb{N}$, $\delta_t \in (0,1)$ such that $\prod_{t=1}^{\infty} \delta_t = 0$, for each $t \in \mathbb{N}$ and $f \in H$, we have a recursive function

$$V_t(f_t, f_{t+1}, \dots) = (1-\delta_t)u(f_t) + \delta_t V_{t+1}(f_{t+1}, f_{t+2}, \dots),$$

with $\limsup_{t\to\infty} |V_t(f_t, f_{t+1}, \dots)| < \infty$ and the recursive solution V_1 represents $\succsim$.

The proof idea for this result is simple: The dynamically consistent dual-self expected utility states that there can be only non-linearities in V_1 when at some time period the utility order for current consumption and future consumption changes. The Choquet expected utility states that there can be only non-linearities in V_1 when the rank of some periods' consumption changes. Especially, at any time period there can be changes in the utility order of current consumption and future consumption but no change in the rank of any periods' consumption. Hence, there is then no change in the discount factor.

4 Conclusion

In this paper, we studied the general dual-self expected utility in the context of intertemporal consumption. We showed that the general representation is simple and tractable under dynamic consistency and is characterized by gain-loss asymmetry in discounting. Additionally, we showed that under stationary this general representation is either convex or concave and that the dynamically consistent version of Choquet expected utility simplifies into discounted expected utility.

We showed that non-convex preferences are tractable and suitable for dynamic programming under dynamic consistency. Generalizing these results to other representations is left for future research. It is an open question if the intertemporal version of dual-self variational expected utility simplifies similarly under dynamic consistency and becomes equally tractable.

Appendix to "Dynamically Consistent Intertemporal Dual-Self Expected Utility"

A Connecting History Independence and Dynamic Consistency

In this section, we connect our history independence axiom to the dynamic consistency axioms from Epstein and Schneider (2003) and Wakai (2008). Here, our primitives are $(\succsim_t)_{t\in\mathbb{N}}$ preferences at each time period t over consumption streams H. The following two axioms are the dynamic consistency axioms from Epstein and Schneider (2003) and Wakai (2008).

Axiom 9 (Conditional Preference) For each t, if for all $\tau \geq t$, $f_\tau = g_\tau$, then $f \sim_t g$.

Axiom 10 (Dynamic Consistency) For each t and for all $f, g \in H$, if $f_\tau = g_\tau$ for all $\tau \leq t$ and if $f \succsim_{t+1} g$, then $f \succsim_t g$; the latter ranking is strict if the former is strict.

First, we show that these two axioms are stronger than the history independence axiom for $\succsim_1$.

Proposition 6 If each $\succsim_t$ is complete and transitive and satisfies Axioms 9 and 10, then $\succsim_1$ satisfies Axiom 5.

Second, we provide a partial converse of the previous result. The next result shows that under the history independence axiom, we can infer preference for consumption streams starting at a time period t using period 1 preferences with a common history. Under dynamic consistency, these inferred preferences for the time period t coincide with the actual preferences at the time period t.

Proposition 7 Define the inferred preferences at each time period t by

$$(f_t, f_{t+1}, \dots)\tilde{\succsim}_t(g_t, g_{t+1}, \dots) \Longleftrightarrow \forall\, h\in H, (h_1, \dots, h_{t-1}, f_t, f_{t+1}, \dots)\succsim_1(h_1, \dots, h_{t-1}, g_t, g_{t+1}, \dots).$$

Assume that each $\succsim_t$ is complete and transitive. If each $\succsim_t$ satisfies Axioms 9 and 10, then for each t, $f, g \in H$

$$f \succsim_t g \iff (f_t, f_{t+1}, \dots) \tilde{\succsim}_t (g_t, g_{t+1}, \dots).$$

B Proofs

B.1 Recursive Dual-Self Expected Utility

Let u be an affine utility and $I : u(H) \to \mathbb{R}$ be a function. We say that

- I is strictly monotonic if for all $\varphi, \psi \in u(H)$ such that for all $t \in \mathbb{N}$, $\varphi_t \geq \psi_t$, $I(\varphi) \geq I(\psi)$. Additionally, if for all $t \in \mathbb{N}$, $\varphi_t \geq \psi_t$ and for some $t' \in \mathbb{N}$, $\varphi_{t'} \geq \psi_{t'}$, then $I(\varphi) > I(\psi)$.
- I is C-additive if for all $\varphi \in u(H), \alpha \geq 0$ such that $\varphi + \alpha\bar{1} \in u(H)$, $I(\varphi + \alpha\bar{1}) = I(\varphi) + \alpha$.
- I is positive homogeneous if for all $\varphi \in u(H), \alpha > 0$ such that $\alpha\varphi \in u(H)$, $I(\alpha\varphi) = \alpha I(\varphi)$.

For each $t \in \mathbb{N}$ and $(f_t, f_{t+1}, \dots), (g_t, g_{t+1}, \dots) \in H$, define

$$(f_t, f_{t+1}, \dots) \succsim_t (g_t, g_{t+1}, \dots) \Longleftrightarrow \forall\, h \in H, (h_1, \dots, h_{t-1}, f_t, f_{t+1}, \dots) \succsim (h_1, \dots, h_{t-1}, g_t, g_{t+1}, \dots).$$

For an affine utility $u : \Delta(X) \to \mathbb{R}$ and $(f_1, f_2, \dots)$, denote

$$u(f_1, f_2, \dots) = \big(u(f_1), u(f_2), \dots\big)$$

Lemma 8 Assume that $t \in \mathbb{N}$, $\succsim$ satisfies Axioms 1-5. u represents the constant preferences, $\succsim_c$. Then there exist V_t that is strictly monotonic, C-additive, and positively homogeneous and $V_t \circ u$ represents $\succsim_t$.

