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Abstract: This paper establishes necessary and sufficient conditions for the existence of a unique strictly stationary and ergodic solution for integer-valued autoregressive conditional heteroscedasticity (INARCH) processes. We also provide conditions that guarantee existence of higher order moments. The results apply to integer-valued GARCH model, and its long-memory versions with hyperbolically decaying coefficients.

KEY WORDS: INARCH processes; Stationarity; Ergodicity; Lyapunov exponent

JEL classification: C1, C4, C5

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1 Introduction

In many real-world situations, we have to deal with non-negative integer-valued time series. Such time series are often produced in fields that include economics, insurance, medicine, epidemiology, queueing systems, communications, and meteorology and so on. Examples for the wide range of practical applications are the daily or monthly number of cases in epidemiology, the number of stock market transactions or stock price changes per minute in finance and the number of photon arrivals per microsecond measured in a biological experiment. Their analysis may present some difficulties, however, and if the analysis is based on stochastic models, these models have to reflect the integer peculiarity of the observed series. Various models have been suggested in the literature to tackle the problem of integer-valued time series analysis. These models include the traditional generalized linear model methodology and the state-of-the-art integer-valued autoregressive moving average (INARMA), and integer-valued generalized autoregressive conditional heteroscedasticity (INGARCH) processes. The first modeling approach is very simple and consists of choosing a suitable distribution for count data and an appropriate link function, (see Kedem and Fokianos, 2002). The second group of models are adaptation of the well-known ARMA and GARCH processes in the modeling of continuous-state and discrete-time series to count settings by means of thinning operators (see Weiß, 2008, for a recent review of the thinning operators). These processes are developed to model stationary count data. Therefore, considerable effort has been devoted to provide and prove general conditions that ensure existence and uniqueness of second-order stationary solutions using Hilbert space techniques (see Ferland et al., 2006; Latour, 1998; Doukhan and Wintenberger, 2008; Doukhan et al., 2012; Neumann, 2011). Recently, Sim et al. (2021) provide conditions for ergodicity and consistency of the maximum likelihood estimator for general-order observation-driven models (ODMs). However, recent empirical observations indicate that some important count data in modeling are strictly stationary, and non square-integrable (see Segnon and Stapper, 2019).

The objective of this paper is to establish conditions for strict stationarity and ergodicity of the INARCH processes and existence of higher order moments. These statistical properties are crucial for deriving large sample properties of the maximum likelihood estimators of the model parameters. We make use of the multiplicative ergodic theorem developed by Ruelle (1982) for bounded operators in Hilbert space and show that the necessary and sufficient conditions for stationarity is the negativity of the Lyapunov exponent associated with these processes. Our result applies to the INGARCH model in Ferland et al. (2006), and INFIGARCH and INHYGARCH models in Segnon and Stapper (2019). Since the seminal paper by Bougerol and Picard (1992) the use of the multiplicative ergodic theorem to study the stationarity of ARCH-type processes has become very popular, see Kazakevicius and Leipus (2002); Zerner (2018).

The rest of the paper is organized as follows. Section 2 describes the modeling framework. The main results are provided in Section 3. Section 4 presents the proofs to the main results. Finally, Section 5 concludes.

2 Poisson INARCH(∞) Processes

2.1 Definition

A sequence of integer-valued random variables $\{Y_t\}_{t\in Z}$ is said to be an INARCH(∞) process if:

(i) the distribution of Y_t conditional on the $\sigma-$field $\Omega_{t-1} = \sigma(Y_l,\, l \leq t-1)$ is Poisson with mean λ_t,

(ii) there exist nonnegative constants c, ψ_i, $1 \leq i \leq \infty$, such that

$$\lambda_t = c + \psi(\mathrm{L})\, Y_t, \tag{1}$$

where $\Pr(\lambda_t > 0) = 1$ and $\psi(\mathrm{L}) = \sum_{i=1}^{\infty} \psi_i \mathrm{L}^i$.

