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Efficiency based measures of inequality*

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Abstract

How should we make value judgments about wealth inequality? Harsanyi (1953) proposes to take an individual who evaluates her well-being by expected utility and ask her to evaluate the wealth possibilities ex-ante (i.e. before she finds her place in society, i.e., under the "veil of ignorance" of Rawls (1971)) assuming that she will be allocated any one of the possible wealth levels with equal probability. We propose a different notion of how wealth levels are allocated, based on a competition or contest. We find that inequality can be captured through the equilibrium properties of such a game. We connect the inequality measures so derived to existing measures of inequality, and demonstrate the conditions under which they satisfy the received key axioms of inequality measures (anonymity, homogeneity and the Pigou-Dalton transfer principle). Our approach also provides a natural way to discuss the tradeoff between greater total wealth and greater inequality.

Keywords: utilitarianism, inequality, contests
JEL codes: C72, C73, D63, D72

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1 Introduction

The "veil of ignorance" is a thought experiment initiated by Harsanyi (1953) and Rawls (1971). There is a society characterized by a set of possible roles an individual may adopt. Each "place in society" is identified by the wealth (or income) it generates for the individual who adopts it. An individual evaluates such a society by assuming she will somehow be put in one of these places by "fate". Under Harsanyi's (1953) formulation, fate assigns the individual to places uniformly at random, whereas for Rawls (1971) the individual believes fate will assign the individual to the worst possible place in society.

We propose an interpretation of fate in which individuals are allocated to places in society through a contest. The contest is modeled as a game. Under our proposal, the individual anticipates the equilibrium play of this game and forms her judgments about the society at the ex ante stage accordingly using expected utility. We are interested in understanding such an individual's views regarding inequality across places in society.

Our main goal is to demonstrate that the equilibrium properties of the contest can be used to generate a measure of inequality. The reason why inequality can be captured noncooperatively is that the contest allows us to model the following intuition.[1] As the places in society become increasingly unequal, each individual becomes increasingly motivated to occupy the more desirable roles rather than the less desirable roles. Thus the incentives of individuals are brought into stronger conflict. The equilibrium actions that individuals take therefore result in larger inefficiencies, as individuals optimally compete more aggressively for the best roles. These inefficiencies lower the ex ante expected payoff to the contest.

Thus, we can use the equilibrium payoff (and other related quantities) of the contest to capture the extent of inequality in a given society of roles. We refer to an inequality measure so derived as *efficiency-based*.

As far as we are aware, the proposal to derive an inequality measure from a noncooperative framework is novel. The modern theory of inequality is instead largely axiomatic, see e.g. Sen & Foster (1997). The efficiency-based approach to inequality is very different from the axiomatic approach. Yet, and this is our second goal, we show that the efficiency-based approach evaluates inequality in much the same way as the axiomatic approach.

Our third goal is to address an important question that is problematic under both the axiomatic approach and the standard veil of ignorance argument. The question is how to take a stance on the tradeoff between higher incomes and greater inequality.[2] The noncooperative formulation on which we rely naturally addresses this question through the equilibrium payoff. A higher individual income in society is good, all else equal, but it also increases competition for it. The net impact is easily characterized in our framework. In this sense, one can view the equilibrium payoff as playing the role of a social welfare function, similar to the approach of Atkinson (1970) who advocates such an approach to take a stance on efficiency-inequality tradeoffs.

Having outlined our main line of argument, let us describe the elements of our approach in more detail. Let us identify the places in society with monetary "prizes". We study symmetric

[1] We are here motivated in part by Frank & Cook (1995), who argue convincingly that many professions/industries, such as the music industry, pay dramatically unequal prizes, leading to very inefficient choices by the people in this industry.

[2] In the utilitarian view, any increase in any prize represents an improvement. In the axiomatic approach, once one imposes homogeneity, there is nothing that can be said about increasing overall payoffs.

contests for which payoffs are affine functions of the monetary prizes and focus on their symmetric equilibria.[3]

The reason we focus on symmetric equilibria of symmetric contests is that the veil of ignorance view is indeed one of symmetry. The evaluation by a player in our contest is deliberately taken at the ex ante stage, before the individual learns anything about her type that might distinguish her from other individuals. She should, therefore, view the game in exactly the same way as any other individual at the ex ante stage. We therefore require the game to be played in a symmetric way. The axiomatic approach nearly universally imposes an *anonymity* axiom, which is to say that the inequality measure should depend only on the list of prizes, and not on any characteristics of the individuals. Due to the symmetry of our environment, our efficiency-based measures of inequality will necessarily satisfy the anonymity axiom.

The main reason we focus on contests in which payoffs are affine functions of the monetary prizes is to clarify the relationship between our approach and the axiomatic approach. Observe that, in this case, if payoffs are all multiplied by a given factor, then neither the symmetries nor the incentives in the game change (see e.g. Alos-Ferrer & Kuzmics (2013, Proposition 4)), and so the equilibria are also unchanged. Accordingly, it is natural to construct an efficiency-based inequality measure that is homogeneous of degree zero.[4] The view in the axiomatic inequality literature is that, as the goal is to study inequality, one deliberately abstracts from issues that arise when changing the total value of all prizes, and this is accomplished by imposing homogeneity.

There is an important auxiliary reason to focus on affine payoffs. The Harsanyi (1953) version of the veil of ignorance argument asks an individual to evaluate, according to her own utility function, a society, assuming that she wins each prize with equal probability. If the individual judges less equal societies to be worse, then it must be that her utility function for money exhibits risk-aversion. The affine payoffs allow us to demonstrate that the efficiency-based measures deliver similar judgments about inequality even when all individuals are risk-neutral. In our setup, then, it is the nature of how the contest captures inefficiencies of competition, rather than individual preferences, that drives a judgment in favor of a more equal society. This illustrates how the noncooperative view of inequality we advance here, even though it operates very differently from an axiomatic approach, permits similar judgments.

We can now summarize what we accomplish in this paper. We proceed by studying two different models of a contest. The first model is based on an "allocation game" taken from Kuzmics, Palfrey & Rogers (2014). In an allocation game, each player demands one of the prizes. If all demands are unique, then players receive their demanded prizes. Otherwise, if there is any mis-coordination, all players receive zero. Allocation games represent, in a sense, the simplest model of a contest that admits the features we require. Notice that, in its unique symmetric equilibrium with positive payoffs, if prizes are all equal then the equilibrium mixture is uniform. As prizes become less equal, the equilibrium strategy places higher probability on better prizes, and this decreases the probability of successful coordination, lowering the expected payoff.

In Section 3, we show first that the equilibrium probability of coordination is, for a fixed number of prizes, a monotone transformation of the mean-log deviation, a well-known entropy-based measure of inequality due to Theil (1967). It thus ranks societies of fixed size in exactly

[3]We suggest that future literature deviate from this abstraction. We shall comment on some potentially interesting departures below.

[4]That is to say, it is unchanged by a common scalar multiple of prizes.

the same way as mean-log deviation. Through this derivation we thereby provide a completely novel justification of this measure, as arising out of strategic behavior of individuals engaged in a particular form of contest behind the veil of ignorance. We next ask about the tradeoff between higher payoffs and less equal payoffs by examining the equilibrium payoff of the symmetric equilibrium. In particular, we characterize the condition under which increasing the value of the highest prize leads to an increase in expected payoff, which captures a net positive effect of such a change to society's roles. Our result shows that such a change is desirable provided that the best prize is not too great compared to the average prize.