Proof. It follows immediately that $\succsim_t$ satisfies Axioms 1-3. We show C-Independence. Let $f, g \in H$, $c \in \Delta(X)$ and $\alpha \in (0,1)$. By Axiom 5, it follows immediately that if $f \succsim_t g$, then $\alpha f + (1-\alpha)c \succsim_t \alpha g + (1-\alpha)c$ and if $f \succ_t g$, then $\alpha f + (1-\alpha)c \succ_t \alpha g + (1-\alpha)c$.

By Ghirardato et al. (2004), there exist $\tilde{u} : \Delta(X) \to \mathbb{R}$ and $\tilde{V}_t$ that is strictly monotonic, C-additive, and positively homogeneous and $\tilde{V}_t \circ \tilde{u}$ represents $\succsim_t$. By the uniqueness of $\tilde{u}$, there exists $a > 0$ and $b \in \mathbb{R}$ such that $\tilde{u} = au + b$. Thus, by the uniqueness theorem from Ghirardato et al. (2004), there exists V_t that is strictly monotonic, C-additive, and positively homogeneous and $V_t \circ u$ represents $\succsim_t$. □

Lemma 9 Assume that $V : [0,1]^2 \to \mathbb{R}$ is a positively homogeneous, C-additive, and strictly monotonic function. Then then there exists $\delta^+, \delta^- \in (0,1)$ such that

$$V(x,y) = \begin{cases} \min\limits_{\delta \in [\delta^+, \delta^-]} (1-\delta)x + \delta y, & \text{if } \delta^+ \leq \delta^- \\ \max\limits_{\delta \in [\delta^-, \delta^+]} (1-\delta)x + \delta y, & \text{if } \delta^+ > \delta^-. \end{cases}$$

Proof. Since V is Lipschitz continuous, it is differentiable almost everywhere. Let (x^1, y^1) differentiability point for V such that $x^1 > y^1$ with derivative (p_1^1, p_2^1). By Ghirardato et al. (2004), $p^1 \in \Delta(\{1,2\})$. By Chandrasekher et al. (2022, Lemma B.2) $V(x^1, y^1) = x^1 p_1^1 + y^1(1 - p_1^1)$. We show that for all $a > b$, $V(a,b) = ap_1^1 + b(1-p_1^1)$. Now,

$$(a,b) = \frac{a-b}{x^1 - y^1}(x^1, y^1) + \frac{a+b}{2} - \frac{a-b}{x^1-y^1}\frac{x^1+y^1}{2}.$$

Thus by positive homogeneity and C-additivity,

$$V(a,b) = \frac{a-b}{x^1-y^1}V(x^1,y^1) + \frac{a+b}{2} - \frac{a-b}{x^1-y^1}\frac{x^1+y^1}{2} = ap_1^1 + b(1-p_1^1).$$

Symmetrically, let (x^2, y^2) differentiability point for V such that $x^2 < y^2$ with derivative (p_1^2, p_2^2). Then for all $a < b$, $V(a,b) = ap_1^2 + b(1-p_1^2)$. Thus by continuity, if $p_1^1 \geq p_1^2$, then for all (a,b)

$$V(a,b) = \max_{p \in [p_1^1, p_1^2]} pa + (1-p)b$$

and we denote $\delta^+ = p_1^2$ and $\delta^- = p_1^1$ and if $p_1^1 \leq p_1^2$, then for all (a,b)

$$V(a,b) = \min_{p \in [p_1^1, p_1^2]} pa + (1-p)b$$

and we denote $\delta^+ = p_1^1$ and $\delta^- = p_1^2$. Finally, by strict monotonicity $\delta^+, \delta^- \in (0,1)$. $\square$

Lemma 10 Assume that $\succsim$ satisfies Axioms 1-5, $t \in \mathbb{N}$, $V_t \circ u$ is a representation for $\succsim_t$, and $V_{t+1} \circ u$ is a representation for $\succsim_{t+1}$ such that V_t and V_{t+1} are C-additive, strictly monotonic, and positively homogeneous. Then there exists $\delta_t^+, \delta_t^- \in (0,1)$ such that for all $(f_t, f_{t+1}, \dots) \in H$

$$V_t\Big(u(f_t, f_{t+1}, \dots)\Big) = \begin{cases} \min_{\delta_t \in [\delta_t^+, \delta_t^-]} (1-\delta_t)u(f_t) + \delta_t V_{t+1}\Big(u(f_{t+1}, f_{t+2}, \dots)\Big), & \text{if } \delta_t^+ \leq \delta_t^- \\ \max_{\delta_t \in [\delta_t^-, \delta_t^+]} (1-\delta_t)u(f_t) + \delta_t V_{t+1}\Big(u(f_{t+1}, f_{t+2}, \dots)\Big), & \text{if } \delta_t^+ > \delta_t^-. \end{cases}$$

Proof. Define for all $(a,b), (x,y) \in \Delta(X)^2$

$$(a,b) \trianglerighteq (x,y) \iff (a,b,b,\dots) \succsim_t (x,y,y,\dots).$$

By Lemma 8, $\trianglerighteq$ satisfies Axioms 1-4. By the representation for $\succsim_t$,

$$(a,b) \trianglerighteq (x,y) \iff V_t\Big(u(a,b,b,\dots)\Big) \geq V_t\Big(u(x,y,y,\dots)\Big).$$

By Lemma 9, there exists $\delta_t^+, \delta_t^- \in (0,1)$ such that

$$V_t\Big(u(a,b,b,\dots)\Big) = \begin{cases} \min\limits_{\delta_t \in [\delta_t^+, \delta_t^-]} (1-\delta_t)u(a) + \delta_t u(b), & \text{if } \delta_t^+ \leq \delta_t^- \\ \max\limits_{\delta_t \in [\delta_t^-, \delta_t^+]} (1-\delta_t)u(a) + \delta_t u(b), & \text{if } \delta_t^+ > \delta_t^-. \end{cases}$$

Let $(f_t, f_{t+1}, \dots) \in H$. By Axioms 2 and 3, there exists $c \in \Delta(X)$ such that $(f_{t+1}, f_{t+2}, \dots) \sim_{t+1} c$. By the representations and the definition of $\succsim_{t+1}$,

$$V_t\Big(u(f_t, f_{t+1}, \dots)\Big) = V_t\Big(u(f_t, c, c, \dots)\Big) \text{ and } V_{t+1}\Big(u(f_{t+1}, f_{t+2}, \dots)\Big) = V_{t+1}\Big(u(c)\Big) = u(c),$$

where the last equality follows from V_{t+1} being C-additive and positively homogeneous.