This class of models also includes:

(a) The integer-valued HYGARCH(p, d, q) model for, c is an appropriately defined constant, and

$$\psi(\mathrm{L}) = \left[1 - \frac{\Phi(\mathrm{L})\left(1 + \eta[(1-\mathrm{L})^d - 1]\right)}{\mathbf{B}(\mathrm{L})}\right] \tag{2}$$
$$= \sum_{i=1}^{\infty} \psi_i \mathrm{L}^i,$$

with $\beta_0 > 0$ and $\phi_1, \ldots, \phi_{m-1} \geq 0, \beta_1, \ldots, \beta_q \geq 0$, and $\psi_i \geq 0$ for all i. In Eq. (2), L denotes the lag operator. The lag polynomials are defined as $\Phi(\mathrm{L}) = [1 - \beta(\mathrm{L}) - \alpha(\mathrm{L})] = \sum_{i=1}^{m-1} \phi_i \mathrm{L}^i$, where $m = \max(p, q)$, $\alpha(\mathrm{L}) = \sum_{i=1}^{p} \alpha_i \mathrm{L}^i$, $\beta(\mathrm{L}) = \sum_{j=1}^{q} \beta_j \mathrm{L}^j$ and $\mathbf{B}(\mathrm{L}) = [1 - \beta(\mathrm{L})]$. $\eta \geq 0$ is an amplitude parameter, $d \in [0, 1]$ and $(1 - \mathrm{L})^d$ is the fractional differencing operator given by

$$(1 - \mathrm{L})^d = \sum_{k=0}^{\infty} \frac{\Gamma(k - d)\mathrm{L}^k}{\Gamma(-d)\Gamma(k + 1)}, \tag{3}$$

where $\Gamma(\cdot)$ is the gamma function.

(b) The integer-valued FIGARCH(p, d, q) model for $\eta = 1$ in Eq. 2.

(c) The integer-valued GARCH(p, q) model for $\eta = 0$ in Eq. 2.

Remark 1. *Segnon and Stapper (2019) show that for $\eta \in (0, 1)$ implies that $\psi(1) < 1$, and thus, the INHYGARCH process is covariance stationary.*

Remark 2. *Ferland et al. (2006) show that the INGARCH(p,q) process exists and is strictly stationary with finite first and second order moments, if and only if the following restriction is met: $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$, which is equivalent to $\sum_{i=1}^{\infty} \psi_i < 1$. In the simple INGARCH$(1, 1)$, $\psi_i = \alpha_1 \beta_1^{i-1}$ for $i \geq 1$ and the stationarity condition is well known to be $\alpha_1 + \beta_1 < 1$, which is equivalent to $\sum_{i=1}^{\infty} \alpha_1 \beta_1^{i-1} < 1$ in the INARCH representation above. The INGARCH$(1, 1)$ reduces to an integrated INGARCH$(1, 1)$ when the sum of the lag coefficients is unity ($\alpha_1 + \beta_1 = 1$). Segnon and Stapper (2019) point out that in the INFIGARCH(p,q) $\sum_{i=1}^{\infty} \psi_i = 1$. Thus, the process is not covariance stationary. We note that the coefficient ψ_i in the INHYGARCH can be approximated by ci^{-1-d}, with c appropriately defined.*

Table 1: Descriptive statistics of simulated data

	INFIGARCH	INHYGARCH	INGARCH
Overdispersion	11.812	3.736	1.522
Skewness	0.210	0.473	0.663
Kurtosis	3.028	3.301	3.452

Note: The statistics reported in the Table are the averages. The results are based on 100 replications of simulated data with size (n=500) with the following parameters: $\beta_0 = 2$, $\alpha_1 = 0.3$, $d = 0.4$, $\beta_1 = 0.2$, $\eta = 0.8$.

Figure 1 illustrates the capacity of the INHYGARCH(1,d,1) model to reproduce various degree of dependence for $\eta = 1$ (INFIGARCH(1,d,1)), $\eta = 0$ (INGARCH(1,1)) and $\eta = 0.8$ (INHYGARCH(1,d,1)). We see that with different values for the amplitude parameter, η, various long range dependencies observed in empirical data can be captured. Furthermore, Table 1 shows that the INHYGARCH(1,d,1) model can reproduce over-dispersion and asymmetry observed in real world data.

