Center for
Mathematical Economics
Working Papers

529

July 2015

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Florian Gauer

Center for Mathematical Economics (IMW)
Bielefeld University
Universitätsstraße 25
D-33615 Bielefeld · Germany

e-mail: imw@uni-bielefeld.de
http://www.imw.uni-bielefeld.de/wp/
ISSN: 0931-6558

Strategic Formation of Homogeneous Bargaining Networks

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First Version: December 2014

This Version: July 2015

Abstract

We analyze a model of strategic network formation prior to a Manea (2011) bargaining game: ex-ante homogeneous players form costly undirected links anticipating expected equilibrium payoffs from the subsequent network bargaining. Assuming patient players, we provide a complete characterization of non-singularly pairwise (Nash) stable networks: specific disjoint unions of separated pairs, odd circles and isolated players constitute this class. We also show that many generic structures are not even singularly pairwise stable. As an important implication, this reveals the diversity of possible bargaining outcomes to be substantially narrowed down provided pairwise stability. Further, for sufficiently high costs the pairwise stable and efficient networks coincide whereas this does not hold if costs are low or at an intermediate level. As a robustness check, we also study the case of time-discounting players.

Keywords: Bargaining, Network Formation, Non-Cooperative Games

JEL-Classification: C72, C78, D85

*Center for Mathematical Economics (IMW), Economic Behavior and Interaction Models (EBIM), Bielefeld University, P.O. Box 100131, D-33501 Bielefeld, Germany. Email: fgauer@uni-bielefeld.de, Phone: +49 521 106-4918.
The author would like to thank Michael Günther, Tim Hellmann, Christoph Kuzmics, Jakob Landwehr, Mihai Manea, and Fernando Vega-Redondo, as well as the participants of various seminars at Bielefeld University and University Paris 1 Panthéon-Sorbonne for valuable comments and suggestions. This research was carried out within the International Research Training Group "Economic Behavior and Interaction Models" (EBIM) financed by the German Research Foundation (DFG) under contract GRK 1134/2.

1 Introduction

People often engage in bi- and multilateral bargaining: firms bargain with workers' unions over contracts, firms with other firms over prices or collaborations, politicians over environmental or trade agreements, or even friends and family members over household duties or other arrangements. However, in most of the situations that come to mind not everyone will be able or willing to bargain with anyone else. This idea can be expressed by means of a network. One's bargaining power in negotiations then commonly depends on the number and types of alternative partners as they present outside options. Agents typically intend to maximize their expected profit from bargaining, which suggests that beforehand they might want to influence and optimize their network of potential bargaining partners. This motivates that the underlying network should not be regarded as being exogenously given but as the outcome of strategic interaction among agents. However, establishing a connection to someone else usually costs some time and effort, which should be taken into account as well. This gives rise to an interesting trade-off between the costs of forming links and potential benefits from it, which is the topic of this paper.

We set up and analyze a sequential model of strategic network formation prior to a Manea (2011) infinite horizon network bargaining game. We consider ex-ante homogeneous players who in the first stage strategically form undirected costly links. In this context, one might think of one-time initiation or communication costs that players have to bear. In the second stage, we take the resulting network as given and players sequentially bargain with a neighbor for the division of a mutually generated unit surplus. According to Manea (2011) all subgame perfect equilibria of the bargaining game are payoff equivalent. Players are supposed to anticipate these outcomes during the preceding network formation game and to choose their actions accordingly. We examine players' strategic behavior regarding network formation, characterize stable and efficient network structures, and determine induced bargaining outcomes.

After giving a description of the model including a summary of the underlying Manea (2011) bargaining game and his decisive results, we consider the concept of pairwise stability established by Jackson and Wolinsky (1996) and first assume players in the bargaining game to be infinitely patient. For all levels of linking costs we state and prove sufficient conditions for a network to be pairwise stable (Theorem 1). While costs are relatively high, the only structures we find to be pairwise stable are specific disjoint unions of separated pairs and isolated players. When costs decrease, odd circles of increasing size can additionally emerge. At a transition point also lines of length three can be contained in a pairwise stable network. This result will also establish global existence of pairwise stable networks. For every combination of the above subnetworks we state precisely for which cost range it is pairwise stable and for which it is not (Corollary 1). To provide a complete characterization of pairwise stable equitable networks, i.e. of structures inducing homogeneous payoffs among players, we establish that any such network which is non-empty must be a disjoint union of separated pairs and odd circles (Theorem 2). Then we focus on the remaining networks inducing heterogeneous payoffs within a component and show that any of these can at most be singularly pairwise stable, that is at most at a single cost level (Theorem 3). This finishes the complete characterization of non-singularly pairwise stable networks, which is a principal achievement of this paper.

As a second main result, we deduce that pairwise stability narrows down the diversity of induced bargaining outcomes substantially (Corollary 3). Specifically, in non-singularly pairwise stable networks any player's payoff is either $\frac{1}{2}$ or 0 while profits are $\frac{1}{2}$ minus once or twice the linking costs or 0. If one only assumes pairwise stability, payoffs of both $\frac{1}{2}$ plus and minus the costs of one link can additionally occur.

What remains are networks which might be singularly pairwise stable. This means that a slight change or marginal variance of the required exact cost level already leads to a destabilization of such a network structure. However, we reveal that networks containing a tree with more than three players or a certain kind of component-connecting player are never pairwise stable, that is not even at a single cost level (Proposition 2 and 3). The observation that non-singularly pairwise stable structures even prove to be pairwise Nash stable concludes the main part (Corollary 4).

Beyond that, we establish that for sufficiently high linking costs the networks being efficient in terms of a utilitarian welfare criterion coincide with those we identified to be pairwise stable. However, as long as costs are low, the efficient networks constitute a proper subset of the pairwise stable ones while there also exists an intermediate cost range which does not even yield such a subset relation (Theorem 4 and Corollary 5). As a robustness check, we also relax the assumption that players are infinitely patient and show that pairwise stability in this framework does not necessarily imply pairwise stability for the original case with infinitely patient players and vice versa (Example 1 and 2).

For a concrete economic application which is captured by our model and which might contribute to a better understanding of the framework one can have the following in mind: Consider a number of similar firms beginning operation at the same time. They can mutually generate an (additional) surplus within bilateral projects by exploiting synergy potentials. For instance, this possibility might be based on capacity constraints or cost-saving opportunities. However, the underlying cooperation network is not existent yet and will therefore be the outcome of strategic interaction between firms. In charge of that are project managers who receive bonus payments proportional to their employer's profit from the project. Here, one-time costs might arise to prepare each two firms for mutual projects (adjustment of IT, joint training for workers etc.). We assume that each project manager keeps her job until she finalizes a joint project successfully by finding an agreement with the corresponding counterpart and leaves or is promoted afterwards and then gets replaced by a successor. Thus, the network remains unchanged after it has initially been established by the first project managers.

To take the suitable framework and convenient results established by Manea (2011) as a starting point in this context is fairly obvious. To my best knowledge, it is the only work which purely focuses on the impact of explicit network structures on players' bargaining power and outcomes in a setting of decentralized bilateral bargaining without ex-ante imposing any restrictions to the class of networks considered. Therefore, there are no distorting effects present in this setting as they might arise from heterogeneity among players and it is more general than buyer-seller scenarios which impose bipartite network structures. Moreover, Manea's (2011) network bargaining game remains analytically tractable and has some important properties. For any level of time discount there may exist several subgame perfect equilibria but he shows that all of these are payoff equivalent. Furthermore, he develops an equally convenient and sophisticated algorithm determining the limit equilibrium payoffs for a given network if players are infinitely patient. We will extensively make use of this algorithm and contribute to a profound understanding of its features throughout the paper.

The analysis of bargaining problems has a long tradition in the economic literature and dates back to the work of Nash (1950). A Nash bargaining solution is based on factors like players' bargaining power and outside options, whereas their origin is not part of the theory. This also applies for Rubinstein (1982), who analyzes perfect equilibrium partitions in a two player framework of sequential bargaining in discrete time with an infinite horizon, as well as for Rubinstein and Wolinsky (1985), who set up a model of bargaining in stationary markets with two populations. The work of Manea (2011) – to which we add a preceding stage of strategic interaction – can be regarded as an extension or microfoundation of these seminal papers. Here, bargaining power is endogenized in a natural and well-defined manner as an outcome of the given network structure and the respective player's position herein. Further important contributions to the literature on decentralized bilateral bargaining in exogenously given networks have been made by Abreu and Manea (2012) and Corominas-Bosch (2004), where the latter considers the special case of buyer-seller networks.

Second, this paper contributes to the more recently emerging literature on strategic network formation, which has mainly been aroused by the seminal works of Jackson and Wolinsky (1996) and Bala and Goyal (2000). Among other lines of research, some effort has been dedicated to gaining rather general insights regarding the existence, uniqueness and structure of stable networks. Hellmann (2013) and Hellmann and Landwehr (2014) are examples for this. However, since crucial conditions are not met in our model, these results do not apply here.

So far there exist only few papers combining these two fields of research. Calvó-Armengol (2003) studies a bargaining framework à la Rubinstein (1982) embedded in a network context and considers stability and efficiency issues. However, the mechanism determining bargaining partners is different from Manea (2011) and the network bargaining game ends after the first agreement has been found. As a consequence, in Calvó-Armengol's (2003) model a player's network position does not affect her bargaining power as such but only the probability that she is selected as proposer or responder. This leads to a characterization of pairwise stable networks in which the players' neighborhood size is the only relevant feature of the network structure. Thus, it differs

substantially from our results though both papers have in common the assumption that links are costly. By contrast, Manea (2011, Online Appendix) abstracts from explicit linking costs when approaching the issue of network formation as an extension of his model. He shows that for zero linking costs a network is pairwise stable if and only if it is equitable. Though results differ and get more complex for positive linking costs, we will see that the paper at hand is in line with this finding such that both works complement one another. Our additional considerations regarding efficiency and time discount further contribute to this.
Furthermore, the paper of Condorelli and Galeotti (2012) is related to some extent. They analyze the strategic network formation process in a trading framework and also provide a comparison between stable and efficient networks. However, they consider an indivisible object being traded via the network according to a particular class of trading protocols, which again implies significant differences regarding the structure of stable and efficient networks compared to our results. Most other papers studying strategic network formation in a bargaining context again focus on buyer-seller networks, which is complementary to our general approach. Kranton and Minehart (2001) and Polanski and Vega-Redondo (2013) are prominent examples for this, whereby the latter also does not involve explicit linking costs.

The rest of the paper is organized as follows: In Section 2 we introduce the model including the decisive results of Manea (2011). The main results of the paper at hand are developed in Section 3, which focuses on the structure of stable networks in the case that players are infinitely patient. In Section 4 we state commonalities and differences regarding stability if players discount time to some degree. In Section 5 we examine efficient networks. Section 6 concludes and discusses different questions for future research. The detailed proofs not provided immediately as well as some interim results are postponed to the Appendix.

2 The Model

Let time be discrete, $t = 0, 1, 2, ...$ For the initial period $t = 0$ consider a set of players $N = \{1, 2, ..., n\}$ connected by an undirected graph $g \subseteq g^N = \{ij \mid i, j \in N, i \neq j\}$. These players are assumed to be homogeneous apart from their potentially differing network positions. Here $ij \in g$ means that players i and j are able to mutually generate a unit surplus. Let $N_i(g) = \{j \in N \mid ij \in g\}$ denote the set of player i's neighbors in g and let $\eta_i(g) = |N_i(g)|$ be its cardinality. Each link is assumed to cause costs of $c > 0$ for both players involved such that player i has to bear total costs of $\eta_i(g)c$ in $t = 0$. We assume players to discount time by a uniform discount factor $\delta \in (0, 1)$.

We take this as a starting point for an infinite horizon bargaining game à la Manea (2011). In each period $t = 0, 1, 2, ...$ nature randomly chooses one link $ij \in g$, which means that i and j are matched to bargain for the mutually generated unit surplus. One of the two players is randomly assigned the role of the proposer, while the other one is selected as responder. Then the proposer makes an offer how to distribute the unit surplus and the responder has the choice: If she rejects, then both receive a payoff of 0 and stay in the game, whereas if she accepts, then both leave with the shares agreed on. In the latter case both players get replaced one-to-one in the next period.[1] This implies that each of the initial players $1,2,...,n$ will bargain successfully one time at most. A player's strategy in this setting lays down the offer she makes as proposer and the answer she gives as responder after each possible history of the game. Based on this, a player's payoff is then specified as her discounted expected agreement gains. A strategy profile is said to be a subgame perfect equilibrium of the bargaining game if it induces Nash equilibria in subgames following every history (cf. Manea, 2011).

From Manea (2011, Theorem 1) we know that all subgame perfect equilibria are payoff equivalent and each player's equilibrium payoff exclusively depends on her network position and the discount factor δ. Moreover, the equilibrium payoff vector which we denote as $v^{*\delta}(g)$ is the unique solution to the equation system

$$v_i = \left(1 - \sum_{j \in N_i(g)} \frac{1}{2d^{\#}(g)}\right) \delta v_i + \sum_{j \in N_i(g)} \frac{1}{2d^{\#}(g)} \max\{1 - \delta v_j, \delta v_i\}, \quad i \in N, \tag{2.1}$$

where $d^{\#}(g)$ denotes the total number of links in the network g. If it is $\delta\left(v_i^{*\delta}(g) + v_j^{*\delta}(g)\right) < 1$ for $ij \in g$, then this means that player i and j will find an agreement when their mutual link is selected, whereas $\delta\left(v_i^{*\delta}(g) + v_j^{*\delta}(g)\right) > 1$ means that they will each prefer to wait for a potentially better deal with a weaker partner.[2] This gives rise to the definition of the so called equilibrium agreement network $g^{*\delta} := \left\{ij \in g : \ \delta\left(v_i^{*\delta}(g) + v_j^{*\delta}(g)\right) \leq 1\right\}$.

We assume that players $1,2,...,n$ know the whole structure of the network g they are part of. Therefore they are able to anticipate equilibrium payoffs and are assumed to play a a subgame perfect equilibrium strategy profile. Given a network g and a discount factor δ, we will for simplicity refer to $v_i^{*\delta}(g)$ as player i's payoff. Throughout this paper it is important to distinguish this precisely from the profit which is payoff minus total linking costs, so for player $i \in N$

$$u_i^{*\delta}(g) = v_i^{*\delta}(g) - \eta_i(g)c.$$

Notice that in our model a non-isolated player's profit is always strictly smaller than her payoff since we assume strictly positive linking costs $c > 0$.

For a large part of this paper we will as a benchmark focus on the limit case of $\delta \to 1$ meaning that players are infinitely patient. For this case Manea (2011, Theorem 2) finds that for all δ being greater than some lower bound the corresponding equilibrium agreement networks are equal. This

[1] This replacement is primarily due to technical reasons since this implies that the network structure does not change over time, which makes the model analytically tractable. However, recalling the motivating example on bilateral project cooperation shows that there are indeed situations in reality captured by that.