Thus

$$\begin{aligned} V_t\Big(u(f_t, f_{t+1}, \dots)\Big) &= \begin{cases} \min\limits_{\delta_t \in [\delta_t^+, \delta_t^-]} (1-\delta_t)u(f_t) + \delta_t u(c), & \text{if } \delta_t^+ \leq \delta_t^- \\ \max\limits_{\delta_t \in [\delta_t^-, \delta_t^+]} (1-\delta_t)u(f_t) + \delta_t u(c), & \text{if } \delta_t^+ > \delta_t^- \end{cases} \\ &= \begin{cases} \min\limits_{\delta_t \in [\delta_t^+, \delta_t^-]} (1-\delta_t)u(f_t) + \delta_t V_{t+1}\Big(u(f_{t+1}, f_{t+2}, \dots)\Big), & \text{if } \delta_t^+ \leq \delta_t^- \\ \max\limits_{\delta_t \in [\delta_t^-, \delta_t^+]} (1-\delta_t)u(f_t) + \delta_t V_{t+1}\Big(u(f_{t+1}, f_{t+2}, \dots)\Big), & \text{if } \delta_t^+ > \delta_t^-. \end{cases} \end{aligned}$$

□

Lemma 11 Assume that $\succsim$ satisfies Axioms 1-6, for each $t \in \mathbb{N}$, $V_t \circ u$ is a representation for $\succsim_t$ such that V_t is C-additive, monotonic, and positively homogeneous and there exists $\delta_t^+, \delta_t^- \in (0,1)$ such that for all $(f_t, f_{t+1}, \dots) \in H$,

$$V_t\Big(u(f_t, f_{t+1}, \dots)\Big) = \begin{cases} \min\limits_{\delta_t \in [\delta_t^+, \delta_t^-]} (1-\delta_t)u(f_t) + \delta_t V_{t+1}\Big(u(f_{t+1}, f_{t+2}, \dots)\Big), & \text{if } \delta_t^+ \leq \delta_t^- \\ \max\limits_{\delta_t \in [\delta_t^-, \delta_t^+]} (1-\delta_t)u(f_t) + \delta_t V_{t+1}\Big(u(f_{t+1}, f_{t+2}, \dots)\Big), & \text{if } \delta_t^+ > \delta_t^-. \end{cases}$$

Then

$$\lim_{T \to \infty} \prod_{t=1}^{T} \max\{\delta_t^+, \delta_t^-\} = 0.$$

Proof. Since the sequence $\left(\prod_{t=1}^{T} \max\{\delta_t^+, \delta_t^-\}\right)_{T \in \mathbb{N}}$ is decreasing and bounded, there exists $c \in [0,1]$ such that

$$\lim_{T \to \infty} \prod_{t=1}^{T} \max\{\delta_t^+, \delta_t^-\} = c.$$

Assume, per contra, $c > 0$. By Second Borel-Cantelli Lemma,

$$\sum_{t=1}^{\infty}(1 - \max\{\delta_t^+, \delta_t^-\}) < \infty.$$

Denote

$$\varepsilon = \min\{1, \frac{\min_{t\in\mathbb{N}} \max\{\delta_t^+, \delta_t^-\}}{\sum_{t=1}^{\infty}(1 - \max\{\delta_t^+, \delta_t^-\})}\}.$$

Since $\max\{\delta_t^+, \delta_t^-\} \to 1$ as $t \to \infty$ and for each t, $\max\{\delta_t^+, \delta_t^-\} > 0$, we have $\varepsilon > 0$.

Assume without loss of generality that $[-2, 2] \subset u(\Delta(X))$. Define a consumption stream f recursively by the following. Step 0: Let $c^0 \in \Delta(X)$ be such that $u(c^0) = 0$.

Next, let $t \in \mathbb{N}$ and assume that for each $t' < t$ $f_{t'}, c^{t'} \in \Delta(X)$ are defined such that for each $t' < t$, $|u(f_{t'}) - u(c^{t'-1})| = \varepsilon$, $(1 - \max\{\delta_{t'}^+, \delta_{t'}^-\})u(f_{t'}) + \max\{\delta_{t'}^+, \delta_{t'}^-\}u(c^{t'}) = u(c^{t'-1})$, and if $\delta_{t'}^+ < \delta_{t'}^-$, then $u(f_{t'}) > u(c^{t'-1}) \geq u(c^{t'})$ and if $\delta_{t'}^+ \geq \delta_{t'}^-$, then $u(c^{t'}) \geq u(c^{t'-1}) > u(f_{t'})$. Let f_t and c^t be such that $|u(f_t) - u(c^{t-1})| = \varepsilon$ and $(1 - \max\{\delta_t^+, \delta_t^-\})u(f_t) + \max\{\delta_t^+, \delta_t^-\}u(c^t) = u(c^{t-1})$, and if $\delta_t^+ < \delta_t^-$, then $u(f_t) > u(c^{t-1}) \geq u(c^t)$ and if $\delta_t^+ \geq \delta_t^-$, then $u(c^t) \geq u(c^{t-1}) > u(f_t)$. This is well-defined since for each t, $\max\{\delta_t^+, \delta_t^-\} > 0$ and