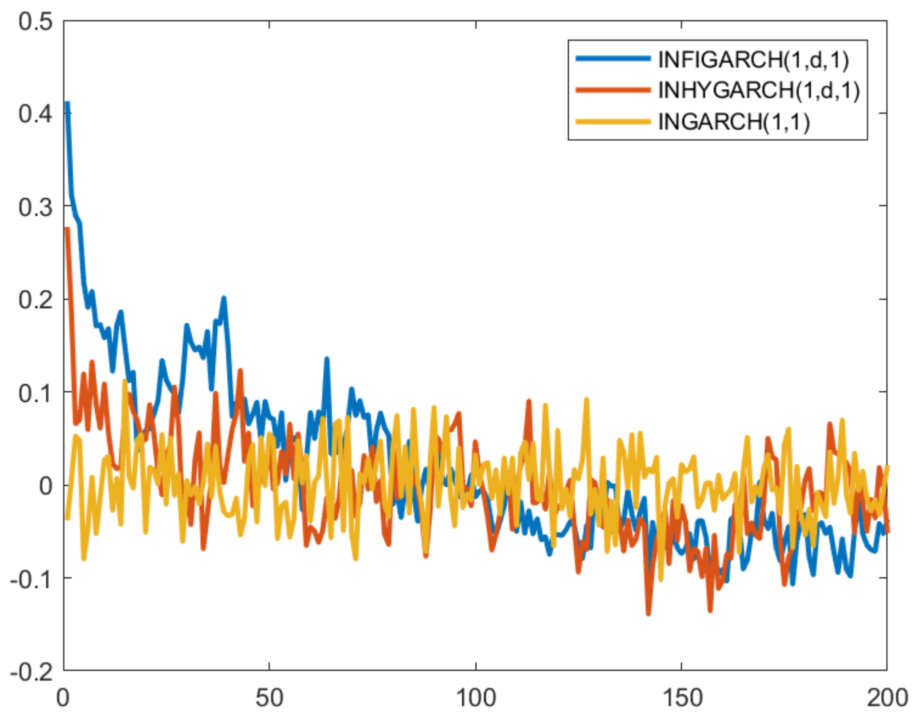

Figure 1: Theoretical ACF for different parameter constellations with baseline setup: $\beta_0 = 2$, $\alpha_1 = 0.3$, $d = 0.4$, $\beta_1 = 0.2$, $\eta = 0.8$ and $n = 500$.

Since the INFIGARCH(p, d, q) process is not covariance stationary, its appears that the INFIGARCH(p, d, q) is not a long memory model in the common sense. However, we aim to show in the next Section that the INARCH representation of the INHYGARCH(p, d, q) and INFIGARCH(p, d, q) processes are strictly stationary and ergodic using a multiplicative ergodic theorem and a Lyapunov exponent. Towards this end, we first look at the construction of an INARCH process.

2.2 Construction

Let $\{u_t\}_{t\in Z}$ be a sequence of independent random variables with values in N (N is the set of non-negative integers) with common mean ω. For each $t \in Z$ and $i \in N$, let $\xi_t^{(i)} = \{\xi_{t,j}^{(i)}\}_{j\in N}$ represent a sequence of independent random variables having a common mean ψ_i. All the variables u_s, $\xi_{t,j}^{(i)}$, $(s \in Z, t \in Z, i \in N \text{ and } j \in N)$ are assumed to be mutually independent. Using these random variables, we introduce a sequence of random variables $\{Y_t^{(n)}\}$ that may be considered as successive approximations of Y_t:

$$Y_t^{(n)} = \begin{cases} 0, & \text{if } n < 0; \\ u_t, & \text{if } n = 0; \\ u_t + \sum_{i=1}^{n} \sum_{j=1}^{Y_{t-i}^{(n-i)}} \xi_{t-i,j}^{(i)} & \text{if } n > 0. \end{cases} \tag{4}$$

From (4) we can see that $Y_t^{(n)}$ is a finite sum of independent Poisson variables. So, the expectation and the variance of $Y_t^{(n)}$ are well defined. In the next Section we want to show that $Y_t^{(n)}$, as $n \to \infty$, admits an almost sure limit Y_t and that the limiting process $\{Y_t\}_{t\in Z}$ satisfies (1), see Proposition 1. Given this result, we want to show that under mild conditions the approximated process is strictly stationary, ergodic and has moments of any order. Then, these statistical properties for the original process is obtained by a limiting argument (Proposition 1) connecting the two representations.

3 Stationarity of INARCH(∞) Processes

To prove the strict stationarity of $\{Y_t\}$ we first show that for any fixed n, $Y_t^{(n)}$ is strictly stationary.