We then generalize the allocation game contest to a repeated version in Section 4. We use this extension to make the point that it is easy to extend the inequality measure from Section 3 to a class of inequality measures that is parameterized by, in this case, the discount factor, δ, in the repeated game. In Proposition 1 we show that every measure in this class satisfies the key defining axiom of an inequality measure: the Pigou-Dalton transfer principle, which requires that a transfer from a lower prize to a higher prize increases inequality (see Pigou (1920), Dalton (1920)).

Beyond this simple illustration, the particular class of measures that corresponds to the repeated allocation game has interesting properties. As δ approaches zero, we of course limit to the MLD measure discussed above. On the other hand, as δ approaches one and individuals become patient, the measure converges to Rawls's (1971) maxmin measure of inequality. In other words, we present a derivation of Rawls's (1971) maxmin measure under Harsanyi's (1953) interpretation of the veil of ignorance that is based on a noncooperative contest with risk-neutral expected utility preferences.[5] Finally, in Proposition 2, we characterize when it is that increasing the highest prize results in a social improvement. It turns out that the answer depends nontrivially on δ, such that for higher values of δ one may reach the opposite conclusion than was reached for the one shot allocation game.

The second contest model we study is more in spirit with the recent literature on contests (see e.g. Konrad (2009) for a survey), in that individuals compete for prizes by exerting unproductive, but costly, effort.[6] This model is general enough that the Pigou-Dalton axiom is not always satisfied. Proposition 3 provides a necessary and sufficient condition for the associated inequality measure to satisfy the Pigou-Dalton axiom. Essentially the condition is that an increase in one individual's effort must lead to a probability distribution over prizes that *dominates* the original distribution in terms of the *likelihood ratio*. Finally, we demonstrate, by studying a version of the Tullock (1980) contest, that increasing the value of the highest prize can have either a positive or negative effect on social welfare.

2 Measures of Inequality

Let $X = \mathbb{R}^n_{++} = \{x \in \mathbb{R}^n | x_i > 0 \forall i\}$ be the set of all vectors in $\mathbb{R}^n$ with strictly positive entries in all coordinates.[7] Let an element x of X be an *allocation* and, thus, X the set of all allocations.

[5] In particular, there is no need to resort to ambiguity aversion to generate maxmin preferences.

[6] While more general then the allocation game, this model is too artificial to be of practical importance. It seems worthwhile to extend the analysis from this section to more general contests, such as those characterized in Siegel (2009) and Siegel (2014).

[7] Throughout the paper we shall consider n fixed. If one wanted to discuss properties such as *replication invariance* one would have to extend this setup to cover the space $\mathcal{X} = \bigcup_{n=2}^{\infty} \mathbb{R}^n_{++}$.

A *measure of inequality* is a mapping $M : X \to \mathbb{R}$ attaching to each allocation a single number, the inequality of the allocation.

The following three properties (or axioms) have been proposed by Fields & Fei (1978) as essential properties that a measure of inequality should satisfy (see also Sen & Foster (1997), and Foster (1983)).

Axiom 1 (Anonymity) *A measure of inequality M is* anonymous *if for any two allocations $x, x' \in X$ such that x' is a permutation of x we have $M(x) = M(x')$.*

In words, the measure of inequality does not depend on any characteristic of a person other than its "income" or "assigned value" in the allocation.

Axiom 2 (Homogeneity) *A measure of inequality M is* homogenous *(of degree zero) if $M(x) = M(\alpha x)$ for any allocation $x \in X$ and any real number $\alpha > 0$.*

The idea behind this axiom is not that more income to all would not be an improvement, but that we want to isolate welfare effects due to a change in inequality from those due to a change in overall payoffs.

The previous two axioms tell us what we want a measure of inequality not to depend on. The next, key, axiom considers a transfer of income from a poorer individual to a richer individual, in which case inequality must increase.

Axiom 3 (Pigou-Dalton) *A measure of inequality M satisfies the* Pigou-Dalton transfer principle *if for any two allocations $x, x' \in X$ such that there are distinct indices i and j and a real number $\Delta > 0$ with $x_i \leq x_j$ and $x'_i = x_i - \Delta$ and $x'_j = x_j + \Delta$ while $x_l = x'_l$ for any $l \neq i, j$ we have $M(x) < M(x')$.*

3 Allocation Games

3.1 The Game

The essential ingredient we require for a contest is that, as a prize becomes more valuable relative to other prizes, each individual will compete harder to win that prize, such that the fiercer competition is inefficient, lowering equilibrium payoffs. With this intuition in mind, we propose the following formulation, which follows Kuzmics, Palfrey & Rogers (2014) in calling an *allocation game* the following game characterized by a vector $x \in X$. There are n players. Each player has the same set of pure strategies (or actions), denoted by $A = \{1, ..., n\}$. The set of action-profiles is denoted by A^n. Payoffs to all players are zero unless an action-profile that is a permutation of $(1, 2, 3, ..., n)$ is played, in which case the player who plays action i receives payoff x_i.

3.2 Symmetric Equilibrium

This game has a unique symmetric equilibrium that generates positive expected payoff to any player.[8] This equilibrium is necessarily in mixed strategies with full support over the set of all actions, placing probability $\frac{x_i}{\sum_{j=1}^{n} x_j}$ on action i.

[8]There are many symmetric equilibria with zero expected payoff to all players. There are also asymmetric equilibria with positive payoffs. Asymmetric equilibria are of no interest to us, as we want to investigate the difficulty of achieving an asymmetric outcome as a function of the inequality in the promised asymmetric allocation.

3.3 Equilibrium Payoff and Induced Measures of Inequality

The expected payoff in this equilibrium is given by

$$V(x) = (n-1)! \frac{\prod_j x_j}{(\sum_j x_j)^{n-1}}.$$

Let us denote the event in which the realized action profile played is a permutation of $(1, 2, ..., n)$ the event of *coordination*. Then the probability of coordination in this symmetric equilibrium is given by

$$P(x) = n! \frac{\prod_j x_j}{(\sum_j x_j)^n}.$$

Note that the probability of coordination is also equal to the normalized expected payoff, i.e. expected payoff $V(x)$ divided by the average prize $\bar{x} = \frac{1}{n} \sum_j x_j$.

3.4 Pigou-Dalton

Note that a higher probability of coordination is of course a good thing, so if we want the probability of coordination to measure inequality we need to reverse it. Note that any monotonic transformation of P will still "rank" different allocations in exactly the same way as the original function P. So let us take the negative natural logarithm. This gives us

$$-\log P(x) = -\log(n!) + n \log(n) - \sum_i \log(\frac{x_i}{\bar{x}}),$$

which, for fixed n, is a monotone transformation of

$$\mathrm{MLD}(x) = -\frac{1}{n} \sum_i \log(\frac{x_i}{\bar{x}}),$$

which is the so-called *mean log deviation*, a well-known entropy measure of inequality due to Theil (1967). The mean log deviation is known to satisfy the Pigou-Dalton transfer principle. It also satisfies, by design, the axioms of anonymity and homogeneity. For a full axiomatic characterization see Shorrocks (1980, 1984). Notice, thus, that our derivation demonstrates that an inequality measure founded on the equilibrium of a contest, and evaluated ex ante behind a veil of ignorance, satisfies all of the axioms that characterize the MLD measure.