[2] In the case $\delta\left(v_i^{*\delta}(g) + v_j^{*\delta}(g)\right) = 1$ both players are indifferent.

network g^* is then defined as limit equilibrium agreement network. We adopt this notation here. Moreover, we deduce again from Manea (2011, Theorem 2) that the so called limit equilibrium payoff vector $v^*(g) = \lim_{\delta \to 1} v^{*\delta}(g)$ exists. Beyond that, Manea (2011, Proposition 2) shows that it is $v_i^*(g) + v_j^*(g) \geq 1$ for all $ij \in g$ and if $ij \in g^*$, then we even have $v_i^*(g) + v_j^*(g) = 1$, which will be important throughout our analysis as well.

Manea (2011) develops a smart algorithm to compute the payoff vector $v^*(g)$ and we will make use of this. To understand the algorithm we need to introduce some additional notation. For any set of players $M \subseteq N$ and any network g let $L^g(M) = \{j \in N \mid ij \in g,\ i \in M\}$ be the set of the corresponding partners in g, that is the set of players having one or more links to players in M in g.[3] A set $M \subseteq N$ is called g-independent if $g|_M = \{ij \in g \mid i, j \in M\} = \emptyset$, that is no two players contained in M are linked in g. Moreover, let $\mathscr{I}(g) \subseteq \mathscr{P}(N)$ denote the set of all nonempty g-independent subsets of N. Then the algorithm determining the payoff vector $v^*(g)$ is the following.

Definition 1 (Manea (2011))**.** *For a given network g on the player set N, the algorithm $\mathscr{A}(g)$ provides a sequence $(r_s, x_s, M_s, L_s, N_s, g_s)_{s=1,\ldots,\bar{s}}$ which is defined recursively as follows: Let $N_1 = N$ and $g_1 = g$. For $s \geq 1$, if $N_s = \emptyset$ then stop and set $\bar{s} = s$. Otherwise, let*

$$r_s = \min_{M \subseteq N_s, M \in \mathscr{I}(g)} \frac{|L^{g_s}(M)|}{|M|}. \tag{2.2}$$

If $r_s \geq 1$ then stop and set $\bar{s} = s$. Otherwise, set $x_s = \frac{r_s}{1+r_s}$. Let M_s be the union of all minimizers M in (2.2). Denote $L_s = L^{g_s}(M_s)$. Let $N_{s+1} = N_s \backslash (M_s \cup L_s)$ and $g_{s+1} = g|_{N_{s+1}}$.

Given such a sequence being the outcome of the described algorithm it is straightforward to calculate the limit equilibrium payoff vector for the network at hand. However, this sophisticated result of Manea (2011, Theorem 4) is absolutely fundamental for our work:

Payoff Computation (Manea (2011))**.** *Let $(r_s, x_s, M_s, L_s, N_s, g_s)_{s=1,\ldots,\bar{s}}$ be the outcome of $\mathscr{A}(g)$ for a given network g. Then the limit equilibrium payoffs are given by*

$$\begin{aligned} v_i^*(g) &= x_s && \forall\, i \in M_s\ \forall\, s < \bar{s}, \\ v_j^*(g) &= 1 - x_s && \forall\, j \in L_s\ \forall\, s < \bar{s}, \\ v_k^*(g) &= \tfrac{1}{2} && \forall\, k \in N_{\bar{s}}. \end{aligned} \tag{2.3}$$

Let us figure out what the algorithm $\mathscr{A}(g)$ in combination with the previous result actually does. Starting with the network g and player set N, at each step s it identifies the so called minimal shortage ratio r_s among the remaining players N_s in the network $g_s = g|_{N_s}$. There is a largest set M_s which minimizes this shortage ratio such that

$$r_s = \frac{|L_s|}{|M_s|},$$

[3] Do not confuse with the notation of Manea (2011) who calls $L^{g^*}(M)$ partner set.

where L_s is the partner set of M_s. The limit equilibrium payoff of all players in M_s is then given by $x_s = \frac{r_s}{1+r_s} = \frac{|L_s|}{|M_s|+|L_s|}$, whereas their partners in L_s receive $1 - x_s = \frac{|M_s|}{|M_s|+|L_s|}$. These players and their links are then deleted from the network and the algorithm moves forward to the next step. It stops when there are either no more players left or if the minimal shortage ratio is greater than or equal to 1. In the latter case the limit equilibrium payoff of all remaining players is $\frac{1}{2}$. Manea (2011, Proposition 3) shows that the sequence of minimal shortage ratios $(r_s)_s$ and therefore also $(x_s)_s$ are strictly increasing.

In the framework of our paper with $\delta \to 1$ the described algorithm $\mathscr{A}(g)$ together with the previous considerations then determines the profit $u_i^*(g) = v_i^*(g) - \eta_i(g)c$ of every player $i \in N$. Simplistically, the algorithm quantifies the main forces that each player likes to be linked to other players if one disregards the linking costs while preferring neighbors not to be connected to others. Notice that the profile of payoffs and therefore also the profile of profits $u^* = \left(u_i^*\right)_{i\in N}$ is obviously component-decomposable, that is $u_i^*(g) = u_i^*\left(g|_{C_i(g)}\right)$ for all $i \in N$ and for all networks g. Here $C_i(g) \subseteq N$ denotes the so called component player i is part of in the network g which is defined as the minimal set of players such that both $i \in C_i(g)$ and no player in $C_i(g)$ has a g-link to a player not contained in $C_i(g)$. This means that the structure of the network within a certain component does not affect the profit of players outside this component. Furthermore, notice that Manea (2011) develops the algorithm $\mathscr{A}(g)$ assuming that there are no isolated players in the underlying network g. However, it is easy to see that the equations (2.3) are still fulfilled if one relaxes this restriction. It is clear that isolated players have a limit equilibrium payoff of 0 since they have no bargaining partner they could generate a unit surplus with . On the other hand, in this case the algorithm $\mathscr{A}(g)$ provides $r_1 = 0$ such that $x_1 = 0$ and M_1 is the set of all isolated players in the network such that $L_1 = \emptyset$. Then according to (2.3) all players in M_1 receive a limit equilibrium payoff of $x_1 = 0$ as required.

Throughout the next Section we will assume that each player can influence the network structure by altering own links before the bargaining game starts. This means that the network is no longer exogenously given as in the work of Manea (2011) but the outcome of strategic interaction between players. This implies questions regarding the stability of networks and leads to the main results of this paper. Our analysis will mainly be based on the seminal concept of pairwise stability, which has been introduced by Jackson and Wolinsky (1996).

Definition 2 (Pairwise Stability, Jackson and Wolinsky (1996))**.** *Given a profile of network utility or profit functions $(u_i)_{i\in N}$, a network g is* ***pairwise stable*** *if both*

(i) for all $ij \in g$: $u_i(g) \geq u_i(g - ij)$ and

(ii) for all $ij \notin g$: $u_i(g+ij) > u_i(g) \Rightarrow u_j(g+ij) < u_j(g)$.

According to the Definition a network is called pairwise stable if no player can improve by deleting a single link and also no two players can both individually benefit from adding a mutual link.

Within the analysis of our model we will distinguish between networks being pairwise stable only at a single cost level and those fulfilling the conditions for two or more choices of the cost parameter. For this purpose we introduce the following novel terminology:

Definition 3 (Singular and Non-Singular Pairwise Stability)**.** *Consider the above framework with $u = u^*$ or $u = u^{*\delta}$. A network g is called*

- ***singularly pairwise stable*** *if g is pairwise stable at a unique cost level $c > 0$ but nowhere else,*
- ***non-singularly pairwise stable*** *if there exist at least two cost levels $c', c'' > 0$, $c' \neq c''$ such that g is pairwise stable.*

For any singularly pairwise stable network it is of course a very special case (even a singularity) to encounter precisely the parametrization where it is indeed pairwise stable. Insofar, this notion is not robust to a slight change of the required cost level. Thus, networks being non-singularly pairwise stable should be of much greater interest in comparison. In the following the focus will be on this stability concept and the corresponding class of networks. It will even turn out that all non-singularly pairwise stable networks are in fact pairwise stable for a continuum of cost levels in our model.

Observe that the notions of stability considered so far focus exclusively on one link deviations. One might argue that in general severing links does not cost any effort since there is no coordination between players required and therefore one should allow for the possibility of multiple link deletion in this context. This gives rise to the concept of pairwise Nash stability where condition (i) of Definition 2 is replaced by

(i)' for all $i \in N$, $l_i \subseteq \{ij \in g\}$: $u_i(g) \geq u_i(g - l_i)$.

As opposed to stability issues, in Section 5 we will concentrate our attention on networks being efficient, that is on structures which yield maximal utilitarian welfare.

Definition 4 (Utilitarian Welfare and Efficiency)**.** *Let N be a set set of players and $(u_i)_{i \in N}$ a profile of network utility or profit functions.*

- *The **utilitarian welfare** yielded by a network g is defined as*

$$U(g) := \sum_{i \in N} u_i(g).$$

- *A network g is called **efficient** if for all $g' \subseteq g^N$*

$$U(g) \geq U(g').$$

So we will denote a network to be efficient if the unweighted sum of individual profits cannot be further increased. Based on these fundamental concepts and definitions we are now able to establish our results in the following Sections 3, 4 and 5.

3 Stability

Throughout this Section we consider infinitely patient players who intend to maximize their profit from the subsequent network bargaining game by link creation and deletion. We examine which network structures turn out to be stable such that none of the players wants to add or delete further links. For all levels of linking costs we identify pairwise stable structures. Afterwards, we gradually rule out the possibility to be pairwise stable for a broad range of networks. In doing so, we provide a complete characterization of non-singularly pairwise stable networks. We show that these networks are even pairwise Nash stable and that the induced payoff and profit structures are in general highly but not completely homogeneous.

As already mentioned, we will first focus on one link deviations, which is captured by the notion of pairwise stability (cf. Definition 2). To get a first impression of the problem let us have a look at the situation for three players. It will turn out that this case already covers many important aspects of the network formation game. Figure 1 illustrates the four types of networks which might appear including the induced profits u_i^* for players $i = 1,2,3$. Notice that all other possible networks can be derived by permuting the three players, which does not provide any further insight since players are ex-ante homogeneous.

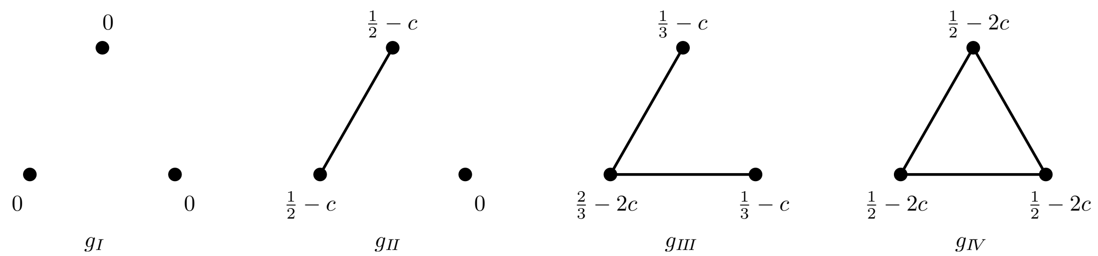

Figure 1: 3-Player Example

We see immediately that the network g_I is pairwise stable if and only if the linking costs c are greater than or equal to $\frac{1}{2}$. Otherwise any two players could both increase their profit from 0 to $\frac{1}{2} - c$ by creating a mutual link. However, for $c = \frac{1}{2}$ also no player wants to delete this link and indeed, the cost range $c \in \left(\frac{1}{6}, \frac{1}{2}\right]$ is the one for which g_{II} is pairwise stable. Link deletion can obviously not be improving and if one of the two connected players creates a link to the third player, she would end up with a profit of $\frac{2}{3} - 2c$ which is strictly smaller than $\frac{1}{2} - c$ for this cost range. These two terms are equal for $c = \frac{1}{6}$, but the third player would improve from 0 to $\frac{1}{6}$.

Therefore in this case and also if costs are even smaller, g_{II} is no longer pairwise stable. But so is g_{III} for $c = \frac{1}{6}$. This is because no player has incentives to delete a link and the two players who are not connected are indifferent between creating a mutual link and refraining from this since for this cost level it is $\frac{1}{3} - c = \frac{1}{6} = \frac{1}{2} - 2c$. However, if linking costs are even smaller, then both would profit from the mutual link. Hence, g_{III} is pairwise stable if and only if $c = \frac{1}{6}$. For $c \in \left(0, \frac{1}{6}\right]$ the network g_{IV} is pairwise stable, which is obviously not true at higher cost levels.

We will see that these mechanisms being crucial in the three-player example transfer similarly to more general cases. The following important Theorem reveals sufficient conditions for pairwise stability. More precisely, for all cost levels it identifies concrete network structures being pairwise stable.

Theorem 1 (Sufficient Conditions for Pairwise Stability)**.** *In the framework as introduced in Section 2 with $\delta \to 1$ the following holds:*

(i) The empty network is pairwise stable if $c \geq \frac{1}{2}$.

(ii) A network which is a disjoint union of separated pairs[4] and at most one isolated player is pairwise stable if $c \in \left(\frac{1}{6}, \frac{1}{2}\right]$. Additionally, if $c = \frac{1}{2}$, then several isolated players can coexist in a pairwise stable network.

(iii) A network which is a disjoint union of odd circles[5] with at most $\frac{1}{2c}$ members and either separated pairs or at most one isolated player is pairwise stable if $c \in \left(0, \frac{1}{6}\right]$. Additionally, if $c = \frac{1}{6}$ and given that there is no isolated player, there can also exist lines of length three[6] in a pairwise stable network.

The formal proof of Theorem 1 as well as all subsequent proofs – unless provided immediately in a detailed and formal manner – can be found in the Appendix. Notice that by disjoint union we do not mean that all of the stated subnetworks must necessarily be present in a network to be pairwise stable. For instance, if it is $c \in \left(0, \frac{1}{6}\right]$, then a network consisting only of separated pairs or of (permissible) odd circles is pairwise stable as well.

A byproduct of Theorem 1 is that it guarantees the existence of a pairwise stable network at each level of costs. Furthermore, we have given a characterization of at least some pairwise stable networks for each level of costs. However, it is not clear at all that the types of networks examined in Theorem 1 are in each case the only pairwise stable ones. Anyway, we can already state some consequences from our observations in Figure 1 and the Proof of Theorem 1:

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[4] A separated pair denotes a subnetwork induced by a two player component.

[5] An odd circle denotes a subnetwork which is induced by a component consisting of an odd number of players and moreover is regular of degree two.

[6] A line of length $K \geq 3$ denotes a subnetwork induced by a K player component which can be transformed to a circle by adding one link.