$$u(c^{t-1}) - u(c^t) = \varepsilon \frac{1 - \max\{\delta_t^+, \delta_t^-\}}{\max\{\delta_t^+, \delta_t^-\}}$$

and so

$$u(c^0) - u(c^t) = \sum_{t'=1}^{t} u(c^{t'-1}) - u(c^{t'}) = \sum_{t'=1}^{t} \varepsilon \frac{1 - \max\{\delta_{t'}^+, \delta_{t'}^-\}}{\max\{\delta_{t'}^+, \delta_{t'}^-\}} \leq \sum_{t'=1}^{\infty} \varepsilon \frac{1 - \max\{\delta_{t'}^+, \delta_{t'}^-\}}{\min_{t'\in\mathbb{N}} \max\{\delta_{t'}^+, \delta_{t'}^-\}} \leq 1.$$

Now for each t, $(f_1, \dots, f_t, c^t, c^t, \dots) \sim (f_1, \dots, f_{t-1}, c^{t-1}, c^{t-1}, \dots) \sim c^0$. Additionally, there exists $x, y \in \Delta(X)$ such that for all $t \in \mathbb{N}$ $x \succsim f_t \succsim y$ and so $f \in H$.

Let $0 < \upsilon < \varepsilon$. Let $h \in H$ be such that for each t, $u(h_t) = u(f_t) + \upsilon$, that exists since $[-2, 2] \subset u(\Delta(X))$. Let $x \in \Delta(X)$ be such that $V_1(u(f)) < u(x) < V_1(u(f)) + c\upsilon$. Now $x \succ f$.

Let $t^0 \in \mathbb{N}$. We show that $(f_1, \dots, f_{t^0-1}, h_{t^0}, h_{t^0+1}, \dots) \succ x$. Now for all t, we have

$$\varepsilon = |u(f_t) - u(c^{t-1})| \leq |u(f_t) - u(c^t)| = \Big|u(f_t) - V_{t+1}\Big(u(f_{t+1}, f_{t+2}, \dots)\Big)\Big|. \qquad (3)$$

Additionally, we have for all t, by C-additivity and monotonicity,

$$V_{t+1}\Big(u(f_{t+1}, \dots, f_{t^0-1}, h_{t^0}, h_{t^0+1}, \dots)\Big) \leq V_{t+1}\Big(u(f_{t+1}, f_{t+1}, \dots)\Big) + \upsilon.$$

Thus for each $t < t^0$, we have by (3)

$$u(f_t) > V_{t+1}\Big(u(f_{t+1}, \dots, f_{t^0-1}, h_{t^0}, h_{t^0+1}, \dots)\Big) \iff u(f_t) > V_{t+1}\Big(u(f_{t+1}, f_{t+1}, \dots)\Big)$$

and

$$u(f_t) < V_{t+1}\Big(u(f_{t+1}, \dots, f_{t^0-1}, h_{t^0}, h_{t^0+1}, \dots)\Big) \iff u(f_t) < V_{t+1}\Big(u(f_{t+1}, f_{t+1}, \dots)\Big).$$

We have,

$$\begin{aligned}
&V_1\Big(u(f_1, \dots, f_{t^0-1}, h_{t^0}, h_{t^0+1}, \dots)\Big)\\
&= \sum_{t=1}^{t^0-1} \prod_{t'=1}^{t-1} \max\{\delta^+_{t'}, \delta^-_{t'}\}(1 - \max\{\delta^+_t, \delta^-_t\})u(f_t) + \prod_{t'=1}^{t^0-1} \max\{\delta^+_{t'}, \delta^-_{t'}\} V_{t_0}\Big(u(h_{t^0}, h_{t^0+1}, \dots)\Big)\\
&\le \sum_{t=1}^{t^0-1} \prod_{t'=1}^{t-1} \max\{\delta^+_{t'}, \delta^-_{t'}\}(1 - \max\{\delta^+_t, \delta^-_t\})u(f_t) + \prod_{t'=1}^{t^0-1} \max\{\delta^+_{t'}, \delta^-_{t'}\}\Big(V_{t_0}\Big(u(f_{t^0}, f_{t^0+1}, \dots)\Big) + \upsilon\Big)\\
&= V_1\Big(u(f)\Big) + \prod_{t'=1}^{t^0-1} \max\{\delta^+_{t'}, \delta^-_{t'}\}\upsilon \le V_1\Big(u(f)\Big) + \prod_{t'=1}^{\infty} \max\{\delta^+_{t'}, \delta^-_{t'}\}\upsilon \le V_1\Big(u(f)\Big) + c\upsilon.
\end{aligned}$$

Thus $(f_1, \dots, f_{t^0-1}, h_{t^0}, h_{t^0+1}, \dots) \succ x$. Since $t^0 \in \mathbb{N}$ was arbitrary, this violates Axiom 6. Thus

$$\lim_{T\to\infty} \prod_{t=1}^{T} \max\{\delta^+_t, \delta^-_t\} = 0.$$

□

Lemma 12 Assume that $\succsim$ satisfies Axioms 1-5, for each $t \in \mathbb{N}$, $V_t \circ u$ is a representation for $\succsim_t$ such that V_t is C-additive, monotonic, and positively homogeneous and there exists $\delta^+_t, \delta^-_t \in (0, 1)$ such that for all $(f_t, f_{t+1}, \dots) \in H$,

$$V_t\Big(u(f_t, f_{t+1}, \dots)\Big) = \begin{cases} \min\limits_{\delta_t \in [\delta^+_t, \delta^-_t]} (1 - \delta_t)u(f_t) + \delta_t V_{t+1}\Big(u(f_{t+1}, f_{t+2}, \dots)\Big), & \text{if } \delta^+_t \le \delta^-_t \\ \max\limits_{\delta_t \in [\delta^-_t, \delta^+_t]} (1 - \delta_t)u(f_t) + \delta_t V_{t+1}\Big(u(f_{t+1}, f_{t+2}, \dots)\Big), & \text{if } \delta^+_t > \delta^-_t. \end{cases}$$

If

$$\lim_{T\to\infty} \prod_{t=1}^{T} \max\{\delta^+_t, \delta^-_t\} = 0,$$

then $\succsim$ satisfies Axiom 6.