3.5 Efficiency–Inequality Tradeoff

Suppose we increase one prize x_i by a small amount, such that inequality is increased. Under what conditions does this actually decrease expected payoff $V(x)$? To answer this we simply need to differentiate $V(x)$ with respect to one coordinate, say x_1. One can show that the sign of this derivative is equal to the sign of the following expression:

$$\sum_j x_j - (n-1)x_1.$$

For the case of $n = 2$ the partial derivative is thus always positive. This means that with only two prizes there is never a tension between inequality and total payoff. A benevolent planner could thus be viewed as having lexicographic preferences: first the planner seeks to maximize the sum of the prizes, then secondarily she seeks to distribute this total as evenly as possible.

When $n \geq 3$ this is no longer universally true. For the case of $n = 3$, for instance, if x_1 already exceeds the sum of x_2 and x_3, increasing x_1 further, even without a corresponding decrease of x_2 and x_3, is detrimental to expected payoff. As a concrete example consider the two allocations $(3,1,1)$ and $(4,1,1)$ with $V(3,1,1) = \frac{6}{25} > \frac{2}{9} = V(4,1,1)$.

The intuition is that if x_1 is already large relative to the other prizes, an increase of x_1 decreases the probability of coordination $P(x)$. In this situation, the lower probability of coordination more than offsets the increase in ex post payoff from the higher x_1. In other words, one may reject an increase in all prizes, if this is done unequally, in favor of the more equal original prize distribution.

4 Repeated allocation games

4.1 The game

Consider a particular version of the repeated allocation game. The game is repeated at discrete points in time until coordination is achieved. There is never any feedback to the agents except for whether or not coordination was achieved. Players discount their payoffs with a common discount factor $\delta < 1$. As for $\delta = 0$ the analysis is the same as in the one-shot allocation games from Section 3, the analysis in this section is a generalization of that of Section 3.

In principle players can condition their play on their private history, but we assume here that they do not. That is we investigate stationary symmetric strategies for this repeated game.

4.2 Symmetric Equilibrium

The stationary symmetric equilibrium of the repeated allocation game is characterized by a (mixed) strategy $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, which each player uses, at each time $t \geq 1$. As in the simple allocation game, we are interested in the (symmetric) equilibrium with positive payoffs, so that all $\alpha_i > 0$. Let us use $v > 0$ to denote the expected continuation payoff of a player in the repeated allocation game, at some point t, given that coordination was not achieved yet. Because $\alpha_i > 0$ for all i and from stationarity, we must have, for each $i = 1, .., n$,

$$v = x_i (n-1)! \Pi_{j \neq i} \alpha_j + \delta v (1 - (n-1)! \Pi_{j \neq i} \alpha_j)$$

So the expected payoff of each action i, which is the average of x_i and the future (discounted) payoff δv, must equal v. These equations can be re-written as:

$$(1-\delta) v = (n-1)! \frac{\Pi_j \alpha_j}{\alpha_i} (x_i - \delta v) \tag{4.1}$$

for each $i = 1, .., n$ and they imply $\delta v < \min_i x_i$. We can now analyse these equations to obtain the following lemma, which is proved in the Appendix.

Lemma 1 *The repeated allocation game with allocation $x \in X$ has a unique stationary symmetric equilibrium that generates positive expected payoff. This expected payoff, denoted by $V^\delta(x)$, is the value v that solves the following (implicit) equation.*[9]

$$(1-\delta)v(\sum_j x_j - n\delta v)^{n-1} = (n-1)!\Pi_j(x_j - \delta v) \tag{4.2}$$

4.3 Equilibrium Payoff and Induced Measures of Inequality

As pointed out in the Introduction, a common affine transformation of the allocation x does not change players' incentives in the allocation game. It also does not change the players' incentives in the repeated game. Thus, $V^\delta(x)$, is homogenous of degree one, i.e., $V^\delta(\lambda x) = \lambda V^\delta(x)$ for all $\lambda > 0$.

To generate a measure of inequality from the expected payoff that is net of total payoff and homogeneous of degree zero, we normalize the expected payoff by dividing by the average payoff in the allocation, $\bar{x}$. Denote this measure by $V_*^\delta(x) = \frac{1}{\bar{x}}V^\delta(x)$.[10]

4.4 Pigou-Dalton

Proposition 1 *The measure of efficiency defined by V_*^δ satisfies the Pigou-Dalton axiom.*

Proof.

See Appendix.

■

Naturally, when $\delta = 0$ we obtain our measure derived for the static game in Section 3, i.e. $V_*^0(x) = P(x) = n!\frac{\Pi_j x_j}{(\sum_j x_j)^n}$. On the other hand, as $\delta \to 1$, V_*^δ converges to $\frac{x_{min}}{\frac{1}{n}\sum_j x_j}$. This is Rawls's (1971) maxmin measure of inequality. From these extreme cases, it is immediate that $V_*^\delta(x)$ depends non-trivially on δ and that, in fact, the choice of $\delta \in [0,1]$ can be taken to parameterize a class of measures. We find it interesting that the inequality measure derived from the repeated allocation game delivers the maxmin measure, even though all players are risk-neutral expected utility maximizers and, in particular, there is no role of ambiguity aversion in the model.

4.5 Efficiency–Inequality Tradeoff

We turn now to investigating the conditions under which an increase in the highest prize delivers a social improvement. The next result provides an implicit, but exact, characterization.

[9] For the special case of $n = 2$ this equation can be solved analytically and yields $V^\delta(x) = \frac{1}{2\delta(2-\delta)}\left((x_1 + x_2) - \sqrt{(x_1+x_2)^2 - 4\delta(2-\delta)x_1x_2}\right)$.

[10] There are some other measures of interest that one could derive from the equilibrium in this repeated game. One is the (constant) per-period probability of coordination, denoted by $Q^\delta(x)$, which is related to the expected payoff as follows.

$$V_*^\delta(x) = \frac{1}{\bar{x}}V^\delta(x) = \frac{Q^\delta(x)}{1-\delta(1-Q^\delta(x))}$$

Note that $V_*^\delta(x)$ is a monotone transformation of $Q^\delta(x)$, so both measures rank societies in the same way. Of course, to make these two measures of inequality one needs to reverse their sign. Another measure of potential interest is the expected time until coordination is achieved in the equilibrium, given by $T^\delta(x) = \frac{1}{Q^\delta(x)}$.

Proposition 2 *The non-normalized expected payoff in the repeated allocation game, $V^\delta(x)$, decreases in the value of a prize x_i if and only if*

$$\frac{n-1}{n}\left(x_i - \delta V^\delta(x)\right) > \bar{x} - \delta V^\delta(x).$$

Proof. See Appendix. ∎

Note that this proposition implies that for $n = 2$, as in the one-shot allocation game, inequality must be a concern that comes lexicographically after a concern for the sum of all prizes.[11] In other words, non-normalized expected payoff can never decrease if the value of a single prize is increased (no matter how unequal the distribution is).

Taking n to infinity we also get the same result as for the one-shot allocation game. If x_i is greater than the average prize and is increased non-normalized expected payoff decreases.

For intermediate n the condition given here is somewhat different than the condition we obtained in the one-shot allocation game. For the one-shot allocation game increasing the value of a prize x_i is detrimental to expected payoff if and only if $\frac{n-1}{n}x_i > \bar{x}$. In the repeated game with discount factor δ the condition is $\frac{n-1}{n}x_i > \bar{x} - \frac{1}{n}\delta V^\delta(x)$. The first condition implies the latter, but not vice versa. Thus, an increase in x_i may be detrimental to expected payoff in the repeated game but not in the one-shot game.