Corollary 1. *In the model with $\delta \to 1$ a network is not pairwise stable if it contains*

(i) more than one isolated player while $c < \frac{1}{2}$.

(ii) a separated pair while $c > \frac{1}{2}$.

(iii) a line of length three while $c \neq \frac{1}{6}$.

(iv) an odd circle with more than $\frac{1}{2c}$ members. In particular this means that there can be no odd circles at all in pairwise stable networks as long as $c > \frac{1}{6}$.

(v) an isolated player combined with a separated pair or a line of length three while $c \leq \frac{1}{6}$.

The statements (i)-(iv) follow immediately from the three-player example (cf. Figure 1) and the proof of Theorem 1. The short proof of part (v) is as usual given in the Appendix. It will turn out that the Corollary in combination with the subsequent results establishes that the conditions stated in Theorem 1 are indeed also necessary for a network to be non-singularly pairwise stable. However, notice that according to the above results a network consisting of several isolated players and at least one separated pair is singularly pairwise stable. Likewise, a network containing a line of length three cannot be non-singularly pairwise stable.

In general, it is clear that a network can only be pairwise stable if any link is profitable for both players involved or at least linking costs are covered by the additional payoff. Therefore, the intuition says that there cannot be so called disagreement links in a pairwise stable network, that is links which are contained in the original network but not in the corresponding limit equilibrium agreement network. One might argue that such a link leads to higher costs for both players connected through it, whereas it seems to be irrelevant regarding payoffs. If it is selected by nature at some point in time, the two players will not find an agreement in the bargaining game. So why should they connect? However, things are a bit more complicated. With regard to the mechanism of the algorithm $\mathscr{A}(g)$ which determines the payoff of each player in a given network, a disagreement link could have a rather global effect. It is conceivable that deleting such a link can change the whole payoff structure induced by the network, which then might also affect the two edge players. For instance, the presence of a link, though giving rise to a disagreement, might prevent one of the players it connects and who receives a payoff of at least $\frac{1}{2}$ from being deleted during the algorithm as part of a g-independent set, which would induce a lower payoff for this player. However, we find that our first intuition is indeed correct:

Proposition 1 (Disagreement Links)**.** *If a network g is pairwise stable for $\delta \to 1$, then $g^* = g$, that is g does not contain disagreement links. In particular, this implies $v_i^*(g) + v_j^*(g) = 1$ for all $ij \in g$.*

In the following this will repeatedly prove to be a valuable insight while ruling out the possibility to be pairwise stable for a broad range of network structures. The idea of the proof (which can again be found in the Appendix) is to assume for contradiction that g is pairwise stable and contains a disagreement link ij. By adapting the proof of Manea (2011, Theorem 4) appropriately we show that it is indeed $v_k^*(g) = v_k^*(g - ij)$ for all $k \in N$ such that players i and j will want to delete their mutual link.

We will now first consider networks with a homogeneous payoff structure. In line with Manea (2011) we call a network equitable if every player receives a payoff of $\frac{1}{2}$. For a given network g with player set N we define the subset $\tilde{N}(g) := \left\{ i \in N \mid v_i^*(g) = \frac{1}{2} \right\}$. We will make use of this notation in the following Theorem. In combination with Proposition 1, it reveals that a network can only be pairwise stable if any player receiving a payoff of $\frac{1}{2}$ is contained in a component which either induces a separated pair or an odd circle.

Theorem 2 (Equitability and Pairwise Stability)**.** *In the model with $\delta \to 1$ consider a network g with player set N. If g is pairwise stable, then $g|_{\tilde{N}(g)}$ must be a disjoint union of separated pairs and odd circles.*

In the proof (in the Appendix) we assume for contradiction that g is pairwise stable but $g|_{\tilde{N}(g)}$ is not a disjoint union of separated pairs and odd circles. Notice that by Proposition 1 a link from a player in $\tilde{N}(g)$ to a player outside this set cannot exist, which implies that it is $v_i^*(g) = v_i^*(g|_{\tilde{N}(g)})$ for all $i \in \tilde{N}(g)$. Further, we make use of both directions of Manea (2011, Theorem 5), which establishes that a network is equitable if and only if it has a so called edge cover g' formed by a disjoint union of separated pairs and odd circles. In this context a network g' is said to be an edge cover of $g|_{\tilde{N}(g)}$ if it is $g' \subseteq g|_{\tilde{N}(g)}$ and no player in $\tilde{N}(g)$ is isolated in g'. This implies that any player in $\tilde{N}(g)$ will want to delete each of her links not contained in g'.

Though statements differ, notice that Theorem 2 is still in line with Manea (2011, Theorem 1(ii) of the Online Appendix). The latter establishes that for zero linking costs a network is pairwise stable if and only if it is equitable. Of course in this case no player can gain anything by deleting redundant links from an equitable network, which gives rise to a larger class of pairwise stable equitable networks. For instance, any even circle is equitable and therefore pairwise stable as long as there are no linking costs whereas Theorem 2 entirely rules out this possibility for $c > 0$. However, as we have seen in Figure 1 and Theorem 1, for positive linking costs there exist also non-equitable structures such as the network consisting of an isolated player combined with separated pairs or odd circles which can be pairwise stable. Here, the line of length three has to be mentioned as well though such a component can only occur if it is exactly $c = \frac{1}{6}$. In what follows, this kind of singularity will be the focal point of our investigation.

Summing up our results so far, for all levels of positive linking costs we achieved a complete characterization of networks which are pairwise stable and induce homogeneous payoffs within each of its components.[7] According to Theorem 1, Corollary 1 and Theorem 2 certain disjoint

[7] Be aware that all payoffs must be equal to either $\frac{1}{2}$ or 0 in such networks by Manea (2011, Proposition 2, Lemma 1).

unions of separated pairs and odd circles together with the empty network constitute this class of networks.

What remains to be considered are therefore structures which induce heterogeneous payoffs within a component. Most of the rest of the Section will be devoted to the examination of such networks and the question whether and in which cases they can be pairwise stable. To begin with, let us make sure to be aware of the following property of pairwise stable non-equitable networks, which is again an immediate consequence of Proposition 1.

Remark 1. *Let $g \neq \emptyset$ be a non-equitable network consisting of only one component and assume that it is pairwise stable for $\delta \to 1$. Then there exists a unique partition $M \dot{\cup} L = N$ with $|M| > |L|$ and $g|_M = g|_L = \emptyset$, meaning that g is bipartite. What is more, payoffs are*

$$v_i^*(g) = x < \frac{1}{2} \;\; \forall i \in M \quad \textit{and} \quad v_j^*(g) = 1 - x > \frac{1}{2} \;\; \forall j \in L,$$

where $x = \frac{|L|}{|M|+|L|}$.

Notice here that according to Manea (2011, Proposition 3) the sequence of minimal shortage ratios provided by the algorithm in Definition 1 is strictly increasing for any network. Thus, Remark 1 implies that in the given case $\mathscr{A}(g)$ already stops after removing all players during the first step. This leads to a heterogeneous payoff distribution with two different payoffs, one below and one above $\frac{1}{2}$. Moreover, we discover the following:

Theorem 3 (Payoff Heterogeneity and Pairwise Stability)**.** *In the model with $\delta \to 1$ a network containing a component in which players receive heterogeneous payoffs can only be pairwise stable if there occur exactly two different payoffs $x \in \left(0, \frac{1}{2}\right)$ and $1 - x \in \left(\frac{1}{2}, 1\right)$ in any such component and it is*

$$x + c = \frac{1}{2}.$$

Based on the Lemmata 2 and 3 in the Appendix the proof of this Theorem is straightforward. In Lemma 2 we show (among other things) that for any two players $i, j \in N$ who receive a payoff of x in a pairwise stable network g it is $v_i^*(g + ij) = v_j^*(g + ij) = \frac{1}{2}$, that is linking to each other always leads to a payoff of $\frac{1}{2}$ for both players. In contrast, Lemma 3 basically establishes that in a pairwise stable network, one link deletion cannot effect a player with original payoff $1 - x$ to fall below $\frac{1}{2}$.

Proof of Theorem 3. Let g be a pairwise stable network containing a component in which there exist two players receiving different payoffs. Let $C \subseteq N$ be one such component and let $(r_1, x, M_1, L_1, N_1, g_1)$ be the outcome of $\mathscr{A}(g|_C)$. Notice that in this case the algorithm already has to stop after the first step by Proposition 1 (cf. Remark 1). Hence, there must be exactly two different payoffs given by $x \in \left(0, \frac{1}{2}\right)$ and $1 - x \in \left(\frac{1}{2}, 1\right)$ within C. Moreover, for $i \in M_1$ Lemma 2 provides the stability condition

$$x - \eta_i(g|_C)c \geq \frac{1}{2} - (\eta_i(g|_C) + 1)c \quad \Leftrightarrow \quad x + c \geq \frac{1}{2}.$$

Similarly, according to Lemma 3 for $j \in L_1$ we need to have

$$(1-x) - \eta_j(g|_C)c \geq \frac{1}{2} - (\eta_j(g|_C) - 1)c \quad \Leftrightarrow \quad \frac{1}{2} \geq x + c.$$

So the payoffs must be $x = \frac{1}{2} - c$ and $1 - x = \frac{1}{2} + c$. Obviously, this has to hold for all components of g in which players receive heterogeneous payoffs. □

Notice by considering the limit case $c \to 0$ that Theorem 3 is in line with Manea's (2011, Online Appendix) result that for zero linking costs any pairwise stable network must be equitable. As an immediate consequence of Theorem 3 and the previous findings we arrive at the first main result of this paper:

Corollary 2 (Complete Characterization)**.** *In the model with $\delta \to 1$, the class of non-singularly pairwise stable networks is characterized completely by Theorem 1 for each cost level $c > 0$.*

To see this, recall first that according to Theorem 2 any network g not mentioned in Theorem 1 can only be pairwise stable if it contains a component in which players receive heterogeneous payoffs. Within such a component each player must either receive a payoff of $x = \frac{1}{2} - c$ or $1 - x = \frac{1}{2} + c$ by Theorem 3. Be aware that these equations do not represent calculation rules determining the payoffs in g but necessary conditions which have to be fulfilled for pairwise stability. Recall here that x is in fact determined by the algorithm $\mathscr{A}(g)$ meaning that x solely depends on the structure of g and that c is an independent parameter of the model. Therefore, such a network g can only be pairwise stable at the single cost level $c = \frac{1}{2} - x$.
Besides, notice that for $c > \frac{1}{4}$ no network apart from the ones mentioned in Theorem 1 can be pairwise stable at all, that is not even singularly pairwise stable. That is because a weaker player receiving a payoff of $x = \frac{1}{2} - c$ could save costs of c when deleting a link while falling back to a payoff of 0 in the worst case. The corresponding stability condition then implies $x \geq c$ which is here equivalent to $c \leq \frac{1}{4}$.

Further, it is not at all clear that a network where each player either receives a payoff of $x \in \left(0, \frac{1}{2}\right)$ or $1 - x$ is actually pairwise stable at $c = \frac{1}{2} - x$. However, even if this is the case, any two players with a payoff of x are indifferent between leaving the network unchanged and adding a mutual link. Also, any player receiving a payoff of $1 - x$ must be indifferent between keeping all of her links and deleting any of them. In this sense, the case that a network which generates heterogeneous payoffs, but does not consist of an isolated player combined with either separated pairs or odd circles, is pairwise stable and does in indeed form is really special and insofar a singularity. Despite all that, we already identified such a network: the line of length three with payoffs $x = \frac{1}{3}$ and $1 - x = \frac{2}{3}$, which is pairwise stable if and only if it is $c = \frac{1}{6}$. [8] As opposed to this, we will even rule out the possibility to be singularly pairwise stable for a broad range of network structures in the further course of this Section.

However, we will now at first discuss and note the implications regarding payoffs and profits that might occur in pairwise stable networks. Beside the characterization of pairwise stable networks this is the main result of the paper. The following Corollary establishes and specifies that provided pairwise stability a high degree of inequality among players cannot appear.

Corollary 3 (Limited Outcome Diversity). *In the framework with $\delta \to 1$ consider a network g which is pairwise stable at a given level of linking costs $c > 0$. It holds that either $v_j^*(g) \in \left\{\frac{1}{2} - c, \frac{1}{2}, \frac{1}{2} + c\right\}$ with $c \in \left(0, \frac{1}{4}\right]$ or $v_j^*(g) \in \left\{0, \frac{1}{2}\right\}$ for all $j \in N$.*
If g is assumed to be non-singularly pairwise stable, then only the latter of these two cases can occur, and furthermore, there exists a set $P \subset \left\{0, \frac{1}{2} - 2c, \frac{1}{2} - c\right\}$ with $|P| \leq 2$ such that $u_j^*(g) \in P$ for all $j \in N$.

A short proof is provided in the Appendix though the Corollary is basically a direct consequence of the Theorems 2 and 3, Corollary 1 and Lemma 2. Meanwhile, it should be clear that in a pairwise stable network there can only occur four kinds of players characterized by their payoffs: isolated players receiving 0, players belonging to a separated pair or an odd circle with a payoff of $\frac{1}{2}$ and players contained in a component with heterogeneous payoffs who receive $\frac{1}{2} + c \in \left(\frac{1}{2}, \frac{3}{4}\right]$ or $\frac{1}{2} - c \in \left[\frac{1}{4}, \frac{1}{2}\right)$. However, Lemma 2 implies that the former and the latter player type cannot coexist in a pairwise stable network. Taken together, we established that the number of different possible bargaining outcomes gets narrowed down substantially compared to the work of Manea (2011) if one allows for strategic network formation in advance. To this end observe that in Manea's (2011) basic framework with $\delta \to 1$ one can induce any payoff which is a rational number from the interval $[0, 1)$ by a suitable complete bipartite network on a sufficiently large player set.
Beyond that, the second part of the Corollary focuses on the crucial class of non-singularly pairwise stable networks and the induced profits. It points out that in any of these cases players can at most be divided into two distinct groups as regarding the network position at most two of the following three kinds of players can coexist: players being part of a separated pair receiving a profit of $\frac{1}{2} - c$, players contained in an odd circle facing twice the costs and isolated players with a profit of zero.

As already announced, we will next rule out the possibility to be pairwise stable at all for certain kinds of subnetworks induced by components in which players receive heterogeneous payoffs, even if the conditions of Theorem 3 are fulfilled. The main idea of the proofs of the following Propositions, which are rather lengthy and can again be found in the Appendix, is to identify

[8] In Section 4 we will additionally reveal that the stability of this particular subnetwork is not robust in another respect either.

network positions in which the player receives a payoff strictly greater than $\frac{1}{2}$ and still does so after deleting a certain link. Using the notation of Theorem 3 this then leads to the unfulfillable stability condition $x+c<\frac{1}{2}$. Another approach we use focuses on players who are in an weak bargaining position but whose loss in payoff from dropping a certain own link is too small to be compatible with the condition $x+c=\frac{1}{2}$.