Proof. Let $f, g, h \in H$ be such that $f \succ g$. Since f and h are bounded, there exist $x^*, x_* \in \Delta(X)$ such that $x^* \succ x_*$, and for each t, $x^* \succsim f_t \succsim x_*$ and $x^* \succsim h_t \succsim x_*$. Denote

$$\varepsilon = \frac{V_1(u(f)) - V_1(u(g))}{\min\{u(x^*) - u(x_*), 1\}}.$$

Since $\lim_{T\to\infty} \prod_{t=1}^{T} \max\{\delta^+_t, \delta^-_t\} = 0$, there exists $t^1 \in \mathbb{N}$ such that $\prod_{t=1}^{t^1} \max\{\delta^+_t, \delta^-_t\} < \varepsilon$.

Denote

$$\upsilon = \frac{1}{2} \max \left\{ \min \left\{ \Big| u(f_t) - V_{t+1}\Big(u(f_{t+1}, f_{t+2}, \dots)\Big) \Big| \Big| 1 \le t \le t^0 \right\}, 1 \right\}.$$

Since for all t, $\max\{\delta_{t^0}^+, \delta_{t^0}^-\} > 0$, there exists $t^2 \in \mathbb{N}$ such that

$$\prod_{t=1}^{t^2} \max\{\delta_t^+, \delta_t^-\} < \prod_{t=1}^{t^1} \max\{\delta_t^+, \delta_t^-\}\varepsilon\upsilon.$$

For each $1 \le t \le t^1$, denote

$$\hat{\delta}^t = \begin{cases} \underset{\delta_t \in [\delta_t^+, \delta_t^-]}{\arg\min} (1-\delta_t)u(f_t) + \delta_t \bigg(V_{t+1}\Big(u(f_{t+1}, f_{t+2}, \dots)\Big) - \upsilon \bigg), & \text{if } \delta_t^+ \le \delta_t^- \\ \underset{\delta_t \in [\delta_t^-, \delta_t^+]}{\arg\max} (1-\delta_t)u(f_t) + \delta_t \bigg(V_{t+1}\Big(u(f_{t+1}, f_{t+2}, \dots)\Big) - \upsilon \bigg), & \text{if } \delta_t^+ > \delta_t^-. \end{cases}$$

By the definition of υ, for each $1 \le t \le t^1$, $\hat{\delta}^t$ is a singleton. By the definitions of υ and t^2, for each $1 \le t \le t^1$

$$V_t\Big(u(f_t, f_{t+1}, \dots)\Big) = (1-\tilde{\delta}^t)u(f_t) + \tilde{\delta}^t V_{t+1}\Big(u(f_{t+1}, f_{t+2}, \dots)\Big)$$

and

$$\Big| V_{t+1}\Big(u(f_{t+1}, f_{t+2}, \dots)\Big) - V_{t+1}\Big(u(f_{t+1}, \dots, f_{t^2-1}, x_*, x_*, \dots)\Big) \Big| < \upsilon.$$

Hence,

$$V_t\Big(u(f_t, \dots, f_{t^2-1}, x_*, x_*, \dots)\Big) = (1-\tilde{\delta}^t)u(f_t) + \tilde{\delta}^t V_{t+1}\Big(u(f_{t+1}, \dots, f_{t^2-1}, x_*, x_*, \dots)\Big).$$

By above,

$$\begin{aligned} & V_1\Big(u(f_1, f_2, \dots)\Big) - V_1\Big(u(f_1, \dots, f_{t^2-1}, x_*, x_*, \dots)\Big) \\ & = \prod_{t'=1}^{t^1} \tilde{\delta}^{t'} \bigg(V_{t^1+1}\Big(u(f_{t^1+1}, f_{t^1+2}, \dots)\Big) - V_{t^1+1}\Big(u(f_{t^1+1}, \dots, f_{t^2-1}, x_*, x_*, \dots)\Big) \bigg) \\ & \le \prod_{t'=1}^{t^1} \tilde{\delta}^{t'} \Big(u(x^*) - u(x_*) \Big) \le \prod_{t'=1}^{t^1} \max\{\delta_{t'}^+, \delta_{t'}^-\} \Big(u(x^*) - u(x_*) \Big) \\ & < \varepsilon \Big(u(x^*) - u(x_*) \Big) \le V_1\Big(u(f)\Big) - V_1\Big(u(g)\Big). \end{aligned}$$

Thus,

$$V_1\Big(u(f_1, \dots, f_{t^2-1}, h_{t^2}, h_{t^2+1}, \dots)\Big) \ge V_1(u((f_1, \dots, f_{t^2-1}, x_*, x_*, \dots)\Big) > V_1\Big(u(g)\Big).$$

So $(f_1, \dots, f_{t^2-1}, h_{t^2}, h_{t^2+1}, \dots) \succ g$.

The other claim that there exists $t^3 \in \mathbb{N}$ such that $f \succ (g_1, \dots, g_{t^3-1}, h_{t^3}, h_{t^3+1}, \dots)$ follows symmetrically.

□

B.2 Rectangularity

Proposition 13 For each $t \in \mathbb{N}$, denote the partition $\Pi_t = \Big\{\{i\}_{i=1}^{t-1}, \{t, t+1, \dots\}\Big\}$, then $\mathcal{D}$ is Π_t-rectangular.