Consider an example with allocations $x = (3,1,1)$ and $x' = (3,3,1)$. Which one of these allocations is preferred? In the one shot game, we have $P(x) = \frac{18}{125} < \frac{54}{343} = P(x')$. Thus, for sufficiently small δ, we also have $V_*^\delta(x) < V_*^\delta(x')$. However, considering $\delta \to 1$, we have that $V_*^1(x) = \frac{3}{5} > \frac{3}{7} = V_*^1(x')$, showing that the social preference is reversed for sufficiently high discount factors.

5 Contests

In this section we offer a more general model in which, as before, n players compete for n prizes, but rather than simply by stating which prize they would like, they instead choose an effort. Effort in this model is, as in Spence's (1973) signalling model, completely unproductive and costly.

5.1 The Game

Let us define an *allocation contest*, parameterized by an allocation $x \in X$, a game with $I = \{1, ..., n\}$ players, with $E = \mathbb{R}_+$ the set of possible non-negative effort levels for each player, and payoffs given by the element of the allocation the player receives minus the cost of producing effort.

For a fixed allocation $x \in X$ let $\mathcal{B}$ denote the set of all bijections (matchings) from the set of prizes $\{x_1, ..., x_n\}$ to the set of players I. A class of allocation contests is then characterized by a function, φ, from the set of all effort profiles, i.e. $\mathbb{R}_+^n$, times the set of allocations X, to the set of all probability distributions over $\mathcal{B}$. Let $\varphi_i^j(e_1, ..., e_n)$ denote the probability, induced by φ, that player i receives prize x_j for the given effort profile and for a fixed allocation $x \in X$.

[11] For $n = 2$ the condition above reduces to $x_{min} < \delta v$, which can never be true.

The cost of effort is assumed to be a function of the sum of all values in the allocation, $\sum_j x_j$, and the chosen level of effort in such a way that the cost is homogeneous of degree one in $\sum_j x_j$.[12] We shall write this cost function as $\sum_j x_j c(e_i)$.

If we add to all bijections also the null-element to indicate that no player gets anything, then the setup as it is so far is flexible enough to accommodate the allocation games of Section 3 as a special case of allocation contests. For this we have to set the effort cost function to zero (thus it is homogeneous of degree one) and make φ such that only for any permutation of a prespecified set of n distinct effort levels does it provide a match.

The utility function to any player, here w.l.o.g. player 1 is given by

$$u_1(e_1, e_{-1}) = \sum_j x_j \varphi_1^j(e_1, e_{-1}) - \sum_j x_j c(e_1).$$

5.1.1 Further assumptions

To obtain a symmetric game we need to assume that φ is *anonymous*, i.e. that the probability that a player gets a certain prize, given that player's effort choice, depends only on the profile of other players' effort choices and not on the identity of which of the other players chooses what. This implies in particular that if all players choose the same level of effort all have an equal chance of getting any given prize.

We also employ the following technical assumptions for convenience:

(i) both φ and c are twice continuously differentiable,

(ii) c is increasing and convex, with $c'(0) = 0$ and $c'(\infty) = \infty$,

(iii) $\sum_j x_j \frac{\partial}{\partial e_1} \varphi_1^j(e, ..., e)$ is positive, and it is a decreasing function in e, with $\sum_j x_j \frac{\partial}{\partial e_1} \varphi_1^j(0, ..., 0) = \infty$ and $\sum_j x_j \frac{\partial}{\partial e_1} \varphi_1^j(\infty, ..., \infty) = 0$,

(iv) $\sum_j x_j \frac{\partial^2}{\partial (e_1)^2} \varphi_1^j(e, ..., e)$ and $\sum_j x_j, \frac{\partial^2}{\partial e_1 \partial e_2} \varphi_1^j(e, ..., e)$ are non-positive.

The interpretation of (iii) is that, at any symmetric effort profile, if a player (here w.l.o.g. player 1) increases her effort she increases the expected value of the prize she will get, where this increase is greater for lower levels of effort from everyone else. Condition (iv) means that the marginal effect of effort of one player on this player's expected value of the prize is decreasing, and it is reduced by an increase of another player's effort.

5.2 Symmetric Equilibrium

Assuming an interior solution (see below) the first order condition for the optimal choice of effort for player 1 is given by

$$\sum_j x_j \frac{\partial}{\partial e_1} \varphi_1^j(e_1, e_{-1}) - \sum_j x_j c'(e_1) = 0.$$

[12] While this assumption may seem odd, we make it in order to best compare our approach of assessing inequality with the utilitarian welfare criterion to the approach of assessing inequality through a different, perhaps axiomatic, welfare criterion.

In a symmetric equilibrium, in which all players choose the same effort level e^* we, therefore, must have

$$\sum_j x_j \left(\frac{\partial}{\partial e_1} \varphi_1^j(e^*, ..., e^*) \right) = \sum_j x_j c'(e^*).$$

By assumption, the left hand side decreases in e^* from ∞ to 0 and the right hand side increases in e^* from 0 to ∞. So there must be a unique solution. Thus, if there exists a symmetric equilibrium of the allocation contest it must be unique.

Our assumptions also imply that the second-order condition (for a maximum) is satisfied at this candidate equilibrium point. We therefore know that at the candidate equilibrium all players use a local best response. This, however, does not mean that players play a global best response. For instance, they might want to deviate to zero effort. For the remainder of this section we shall simply assume that the unique solution above is in fact an equilibrium. Producing exact conditions on the parameters of the game that guarantee such existence would not contribute much to the purpose of this article. We will, however, be careful about this point in the example of Section 5.5.1 below.

5.3 Equilibrium Payoff and Induced Measures of Inequality

Note that the expected payoff (to any player) in the interior symmetric equilibrium is given by

$$U(x) = \frac{1}{n} \sum_j x_j - \sum_j x_j c(e^*).$$

Notice that $U(x)$ is homogeneous of degree 1. If we desire instead a measure that is homogeneous of degree zero, so as to satisfy the homogeneity axiom, we could look at the cost of effort. This, however, is simply a monotone transformation of equilibrium effort, denoted $E(x)$.

5.4 Pigou-Dalton

We can now provide necessary and sufficient conditions on φ under which $E(x)$ satisfies the Pigou-Dalton axiom.

Let p and q be two discrete probability distributions with support in $\{x_1, ..., x_n\}$. Distribution p *dominates* distribution q in terms of the *likelihood ratio* if $\frac{p(x_i)}{q(x_i)} < \frac{p(x_j)}{q(x_j)}$ if and only if $x_i < x_j$. This form of dominance requires that for any two prizes, the relative likelihood of obtaining the higher prize compared to the lower prize increases.[13]

Proposition 3 *The measure $E(x)$ (and $U(x)$) satisfies the Pigou-Dalton axiom if and only if, for all possible symmetric equilibrium effort levels e^* there is an $\bar{\epsilon} > 0$ such that for every $\epsilon \in (0, \bar{\epsilon})$ the distribution $\varphi_1(e^* + \epsilon, e^*, ..., e^*)$ dominates the uniform distribution in terms of the likelihood-ratio.*[14]

Proof. See Appendix. ■

[13]See, for instance, Krishna (2010, Appendix B) for this definition. There it is also stated that likelihood ratio dominance implies among other notions of dominance also first-order stochastic dominance.

[14]By the assumed smoothness of φ we then also have that for small $\epsilon < 0$ the distribution $\varphi_1(e^* - \epsilon, e^*, ..., e^*)$ is dominated by the uniform distribution in terms of the likelihood-ratio.