We show first that all networks containing a tree component[9] not considered in Theorem 1 cannot be pairwise stable.

Proposition 2 (Trees)**.** *If a network g is pairwise stable for $\delta \to 1$, then it does not contain a component of more than three players which has a tree structure.*

Proposition 2 further reduces the class of potentially pairwise stable networks. It implies that any component of a pairwise stable network either contains at most three players or induces a subnetwork which has a cycle[10]. The former case has been analyzed completely by Theorem 1 and Corollary 1. This gives rise to the following examination of structures which have a cycle and in which players receive heterogeneous payoffs.

Proposition 3 (Cycles and Component-Connecting Players)**.** *Consider a network g containing a player $k \in N$ who is part of a cycle, receives a payoff $v_k^*(g) > \frac{1}{2}$ and is such that $g|_{N\setminus\{k\}}$ consists of more components than g. Then g is not pairwise stable for $\delta \to 1$.*

This Proposition might seem somewhat artificial but it rules out the possibility to be pairwise stable for several generic kinds of networks. For instance many components containing a cycle and a loose-end player, i.e. a player having only one link are excluded. For an illustration of exemplary structures which cannot be pairwise stable according to Proposition 3 see Figure 2.

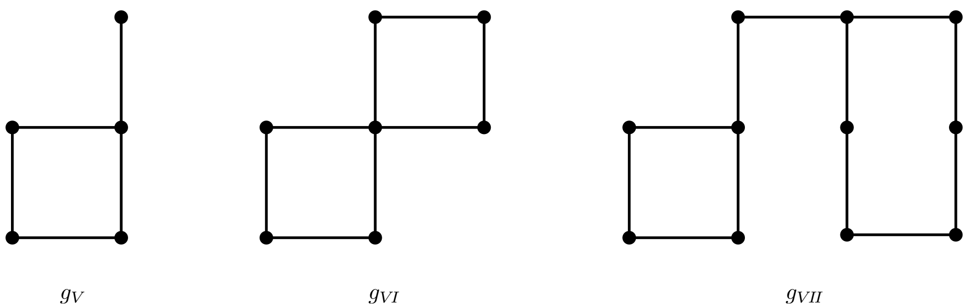


Figure 2: Unstable Network Structures

However, there exist other networks not captured by Proposition 3 which have a cycle and could potentially be pairwise stable. Two examples for this are given in Figure 3. Though a further

[9] A tree component denotes a subnetwork induced by a component of players which is minimally connected.

[10] A network g is said to have a cycle if there exist distinct players $i_1, i_2, ..., i_K \in N$, $K \geq 3$ such that $i_1 i_K \in g$ and $i_k i_{k+1} \in g$ for all $k = 1, 2, ..., K-1$

generalization is not reached in this paper, it is certainly straightforward to check that the concrete networks g_{VIII} and g_{IX} cannot be pairwise stable. Also, recall that any remaining network can at most be singularly pairwise stable.

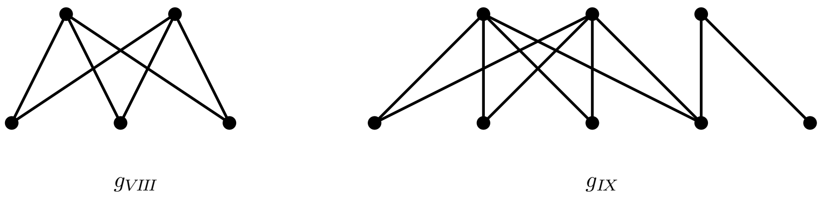


Figure 3: Cylic Network Structures of Different Kind

So far we have considered pure one link deviations and the concept of pairwise stability. Finally, we will now relax this assumption and allow for multiple link deletion as it is captured by the notion of pairwise Nash stability which has been introduced in Section 2. It is clear that every pairwise Nash stable network is also pairwise stable whereas the reverse is in general not true. This gives rise to the question whether in the model at hand there exist pairwise stable networks which are not pairwise Nash stable. The following Corollary constitutes that this is not the case at least for networks being non-singularly pairwise stable.

Corollary 4 (Pairwise Nash Stability)**.** *In the framework with $\delta \to 1$ consider a non-singularly pairwise stable network g. Then g is pairwise Nash stable at each cost level for which it is pairwise stable.*

This result is easy to prove. According to our previous results there exist players with more than one link in a non-singularly pairwise stable network if only if it contains odd circles. Hence, only in this case the definitions of pairwise Nash stability and pairwise stability differ. Recall that odd circles can only occur in pairwise stable networks if it is $c \leq \frac{1}{6}$ and that each player contained in such a circle receives a payoff of $\frac{1}{2}$. Hence, each player's profit must at least be $\frac{1}{6}$. On the contrary, multiple link deletion would lead to a profit of zero since the player would be isolated afterwards. Furthermore, observe that the three player line is also even pairwise Nash stable at $c = \frac{1}{6}$: the central player would get isolated by deleting both of her links whereas she receives a profit of $\frac{1}{3}$ if she keeps them. Finally, notice that the equivalence of pairwise stability and pairwise Nash stability is even generally fulfilled as long as $c > \frac{1}{4}$. These additional considerations conclude this Section.

4 Effects of Time Discount

So far, we examined the case $\delta \to 1$ in which players are infinitely patient. However, in many situations it might be reasonable to assume that players are rather impatient, meaning that in the network bargaining game they discount time at least to some degree. In the underlying model this is captured by a parametrization with $\delta \in (0,1)$. In this Section we will reveal some important commonalities and differences between these two cases with regard to strategic network formation and stability.

In Proposition 1 we proved that there are no disagreement links in pairwise stable networks if it is $\delta \to 1$. For two reasons it is intuitively clear that this must still hold if we set $\delta \in (0,1)$. On the one hand, if $ik \in g$ is a disagreement link, then it is $\delta v_i^{*\delta}(g) > 1 - \delta v_k^{*\delta}(g)$ by definition and therefore the ith equation of the system (2.1) determining the equilibrium payoffs is equivalent to

$$v_i = \left(1 - \sum_{j \in N_i(g-ik)} \frac{1}{2d^{\#}(g)}\right) \delta v_i + \sum_{j \in N_i(g-ik)} \frac{1}{2d^{\#}(g)} \max\{1 - \delta v_j, \delta v_i\}.$$

This means that from player i's perspective it does not make a difference whether she can get selected to bargain with player k or not since they will either way not find an agreement. This is of course similarly true from player k's and also any other player's point of view. On the other hand, an additional amplifying effect comes into play when players are impatient. In this case, players care about the time they have to wait until a certain outcome of a bargain is achieved. They discount these payments by δ when calculating their expected payoffs. The existence of a disagreement link prolongs the expected time until any other link gets selected in the bargaining game and therefore has a negative impact on any player's payoff.

Next, we will see that there are networks which are pairwise stable for a certain level of costs if players are infinitely patient while this possibility can be ruled out if there is some time discount. The converse will turn out to be true as well.

Example 1. *Let g be a line of length three. Then g is pairwise stable if $\delta \to 1$ and $c = \frac{1}{6}$. However, for $\delta \in (0,1)$ it is not pairwise stable for any $c \in (0,+\infty)$.*

The first statement of Example 1 has been established by Theorem 1(iii). So let us deliberate why a line of length three cannot be pairwise stable if players are impatient to some degree. By applying the equation system (2.1) to g we find that the payoff of player 1 – who is supposed to be the player having two links – is $v_1^{*\delta}(g) = \frac{2}{4-\delta}$ and $v_2^{*\delta}(g) = v_3^{*\delta}(g) = \frac{1}{4-\delta}$ for the two loose-end players 2 and 3. Similarly, for $g-12$ and $g+23$ we calculate $v_1^{*\delta}(g-12) = \frac{1}{2}$, $v_2^{*\delta}(g-12) = 0$ and $v_2^{*\delta}(g+23) = v_3^{*\delta}(g+23) = \frac{1}{3-\delta}$. Consequently, for g to be pairwise stable the following three conditions would have to be satisfied simultaneously:

$$u_2^{*\delta}(g) \geq u_2^{*\delta}(g-12) \;\Leftrightarrow\; v_2^{*\delta}(g) \geq c \;\Leftrightarrow\; \frac{1}{4-\delta} \geq c, \tag{4.1}$$

$$u_1^{*\delta}(g) \geq u_1^{*\delta}(g-12) \;\Leftrightarrow\; v_1^{*\delta}(g) - v_1^{*\delta}(g-12) \geq c \;\Leftrightarrow\; \frac{\delta}{2(4-\delta)} \geq c, \tag{4.2}$$

$$u_2^{*\delta}(g) \geq u_2^{*\delta}(g+23) \;\Leftrightarrow\; v_2^{*\delta}(g+23) - v_2^{*\delta}(g) \leq c \;\Leftrightarrow\; \frac{1}{(3-\delta)(4-\delta)} \leq c. \tag{4.3}$$

However, one can show by simple transformations that conditions (4.2) and (4.3) cannot be fulfilled at the same time. Figure 4 illustrates this. According to condition (4.2) the level of costs must be below the blue line and (4.3) requires that c is above the red line, which is obviously not possible simultaneously for $\delta < 1$.

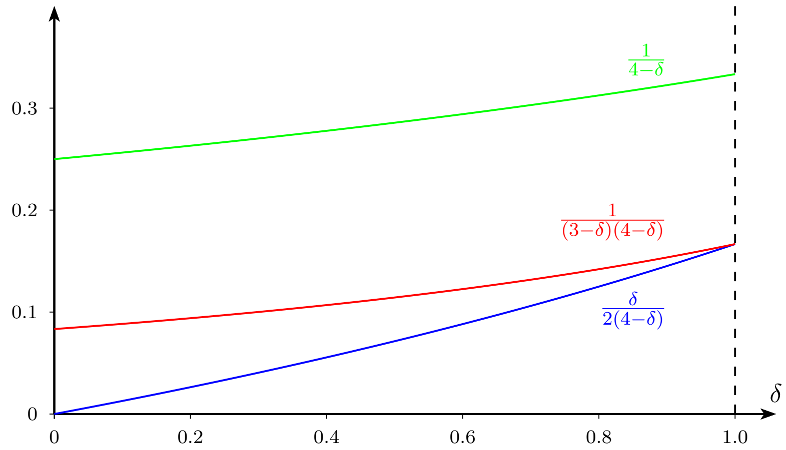


Figure 4: Stability Conditions Example 1

Thus, we notice that the existence of lines of length three within pairwise stable networks of patient players is not robust in two respects. We already know that these components may only occur if linking costs are exactly $c = \frac{1}{6}$ meaning that such networks are (at most) singularly pairwise stable. Now, we have additionally seen that already a marginal decrease of δ – meaning however that players are still almost completely patient – causes global instability for this kind of networks.

On the contrary, given any $\delta \in (0,1)$ there exist networks which are pairwise stable at some level of costs, whereas such $c \in (0,+\infty)$ does not exist if players are infinitely patient, that is if we let $\delta \to 1$.

Example 2. *Let g^N be the complete network with $n \geq 4$ players. Then for all $\delta \in (0,1)$ there exists $c > 0$ such that $u_i^{*\delta}(g^N) \geq u_i^{*\delta}(g^N - ij)$ for all $i, j \in N$. However, it is $u_i^*(g^N) < u_i^*(g^N - ij)$ for all $i, j \in N$ and all $c \in (0,+\infty)$.*

As usual a formal proof can be found in the Appendix. However, the second part should be clear, for instance by Theorem 2. To establish the first part we basically solve the equation system (2.1) for g^N and $g^N - ij$ and show that for sufficiently small costs it is profitable for any two players i and j to keep their mutual link.
At this, notice that the cost range for which the complete network of impatient players is pairwise stable gets arbitrarily small and close to 0 as δ approaches 1. In this sense, the limit case $\delta \to 1$ does not constitute a discontinuity regarding our previous results as it might seem at first sight in the light of Example 2.

5 Efficiency

Beside different stability concepts, it is of importance to ask for efficiency of the bargaining networks. From the perspective of a social planner it is of particular interest to understand the connection between pairwise stable network structures on the one hand and efficient ones on the other. In this light, Polanski and Vega-Redondo (2013) argue that the discrepancy between pairwise stability and efficiency in their model is due to the ex-ante heterogeneity between players. In the following we establish that in our model the two sets of pairwise stable and efficient networks do not coincide in general either though players are ex-ante homogeneous. Our analysis will be based on the concept of utilitarian welfare, which postulates that a society's welfare is simply given by the sum of the individual profits (cf. Definition 4). Notice that by using this notion we indeed solely consider profits of the initial players who are in charge of forming the network. One might argue that this is somewhat short-sighted, but it is these players who are present today and it is in general uncertain whether or when they will get replaced during the subsequent bargaining game. And not least, these considerations can be of interest from a purely technical perspective as well. Applying Definition 4 to the model with infinitely patient players reveals the following result.

Theorem 4 (Efficiency). *Consider the model with $\delta \to 1$. For $c > \frac{1}{2}$, $g = \emptyset$ is the unique efficient network. For $c = \frac{1}{2}$, a network g is efficient if and only if it is a disjoint union of a number of separated pairs and isolated players. And for $c \in \left(0, \frac{1}{2}\right)$, a network g is efficient if and only if it is a disjoint union of separated pairs, in case that n is odd either supplemented by*

- *an isolated player if $c \in \left[\frac{1}{6}, \frac{1}{2}\right)$,*
- *a line of length three if $c \in \left[\frac{1}{12}, \frac{1}{6}\right]$ or*
- *a three player circle if $c \in \left(0, \frac{1}{12}\right]$.*

In the following we will comprehend the situation for an even number of players. The proof of the case n odd is as usual given in the Appendix.