3.1 Some basic definitions and results

Definition 1. *Let $\{z_j\}_{j\in N}$ be a sequence of independent and identically distributed non-negative integer-valued random variables with mean ψ and finite variance σ^2 which is independent of a non-negative integer-valued random variable y. The generalized Steutel and van Harn operator, $\psi\diamond$, is defined as*

$$\psi \diamond y = \begin{cases} \sum_{i=1}^{y} z_i & \text{if } y > 0; \\ 0 & \text{if } y = 0. \end{cases} \tag{5}$$

Remark 3. *The sequence $\{z_j\}_{j\in N}$ is called a counting sequence. Let $\alpha\diamond$ be another operator based on a counting sequence $\{x_j\}_{j\in N}$. Both operators $\psi\diamond$ and $\alpha\diamond$ are said to be independent if and only if the counting sequences $\{z_j\}_{j\in N}$ and $\{x_j\}_{j\in N}$ are mutually independent.*

Using the operator from Eq. 5, we may rewrite the sequence of random variables $\{Y_t^{(n)}\}_{n\in N}$ as

$$Y_t^{(n)} = \sum_{i=1}^{n} E\left(\xi_{t-i}^{(i)}\right) \diamond Y_{t-i}^{(n-i)} + u_t, \quad n > 0, \tag{6}$$

where $E\left(\xi_{t-i}^{(i)}\right) = \psi_i$.

Proposition 1. *If $\psi(1) < 1$ then the sequence $\{Y_t^{(n)}\}_{n\in N}$ has an almost sure limit.*

For the state space representation of Eq. 6, we consider the following nonnegative multidimensional autoregressive process $\mathbf{Z} = (\mathbf{Z}_n)_{n\geq 0}$ of order one with random coefficient matrix. For a fix dimension $d \in N$, let $\mathbf{U} = (\mathbf{U}_n)_{n\geq 0}$ be a sequence of $[0,\infty)^d$-valued random vectors, and let $(\mathbf{C}_n)_{n\geq 1}$ be an i.i.d. sequence of $[0,\infty)^{d\times d}$-valued random matrices that are given by

$$\mathbf{C}_n = \begin{pmatrix} \xi_{t-1}^{(1)} & \xi_{t-2}^{(2)} & \dots & \xi_{t-n}^{(n)} \\ 1 & 0 & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \dots & 1 & 0 \end{pmatrix}. \tag{7}$$

We have

$$\mathbf{Z}_n = E(\mathbf{C}_n) \diamond \mathbf{Z}_{n-1} + \mathbf{U}_n, \quad n \geq 1, \tag{8}$$

where $\mathbf{Z}_n = \left(Y_t^{(n)}, Y_{t-1}^{(n-1)}, Y_{t-2}^{(n-2)}, \dots Y_{t-n}^{(0)}\right)'$ and $\mathbf{U}_n = (u_t, 0, \dots 0)'$.

Proposition 2. *Then, (6) has a stationary and ergodic solution if and only if (8) has a stationary and ergodic solution.*

Lemma 1. *Let $\psi(z) = z^n - \alpha_1 z^{n-1} - \cdots - \alpha_{n-1} z - \alpha_n$ with $\sum_{k=1}^n |\alpha_k| \leq 1$ and $\alpha_n > 0$. Then the roots of $\psi(z)$ are all inside the unit circle.*

Lemma 2. *(Lemma 5.2 in Bougerol (1987)) Let $\{\mathbf{A}_n, n \in Z\}$ be a sequence of independent, identically distributed, random matrices such that $E(\log^+ \|\mathbf{A}_1\|)$ is finite. If, almost surely,*

$$\lim_{n\to+\infty} \frac{1}{n} \log \|\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_n\| = 0,$$

then the top Lyapunov exponent associated with this sequence is strictly negative.