To understand the result, recall first that at a symmetric equilibrium all players obtain the same distribution over prizes, which implies that $\varphi_1(e^*, \ldots, e^*)$ is uniform. The condition expressed in this result requires that, at the interior equilibrium, when a player increases her effort slightly, she obtains a better distribution over prizes, where "better" is in the sense of likelihood ratio dominance. An inequality-increasing Pigou-Dalton transfer casues equilibrium effort to increase, exactly when such an increase in effort leads to unambigously better outcomes.

5.5 Efficiency-Inequality Tradeoff

We provide here two simple results that allow us to conclude that, in general the efficiency-inequality tradeoff is nontrivial for the allocation contest model. We then illustrate this finding by way of example in the next subsection.

Differentiating $U(x) = \frac{1}{n}\sum_j x_j - \sum_j x_j c(e^*)$ with respect to x_1 we have, immediately:

Lemma 2 *Non-normalized expected equilibrium payoff $U(x)$ increases in x_1 if and only if*

$$\frac{1}{n} - c(e^*) - \frac{\partial e^*}{\partial x_1} c'(e^*) \sum_j x_j > 0.$$

Next, by considering the characterization of the Pigou-Dalton axiom from the previous section, we have:

Lemma 3 *Suppose $E(x)$ satisfies the Pigou-Dalton axiom. If $x_j > x_i$ then $\frac{\partial e^*}{\partial x_j} > \frac{\partial e^*}{\partial x_i}$.*

Proof. Let $g(e) = \sum_l x_l a_l(e)$, where $a_l(e) = \frac{\partial}{\partial e_1}\varphi_1^l(e, ..., e) - c'(e)$.From the equilibrium condition we have $g(e^*) = \sum_l x_l a_l(e^*) = 0$. Differentiating with respect to x_j and x_i we obtain the following set of equations:

$$g'(e^*)\frac{\partial e^*}{\partial x_j} + a_j(e^*) = 0$$
$$g'(e^*)\frac{\partial e^*}{\partial x_i} + a_i(e^*) = 0$$

Because $x_j > x_i$, Proposition 3 says that $a_j(e^*) > a_i(e^*)$, and so $-g'(e^*)\frac{\partial e^*}{\partial x_j} > -g'(e^*)\frac{\partial e^*}{\partial x_i}$. Since our assumptions imply that $g'(e^*) < 0$, we have that $\frac{\partial e^*}{\partial x_j} > \frac{\partial e^*}{\partial x_i}$, as desired. ■

Taking these lemmas together and using the fact, from the proof of the latter, that for any l, $\frac{\partial e^*}{\partial x_l} = -\frac{a_l(e^*)}{g'(e^*)}$ it is possible to express the necessary and sufficient condition of Lemma 2, under which an increase in the highest prize leads to increase in the ex-ante expected payoff, in terms of parameters of the allocation contest. However, since the result is not easy to interpret, we point out how it can happen that an increase in the highest prize will lead to a reduction in the ex-ante expected equilibrium payoff.

Consider a situation in which the cost of effort approaches $\frac{1}{n}$, i.e. close to the point where an individual would abandon the candidate equilibrium and deviate to zero effort. Then the condition in Lemma 2 is dominated by the term $-\frac{\partial e^*}{\partial x_1}c'(e^*)\sum_j x_j$. Taking x_1 to be the largest prize, from Lemma 3 we know that $\frac{\partial e^*}{\partial x_1}$ is positive. By assumption $c'(e^*)$ is also positive. Thus, at such a point Lemma 2 implies that increasing the value of the highest prize reduces the ex-ante expected equilibrium payoff. The next subsection provides a concrete example to better demonstrate the various possibilities.

5.5.1 An example

Let $n = 3$ and, without loss, $x_1 \geq x_2 \geq x_3$. Suppose the contest is in the spirit of Tullock (1980) but generalized to multiple prizes, following Clark & Riis (1996), Clark & Riis (1998), and, more recently, Fu & Lu (2012). The probabilities φ of, w.l.o.g. player 1 getting prizes x_1, x_2 and x_3, respectively, are as follows.

$$\begin{aligned}
\varphi_1^1(e_1, e_2, e_3) &= \frac{(e_1)^\alpha}{(e_1)^\alpha + (e_2)^\alpha + (e_3)^\alpha}, \\
\varphi_1^2(e_1, e_2, e_3) &= \frac{(e_2)^\alpha}{(e_1)^\alpha + (e_2)^\alpha + (e_3)^\alpha} \frac{(e_1)^\alpha}{(e_1)^\alpha + (e_3)^\alpha} + \frac{(e_3)^\alpha}{(e_1)^\alpha + (e_2)^\alpha + (e_3)^\alpha} \frac{(e_1)^\alpha}{(e_1)^\alpha + (e_2)^\alpha} \text{ and} \\
\varphi_1^3(e_1, e_2, e_3) &= \frac{(e_2)^\alpha}{(e_1)^\alpha + (e_2)^\alpha + (e_3)^\alpha} \frac{(e_3)^\alpha}{(e_1)^\alpha + (e_3)^\alpha} + \frac{(e_3)^\alpha}{(e_1)^\alpha + (e_2)^\alpha + (e_3)^\alpha} \frac{(e_2)^\alpha}{(e_1)^\alpha + (e_2)^\alpha}
\end{aligned}$$

where $\alpha > 0$ is a parameter. The interpretation is that prizes are competed for in order of their desirability. Each player i wins x_1 with probability proportional to $(e_i)^\alpha$. Given the winner of x_1, between the two players who did not win x_1, each wins x_2 with probability proportional to $(e_i)^\alpha$. The remaining player wins x_3.

The cost function is $c(e_1) = (x_1 + x_2 + x_3)\frac{1}{2}(e_1)^2$. In order to ease the computations, we let $f_j = (e_j)^\alpha$, for $j = 1, 2, 3$.

If a symmetric equilibrium $\{f^*, f^*, f^*\}$ exists, it must satisfy:

$$x_1 \frac{\partial}{\partial f_1}\varphi_1^1(f^*, f^*, f^*) + x_2 \frac{\partial}{\partial f_1}\varphi_1^2(f^*, f^*, f^*) + x_3 \frac{\partial}{\partial f_1}\varphi_1^3(f^*, f^*, f^*) = (x_1 + x_2 + x_3)\frac{1}{\alpha}(f^*)^{\frac{2}{\alpha}-1}.$$

Evaluating the partial derivaties yields the equilibrium equation:

$$\frac{4}{18f^*}x_1 + \frac{1}{18f^*}x_2 - \frac{5}{18f^*}x_3 = (x_1 + x_2 + x_3)\frac{1}{\alpha}(f^*)^{\frac{2}{\alpha}-1}. \tag{5.1}$$

Solving for f^* we obtain: $(f^*)^{\frac{2}{\alpha}} = \frac{\alpha(4x_1 + x_2 - 5x_3)}{18(x_1 + x_2 + x_3)}$. The corresponding ex-ante expected equilibrium payoff is:

$$\begin{aligned}
U(x) &= \frac{1}{3}(x_1 + x_2 + x_3) - (x_1 + x_2 + x_3)\frac{1}{2}(f^*)^{\frac{2}{\alpha}} \\
&= \frac{1}{3}((1 - \frac{\alpha}{3})x_1 + (1 - \frac{\alpha}{12})x_2 + (1 + \frac{5\alpha}{12})x_3)
\end{aligned}$$

In the appendix we show that for $\alpha \in (0, \frac{24}{5})$ (with some restrictions on the prizes x_1, x_2, x_3 for $\alpha > 3$) this is indeed the (unique) equilibrium of the allocation contest. That is, we verify that all our assumptions are satisfied in this range.