Proof of Theorem 4 (n even). First, we need to introduce some additional notation. Given the player set N and a network g, let $N'(g) := \{i \in N : \ \eta_i(g) \geq 1\}$ comprise all players being not isolated and let $g' := g|_{N'(g)}$ be the induced network with player set $N'(g)$. Moreover, let $g^{SP}_{N'(g)}$ denote the network on $N'(g)$ consisting of $\frac{|N'(g)|}{2}$ separated pairs.[11] Notice that this network is only well-defined for $|N'(g)|$ even. Finally, let $(r'_s, x'_s, M'_s, L'_s, N'_s, g'_s)_{s=1,\dots,\bar{s}}$ be the outcome of the algorithm $\mathscr{A}(g')$. Additionally, notice that for any $y, z \in \mathbb{R}$ it is $y \cdot z \leq \frac{1}{4}(y+z)^2$ and that this holds

[11] Strictly speaking, there are of course many networks of this kind. Since any two of these can be converted into each other by a permutation of players, they are all payoff respectively welfare equivalent. This justifies the simplistic consideration of one such representative network in this context.

strictly as long as $y \neq z$. Using this, we calculate

$$
\begin{aligned}
U^*(g) = U^*(g') = \sum_{i \in N'(g)} \left(v_i^*(g') - \eta_i(g')c\right) &= \sum_{i \in N'(g)} v_i^*(g') - 2d^{\#}(g')c \\
&= \sum_{s=1}^{\bar{s}-1} \left(x'_s|M'_s| + (1-x'_s)|L'_s|\right) + \frac{1}{2}|N'_{\bar{s}}| - 2d^{\#}(g')c \\
&= 2\sum_{s=1}^{\bar{s}-1} \frac{|M'_s||L'_s|}{|M'_s| + |L'_s|} + \frac{1}{2}|N'_{\bar{s}}| - 2d^{\#}(g')c \\
&\leq \frac{1}{2}\sum_{s=1}^{\bar{s}-1} \left(|M'_s| + |L'_s|\right) + \frac{1}{2}|N'_{\bar{s}}| - 2d^{\#}(g')c \\
&= \frac{1}{2}|N'(g)| - 2d^{\#}(g')c
\end{aligned} \tag{5.1}
$$

Since it is $\eta_i(g') \geq 1$ for all $i \in N'(g)$, we have that $d^{\#}(g') \geq \frac{1}{2}|N'(g)|$. Moreover, if $d^{\#}(g') = \frac{1}{2}|N'(g)|$, then this obviously implies that $|N'(g)|$ even and $g' = g^{SP}_{N'(g)}$. Hence, according to (5.1) for a network g inducing $g' \neq g^{SP}_{N'(g)}$ it holds that

$$
U^*(g) < \frac{1}{2}|N'(g)| - |N'(g)|c = |N'(g)|\left(\frac{1}{2} - c\right) \leq \begin{cases} 0 = U^*(g^{SP}_{\emptyset}) = U^*(\emptyset) & \text{if} \quad c \geq \frac{1}{2} \\ |N|\left(\frac{1}{2} - c\right) = U^*(g^{SP}_N) & \text{if} \quad c \leq \frac{1}{2} \end{cases}
$$

This means that only networks which are disjoint unions of separated pairs and isolated players are potentially efficient. It remains to identify those networks among these candidates which induce maximal utilitarian welfare. For $c > \frac{1}{2}$ this is obviously solely the network with minimal $|N'(g)|$, namely the empty network, whereas for $c \in (0, \frac{1}{2})$ it is the one with maximal $|N'(g)|$, namely g^{SP}_N. For $c = \frac{1}{2}$ all candidates provide the same welfare of 0. □

A comparison of Theorem 4 with the results of Section 3 reveals some interesting insights concerning the relationship between efficient and pairwise stable networks. They are summarized in the following Corollary.

Corollary 5 (Efficiency vs. Pairwise Stability)**.** *In the model with $\delta \to 1$ it applies*

(i) for $c > \frac{1}{6}$ that a network is efficient if and only if it is pairwise stable according to Theorem 1,

(ii) for $c \in \left[\frac{1}{12}, \frac{1}{6}\right]$ that there exists both efficient networks being not pairwise stable and pairwise stable networks being not efficient and

(iii) for $c \in \left(0, \frac{1}{12}\right)$ that every efficient network is also pairwise stable, but there exist pairwise stable networks being not efficient.

Notice that in Corollary 5(i) we only take the networks into account which we identified to be pairwise stable in Section 3. In principle, there could be non-efficient networks being pairwise stable, however only singularly in this case. Thus, we can constitute that as long as linking costs are high enough, efficient and pairwise stable networks coincide to the greatest extent. However, there is an intermediate level of costs for which a general statement is not possible at all whereas for low costs the efficient networks constitute a proper subset of the pairwise stable ones. This confirms the intuition that as long as linking costs are relatively low, there might be incentives for players to implement individually beneficial but non-efficient outside options. For an illustration consider the examples of networks given in Figure 5, which are efficient for certain cost ranges but not pairwise stable or vice versa.

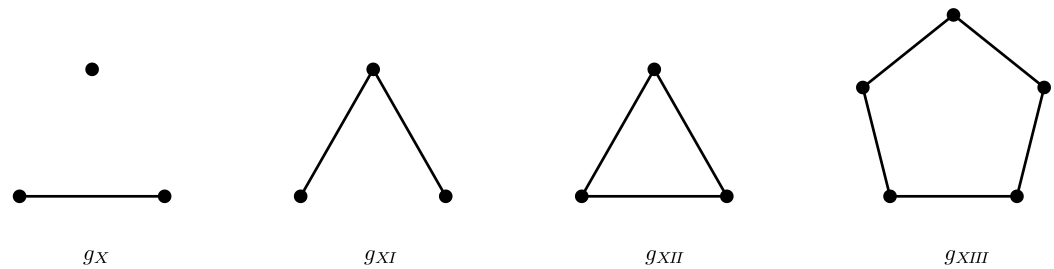


Figure 5: Efficiency vs. Stability

Observe that for $c = \frac{1}{6}$ the network g_X is efficient according to Theorem 4 but not pairwise stable (cf. Figure 1 or Corollary 1). The same is true for g_{XI} and the cost range $c \in \left[\frac{1}{12}, \frac{1}{6}\right)$. On the other hand, g_{XII} is pairwise stable for $c \in \left(\frac{1}{12}, \frac{1}{6}\right]$ but not efficient. And finally, for $c \in \left(0, \frac{1}{12}\right]$ the network g_{XIII} is pairwise stable – as larger odd circles are as well for smaller costs – but circles containing more than three players are never efficient.

Altogether, we find that efficiency does in general not coincide with pairwise stability although we deal with a setting of ex-ante homogeneous players. Notice however that the efficient networks are a subset of the pairwise stable ones at each level of linking costs if we restrict our attention to player sets with an even number of players.

6 Conclusion

Throughout this paper we developed a well-founded and analytically tractable model of strategic network formation in the context of decentralized bilateral bargaining involving ex-ante homogeneous players and explicit linking costs. One reasonable application of our model is constituted by the stylized example of project collaboration between firms we introduced at the beginning.
In the case that players are infinitely patient, we developed a complete characterization of non-singularly pairwise (Nash) stable networks: specific disjoint unions of separated pairs, odd circles and isolated players exclusively have this property depending on the cost level. The induced bargaining outcomes are mostly homogeneous but a certain level of diversity regarding players' payoffs and profits can still occur. Besides, we studied the remaining networks which could pos-

sibly be singularly pairwise stable and succeeded in ruling out this possibility for a broad range of structures. These results are complementary to Manea (2011, Online Appendix). As a robustness check, we relaxed the assumption that players are infinitely patient and gained important insights regarding commonalities and differences between the two cases. Finally, we provided a complete characterization of networks being efficient in terms of a utilitarian welfare criterion and revealed that these coincide only partially with the class of the pairwise stable ones.
Altogether, our work contributes to a better understanding of the behavior of players in a non-cooperative setting of decentralized bilateral bargaining when the underlying network is not exogenously given but the outcome of preceding strategic interaction. We gained important insights concerning the structure of the resulting networks, induced bargaining outcomes and regarding the effects which influence players aiming at an optimization of their bargaining position in the network.

Regarding future research, it would be a reasonable next step to approach a complete characterization of pairwise stable networks in general for our model. This would call for a further discussion of networks which – according to the results of the paper at hand – might be singularly pairwise stable, that is pairwise stable at one particular cost level. If it is not feasible to rule out this possibility in general, that is for all such structures apart from the three player line, one could instead work towards a generalization of Example 1 as an additional robustness check. Notwithstanding the above, it could be enriching to thoroughly analyze the class of stable and efficient networks when allowing players to discount time to some degree. A consideration of alternative stability concepts such as pairwise stability with transfers, which seems quite natural in a bargaining context, could generate further important insights. Beyond that, it would surely be interesting to set up an analytically tractable model of network formation in a bargaining framework, in which players do not get replaced one-to-one after dropping out. Due to the resulting stochastic change of the network structure over time, this would certainly constitute a challenging research topic.

A Appendix

Proof of Theorem 1. From the three-player example it is straightforward to see that the empty network is pairwise stable for $c \geq \frac{1}{2}$. In the empty network every player makes a zero profit. Adding a link would lead to a profit of $\frac{1}{2} - c \leq 0$ for both players involved. This proves part (i) of the Theorem.

Conversely, if $c \leq \frac{1}{2}$, then no player in a separated pair has an incentive to delete her link. Moreover, if two players who are members of two different separated pairs create a mutual link, then restricted to this four player component, the algorithm $\mathscr{A}(g)$ stops immediately and hence the two newly connected players receive an unchanged payoff of $\frac{1}{2}$, but have to bear higher costs of $2c$ instead of c. Therefore, such a link will not be formed.

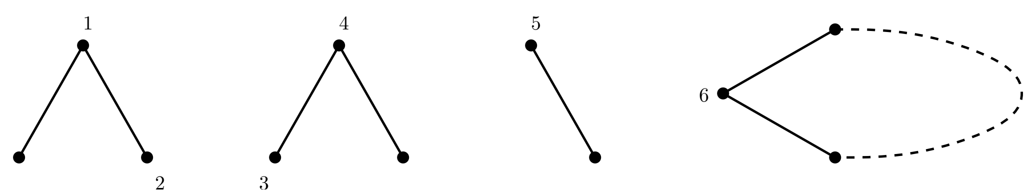
Let now $c \in \left(\frac{1}{6}, \frac{1}{2}\right]$ and consider a network containing at least one separated pair and an isolated player. As we have seen in the three-player example, a member of a separated pair and an isolated player will not create a mutual link in this situation. Beyond that, if $c = \frac{1}{2}$, then two isolated players are indifferent between linking to each other and receiving $\frac{1}{2} - c$ or refraining and having as well a profit of 0. This completes the proof of part (ii).

With regard to (iii) we consider first odd circles. Let $k \in \mathbb{N}$ be the number of players within the circle g. Then k must be odd and greater than or equal to 3. One can see immediately that it is impossible to find a g-independent set of players such that the corresponding partner set is smaller than the independent set itself. Hence $\mathscr{A}(g)$ stops in the first step and every member of the circle receives a profit of $\frac{1}{2} - 2c$. Adding a link between any two members of the circle who are not connected yet does not change the situation with regard to the algorithm and hence both players would be worse off due to higher costs. Next, consider two players i and j who are neighbors within the circle. If they delete their link, the circle will turn into a line network with an odd number of players. Let M be the independent set of players which includes i, j and every second player within the line network. Obviously this set minimizes the shortage ratio and it is $|M| = \frac{k+1}{2}$, $|L^{g-ij}(M)| = \frac{k-1}{2}$. Hence, it is $v_i^*(g - ij) = \frac{k-1}{2k}$ and therefore

$$u_i^*(g) - u_i^*(g - ij) = \frac{1}{2} - 2c - \left(\frac{k-1}{2k} - c\right) \geq 0 \iff \frac{1}{2k} \geq c \iff k \leq \frac{1}{2c}$$

The same holds for player j and hence an odd circle is indeed pairwise stable if and only if it has at most $\frac{1}{2c}$ members. Let now $c \in \left(0, \frac{1}{6}\right]$. If besides one or several odd circles there is one isolated player, then adding a link between a member of a circle and the yet isolated player will lead to a payoff of $\frac{1}{2}$ for all players in the network. This means that the player with three links would be worse off. Therefore, such a link will not be formed. Similarly, if two players connect who are members of different circles, according to the algorithm $\mathscr{A}(g)$ they will stay with a payoff of $\frac{1}{2}$ facing higher costs. And for the same reason a player being part of a separated pair and a player in an odd circle will not create a mutual link as well.

Finally, consider the case $c = \frac{1}{6}$ and that – possibly beside some odd circles and separated pairs – there are one or several components in the network g which are lines of length three. We already know from the three-player example that g restricted to such a component is pairwise stable. It remains to check whether a line of length three is also stable with regard to link addition to players in other components. For that purpose consider w.l.o.g. the following network g



where the last component is an odd circle. The labeled players $1,\ldots,6$ will be of interest. The algorithm $\mathscr{A}(g)$ provides the following profits for the network g:

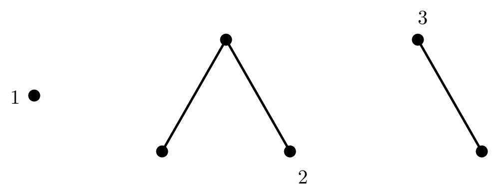
$$u_1^*(g) = \frac{2}{3} - 2c = \frac{1}{3}, \quad u_2^*(g) = u_3^*(g) = \frac{1}{3} - c = \frac{1}{6}, \quad u_6^*(g) = \frac{1}{2} - 2c = \frac{1}{6}$$

Based on this, we will see next that link addition either leads to a worsening for at least one of the two players involved or both are indifferent. It is

$$\begin{aligned}
&u_2^*(g+23) = u_3^*(g+23) = \frac{1}{2} - 2c = \frac{1}{6} = u_2^*(g) = u_3^*(g),\\
&u_1^*(g+13) = u_1^*(g+14) = u_1^*(g+15) = u_1^*(g+16) = \frac{2}{3} - 3c = \frac{1}{6} < \frac{1}{3} = u_1^*(g),\\
&u_2^*(g+25) = \frac{2}{5} - 2c = \frac{1}{15} < \frac{1}{6} = u_2^*(g),\\
&u_6^*(g+26) = \frac{1}{2} - 3c = 0 < \frac{1}{6} = u_6^*(g).
\end{aligned}$$

This completes the proof of part (iii) and the whole Theorem. □

Proof of Corollary 1 (v). Consider w.l.o.g. the following network g:



By using again the algorithm $\mathscr{A}(g)$ we get for $c \leq \frac{1}{6}$

$$\begin{aligned}
1.\quad & u_1^*(g+12) = \frac{1}{2} - c > 0 = u_1^*(g),\\
& u_2^*(g+12) = \frac{1}{2} - 2c \geq \frac{1}{3} - c = u_2^*(g) \quad \text{and}\\
2.\quad & u_1^*(g+13) = \frac{1}{3} - c > 0 = u_1^*(g),\\
& u_3^*(g+13) = \frac{2}{3} - 2c \geq \frac{1}{2} - c = u_3^*(g).
\end{aligned}$$

Hence, for this cost range a network containing an isolated player combined with a separated pair or a line of length three is not pairwise stable. □

Proof of Proposition 1. For ease of notation consider a network g' on player set N and assume that it is pairwise stable. Moreover, assume that there is a disagreement link in the network, that is $g' \backslash g'^* \neq \emptyset$. Let w.l.o.g. $12 \in g' \backslash g'^*$ be such a link and define $g := g' - 12$. This implies $g'^* \subseteq g$. Furthermore assume w.l.o.g. that every player has at least one link in g' (otherwise neglect isolated players, which is permissible since the utility function is component-decomposable). According to Manea (2011, Lemma 1) every player has at least one link in g'^* and therefore also in g.