Proposition 3. *The process defined in Eq. (8) has a unique strictly stationary and ergodic solution if and only if the top Lyapunov exponent γ associated with the random matrices $\{\mathbf{C}_n\}_{n\geq 1}$ is strictly negative. The unique strictly stationary solution $(\mathbf{Z}_n)_{n\geq 0}$ of (8) is given by*

$$\mathbf{Z}_n = \mathbf{U}_n + \sum_{k=1}^n E(\mathbf{C}_n\mathbf{C}_{n-1} \dots \mathbf{C}_{n-k+1}) \diamond \mathbf{U}_{n-k}. \tag{9}$$

Corollary 1. *Let assume that the support of the law of $\mathbf{U}_1$ is unbounded and all the coefficients ψ are nonnegative. Then, if $\sum_{i=1}^n \psi_i = 1$, then the INARCH process defined in Eq. (8) has a unique stationary solution.*

3.2 Moments of the INARCH Processes

It is crucial for statistical inference to know whether the unique stationary solution has moments of higher order. To derive conditions for the existence of higher order moments for INARCH models we use the state space representation of the successive approximated process in (8). The following proposition guarantees the existence of higher order moments.

Proposition 4. *Let $m \in N^*$. Then the mth moment of $Y_t^{(n)}$ is finite if and only if the spectral radius of the matrix $E\left(\mathbf{C}_n^{\otimes m}\right)$ is strictly less than 1, where $\mathbf{C}^{\otimes m} = \mathbf{C}\otimes\mathbf{C}\otimes\cdots\otimes\mathbf{C}$ (m factors), $\rho(\mathbf{C}) = \min\{|$eigenvalues of a matrix $\mathbf{C})\}$ and $\mathbf{C}_n$ is defined by Eq. (7).*

4 Proofs

Proof of Proposition 1. We closely follow the Proof of Proposition 2 in Ferland et al. (2006), Page 928. It follows from Eq. (4) that $Y_t^{(n)}$ is obtained through a cascade of thinning operations along the sequence $\{u_t\}_{t\in Z}$. So, the expectation and the variance of $Y_t^{(n)}$ are well defined and given by

$$\begin{aligned} \mu_n &= E\left(u_t + \sum_{i=1}^{n} \sum_{j=1}^{Y_{t-i}^{(n-i)}} \xi_{t-i,j}^{(i)}\right) \\ &= \omega + \sum_{i=1}^{n} E\left(\sum_{j=1}^{Y_{t-i}^{(n-i)}} \xi_{t-i,j}^{(i)}\right) \end{aligned} \tag{10}$$

Let (Ω, F, P) be the common probability space on which the relevant random variables are defined. Because $Y_j^{(n)}$ is a non-decreasing sequence of non-negative integers, we have

$$\forall \omega \in \Omega, \lim_{n\to\infty} Y_t^{(n)}(\omega) = Y_t \tag{11}$$

which is either finite or infinite. We will show that the set

$$A_\infty = \{\omega : Y_t(\omega) = \infty\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \limsup_n A_n \tag{12}$$

is of probability zero, where

$$A_n = \left\{\omega : Y_t^{(n)}(\omega) - Y_t^{(n-1)}(\omega) > 0\right\}, \text{ for } n > 1. \tag{13}$$

On the one hand:

$$E\left(Y_t^{(n)} - Y_t^{(n-1)}\right) \geq \sum_{k=1}^{\infty} \Pr\left\{\omega : Y_t^{(n)}(\omega) - Y_t^{(n-1)}(\omega) = k\right\} = \Pr(A_n). \tag{14}$$

On the other hand:

$$E\left(Y_t^{(n)} - Y_t^{(n-1)}\right) = \mu_n - \mu_{n-1} \equiv \nu_n. \tag{15}$$

Obviously, the sequence $\{\nu_n\}$ satisfies a homogeneous finite difference equation with a characteristic polynomial, namely $\psi(z)$, that has all its roots outside the unit circle. As shown in Brockwell and Davis (1991), Section 3.6, sequence $\{\nu_n\}$ tends towards zero with a geometric rate as $n \to \infty$. In other words, a constant $Q \geq 0$ and a constant $0 < \alpha < 1$ exist such that $\nu_n \leq Q\alpha^n$. Since $\Pr\{A_n\} \leq \nu_n$ we get

$$\sum_{n=1}^{\infty} \Pr\{A_n\} \leq Q \sum_{n=1}^{\infty} \alpha^n < \infty. \tag{16}$$

By the Borel-Cantelli lemma, $\Pr\{A_\infty\} = 0$. ∎

Proof of Proposition 2. Eq. (8) is a state-space representation of (6), and thus, any stationary solution of (8) is also a stationary solution of (6) and vice versa. Analogously, any ergodic solution of (8) is also an ergodic solution of (6), and vice versa. The proof of the ergodicity follows from Lemma A 1.2.7 in Brandt et al. (1990). ∎