Proof. Let $n \in \mathbb{N}$. Let $r, q_1, \dots, q_n \in \mathcal{D}$. Define the measure p on $\mathbb{N}$ by for all $A \subset N$,

$$\begin{aligned} p(A) &= \sum_{i=1}^{n-1} r(i) q_i(A|\{i\}) + r\Big(\{n, n+1, \dots\}\Big) q_n\Big(A\Big|\{n, n+1, \dots\}\Big) \\ &= r\Big(A \cap \{1, \dots, n-1\}\Big) + r\Big(\{n, n+1, \dots\}\Big) q_n\Big(A\Big|\{n, n+1, \dots\}\Big). \end{aligned}$$

Let $(\delta_t^r)_{t\in\mathbb{N}}$ and $(\delta_t^q)_{t\in\mathbb{N}}$ be such that

$$\left((1-\delta_t^r)\prod_{t'=1}^{t-1}\delta_{t'}^r\right)_{t\in\mathbb{N}} = r \text{ and } \left((1-\delta_t^q)\prod_{t'=1}^{t-1}\delta_{t'}^q\right)_{t\in\mathbb{N}} = q_n.$$

Define $(\delta_t^p)_{t\in\mathbb{N}}$ by $\delta_t^p = \delta_t^r$ for all $t < n$ and $\delta_t^p = \delta_t^q$ for all $t \leq n$. Now

$$\left((1-\delta_t^p)\prod_{t'=1}^{t-1}\delta_{t'}^p\right)_{t\in\mathbb{N}} \in \mathcal{D}.$$

Additionally for all $A \subset \mathbb{N}$, we have

$$\begin{aligned} p(A) &= \sum_{t=1}^{n-1} \mathbb{1}(t \in A)(1-\delta_t^r)\prod_{t'=1}^{t-1}\delta_{t'}^r + \frac{\prod_{t=1}^{n-1}\delta_{t'}^r}{\prod_{t=1}^{n-1}\delta_{t'}^q}\sum_{t=n}^{\infty}\mathbb{1}(t\in A)(1-\delta_t^q)\prod_{t'=1}^{t-1}\delta_{t'}^q \\ &= \sum_{t=1}^{n-1} \mathbb{1}(t \in A)(1-\delta_t^r)\prod_{t'=1}^{t-1}\delta_{t'}^r + \sum_{t=n}^{\infty}\mathbb{1}(t\in A)(1-\delta_t^q)\prod_{t=1}^{n-1}\delta_{t'}^r\prod_{t'=n}^{t-1}\delta_{t'}^q \\ &= \sum_{t=1}^{\infty}\mathbb{1}(t\in A)(1-\delta_t^p)\prod_{t'=1}^{t-1}\delta_{t'}^p. \end{aligned}$$

Thus

$$p = \left((1-\delta_t^p)\prod_{t'=1}^{t-1}\delta_{t'}^p\right)_{t\in\mathbb{N}} \in \mathcal{D}.$$

□

B.3 Stationarity

Lemma 14 Assume that $\succsim$ satisfies Axioms 1-5 and 7, for each $t\in\mathbb{N}$, $V_t \circ u$ is a representation for $\succsim_t$ such that V_t is C-additive, monotonic, and positively homogeneous and there exists

$\delta_t^+, \delta_t^- \in (0,1)$ such that for all $(f_t, f_{t+1}, \dots) \in H$,

$$V_t\Big(u(f_t, f_{t+1}, \dots)\Big) = \begin{cases} \min_{\delta_t \in [\delta_t^+, \delta_t^-]} (1-\delta_t)u(f_t) + \delta_t V_{t+1}\Big(u(f_{t+1}, f_{t+2}, \dots)\Big), & \text{if } \delta_t^+ \leq \delta_t^- \\ \max_{\delta_t \in [\delta_t^-, \delta_t^+]} (1-\delta_t)u(f_t) + \delta_t V_{t+1}\Big(u(f_{t+1}, f_{t+2}, \dots)\Big), & \text{if } \delta_t^+ > \delta_t^-. \end{cases}$$

Then for each $t \in \mathbb{N}$, $\delta_t^+ = \delta_{t+1}^+$ and $\delta_t^- = \delta_{t+1}^-$.

Proof. Define for all $(a,b),(x,y) \in \Delta(X)^2$

$$(a,b) \trianglerighteq^1 (x,y) \iff (a,b,b,\dots) \succsim_t (x,y,y,\dots)$$

and

$$(a,b) \trianglerighteq^2 (x,y) \iff (a,b,b,\dots) \succsim_{t+1} (x,y,y,\dots).$$

We show that $\trianglerighteq^1 = \trianglerighteq^2$. This shows the claim by the proof of Lemma 10. Assume that

$$(a,b) \trianglerighteq^1 (x,y).$$

Now $(a,b,b,\dots) \succsim_t (x,y,y,\dots)$. So there exists $f \in H$ such that

$$(f_1, \dots, f_{t-1}, a, b, b, \dots) \succsim (f_1, \dots, f_{t-1}, x, y, y, \dots).$$

For $z \in \Delta(X)$, by Axiom 7,

$$(z, f_1, \dots, f_{t-1}, a, b, b, \dots) \succsim (z, f_1, \dots, f_{t-1}, x, y, y, \dots).$$

Thus by the definition of $\succsim_{t+1}$, $(a,b) \trianglerighteq^2 (x,y)$.