Compare the cases $\alpha = 2$ and $\alpha = 4$. For $\alpha = 2$, increasing any of the three prizes is always beneficial. Increasing the lowest prize is more beneficial than increasing the middle prize, which in turn is more beneficial than increasing the highest prize. Nevertheless, for $\alpha = 2$ the first concern would be to make the three prizes as high as possible and only then would one be concerned with inequality. For $\alpha = 4$, however, while an increase in the middle and lowest prize would be beneficial, increasing the highest prize is detrimental. Thus, for $\alpha = 4$, the choice of the total value of the prizes and their distribution cannot be separated.

6 Conclusion

We propose a novel method with which to consider inequality. Our method operationalizes the assignment of individuals to places in society under the veil of ignorance thought experiment. This assignment is modeled explicitly through a game. The individual, at the ex ante stage, when evaluating a society, rests her judgments upon the equilibrium properties of this game, which we think of as a contest. The essential ingredient of the contest – that which allows it to serve as a medium for inequality judgments – is that, as a prize becomes increasingly valuable relative to the other possible prizes, individuals optimally compete harder to be assigned that prize. The increased competition results in inefficiencies, modeled here either as a possibility of mis-coordination or as an unproductive costly investment, which, in either case, decrease the equilibrium payoff. It is for this reason that one can use properties of the equilibrium to serve as measures of inequality.

From this perspective, we show, perhaps surprisingly, that natural models of contests result in inequality measures that are closely related to measures that have been studied axiomatically. One can view this either as an independent justification for such measures, or as a way to better understand the properties of the contest model we propose.

We hope that these ideas may lead to further research. Obviously, the inequality measure depends crucially on how the contest is modeled. Just as there are many different axioms and measures in the literature, there may be many other models of contests that prove interesting.

A Proof of lemma 1

Summing equations (4.1) for all $i = 1, 2, ..., n$, we obtain $(1-\delta)v = (n-1)!\Pi_j \alpha_j (\sum_j x_j - n\delta v)$. Using $\Pi_j \alpha_j$ from this equation in each of the equations (4.1), we have: $\alpha_i = \frac{x_i - \delta v}{(\sum x_j - n\delta v)}$. Now, given this expression for α_i's as a function of v, we can use it in equation (4.1) to obtain 4.2.

As we already noted, we have $\delta v < x_{min}$, where $x_{min} = \min_j x_j$. To show that there is a unique v that solves equation (4.2) let us define

$$g(v) = \ln(1-\delta) + \ln v + (n-1)\ln\left(\sum_j x_j - n\delta v\right) - \ln((n-1)!) - \sum_j \ln(x_j - \delta v),$$

which is obtained from (4.2) by taking logs. Thus we must show there exists a unique zero of $g(v)$ on $v \in [0, \frac{1}{\delta}x_{min}]$. It is easy to check that $g(0) = -\infty$ and $g(\frac{1}{\delta}x_{min}) = +\infty$, so there exists a solution because $g()$ is continuous. To show uniqueness we demonstrate monotonicity of g. We have

$$\frac{dg(v)}{dv} = \frac{1}{v} - \frac{(n-1)\delta}{\frac{1}{n}\sum_j (x_j - \delta v)} + \delta \sum_j \frac{1}{x_j - \delta v},$$

which is positive as $(\frac{1}{n}\sum_j x_j - \delta v)\sum_j \frac{1}{x_j - \delta v} \geq n$, which in turn follows from an application of the Cauchy-Schwarz inequality $\sum_j (a_j)^2 \sum_j (b_j)^2 \geq (\sum a_j b_j)^2$ where $a_j = \sqrt{x_j - \delta v}$ and $b_j = \frac{1}{\sqrt{x_j - \delta v}}$.

B Proof of Proposition 1

Take an allocation x with $x_i < x_k$, for some $i \neq k$. We will prove that $V^{\delta}(x)$ increases if some $\Delta > 0$ is transferred from x_k to x_i (thus, not changing $\sum_j x_j$). Consider function $g(\cdot)$ from the proof to Lemma 1. We observe that the transfer Δ decreases g due to the concavity of the last term in x. Because g increases in v, we then have that $V^{\delta}(x)$, which is the unique v that satisfies $g(v) = 0$, must increase.

C Proof of proposition 2

To see this take logs on both sides of equation 4.2. Fixing $v = V^{\delta}(x)$ take derivatives of the left-hand side and the right hand side with respect to x_i. For the left hand side this is $\frac{n-1}{\sum_j x_j - n\delta V}$. For the right hand side this is $\frac{1}{x_i - \delta v}$. Thus an increase in x_i (not changing v) increases the left-hand side more than the right hand side if and only if $\frac{n-1}{n}(x_i - \delta v) > \frac{1}{n}\sum_j x_j - \delta v$. In this case the value of the function $g(v)$ (defined in the proof of Lemma (1)) increases as x_i increases, while v is kept fixed. As g is an increasing function and as we need $g(v) = 0$ in equilibrium we must have that $V^{\delta}(x)$ decreases as a response to an increase in x_i under the stated condition.

D Proof of proposition 3

Consider allocations x and x' and $\Delta > 0$ such that $x'_i = x_i - \Delta$ and $x'_j = x_j + \Delta$ and $x'_l = x_l$ if $l \neq i, j$. We assume that Δ is small so the ranking of prizes is unchanged. At $e^* = E(x)$ we must have that the marginal utility of player 1 in her effort choice is $\sum_j x_j \left(\frac{\partial}{\partial e_1}\varphi_1^j(e^*, ..., e^*) - c'(e^*) \right) = 0$. We shall now investigate under what conditions player 1's marginal utility, given allocation x', at the same effort level e^* is positive. This would mean that $E(x') > E(x)$. Let, for any l, $a_l = \frac{\partial}{\partial e_1}\varphi_1^l(e^*, ..., e^*) - c'(e^*)$. Player 1's marginal utility, given allocation x' and given effort level e^* is then given by $\sum_l x'_l a_l$ which can be written as $\sum_l x_l a_l - a_i\Delta + a_j\Delta$. As $\sum_l x_l a_l = 0$ the marginal utility is simply $-a_i\Delta + a_j\Delta$ and, thus, positive if and only if $a_j > a_i$, i.e.,

$$\frac{\partial}{\partial e_1}\varphi_1^j(e^*, ..., e^*) > \frac{\partial}{\partial e_1}\varphi_1^i(e^*, ..., e^*).$$

As φ is assumed to be smooth, the last inequality is equivalent to:

$$\frac{\varphi_1^j(e^* + \epsilon, e^*, ..., e^*) - \varphi_1^j(e^*, e^*, ..., e^*)}{\epsilon} > \frac{\varphi_1^i(e^* + \epsilon, e^*, ..., e^*) - \varphi_1^i(e^*, e^*, ..., e^*)}{\epsilon}$$

for ϵ small. Because $\varphi_1^k(e^*, e^*, ..., e^*) = \frac{1}{n}$ for all k , for $\epsilon > 0$ and small, this is the same as:

$$\frac{\varphi_1^j(e^* + \epsilon, e^*, ..., e^*)}{1/n} > \frac{\varphi_1^i(e^* + \epsilon, e^*, ..., e^*)}{1/n}$$

. Thus, if $x_j \geq x_i$, Pigou-Dalton requires $E(x') > E(x)$ which is equivalent to $\varphi_1(e^* + \epsilon, e^*, ..., e^*)$ dominating the uniform distribution in terms of the likelihood ratio, and conversely for $x_j < x_i$.