Proof of Lemma 1. Let us consider the unit circle $\zeta = \{z : |z| = 1\}$ and suppose $\sum_{k=1}^{n} |\alpha_k| < 1$. The functions $h(z) = z^n$ and $T(z) = -(\alpha_1 z^{n-1} + \alpha_2 z^{n-2} + \cdots + \alpha_n)$ are both analytic inside and on ζ. Hence, on ζ,

$$|T| \leq \sum_{k=0}^{r-1} |\alpha_{n-k} z^k| \leq \sum_{k=0}^{n-1} |\alpha_{n-k}| < 1 = |h|.$$

Based on the theorem of Rouché, $h(z)$ and $h(z) + T(z)$ have the same number of zeros inside ζ. But h has n zeros inside ζ. Therefore, we conclude that all roots of $\alpha(z)$ are inside the unit circle. ■

Proof of Proposition 3. Suppose that the top Lyapunov exponent γ is strictly negative. We can see that the random matrices $\{\mathbf{C}_n\}$ in Eq. 7 consist of independent and identically distributed non-negative integer-valued random variables, $\xi_t^{(i)}$, with the baseline distribution f (Poisson) and with a finite mean, ψ_i and variance. This means that all the coefficients of these matrices are integrable. Furthermore, the random vectors $\{\mathbf{U}_n\}_{n\in N}$ contain *i.i.d.* non-negative integer-valued random variables and therefore are also integrable. All these imply that $E(\log^+ \|\mathbf{C}_1\|)$ and $E(\log^+ \|\mathbf{U}_1\|)$ are finite and therefore, the process (8) has a strictly stationary solution that is given by

$$\mathbf{Z}_n = \mathbf{U}_n + \sum_{k=1}^{\infty} E\left(\mathbf{C}_n\mathbf{C}_{n-1}\ldots\mathbf{C}_{n-k+1}\right) \diamond \mathbf{U}_{n-k}. \tag{17}$$

Conversely, let assume that there exists a strictly stationary solution $\{\mathbf{Z}_n\}_{n\in N}$ of Eq. (6). By iterating Eq. (8), we have for $n > 0$,

$$\begin{aligned}
\mathbf{Z}_n &= \mathrm{E}\left(\mathbf{C}_n\right) \diamond \mathbf{Z}_{n-1} + \mathbf{U}_n \\
&= \mathrm{E}\left(\mathbf{C}_n\mathbf{C}_{n-1}\right) \diamond \mathbf{Z}_{n-2} + \mathbf{U}_n + \mathrm{E}\left(\mathbf{C}_n\right) \diamond \mathbf{U}_{n-1} \\
&= \ldots \\
&= \mathrm{E}\left(\prod_{j=0}^{n-1} \mathbf{C}_{n-j}\right) \diamond \mathbf{Z}_0 + \sum_{j=0}^{n} \mathrm{E}\left(\prod_{i=0}^{j-1} \mathbf{C}_{n-i}\right) \diamond \mathbf{U}_{n-j} \\
\mathbf{Z}_n &= E\left(\mathbf{C}^{(n)}\right) \diamond \mathbf{Z}_0 + \mathbf{U}^{(n)}
\end{aligned}$$

with $n \in N_0$ and where $\prod_{i=0}^{-1} \mathbf{C}_{n-i} = 1$.

All the coefficients of $\mathbf{C}_n$, $\mathbf{Z}_n$ and $\mathbf{U}_n$ are nonnegative. The characteristic polynomial of $E(\mathbf{C}_n)$ is $\Psi(z) = z^n - \psi_1 z^{n-1} - \cdots - \psi_{n-1}z - \psi_n$. By Lemma 1, the roots of $\Psi(z)$ are all inside the unit circle, then $\lim_{t\to\infty} E\left(\left(\prod_{j=0}^{n} \mathbf{C}_{n-j}\right) e_i\right) = 0$ a.s. where e_i denotes the canonical basis of R^n. We have