Next assume that $(a,b) \triangleright^1 (x,y)$. By Axiom 3, there exist $c,d \in \Delta(X)$ such that $(c,c) \triangleright^2 (d,d)$. By the negation of Axiom 7, we have $(c,c) \triangleright^1 (d,d)$. Assume first that $(x_1,x_2) \trianglerighteq^1 (c,c)$. By Axioms 2 and 4, there exists $\alpha \in (0,1)$ such that

$$(a,b) \trianglerighteq^1 \alpha(c,c) + (1-\alpha)(a,b) \triangleright^1 \alpha(d,d) + (1-\alpha)(a,b) \triangleright^1 (x,y).$$

Thus by the above

$$(a,b) \trianglerighteq^2 \alpha(c,c) + (1-\alpha)(a,b) \trianglerighteq^2 \alpha(d,d) + (1-\alpha)(a,b) \trianglerighteq^2 (x,y).$$

By Axiom 4, $\alpha(c,c) + (1-\alpha)(a,b) \triangleright^2 \alpha(d,d) + (1-\alpha)(a,b)$ and so $(a,b) \triangleright^2 (x,y)$.

□

Proposition 15 Assume that there exist an affine $u : \Delta(X) \to \mathbb{R}$ and for each $t \in \mathbb{N}$, $\delta_t^+, \delta_t^- \in (0,1)$ such that for all $t \in \mathbb{N}$

$$V_t(f) = u(f_t) + \delta_t^+ \max\{V_{t+1}(f) - u(f_t), 0\} + \delta_t^- \min\{V_{t+1}(f) - u(f_t), 0\},$$

where the recursive solution V_1 represents $\succsim$. For all $t \in \mathbb{N}$, $\delta_t^+ \geq \delta_{t+1}^+$ and $\delta_{t+1}^- \geq \delta_t^-$ iff for all $f \in H, x \in X$,

$$x \succsim (f_1, f_2, \dots) \Longrightarrow x \succsim (x, f_1, f_2, \dots). \tag{4}$$

Proof. Assume that for all $f \in H, x \in X$,

$$x \succsim (f_1, f_2, f_3, \dots) \Longrightarrow x \succsim (x, f_1, f_2, f_3, \dots).$$

Let $t \in \mathbb{N}$ and $h \in H$. Define $\trianglerighteq_1$ and $\trianglerighteq_2$ by for all $x, y, x', y' \in \Delta(X)$,

$$(a, b) \trianglerighteq_1 (x, y) \iff (h_1, \dots, h_{t-1}, a, b, b, \dots) \succsim (h_1, \dots, h_{t-1}, x, y, y, \dots)$$

and

$$(a, b) \trianglerighteq_2 (x, y) \iff (h_1, \dots, h_t, a, b, b, \dots) \succsim (h_1, \dots, h_t, x, y, y, \dots).$$

By (4), $\trianglerighteq_2$ is more uncertainty averse than $\trianglerighteq_1$ that gives the condition.

□

B.4 Recursive Choquet Expected Utility

Theorem 16 (Recursive Choquet EU) $\succsim$ satisfies Axioms 1-3, 5, 6, and 8 iff. there exist an affine $u : \Delta(X) \to \mathbb{R}$ and for each $t \in \mathbb{N}$, $\delta_t \in (0,1)$ such that for each $t \in \mathbb{N}$

$$V_t(f) = (1 - \delta_t)u(f_t) + \delta_t V_{t+1}(f).$$

Then the recursive solution V_1 represents $\succsim$.

Proof. By Theorem 1, there exist an affine $u : \Delta(X) \to \mathbb{R}$ and for each $t \in \mathbb{N}$, $\delta_t^+, \delta_t^- \in (0,1)$ such that for each $t \in \mathbb{N}$,

$$V_t\Big(u(f_t, f_{t+1}, \dots)\Big) = \begin{cases} \min\limits_{\delta_t \in [\delta_t^+, \delta_t^-]} (1 - \delta_t)u(f_t) + \delta_t V_{t+1}\Big(u(f_{t+1}, f_{t+2}, \dots)\Big), & \text{if } \delta_t^+ \leq \delta_t^- \\ \max\limits_{\delta_t \in [\delta_t^-, \delta_t^+]} (1 - \delta_t)u(f_t) + \delta_t V_{t+1}\Big(u(f_{t+1}, f_{t+2}, \dots)\Big), & \text{if } \delta_t^+ > \delta_t^-. \end{cases}$$

and the recursive solution V_1 represents $\succsim$.

Let $t \in \mathbb{N}$. We show that $\delta_t^+ = \delta_t^-$. Let $c \in \Delta(X)$ be such that $u(c) \in \operatorname{int} u(\Delta(X))$. Now there exist $f \in H$ and $\gamma > 0$ such that for all $t' > t$, $|u(f_{t'}) - u(c)| > \gamma$ and for all $t' \leq t$, $u(f_{t'}) = u(c) + \gamma$ and $V_{t+1}(f) = u(c)$. Let $\zeta, \eta \in \Delta(X)$ be such that $u(\zeta) = u(c) + \gamma$ and $u(\eta) = u(c) - \gamma$. Let $\theta, \kappa \in \Delta(X)$ be such that

$$u(\theta) = V_1\Big(u(f_{-t}, \zeta)\Big) \text{ and } u(\kappa) = V_1\Big(u(f_{-t}, \eta)\Big).$$

Now $(f_{-t}, \zeta), (f_{-t}, \eta), \theta, \kappa$ are pairwise comonotonic. Thus for all $\alpha \in (0, 1)$,

$$\alpha(f_{-t}, \zeta) + (1-\alpha)(f_{-t}, \eta) \sim \alpha\theta + (1-\alpha)(f_{-t}, \eta) \sim \alpha\theta + (1-\alpha)\kappa. \tag{5}$$