Take the network g as a basis and let $(r_s, x_s, M_s, L_s, N_s, g_s)_{s=1,2,\dots,\bar{s}}$ be the outcome of $\mathscr{A}(g)$ (cf. Definition 1). Then by equations (2.3) the limit equilibrium payoffs $v^*(g)$ are given by

$$\begin{aligned} v_i^*(g) &= x_s && \forall\, i \in M_s\ \forall\, s < \bar{s}, \\ v_j^*(g) &= 1 - x_s && \forall\, j \in L_s\ \forall\, s < \bar{s}, \\ v_k^*(g) &= \tfrac{1}{2} && \forall\, k \in N_{\bar{s}}. \end{aligned}$$

Now consider g'^*. The following findings being equivalent to Manea (2011, Proposition 2, Theorem 3) are important:

- From Manea (2011, Proposition 2) we have that if $ij \in g$, then $v_i^*(g') + v_j^*(g') \geq 1$ and if $ij \in g'^*$, then $v_i^*(g') + v_j^*(g') = 1$.
- By Manea (2011, Theorem 3) for all $M \in \mathscr{I}(g'^*)$ the following bounds on limit equilibrium payoffs hold:

$$\begin{aligned} \min_{i \in M} v_i^*(g') &\leq \frac{|L^{g'^*}(M)|}{|M| + |L^{g'^*}(M)|} \\ \max_{j \in L^{g'^*}(M)} v_j^*(g') &\geq \frac{|M|}{|M| + |L^{g'^*}(M)|} \end{aligned}$$

If in Manea's (2011, Theorem 4) proof of the payoff computation rule (2.3) one replaces g^* by g'^*, v_i^* by $v_i^*(g')$, v_j^* by $v_j^*(g')$, v_k^* by $v_k^*(g')$ and Proposition 2, Lemma 1 and Theorem 3 (Manea, 2011) by the corresponding statements from above, then this leads to the result that also

$$\begin{aligned} v_i^*(g') &= x_s && \forall\, i \in M_s\ \forall\, s < \bar{s}, \\ v_j^*(g') &= 1 - x_s && \forall\, j \in L_s\ \forall\, s < \bar{s}, \\ v_k^*(g') &= \tfrac{1}{2} && \forall\, k \in N_{\bar{s}}. \end{aligned}$$

Thus, it is $v^*(g') = v^*(g)$ and hence

$$u_1^*(g') = v_1^*(g') - \eta_1(g')c = v_1^*(g) - (\eta_1(g) + 1)c < v_1^*(g) - \eta_1(g)c = u_1^*(g' - 12).$$

This is a contradiction to pairwise stability and proves that a pairwise stable network cannot contain a disagreement link.
Finally notice that for any network g it is $v_i^*(g) + v_j^*(g) = 1$ for all $ij \in g^*$ as we know from Manea (2011, Proposition 2). Based on the above result this implies $v_i^*(g) + v_j^*(g) = 1$ for all $ij \in g$ if g is pairwise stable. □

Proof of Theorem 2. Consider a pairwise stable network g and assume that $g|_{\tilde{N}(g)}$ is not a disjoint union of separated pairs and odd circles. Notice that due to Proposition 1 for any component $C \subseteq N$ in g it must either be $C \subseteq \tilde{N}(g)$ or $C \subseteq \tilde{N}(g)^{\complement}$. Furthermore and as we already know, the profile of the payoffs is component-decomposable such that it is $v_i^*(g) = v_i^*\big(g|_{\tilde{N}(g)}\big)$ for all $i \in \tilde{N}(g)$. Consequently, the network $g|_{\tilde{N}(g)}$ is equitable and by Manea (2011, Theorem 5), Berge (1981) it therefore has a so called edge cover formed by a disjoint union of separated pairs and odd circles. This means that there exists a disjoint union of separated pairs and odd circles $g' \subseteq g|_{\tilde{N}(g)}$ such that no player $i \in \tilde{N}(g)$ is isolated in g'. Now consider a link $ij \in g|_{\tilde{N}(g)} \backslash g'$, which must exist by assumption. Obviously, the network g' is also an edge cover of the network $g|_{\tilde{N}(g)} - ij$. Again from Manea (2011, Theorem 5), Berge (1981) it then follows that $g|_{\tilde{N}(g)} - ij$ is still equitable. Hence, recalling the implication of Proposition 1 mentioned above it is

$$\begin{aligned} u_i^*(g) = v_i^*\big(g|_{\tilde{N}(g)}\big) - \eta_i\big(g|_{\tilde{N}(g)}\big)c = \frac{1}{2} - \eta_i\big(g|_{\tilde{N}(g)}\big)c &< \frac{1}{2} - \big(\eta_i\big(g|_{\tilde{N}(g)}\big) - 1\big)c \\ &= v_i^*\big(g|_{\tilde{N}(g)} - ij\big) - \eta_i\big(g|_{\tilde{N}(g)} - ij\big)c \\ &= u_i^*(g - ij). \end{aligned}$$

This obviously constitutes a contradiction to g being pairwise stable. □

Lemma 1. *Let $\tilde{g}$ be a network with $\mathscr{A}(\tilde{g})$ providing $(\tilde{r}_s, \tilde{x}_s, \tilde{M}_s, \tilde{L}_s, \tilde{N}_s, \tilde{g}_s)_s$. For any step $s < \bar{s}$ and any set $I \subseteq N$ the following implications must apply:*

$$\begin{aligned} &(i) \quad 1 \le |\tilde{M}_s \cap I| \le |\tilde{L}_s \cap I| \quad \Rightarrow \quad |L^{\tilde{g}_s}(\tilde{L}_s \cap I) \cap \tilde{M}_s \cap I^{\complement}| \ge 1, \\ &(ii) \quad 1 \le |\tilde{M}_s \cap I| < |\tilde{L}_s \cap I| \quad \Rightarrow \quad |L^{\tilde{g}_s}(\tilde{L}_s \cap I) \cap \tilde{M}_s \cap I^{\complement}| \ge 2. \end{aligned}$$

Proof of Lemma 1. To see part (i) assume that we had $1 \le |\tilde{M}_s \cap I| \le |\tilde{L}_s \cap I|$ and $L^{\tilde{g}_s}(\tilde{L}_s \cap I) \cap \tilde{M}_s \cap I^{\complement} = \emptyset$ in some step $s < \bar{s}$ and for some set $I \subseteq N$. It obviously is

$$\frac{|\tilde{L}_s|}{|\tilde{M}_s|} < 1 \le \frac{|\tilde{L}_s \cap I|}{|\tilde{M}_s \cap I|}.$$

Additionally, we have that $\tilde{M}_s = (\tilde{M}_s \cap I) \,\dot{\cup}\, (\tilde{M}_s \backslash I)$ and $\tilde{L}_s = (\tilde{L}_s \cap I) \,\dot{\cup}\, (\tilde{L}_s \backslash I)$. This induces that $\tilde{M}_s \backslash I \neq \emptyset$ since it is $|\tilde{M}_s \cap I| \le |\tilde{L}_s \cap I| \le |\tilde{L}_s|$ but $|\tilde{M}_s| > |\tilde{L}_s|$. It follows that

$$\frac{|\tilde{L}_s \backslash I|}{|\tilde{M}_s \backslash I|} < \frac{|\tilde{L}_s|}{|\tilde{M}_s|}.$$

Moreover, it is $L^{\tilde{g}_s}(\tilde{M}_s \backslash I) \subseteq \tilde{L}_s \backslash I$ since by assumption $L^{\tilde{g}_s}(\tilde{L}_s \cap I) \cap \tilde{M}_s \subseteq I$. Taken together, this then gives

$$\frac{|L^{\tilde{g}_s}(\tilde{M}_s \backslash I)|}{|\tilde{M}_s \backslash I|} \le \frac{|\tilde{L}_s \backslash I|}{|\tilde{M}_s \backslash I|} < \frac{|\tilde{L}_s|}{|\tilde{M}_s|} = \tilde{r}_s,$$

which contradicts the minimality of $\tilde{r}_s$.

For part (ii) it remains to show that having $1 \leq |\tilde{M}_s \cap I| < |\tilde{L}_s \cap I|$ and $|L^{\tilde{g}_s}(\tilde{L}_s \cap I) \cap \tilde{M}_s \cap I^{\complement}| = 1$ in some step $s < \bar{s}$ and for some set $I \subseteq N$ leads to a contradiction as well. Somewhat different from part (i) we here have

$$\frac{|\tilde{L}_s|}{|\tilde{M}_s|} < 1 \leq \frac{|\tilde{L}_s \cap I|}{|\tilde{M}_s \cap I| + 1}.$$

Again, it holds that $\tilde{M}_s = (\tilde{M}_s \cap I) \,\dot{\cup}\, (\tilde{M}_s \backslash I)$ and $\tilde{L}_s = (\tilde{L}_s \cap I) \,\dot{\cup}\, (\tilde{L}_s \backslash I)$, which in this case even guarantees that $|\tilde{M}_s \backslash I| \geq 2$ since it is $|\tilde{M}_s \cap I| < |\tilde{L}_s \cap I| \leq |\tilde{L}_s|$, but $|\tilde{M}_s| > |\tilde{L}_s|$. This gives

$$\frac{|\tilde{L}_s \backslash I|}{|\tilde{M}_s \backslash I| - 1} < \frac{|\tilde{L}_s|}{|\tilde{M}_s|}.$$

Beyond that we have that there exists a unique player $\tilde{i} \in L^{\tilde{g}_s}(\tilde{L}_s \cap I) \cap \tilde{M}_s \cap I^{\complement}$. Similarly to part (i) this implies that it is $L^{\tilde{g}_s}(\tilde{M}_s \backslash (I \cup \tilde{i})) \subseteq \tilde{L}_s \backslash I$, which combined with the above leads to

$$\frac{|L^{\tilde{g}_s}(\tilde{M}_s \backslash (I \cup \tilde{i}))|}{|\tilde{M}_s \backslash (I \cup \tilde{i})|} \leq \frac{|\tilde{L}_s \backslash I|}{|\tilde{M}_s \backslash I| - 1} < \frac{|\tilde{L}_s|}{|\tilde{M}_s|} = \tilde{r}_s,$$

which again contradicts the minimality of $\tilde{r}_s$. □

Lemma 2. *In the framework with $\delta \to 1$ consider a pairwise stable network g with player set N for which the algorithm $\mathscr{A}(g)$ provides $(r_1, x_1, M_1, L_1, N_1, g_1)$ with $r_1 \in (0,1)$. Then for all $i, j \in M_1$ it is*

$$v_i^*(g + ij) = v_j^*(g + ij) = \frac{1}{2}.$$

Further, if the player set is extended by an isolated player $n+1$ while the network g contains the same links as before, it similarly is $v_i^*(g + i(n+1)) = v_{n+1}^*(g + i(n+1)) = \frac{1}{2}$.

Proof of Lemma 2. For $i, j \in M_1$ consider the network $g' = g + ij$. Let $(r'_s, x'_s, M'_s, L'_s, N'_s, g'_s)_{s=1,\ldots,\bar{s}'}$ be the outcome of $\mathscr{A}(g')$. Assume for contradiction that there exists a step $\hat{s} \in \{1, \ldots, \bar{s}' - 1\}$ such that $L_1 \cap M'_s = M_1 \cap L'_s = \emptyset$ for all $s \in \{1, \ldots, \hat{s} - 1\}$ but $M_1 \cap L'_{\hat{s}} \neq \emptyset$. Observe that $L_1 \cap M'_{\hat{s}} \neq \emptyset$ would also entail $M_1 \cap L'_{\hat{s}} \neq \emptyset$ since due to the minimality of $r'_{\hat{s}}$ any player $k \in L_1 \cap M'_{\hat{s}}$ needs to have at least one neighbor in the network $g'_{\hat{s}}$ who then must have been a neighbor in g as well. In the following, we will construct a sequence of players $(i_0, i_1, i_2, \ldots)$ and show by induction that the underlying procedure which sequentially adds players to it will never break up so that we get a contradiction to the finiteness of the player set N. For $m \in \mathbb{N}$ let $I_m = \{i_0, i_1, \ldots, i_m\} \subseteq N$ denote the players of the sequence up to the mth one. We need to distinguish two cases.

First consider the case that $i \in L'_{\hat{s}}$ and set $i_0 = i$. It then must be $|N_{i_0}(g'_{\hat{s}}) \cap M'_{\hat{s}}| \geq 2$ since otherwise one could reduce $r'_{\hat{s}}$ by not including i_0 and possibly her one contact belonging to $M'_{\hat{s}}$. This guarantees that there exists $i_1 \in N_{i_0}(g'_{\hat{s}}) \cap M'_{\hat{s}} \backslash \{j\}$. So it is $i_0 \in M_1 \cap L'_{\hat{s}}$ and $i_1 \in L_1 \cap M'_{\hat{s}}$. Let $I_1 = \{i_0, i_1\}$. Now consider some odd number $m \in \mathbb{N}$. Assume that $L_1 \cap I_m = M'_{\hat{s}} \cap I_m$, $M_1 \cap I_m = L'_{\hat{s}} \cap I_m$ and that the cardinalities of these two sets are equal. We then have:

- It is $1 \leq |M_1 \cap I_m| = |L_1 \cap I_m|$ and therefore by Lemma 1(i) there exists a player $i_{m+1} \in L^g(L_1 \cap I_m) \cap M_1 \cap I_m^{\complement}$. For this player it must hold that $i_{m+1} \in M_1 \cap L'_{\hat{s}} \backslash I_m$ since $L_1 \cap I_m \subseteq M'_{\hat{s}}$ and $M_1 \cap L'_s = \emptyset$ for all $s \in \{1,...,\hat{s}-1\}$.
- It then is $1 \leq |M'_{\hat{s}} \cap I_{m+1}| < |L'_{\hat{s}} \cap I_{m+1}|$ and therefore by Lemma 1(ii) there exists a player $i_{m+2} \in L^{g'_{\hat{s}}}(L'_{\hat{s}} \cap I_{m+1}) \cap M'_{\hat{s}} \cap I_{m+1}^{\complement} \backslash \{j\}$. For this player it must hold that $i_{m+2} \in L_1 \cap M'_{\hat{s}} \backslash I_{m+1}$ since $L'_{\hat{s}} \cap I_{m+1} \subseteq M_1$ and $i_{m+2} \neq j$.

Thus it is $L_1 \cap I_{m+2} = M'_{\hat{s}} \cap I_{m+2}$, $M_1 \cap I_{m+2} = L'_{\hat{s}} \cap I_{m+2}$ and also the cardinalities of these two sets are equal. Moreover, it is $|I_{m+2}| = |I_m| + 2$. By induction it follows as a contradiction that the player set N must be infinitely large.