E Proving existence of equilibrium for the example of Section 5.5.1

In this section, we show that $\{f^*, f^*, f^*\}$ is a (symmetric) equilibrium of the example from Section 5.5.1 if $\alpha \in (0, \frac{24}{5})$. We will verify that, for example, for player 1 $f_1 = f^*$ maximizes $U_1(f_1, f^*, f^*)$ subject to $f_1 \in [0, +\infty)$ (so $f_1 = f^*$ is a best response to $f_2 = f_3 = f^*$). We will do this in (three) steps. First, we verify that $f_1 = f^*$ is a local maximum of $U_1(f_1, f^*, f^*)$, that is, we have the local second order condition:

$$\frac{\partial^2}{\partial (f_1)^2} U_1(f^*, f^*, f^*) < 0$$

Computing the second order derivatives of the probabilities $\varphi_1()$ with respect to f_1 we obtain:

$$\begin{aligned}
\frac{\partial^2}{\partial (f_1)^2} \varphi_1^1(f^*, f^*, f^*) &= -\frac{4}{27(f^*)^2} \\
\frac{\partial^2}{\partial (f_1)^2} \varphi_1^2(f^*, f^*, f^*) &= -\frac{11}{54(f^*)^2} \\
\frac{\partial^2}{\partial (f_1)^2} \varphi_1^3(f^*, f^*, f^*) &= \frac{19}{54(f^*)^2}
\end{aligned}$$

Given the above second order derivatives, and using the expression of f^* from equation 5.1, we can write:

$$\begin{aligned}
\frac{\partial^2}{\partial (f_1)^2} U_1(f^*, f^*, f^*) &= -\frac{4}{27(f^*)^2} x_1 - \frac{11}{54(f^*)^2} x_2 + \frac{19}{54(f^*)^2} x_3 - (x_1 + x_2 + x_3) \frac{1}{\alpha} (\frac{2}{\alpha} - 1)(f^*)^{\frac{2}{\alpha} - 2} \\
&= \frac{1}{54(f^*)^2} ((4 - \frac{24}{\alpha})(x_1 - x_3) - (8 + \frac{6}{\alpha})(x_2 - x_3))
\end{aligned}$$

A sufficient condition for $\frac{\partial^2}{\partial (f_1)^2} U_1(f^*, f^*, f^*) < 0$ is $\alpha < 6$. (As we will show later (in step 2), actually $\alpha < 6$ is necessary for f^* to be an equilibrium.) So, here we record that if $\alpha < 6$ then indeed $f_1 = f^*$ is a local maximum.

Now, in the second step we verify that $f_1 = f^*$ yields a larger utility than $f_1 = 0$. We already computed the utility of player 1 at $f_1 = f^*$, ie. $U_1^* = \frac{1}{3}((1 - \frac{\alpha}{3})x_1 + (1 - \frac{\alpha}{12})x_2 + (1 + \frac{5\alpha}{12})x_3)$, and since $U_1(f_1 = 0, f^*, f^*) = x_3$, the condition is:

$$\frac{1}{3}((1 - \frac{\alpha}{3})x_1 + (1 - \frac{\alpha}{12})x_2 + (1 + \frac{5\alpha}{12})x_3) \geq x_3$$

which, on re-arranging, becomes:

$$(1 - \frac{\alpha}{3})(x_1 - x_3) + (1 - \frac{\alpha}{12})(x_2 - x_3) \geq 0$$

To check whether this inequality can possibly be satisfied, we consider the following three cases:

(a) If $\alpha \leq 3$ then the above inequality is satisfied for any values x_1, x_2, x_3 of the three prizes.

(b) If $3 < \alpha \leq \frac{24}{5}$, then the above inequality is satisfied only if the value of the second highest prize (ie. x_2) is large enough. More precisely, we need $x_2 \geq x_3 + \frac{\alpha - 3}{3 - \frac{\alpha}{4}}(x_1 - x_3)$ in this case.

(c) Finally, if $\alpha > \frac{24}{5}$, then for any values x_1, x_2, x_3 of the three prizes, the inequality cannot be satisfied.

Therefore, we necessarily need to have $\alpha \leq \frac{24}{5}$. We observe that if $\alpha < 3$ then U_1^* is increasing in x_1, while if $3 < \alpha < \frac{24}{5}$, U_1^* is decreasing in x_1. We also note that given that $\alpha \leq \frac{24}{5}$, the local second order condition is then satisfied too (because $\frac{24}{5} < 6$).

Finally, in step 3, we want to verify there are no other local maxima of the utility function $U_1(f_1, f^*, f^*)$ that would yield larger utility than $f_1 = f^*$. Let us compute $\frac{\partial}{\partial f_1} U_1(f_1, f^*, f^*)$, and for this, let us derive first the derivatives of the probabilities $\varphi_1()$. These are:

$$
\begin{aligned}
\frac{\partial}{\partial f_1}\varphi_1^1(f_1, f^*, f^*) &= \frac{2f^*}{(f_1 + 2f^*)^2} \\
\frac{\partial}{\partial f_1}\varphi_1^2(f_1, f^*, f^*) &= \frac{2f^*}{(f_1 + 2f^*)^2(f_1 + f^*)^2}(2(f^*)^2 - (f_1)^2) \\
\frac{\partial}{\partial f_1}\varphi_1^3(f_1, f^*, f^*) &= -\frac{\partial}{\partial f_1}\varphi_1^1(f_1, f^*, f^*) - \frac{\partial}{\partial f_1}\varphi_1^2(f_1, f^*, f^*)
\end{aligned}
$$

Therefore, we have:

$$
\begin{aligned}
\frac{\partial}{\partial f_1}U_1(f_1, f^*, f^*) &= \frac{2f^*}{(f_1 + 2f^*)^2}(x_1 - x_3) + \frac{2f^*}{(f_1 + 2f^*)^2(f_1 + f^*)^2}(2(f^*)^2 - (f_1)^2)(x_2 - x_3) \\
&- (x_1 + x_2 + x_3)\frac{1}{\alpha}(f_1)^{\frac{2}{\alpha}-1}
\end{aligned}
$$

We claim that if $\alpha > 0$ (and, of course, $\alpha < \frac{24}{5}$), then there are no other local maxima to $U_1(f_1, f^*, f^*)$ besides $f_1 = f^*$. This is an implication of the following two lemmas, which establish the monotonicity of $U_1(f_1, f^*, f^*)$ when $\alpha > 2$ and $\alpha \leq 2$ respectively.

Lemma 4 *If $\alpha > 2$, then there exists an $f_1^* < f^*$ such that:*

$$
U_1(f_1, f^*, f^*) = \begin{cases} \textit{decreases over} & f_1 \in (0, f_1^*) \\ \textit{increases over} & f_1 \in (f_1^*, f^*) \\ \textit{decreases over} & f_1 \in (f^*, +\infty) \end{cases}
$$

So, $U_1(f_1, f^*, f^*)$ decreases over $(0, f_1^*)$ reaching a local minimum at $f_1 = f_1^*$, increases over (f_1^*, f^*) reaching a local maximum at $f_1 = f^*$, and finally decreases over $(f^*, +\infty)$.