For all $\alpha \in (0, 1)$ and $t' < t$, we have

$$V_{t'}\Big(u\Big(\alpha(f_{-t}, \zeta) + (1-\alpha)(f_{-t}, \eta)\Big)\Big) = (1 - \delta_{t'}^-)u(f_{t'}) + \delta_{t'}^- V_{t'+1}\Big(u\Big(\alpha(f_{-t}, \zeta) + (1-\alpha)(f_{-t}, \eta)\Big)\Big).$$

by the choice of f. Additionally for all $\alpha \in [\frac{1}{2}, 1]$, we have

$$V_t\Big(u\Big(\alpha(f_{-t}, \zeta) + (1-\alpha)(f_{-t}, \eta)\Big)\Big) = (1 - \delta_t^-)\Big(u(c) + (2\alpha - 1)\gamma\Big) + \delta_t^- u(c)$$

and for all $\alpha \in [0, \frac{1}{2}]$

$$V_t\Big(u\Big(\alpha(f_{-t}, \zeta) + (1-\alpha)(f_{-t}, \eta)\Big)\Big) = (1 - \delta_t^+)\Big(u(c) + (2\alpha - 1)\gamma\Big) + \delta_t^+ u(c)$$

By (5), we have for all $\alpha \in [\frac{1}{2}, 1]$

$$\sum_{t'=1}^{t-1} \prod_{t''=1}^{t'-1} \delta_{t''}^-(1 - \delta_{t'}^-)u(f_{t'}) + \prod_{t'=1}^{t-1} \delta_{t'}^- \Big((1 - \delta_t^-)\Big(u(c) + (2\alpha - 1)\gamma\Big) + \delta_t^+ u(c)\Big)$$
$$= \alpha u(\theta) + (1-\alpha)u(\kappa)$$

and for all $\alpha \in [0, \frac{1}{2}]$

$$\sum_{t'=1}^{t-1} \prod_{t''=1}^{t'-1} \delta_{t''}^-(1 - \delta_{t'}^-)u(f_{t'}) + \prod_{t'=1}^{t-1} \delta_{t'}^- \Big((1 - \delta_t^+)\Big(u(c) + (2\alpha - 1)\gamma\Big) + \delta_t^+ u(c)\Big)$$
$$= \alpha u(\theta) + (1-\alpha)u(\kappa).$$

Since for each t', $\delta_{t'}^+, \delta_{t'}^- \in (0, 1)$ and $\gamma > 0$, we have $\delta_t^+ = \delta_t^-$. □

B.5 Connecting History Independence and Dynamic Consistency

Proposition 17 Assume that each $\succsim_t$ are complete and transitive. If each $\succsim_t$ satisfies Axioms 9 and 10, then $\succsim_1$ satisfies Axiom 5.

Proof. Let $t \in \mathbb{N}$, and $f, g, a, b \in H$. Assume by completeness, w.l.o.g.

$$(a_1, \dots, a_{t-1}, f_t, f_{t+1}, \dots) \succsim_t (a_1, \dots, a_{t-1}, g_t, g_{t+1}, \dots).$$

By Axiom 9,

$$(a_1, \dots, a_{t-1}, f_t, f_{t+1}, \dots) \sim_t (b_1, \dots, b_{t-1}, f_t, f_{t+1}, \dots)$$

and

$$(a_1, \dots, a_{t-1}, g_t, g_{t+1}, \dots) \sim_t (b_1, \dots, b_{t-1}, g_t, g_{t+1}, \dots).$$

By applying Axiom 10, $t-1$ times, we have

$$(a_1, \dots, a_{t-1}, f_t, f_{t+1}, \dots) \succsim_1 (a_1, \dots, a_{t-1}, g_t, g_{t+1}, \dots),$$

$$(a_1, \dots, a_{t-1}, f_t, f_{t+1}, \dots) \sim_1 (b_1, \dots, b_{t-1}, f_t, f_{t+1}, \dots),$$

and

$$(a_1, \dots, a_{t-1}, g_t, g_{t+1}, \dots) \sim_1 (b_1, \dots, b_{t-1}, g_t, g_{t+1}, \dots).$$

Thus by transitivity of $\succsim_1$,

$$(b_1, \dots, b_{t-1}, f_t, f_{t+1}, \dots) \succsim_1 (b_1, \dots, b_{t-1}, g_t, g_{t+1}, \dots).$$

$\square$

Proposition 18 Define the inferred preferences at each time period t by

$$(f_t, f_{t+1}, \dots) \tilde{\succsim}_t (g_t, g_{t+1}, \dots) \Longleftrightarrow \forall\, h \in H, (h_1, \dots, h_{t-1}, f_t, f_{t+1}, \dots) \succsim_1 (h_1, \dots, h_{t-1}, g_t, g_{t+1}, \dots).$$

Assume that each $\succsim_t$ are complete and transitive. If each $\succsim_t$ satisfies Axioms 9 and 10, then for each t, $f, g \in H$

$$f \succsim_t g \iff (f_t, f_{t+1}, \dots) \tilde{\succsim}_t (g_t, g_{t+1}, \dots).$$

Proof. Assume first that $f \succsim_t g$. Let $h \in H$. By Axiom 9, we have

$$f \sim_t (h_1, \dots, h_{t-1}, f_t, f_{t+1}) \text{ and } g \sim_t (h_1, \dots, h_{t-1}, g_t, g_{t+1}).$$

Thus by transitivity,

$$(h_1, \dots, h_{t-1}, f_t, f_{t+1}) \succsim_t (h_1, \dots, h_{t-1}, g_t, g_{t+1}).$$

By applying Axiom 10 $t-1$ times, we have

$$(h_1, \dots, h_{t-1}, f_t, f_{t+1}) \succsim_1 (h_1, \dots, h_{t-1}, g_t, g_{t+1}).$$

By Proposition 17, we have

$$(f_t, f_{t+1}, \dots) \tilde{\succsim}_t (g_t, g_{t+1}, \dots).$$

Symmetrically, if $f \succ_t g$, then $(f_t, f_{t+1}, \dots) \tilde{\succ}_t (g_t, g_{t+1}, \dots)$. This shows the claim since $\succsim_t$ and $\tilde{\succsim}_t$ are complete. □

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