Second we analyze the situation for $i \notin L'_{\hat{s}}$. This implies $j \notin M'_{\hat{s}}$ since by assumption $M_1 \cap L'_s = \emptyset$ for all $s \in \{1,...,\hat{s}-1\}$. For the same reason $i \in M'_{\hat{s}}$ would imply $j \in L'_{\hat{s}}$, a case which is equivalent to the first one. This holds likewise for $i \notin M'_{\hat{s}}$ and $j \in L'_{\hat{s}}$. So it remains to consider the case that $i, j \notin (M'_{\hat{s}} \cup L'_{\hat{s}})$.
However, by assumption there must be a player $i_0 \in M_1 \cap L'_{\hat{s}}$. As in the previous case, existence of another player $i_1 \in N_{i_0}(g'_{\hat{s}}) \cap M'_{\hat{s}}$ is then guaranteed and it must be $i_1 \notin \{i, j\}$ since $i, j \notin M'_{\hat{s}}$. Therefore it is $i_1 \in L_1 \cap M'_{\hat{s}}$. Let again $I_1 = \{i_0, i_1\}$ and assume for some odd number $m \in \mathbb{N}$ that $L_1 \cap I_m = M'_{\hat{s}} \cap I_m$, $M_1 \cap I_m = L'_{\hat{s}} \cap I_m$ and that the cardinalities of these two sets are equal. Furthermore, assume that $i, j \notin I_m$. Similarly to the first case we have:

- There exists $i_{m+1} \in M_1 \cap L'_{\hat{s}} \backslash I_m$ for the stated reasons.
- By Lemma 1(ii) there then exists a player $i_{m+2} \in L^{g'_{\hat{s}}}(L'_{\hat{s}} \cap I_{m+1}) \cap M'_{\hat{s}} \cap I_{m+1}^{\complement}$. For this player it must hold that $i_{m+2} \in L_1 \cap M'_{\hat{s}} \backslash I_{m+1}$ since $L'_{\hat{s}} \cap I_{m+1} \subseteq M_1 \backslash \{i, j\}$.

Thus it is again $L_1 \cap I_{m+2} = M'_{\hat{s}} \cap I_{m+2}$, $M_1 \cap I_{m+2} = L'_{\hat{s}} \cap I_{m+2}$ and also the cardinalities of these two sets are equal. Beyond that, we have $i, j \notin I_{m+2}$. By induction this leads again to a contradiction to the finiteness of the player set N.

Summing up, we have that $L_1 \cap M'_s = M_1 \cap L'_s = \emptyset$ for all $s < \bar{s}'$. Therefore, it must be $v_i^*(g'), v_j^*(g') \leq \frac{1}{2}$. On the other hand, we know by Manea (2011, Proposition 2) that $v_i^*(g') + v_j^*(g') \geq 1$. Taken together, this implies $v_i^*(g') = v_j^*(g') = \frac{1}{2}$.

With regard to the second part of the Lemma consider the network $g' = g + i(n+1)$ and let $(r'_s, x'_s, M'_s, L'_s, N'_s, g'_s)_{s=1,...,\bar{s}'}$ be the outcome of $\mathscr{A}(g')$. It is clear that $n+1 \notin L'_s$ for all $s < \bar{s}'$ since otherwise one could simply reduce r'_s by deleting $n+1$ from L'_s and possibly her one neighbor i from M'_s. The possibility that $i \in L'_s$ for some $s < \bar{s}'$ can be ruled out by a line of argumentation which is equivalent to the proof of the first part if one substitutes $n+1$ for j, M_2 for M_1 and L_2 for L_1 (while taking into account that $\mathscr{A}(g)$ provides $M_1 = \{n+1\}$ and $L_1 = \emptyset$ in this case). □

Lemma 3. *In the model with $\delta \to 1$ let g be a pairwise stable network consisting of one single component such that $\mathscr{A}(g)$ provides $(r_1, x_1, M_1, L_1, N_1, g_1)$ with $r_1 < 1$. It then is*

$$v_j^*(g - kl) \geq \frac{1}{2} \geq v_i^*(g - kl)$$

for all $j \in L_1$, $i \in M_1$ and $kl \in g$.

Proof of Lemma 3. Beside g consider the network $g' := g - kl$ and let $(r'_s, x'_s, M'_s, L'_s, N'_s, g'_s)_{s=1,...,\bar{s}'}$ be the outcome of $\mathscr{A}(g')$. Similarly to the proof of Lemma 2 assume for contradiction that there exists a step $\hat{s} \in \{1, ..., \bar{s}' - 1\}$ such that $L_1 \cap M'_s = M_1 \cap L'_s = \emptyset$ for all $s \in \{1, ..., \hat{s} - 1\}$, but $L_1 \cap M'_{\hat{s}} \neq \emptyset$. Observe that $M_1 \cap L'_{\hat{s}} \neq \emptyset$ would also entail $L_1 \cap M'_{\hat{s}} \neq \emptyset$ since due to the minimality of $r'_{\hat{s}}$ any player in $M_1 \cap L'_{\hat{s}}$ needs to have a g'-neighbor in $M'_{\hat{s}}$ who then must have been a neighbor in g as well. We will again construct a sequence of players $(i_0, i_1, i_2, ...)$ and show by induction that the underlying procedure which sequentially adds players to it will never break up so that we get a contradiction to the finiteness of the player set N. For $m \in \mathbb{N}$ let $I_m = \{i_0, i_1, ..., i_m\} \subseteq N$ denote the players of the sequence up to the mth one.

Initially, select some player $i_0 \in L_1 \cap M'_{\hat{s}}$. i_0 cannot be isolated or a loose-end player in g since otherwise one could reduce r_1 by not including i_0 in L_1 and possibly her one contact in M_1. This guarantees that there exists $i_1 \in N_{i_0}(g')$. It must be $i_1 \in M_1 \cap L'_{\hat{s}}$ since by assumption $M_1 \cap L'_s = \emptyset$ for all $s \in \{1, ..., \hat{s} - 1\}$. Let $I_1 = \{i_0, i_1\}$. Now consider some odd number $m \in \mathbb{N}$. Assume that $L_1 \cap I_m = M'_{\hat{s}} \cap I_m$, $M_1 \cap I_m = L'_{\hat{s}} \cap I_m$ and that the cardinalities of these two sets are equal. We then have:

- It is $1 \leq |M'_{\hat{s}} \cap I_m| = |L'_{\hat{s}} \cap I_m|$ and therefore by Lemma 1(i) there exists a player $i_{m+1} \in L^{g'_{\hat{s}}}(L'_{\hat{s}} \cap I_m) \cap M'_{\hat{s}} \cap I_m^{\complement}$. For this player it must hold that $i_{m+1} \in L_1 \cap M'_{\hat{s}} \backslash I_m$ since it is $L'_{\hat{s}} \cap I_m \subseteq M_1$.
- It then is $1 \leq |M_1 \cap I_{m+1}| < |L_1 \cap I_{m+1}|$ and therefore by Lemma 1(ii) there exists a player $i_{m+2} \in L^g(L_1 \cap I_{m+1}) \cap M_1 \cap I_{m+1}^{\complement} \cap L^{g'_{\hat{s}}}(L_1 \cap I_{m+1})$ since g' arose from g by a single link deletion and additionally $M_1 \cap L'_s = \emptyset$ for all $s \in \{1, ..., \hat{s} - 1\}$ and $L_1 \cap I_{m+1} \subseteq M'_{\hat{s}}$. These arguments then also imply that $i_{m+2} \in M_1 \cap L'_{\hat{s}} \backslash I_{m+1}$.

Thus it is $L_1 \cap I_{m+2} = M'_{\hat{s}} \cap I_{m+2}$, $M_1 \cap I_{m+2} = L'_{\hat{s}} \cap I_{m+2}$ and also the cardinalities of these two sets are equal. Moreover, it is $|I_{m+2}| = |I_m| + 2$. By induction this here leads as well to a contradiction to the finiteness of the player set N. Consequently, it must be $L_1 \cap M'_s = M_1 \cap L'_s = \emptyset$ for all $s < \bar{s}'$. □

Proof of Corollary 3. For a pairwise stable network g according to Theorem 3 and Proposition 1 it must be $v_j^*(g) \in \left\{0, \frac{1}{2} - c, \frac{1}{2}, \frac{1}{2} + c\right\}$ for all $j \in N$. Assume for contradiction that there exist players $k, i \in N$ such that $v_k^*(g) = 0 \neq \frac{1}{2} - c = v_i^*(g)$. Notice that $c > \frac{1}{2}$ would imply $v_i^*(g) < 0$ and $c = \frac{1}{2}$ would require to have a player with a payoff of $\frac{1}{2} + c = 1$ to generate a contradiction, which is both generally impossible. Therefore we need to have $c < \frac{1}{2}$. Consider the set $N' = \{ j \in N \mid |v_j^*(g) - \frac{1}{2}| = c \}$. Notice that there cannot be a link from a player in N' to a player in $N \backslash N'$ and hence $g|_{N'}$ must be pairwise stable as well. Then Lemma 2 gives that $v_k^*(g|_{N' \dot\cup \{k\}} + ik) = \frac{1}{2}$. Since the profile of profits is component-decomposable this leads to

$$u_k^*(g) = 0 < \frac{1}{2} - c = u_k^*(g|_{N' \dot\cup \{k\}} + ik) = u_k^*(g + ik),$$

which is a contradiction to g being pairwise stable. So it must indeed either be $v_j^*(g) \in \{ \frac{1}{2} - c, \frac{1}{2}, \frac{1}{2} + c \}$ or $v_j^*(g) \in \{ 0, \frac{1}{2} \}$ for all $i \in N$.
Let now g be pairwise stable at more than only a single cost level. For all $j \in N$ by Theorem 3 it then must be $v_j^*(g) \in \{ 0, \frac{1}{2} \}$ and moreover by Theorem 2 $u_j^*(g) \in \{ 0, \frac{1}{2} - 2c, \frac{1}{2} - c \}$. Furthermore, Corollary 1(iv) implies that $u_j^*(g) \in \{ 0, \frac{1}{2} - c \}$ for $c > \frac{1}{6}$ while Corollary 1(i) and (v) imply that $u_j^*(g) \in \{ \frac{1}{2} - 2c, \frac{1}{2} - c \}$ for $c \leq \frac{1}{6}$. □

Proof of Proposition 2. Consider a network g which is a tree with $n > 3$ players and assume that it is pairwise stable. It is permissible to assume w.l.o.g. that g consists of only one component since the utility function is component-decomposable. Based on Theorem 2 we can immediately rule out the possibility that all players receive a payoff of $\frac{1}{2}$ in g. Therefore and in line with Proposition 1 and Remark 1 the algorithm $\mathscr{A}(g)$ has to stop after the first step providing an outcome $(r_1, x_1, M_1, L_1, N_1, g_1)$ with $M_1 \dot\cup L_1 = N$, $|M_1| > |L_1|$ and $g|_{M_1} = g|_{L_1} = \emptyset$. So it is $r_1 \in (0,1)$ and $v_i^*(g) = 1 - v_j^*(g) = x_1 \in (0, \frac{1}{2})$ for all $i \in M_1,\ j \in L_1$. Theorem 3 then implies that it is

$$x_1 + c = \frac{1}{2}. \tag{A.1}$$

We divide the tree networks considered here into the following three classes. Notice that a player $i \in N$ is called loose-end player if it is $\eta_i(g) = 1$, that is if she has exactly one link.

(a) No player has more than two links in g (line networks).

(b) There is a player who has at least three neighbors in g including at least two loose-end players.

(c) There is a player who has more than two links in g but no player has more than one loose-end contact.

In the following, we will distinguish between these three cases and show separately that there arises a contradiction to pairwise stability.

Class (a):
W.l.o.g. let $g = \{12, 23, ..., (n-1)n\}$. Here n must be odd since otherwise it would obviously be $\frac{|L^g(M)|}{|M|} \geq 1$ for all g-independent sets $M \subseteq N$ inducing a payoff of $\frac{1}{2}$ for every player. So by assumption it must be $n \geq 5$. Considering the algorithm $\mathscr{A}(g)$, we find that the shortage ratio is minimized by the g-independent set which contains the players $1, 3, ..., n-2, n$. Therefore, it is $r_1 = \frac{n-1}{n+1}$ and $x = \frac{n-1}{2n}$. Hence, here condition (A.1) is equivalent to

$$c = \frac{1}{2n} \tag{A.2}$$

Now, if player 3 deletes her link to player 2, then she becomes a loose-end player in a component which is again a line with an odd number of $n-2$ players in the network $g-23$. Hence, it is $v_3^*(g-23) = \frac{n-3}{2(n-2)}$. Taking equation (A.2) into account the corresponding stability condition becomes

$$\begin{aligned} u_3^*(g) - u_3^*(g-23) \geq 0 \quad &\Leftrightarrow \quad v_3^*(g) - v_3^*(g-23) - c \geq 0 \\ &\Leftrightarrow \quad \frac{n-1}{2n} - \frac{n-3}{2(n-2)} - \frac{1}{2n} \geq 0 \\ &\Leftrightarrow \quad \frac{4-n}{2n(n-2)} \geq 0. \end{aligned}$$

Obviously, this is never fulfilled for $n \geq 5$ meaning that a line network cannot be pairwise stable.

Class (b):
Let $k \in N$ be a player with at least three neighbors including two or more loose-end players. Then Manea (2011, Theorem 3) implies that it is $v_k^*(g) \geq \frac{2}{3}$. So it must be $k \in L_1$. Select a player $i \in N_k(g)$ such that $\eta_i(g) \geq \eta_{i'}(g)$ for all $i' \in N_k(g)$. Notice that in the network $g - ki$ player k still has at least two loose-end contacts such that again according to Manea (2011, Theorem 3) it is $v_k^*(g-ki) \geq \frac{2}{3}$. The corresponding stability condition then gives

$$u_k^*(g) \geq u_k^*(g-ki) \quad \Leftrightarrow \quad v_k^*(g) - c \geq v_k^*(g-ki) \quad \Rightarrow \quad 1 - x_1 - c \geq \frac{2}{3} \quad \Leftrightarrow \quad x_1 + c \leq \frac{1}{3}.$$

This obviously contradicts equation (A.1). So a network g of class (b) cannot be pairwise stable.

Class (c):
For this purpose, first deliberate the following: For any tree network $\tilde{g}$ and any player $k \in N$ there exists a unique partition $\left(Br_\nu^k\right)_{\nu \in N_k(g)}$ of $N \backslash \{k\}$ such that for all $\nu \in N_k(g)$ it is $\nu \in Br_\nu^k$ and $g|_{Br_\nu^k}$ is connected, i.e. $g|_{Br_\nu^k}$ consists of a single component. Based on this observation, we define the networks $\left(g|_{Br_\nu^k}\right)_{\nu \in N_k(g)}$ as the **branches of player** k **in** g and $\nu \in N_k(g)$ will be called the **fork player of** $g|_{Br_\nu^k}$.