Proof. Let us denote by:

$$
\begin{aligned}
V(f_1, f^*, f^*) &= (f_1 + 2f^*)^2 \frac{\partial}{\partial f_1}U_1(f_1, f^*, f^*) \\
&= 2f^*(x_1 - x_3) + 2f^*\frac{2(f^*)^2 - (f_1)^2}{(f_1 + f^*)^2}(x_2 - x_3) - (x_1 + x_2 + x_3)\frac{1}{\alpha}(f_1)^{\frac{2}{\alpha}-1}(f_1 + 2f^*)^2
\end{aligned}
$$

In order to determine the sign of $\frac{\partial}{\partial f_1}U_1(f_1, f^*, f^*)$, it is enough to study the sign of the function $V()$ as previously defined. We note a number of four facts on the function $V()$ that will be useful later:

(1) $V(f_1 = 0, f^*, f^*) = -\infty$

(2) $V(f_1 = \infty, f^*, f^*) = -\infty$

(3) $V(f_1 = f^*, f^*, f^*) = (3f^*)^2 \frac{\partial}{\partial f_1} U_1(f_1 = f^*, f^*, f^*) = 0$

(4) $\frac{\partial}{\partial f_1} V(f_1 = f^*, f^*, f^*) = (3f^*)^2 \frac{\partial^2}{\partial (f_1)^2} U(f_1 = f^*, f^*, f^*) < 0.$

Computing the derivative of $V()$ with respect to f_1, we obtain:

$$\begin{aligned}\frac{\partial}{\partial f_1} V(f_1, f^*, f^*) &= 2f^* \frac{-2f^*(f_1 + 2f^*)}{(f_1 + f^*)^3}(x_2 - x_3) \\ &- (x_1 + x_2 + x_3)\frac{1}{\alpha}(f_1)^{\frac{2}{\alpha}-2}(f_1 + 2f^*)((\frac{2}{\alpha}+1)f_1 + 2(\frac{2}{\alpha}-1)f^*)\end{aligned}$$

As we are interested only in the sign of $\frac{\partial}{\partial f_1} V(f_1, f^*, f^*)$, we can work instead with

$$\begin{aligned}W(f_1, f^*, f^*) &= (f_1 + f^*)^3 \frac{\partial}{\partial f_1} V(f_1, f^*, f^*) \\ &= -4(f^*)^2 (f_1 + 2f^*)(x_2 - x_3) \\ &- (x_1 + x_2 + x_3)\frac{1}{\alpha}(f_1)^{\frac{2}{\alpha}-2}(f_1 + f^*)^3 (f_1 + 2f^*)((\frac{2}{\alpha}+1)f_1 + 2(\frac{2}{\alpha}-1)f^*)\end{aligned}$$

We note that the first term of $W()$ is always negative, while the second term is negative if $f_1 > \frac{2(1-\frac{2}{\alpha})f^*}{\frac{2}{\alpha}+1}$. Therefore, if $f_1 \geq \frac{2(1-\frac{2}{\alpha})f^*}{\frac{2}{\alpha}+1}$, we have $W(f_1, f^*, f^*) < 0$. In what follows, we will show that $W(f_1, f^*, f^*)$ is decreasing over interval $(0, \frac{2(1-\frac{2}{\alpha})f^*}{\frac{2}{\alpha}+1})$, and since $W(f_1 = 0, f^*, f^*) = +\infty$, there must exist a f_w such that:

$$W(f_1, f^*, f^*) = \begin{cases} > 0 \text{ if } f_1 < f_w \\ < 0 \text{ if } f_1 > f_w \end{cases}$$

To show that $W(f_1, f^*, f^*)$ decreases for $f_1 < \frac{2(1-\frac{2}{\alpha})f^*}{\frac{2}{\alpha}+1}$, we take the derivative of $W()$ to obtain, after some re-arranging:

$$\begin{aligned}&\frac{\partial}{\partial f_1} W(f_1, f^*, f^*) = -4(f^*)^2 (x_2 - x_3) \\ - &\ (x_1 + x_2 + x_3)\frac{1}{\alpha}\{((\frac{2}{\alpha}+1)f_1 + (\frac{2}{\alpha}-2)f^*)(f_1)^{\frac{2}{\alpha}-3}(f_1 + f^*)^2 (f_1 + 2f^*)((\frac{2}{\alpha}+1)f_1 + 2(\frac{2}{\alpha}-1)f^*) \\ + &\ (f_1)^{\frac{2}{\alpha}-2}(f_1 + f^*)^3 (2(\frac{2}{\alpha}+1)f_1 + \frac{8}{\alpha}f^*)\}\end{aligned}$$

Because $f_1 < \frac{2(1-\frac{2}{\alpha})f^*}{\frac{2}{\alpha}+1}$ the two additive terms in curly brackets are both positive, therefore we have $\frac{\partial}{\partial f_1} W(f_1, f^*, f^*) < 0$ for all $f_1 < \frac{2(1-\frac{2}{\alpha})f^*}{\frac{2}{\alpha}+1}$ as claimed.

Now going back to $\frac{\partial}{\partial f_1} V(f_1, f^*, f^*)$ we have that

$$\frac{\partial}{\partial f_1} V(f_1, f^*, f^*) = \begin{cases} > 0, & \text{if } f_1 < f_w \\ < 0, & \text{if } f_1 > f_w \end{cases}$$

Therefore, $V(f_1, f^*, f^*)$ is increasing for $f_1 < f_w$, reaches a maximum at $f_1 = f_w$, after which decreases for all $f_1 > f_w$. Combining this with the previous four observations on $V()$, we can pin down the sign of $V()$ as follows:

$$V(f_1, f^*, f^*) = \begin{cases} < 0, & \text{if } \ f_1 < f_1^* \\ > 0, & \text{if } \ f_1^* < f_1 < f^* \\ < 0, & \text{if } \ f^* < f_1 \end{cases}$$

where $0 < f_1^* < f_w < f^*$. Finally, since $V(f_1, f^*, f^*)$ and $\frac{\partial}{\partial f_1} U_1(f_1, f^*, f^*)$ have the same sign, we get the claim from the lemma.

Finally, the case of $\alpha \leq 2$ is dealt with in the following lemma.

Lemma 5 *If $\alpha \leq 2$, then:*

$$U_1(f_1, f^*, f^*) = \begin{cases} \textit{increases over} & f_1 \in (0, f^*) \\ \textit{decreases over} & f_1 \in (f^*, +\infty) \end{cases}$$

So, $U_1(f_1, f^*, f^*)$ increases over $(0, f^*)$ reaching a local maximum at $f_1 = f^*$, and decreases over $(f^*, +\infty)$.

Proof. Using the notation of the previous lemma, we note that if $\alpha \leq 2$ then $\frac{\partial}{\partial f_1} V(f_1, f^*, f^*) < 0$ for all $f_1 \geq 0$. In other words, $V(f_1, f^*, f^*)$ is decreasing over $f_1 \in [0, +\infty)$. But, since $V(f_1 = f^*, f^*, f^*) = 0$, we then must have:

$$V(f_1, f^*, f^*) = \begin{cases} > 0 & \text{if } f_1 \in (0, f^*) \\ < 0 & \text{if } f_1 \in (f^*, +\infty) \end{cases}$$

As $V(f_1, f^*, f^*)$ and $\frac{\partial}{\partial f_1} U_1(f_1, f^*, f^*)$ have the same sign, the proof is finished.

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