$\lim_{n\to\infty} E\left(\prod_{j=0}^{n} \mathbf{C}_{n-j}\right) = 0$

$\lim_{n\to\infty} E\left(\| \prod_{j=0}^{n} \mathbf{C}_{n-j}\|\right) = 0$

then $\lim_{n\to\infty} E\left(\prod_{j=0}^{n} \mathbf{C}_{n-j}\right) \diamond \mathbf{Z}_0 = 0$ a.s. According to Lemma 2 the associated top Lyapunov exponent γ is strictly negative, so that the series $\sum_{j=0}^{n} \mathrm{E}\left(\prod_{i=0}^{j-1} \mathbf{C}_{n-i}\right) \diamond \mathbf{U}_{n-j}$ converges a.s. Therefore, $\{\mathbf{Z}_n\}_{n\in Z}$ is a strictly stationary solution of Eq. (6). Furthermore, we can write the solution in (9) as $\mathbf{Z}_n = F(\mathbf{C}_{n-1}, \mathbf{C}_{n-1}, \ldots, \mathbf{U}_n)$ for some measurable function F independent of n. It follows that the strictly stationary solution is also ergodic because $\mathbf{C}_n$ and $\mathbf{U}_n$ are ergodic, see Brandt et al. (1990) Lemma A 1.2.7.

Now, we aim to prove the uniqueness of the strictly stationary solution. Let $\{\mathbf{W}_n\}_{n\in Z}$ be another strictly stationarity solution of Eq. (8). The norm of the following difference for $n > 0$

$$\begin{aligned}
\|\mathbf{Z}_n - \mathbf{W}_n\| &= \|E(\mathbf{C}_1\mathbf{C}_2\ldots\mathbf{C}_n) \diamond (\mathbf{Z}_0 - \mathbf{W}_0)\| \\
&\le \|E(\mathbf{C}_1\mathbf{C}_2\ldots\mathbf{C}_n) \diamond \| \|\mathbf{Z}_0 - \mathbf{W}_0\| \\
&\le \|E\left(\mathbf{C}^{(n)}\right) \diamond \| \|\mathbf{Z}_0 - \mathbf{W}_0\|,
\end{aligned}$$

by Lemma 1, $\|E(\mathbf{C}_1\mathbf{C}_2\ldots\mathbf{C}_n)\|$ converges to 0, a.s. and the fact that the law of the difference $(\mathbf{Z}_0 - \mathbf{W}_0)$ is independent of n, imply that $\mathbf{Z}_n - \mathbf{W}_n$ converges to 0 in probability. We conclude that $\mathbf{Z}_n = \mathbf{W}_n$ and that Eq. (6) has a unique solution, once the counting process are known. ∎

Proof of corollary 1. By induction on n, we have

$$\det(zIn - E(\mathbf{C}_1)) = z^n \left(1 - \sum_{i=1}^{n} \psi_i z^{-i}\right).$$

The inequality $|a - b| \ge |(|a| - |b|)|$ implies that if $|z| > 1$, then

$$\det\left(zI_n - E(\mathbf{C}_1)\right) > 1 - \sum_{i=1}^{n} \psi_i. \tag{18}$$

Since the right-hand side is zero and since $\det\left(zI_n - E(\mathbf{C}_1)\right) = 0$, we conclude that the spectral radius ρ of the matrix $E(\mathbf{C}_1)$ is 1. Furthermore, all the coefficients of the matrix $\mathbf{C}_2\mathbf{C}_1$ are almost surely positive and $\mathbf{C}_1$ has no zero column nor zero row. Since $\mathbf{C}_1$ is not a.s. bounded, these properties imply by theorem 2 in Kesten and Spitzer (1984) that the top Lyapunov exponent γ satisfies $\gamma < \log\rho$. As result, $\gamma < 0$ and the corollary follows from Proposition 3. ■

Proof of Proposition 4. We note that all the elements in $\mathbf{Z}_n$, $\mathbf{C}_n$ and $\mathbf{U}_n$ are strictly positive, thus the model is irreducible, see Bougerol and Picard (1992) for more details.

Now let go back to the model in (8) and show that it has higher order moment: We have

$$\begin{aligned} E\left(\mathbf{Z}_n^{\otimes m}\right) &\geq E\left(E(\mathbf{C}_n) \diamond \mathbf{Z}_{n-1}\right)^{\otimes m} + E\left(\mathbf{U}_n^{\otimes m}\right) \\ &= E\left(\mathbf{C}_n\right)^{\otimes m} E\left(\mathbf{Z}_{n-1}^{\otimes m}\right) G_1 R_1^{\otimes m} \\ &\geq G_1 \sum_{j=0}^{n} \left[E\left(\mathbf{C}_n\right)^{\otimes m}\right]^j R_1^{\otimes m} \end{aligned} \tag{19}$$

$G_1 = \min\{$all the positive elements of $E\left(\mathbf{U}_n^{\otimes m}\right)\}$, $R_1 = (1, 0, 0 \ldots, 0)'$. A vector $A >$ a vector B means that each element of A exceeds the corresponding element of B.

If n tends to infinity, from (19) we have

$$\sum_{j=0}^{n} \left[E\left(\mathbf{C}_n\right)^{\otimes m}\right]^j R_1^{\otimes m} < \infty. \tag{20}$$

The idea here is to make use of the nonnegativity of the elements of $E\left(\mathbf{C}_n\right)^{\otimes m}$ and $R_1^{\otimes m}$. We first show that

$$\left[E\left(\mathbf{C}_n^{\otimes m}\right)\right]^Q R_1^{\otimes m} > 0. \tag{21}$$

We will prove that (21) holds. First $E\left(\mathbf{C}_n\right)^{\otimes m} R_1^{\otimes m} = E\left(\mathbf{C}_n R_1\right)^{\otimes m}$, where $\mathbf{C}_n R_1 = (\xi_{t-1}^{(1)}, 1, 0 \ldots, 0)$. Let $G_2 = \min\{$all the positive elements of $E\left(\mathbf{C}_n R_1\right)^{\otimes m}\}$ and $R_2 =$

$(1, 1, 0, 0, \ldots, 0)'$. It follows that

$$E\left(\mathbf{C}_n\right)^{\otimes m} R_1^{\otimes m} \geq G_2 R_2^{\otimes m}. \tag{22}$$

From (22), we have

$$\left[E\left(\mathbf{C}_n\right)^{\otimes m}\right]^2 R_1^{\otimes m} \geq G_2 E\left(\mathbf{C}_n\right)^{\otimes m} R_2^{\otimes m} = G_2 E\left(\mathbf{C}_n R_2\right)^{\otimes m}. \tag{23}$$

Now, $\mathbf{C}_n R_2 = (\xi_{t-1}^{(1)} + \xi_{t-2}^{(2)}, 1, 1, 0 \ldots, 0)$. Let $G_3 = \min\{\text{all the positive elements of } E\left(\mathbf{C}_n R_2\right)^{\otimes m}\}$ and $R_3 = (1, 1, 1, 0, \ldots, 0)'$. From (23), we have

$$\left[E\left(\mathbf{C}_n\right)^{\otimes m}\right]^2 R_1^{\otimes m} \geq G_2 G_3 R_3^{\otimes m}. \tag{24}$$

Repeating the preceding procedure Q times, we can show that

$$\left[E\left(\mathbf{C}_n\right)^{\otimes m}\right]^Q R_1^{\otimes m} \geq \left(\prod_{j=2}^{Q} G_j\right) R_Q^{\otimes m}, \tag{25}$$

where $G_j > 0$ and $R_Q = (1, 1, 1, \ldots, 1)$. Thus, (21) holds. From (20) and (21), we have

$$\sum_{j=0}^{\infty} \left[E\left(\mathbf{C}_n\right)^{\otimes m}\right]^j \left[E\left(\mathbf{C}_n\right)^{\otimes m}\right]^Q R_1^{\otimes m} < \infty. \tag{26}$$

Let c_{kl}^j be the $(k, l)th$ element of $\left[E\left(\mathbf{C}_n\right)^{\otimes m}\right]^j$. From (26), we know that $\sum_{j=0}^{\infty} c_{kl}^j < \infty$ for all $k, j = 1, \ldots, n^m$, i.e.,

$$\sum_{j=0}^{\infty} \left[E\left(\mathbf{C}_n\right)^{\otimes m}\right]^j < \infty, \tag{27}$$

and hence $\rho\left[E\left(\mathbf{C}_n^{\otimes m}\right)\right] < 1$. ■

5 Conclusion

We have shown that the INARCH(∞) processes admit solutions that are strictly stationary and ergodic. Furthermore, we have proven that the process exhibits higher order moments under certain conditions. These results are crucial for proving the asymptotic normality of the conditional maximum likelihood estimates of the parameters in models.

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