Notice that if g belongs to class (c), then there exists a player $k \in N$ who has three or more links such that for at least all but one of her branches, all players contained in these have at most two

links in g. If this would not be the case, the following procedure would never stop, meaning that there would have to be infinitely many players in N: Initially, select a player k_0 having more than two links and one of her branches containing another player k_1 with more than two links. Then by assumption player k_1 must have a branch in g which does not contain player k_0 but a player k_2 who also has more than two links. For this player k_2 there must again be a branch in g not containing k_0 and k_1 but a player k_3 having more than two links. Continuing this way, for any $m \in \mathbb{N}$ there is a player $k_{m+1} \in N \backslash \{k_0, ..., k_m\}$, which then gives a contradiction by induction. Consequently, a player k as mentioned above must indeed exist.
In the following we will distinguish between two subcases, namely whether it is $k \in M_1$ or $k \in L_1$.

First consider the case where $k \in L_1$. If there are one or more other players having more than two links, let $i \in N$ be the fork player of player k's branch which contains all of them. Otherwise, arbitrarily pick some $i \in N_k(g)$. In both cases consider the component $C \subset N$ to which player k belongs in the network $g - ki$. In the network $g|_C$, there is only player k who might have more than two links. Furthermore, every branch of player k in $g|_C$ must be a line of odd length since Manea (2011, Theorem 3) implies that any loose-end player in g belongs to M_1 in the case at hand. This in turn implies that for any $g|_C$-independent set M with $\frac{|L^{g|_C}(M)|}{|M|} < 1$ it is $k \in L^{g|_C}(M)$. One example for such a set is $M_1 \cap C$ with partner set $L_1 \cap C$. Hence, it must be $v_k^*(g - ki) > \frac{1}{2}$. The corresponding stability condition then gives

$$u_k^*(g) \geq u_k^*(g - ki) \quad \Leftrightarrow \quad v_k^*(g) - c \geq v_k^*(g - ki) \quad \Rightarrow \quad 1 - x_1 - c > \frac{1}{2} \quad \Leftrightarrow \quad x_1 + c < \frac{1}{2}.$$

This obviously again contradicts equation (A.1). Consequently, a network g of class (c) with $k \in L_1$ cannot be pairwise stable.

This means that solely the subcase $k \in M_1$ is remaining. For this purpose we need to introduce some additional notation. Identify a branch of player k which is a line network with minimal length among all of these line branches. The set of the players contained in this branch will be denoted $C^1 \subset N$. Notice that any branch of player k which is a line network is of even length. Let $\hat{M}^1 = M_1 \cap C^1$ and $\hat{L}^1 = L_1 \cap C^1$. Then it is $|\hat{M}^1| = |\hat{L}^1|$. Let j denote the fork player of this branch. In addition, let $C^2 \subset N$ denote the set of all players contained in the other line branch(es) of player k. Let similarly $\hat{M}^2 = M_1 \cap C^2$ and $\hat{L}^2 = L_1 \cap C^2$. Then $|\hat{M}^2| = |\hat{L}^2| \geq |\hat{M}^1|$. Finally, let $C^3 = N \backslash (C^1 \dot{\cup} C^2 \dot{\cup} \{k\})$ and $\hat{M}^3 := M_1 \cap C^3$, $\hat{L}^3 := L_1 \cap C^3$. Then it must be $|\hat{M}^3| \geq |\hat{L}^3|$ since it was $|M_1| > |L_1|$.
We must have $r_1 = \frac{|L_1|}{|M_1|} \leq \frac{|\hat{L}^3|}{|\hat{M}^3|}$ as relationship between these two shortage ratios since r_1 is minimal for g and obviously $L^g(\hat{M}^3) = \hat{L}^3$. Hence, with this notation, it is

$$x_1 = \frac{|L_1|}{|M_1| + |L_1|} = \frac{|\hat{M}^1| + |\hat{M}^2| + |\hat{L}^3|}{2|\hat{M}^1| + 2|\hat{M}^2| + |\hat{M}^3| + |\hat{L}^3| + 1}.$$

Now consider the network $g' := g - kj$. Let $(r'_s, x'_s, M'_s, L'_s, N'_s, g'_s)_{s=1,...,\bar{s}'}$ be the outcome of the algorithm $\mathscr{A}(g')$. Notice first that the set $\hat{M}^2 \dot\cup \hat{M}^3 \dot\cup \{k\} \subset M_1$ is g'-independent and $\hat{L}^2 \dot\cup \hat{L}^3$ is the corresponding partner set in g'. Furthermore, it is

$$\frac{|\hat{L}^2|+|\hat{L}^3|}{|\hat{M}^2|+|\hat{M}^3|+1} = \frac{|\hat{M}^2|+|\hat{L}^3|}{|\hat{M}^2|+|\hat{M}^3|+1} < 1.$$

Now assume for contradiction that there is another g'-independent set $M' \subseteq N$ with partner set $L' = L^{g'}(M') \subseteq N$ which is shortage ratio minimizing in step $s = 1$ of $\mathscr{A}(g')$. Since the set C^1 is a component in g' and $g|_{C^1}$ is a line network of even length where every player receives a payoff of $\frac{1}{2}$, it is $(M' \dot\cup L') \cap C^1 = \emptyset$ and $\bar{s}' \geq 2$. Moreover, according to Lemma 3 it is $M_1 \cap L'_s = L_1 \cap M'_s = \emptyset$ for all $s < \bar{s}'$. Hence, it must be $M' \subset \hat{M}^2 \dot\cup \hat{M}^3 \dot\cup \{k\}$ and $L' \subset \hat{L}^2 \dot\cup \hat{L}^3$ and therefore

$$\frac{|L'|}{|M'|} < \frac{|\hat{M}^2|+|\hat{L}^3|}{|\hat{M}^2|+|\hat{M}^3|+1} < 1.$$

On the other hand, $M' \dot\cup \hat{M}^1 \subset M_1$ is also g-independent and $L^g(M' \dot\cup \hat{M}^1) = L' \dot\cup \hat{L}^1$. The minimality of $r_1 = \frac{|L_1|}{|M_1|}$ in $\mathscr{A}(g)$ then implies

$$r_1 = \frac{|\hat{M}^2|+|\hat{L}^3|+|\hat{M}^1|}{|\hat{M}^2|+|\hat{M}^3|+1+|\hat{M}^1|} \leq \frac{|L'|+|\hat{M}^1|}{|M'|+|\hat{M}^1|} < 1 \quad \Rightarrow \quad \frac{|\hat{M}^2|+|\hat{L}^3|}{|\hat{M}^2|+|\hat{M}^3|+1} \leq \frac{|L'|}{|M'|},$$

which is obviously a contradiction. Thus it is

$$v_k^*(g') = \frac{|\hat{M}^2|+|\hat{L}^3|}{2|\hat{M}^2|+|\hat{M}^3|+|\hat{L}^3|+1}.$$

Taking again equation (A.2) into account the corresponding stability condition becomes

$$\begin{aligned} u_k^*(g) \geq u_k^*(g-kj) \quad &\Leftrightarrow \quad v_k^*(g) - \eta_k(g)c \geq v_k^*(g-kj) - \eta_k(g-kj)c \\ &\Leftrightarrow \quad x_1 \geq v_k^*(g-kj) + \frac{1}{2} - x_1 \\ &\Leftrightarrow \quad 2x_1 - v_k^*(g-kj) \geq \frac{1}{2} \end{aligned} \tag{A.3}$$

However, as we will now finally establish, it always is $2x_1 - v_k^*(g') < \frac{1}{2}$. Some calculations yield

$$\begin{aligned} 2x_1 - v_k^*(g-kj) = &\frac{2|\hat{M}^1|+2(|\hat{M}^2|+|\hat{L}^3|)}{2|\hat{M}^1|+(2|\hat{M}^2|+|\hat{M}^3|+|\hat{L}^3|+1)} - \frac{(|\hat{M}^2|+|\hat{L}^3|)}{(2|\hat{M}^2|+|\hat{M}^3|+|\hat{L}^3|+1)} \\ = &\frac{2|\hat{M}^1|(|\hat{M}^2|+|\hat{M}^3|+1)+(|\hat{M}^2|+|\hat{L}^3|)(2|\hat{M}^2|+|\hat{M}^3|+|\hat{L}^3|+1)}{2|\hat{M}^1|(2|\hat{M}^2|+|\hat{M}^3|+|\hat{L}^3|+1)+(2|\hat{M}^2|+|\hat{M}^3|+|\hat{L}^3|+1)^2} \\ = &\frac{D-R}{2D}, \end{aligned}$$

where

$$D = 2|\hat{M}^1|(2|\hat{M}^2|+|\hat{M}^3|+|\hat{L}^3|+1)+(2|\hat{M}^2|+|\hat{M}^3|+|\hat{L}^3|+1)^2 > 0$$

is the denominator of the preceded line and

$$
\begin{aligned}
R &= -2|\hat{M}^1||\hat{M}^3| + 2|\hat{M}^1||\hat{L}^3| - 2|\hat{M}^1| + 2|\hat{M}^2||\hat{M}^3| - 2|\hat{M}^2||\hat{L}^3| + 2|\hat{M}^2| + |\hat{M}^3|^2 + 2|\hat{M}^3| - |\hat{L}^3|^2 + 1 \\
&= 2(\underbrace{|\hat{M}^2| - |\hat{M}^1|}_{\geq 0}) + 2(\underbrace{|\hat{M}^3| - |\hat{L}^3|}_{\geq 0})(\underbrace{|\hat{M}^2| - |\hat{M}^1|}_{\geq 0}) + (\underbrace{|\hat{M}^3|^2 - |\hat{L}^3|^2}_{\geq 0}) + 2|\hat{M}^3| + 1 \\
&\geq 2|\hat{M}^3| + 1 \\
&> 0.
\end{aligned}
$$

Hence, it is indeed $2x - v_k^*(g - kj) = \frac{D-R}{2D} < \frac{1}{2}$. This concludes the proof of case (c) and the whole Proposition. □

Proof of Proposition 3. Assume for contradiction that g is pairwise stable and w.l.o.g. that it consists of only one component. Since there is player k receiving a payoff $v_k^*(g) > \frac{1}{2}$, the algorithm $\mathscr{A}(g)$ must provide $(r_1, x_1, M_1, L_1, N_1, g_1)$ with $r_1 = \frac{|L_1|}{|M_1|} < 1$ and $M_1 \cup L_1 = N$ and of course $k \in L_1$. From Theorem 3 we already know that $x_1 + c = \frac{1}{2}$ has to be satisfied for g to be pairwise stable. We will prove that player k can delete a certain link such that in the resulting network she will still receive a payoff greater than $\frac{1}{2}$. As we will see, this constitutes a contradiction.

To start with, based on the assumptions of the Proposition there must be a set $K \subset N$ such that $k \in K$ fulfilling

- $L^g(K \backslash \{k\}) = K$ and $L^g(K^{\complement}) = K^{\complement} \cup \{k\}$,
- k is contained in a cycle in $g|_{K^{\complement} \cup \{k\}}$,
- $g|_{K^{\complement}}$ consists of one component.

Since it is $k \in L^g(K \backslash \{k\})$, it must be $N_k(g) \cap K \neq \emptyset$. Also, there exists $i' \in N_k(g) \backslash K$ such that k and i' belong to the same cycle in g. Now consider the network $g' := g - ki'$ and let $(r'_s, x'_s, M'_s, L'_s, N'_s, g'_s)_{s=1,\ldots,\bar{s}'}$ be the outcome of $\mathscr{A}(g')$. Lemma 3 guarantees that it is $v_k^*(g') \geq \frac{1}{2}$. So assume that we had $v_k^*(g') = \frac{1}{2}$ meaning that $k \in N'_{\bar{s}'}$.

As a first step, we establish that given this assumption it is

$$\frac{|L_1 \cap C_k(g|_{N'_{\bar{s}'} \cap K})|}{|M_1 \cap C_k(g|_{N'_{\bar{s}'} \cap K})|} = 1, \tag{A.4}$$

where $C_k(g|_{N'_{\bar{s}'} \cap K})$ denotes the component to which player k belongs in the network g restricted to the set $N'_{\bar{s}'} \cap K$. Notice first that it is $N_k(g'|_K) \neq \emptyset$ and it must be $N_k(g') \subseteq M_1 \cap N'_{\bar{s}'}$ since according to Lemma 3 it is $M_1 \cap L'_s = \emptyset$ for all $s < \bar{s}'$. This guarantees $M_1 \cap C_k(g|_{N'_{\bar{s}'} \cap K}) \neq \emptyset$. Based on this, we can immediately rule out the possibility that the left-hand side of (A.4) is strictly smaller than 1 since $M_1 \cap C_k(g|_{N'_{\bar{s}'} \cap K})$ is g'-independent and clearly $L^{g'_{\bar{s}'}}(M_1 \cap C_k(g|_{N'_{\bar{s}'} \cap K})) \subseteq L_1 \cap C_k(g|_{N'_{\bar{s}'} \cap K})$.

To construct a contradiction assume that the left-hand side of (A.4) is strictly greater than 1. We make use of the following implication, which we will verify at the end of the proof:

$$|\hat{L}| = |\hat{M}| \geq 1 \text{ for } \hat{L} \subseteq L_1 \cap C_k(g|_{N'_{\tilde{s}'} \cap K}) \backslash \{k\},\ N_k(g) \cap K \subseteq \hat{M} \subseteq M_1 \cap C_k(g|_{N'_{\tilde{s}'} \cap K}) \quad \Rightarrow \quad L^{g'_{\tilde{s}'}}(\hat{L}) \backslash \hat{M} \neq \emptyset \tag{A.5}$$

We know that it is $\emptyset \neq N_k(g) \cap K \subseteq N'_{\tilde{s}'}$. Let $\hat{M}^0 = N_k(g) \cap K$ such that $\hat{M}^0 \subseteq M_1 \cap C_k(g|_{N'_{\tilde{s}'} \cap K})$. Hence, it must be $|L_1 \cap C_k(g|_{N'_{\tilde{s}'} \cap K}) \backslash \{k\}| \geq |\hat{M}^0|$ since otherwise we had

$$\frac{|L_1 \cap C_k(g|_{N'_{\tilde{s}'} \cap K})|}{|M_1 \cap C_k(g|_{N'_{\tilde{s}'} \cap K})|} \leq \frac{|L_1 \cap C_k(g|_{N'_{\tilde{s}'} \cap K})|}{|\hat{M}^0|} \leq 1,$$

contradicting our assumption. Select a set of players $\hat{L}^0 \subseteq L_1 \cap C_k(g|_{N'_{\tilde{s}'} \cap K}) \backslash \{k\}$ with $|\hat{L}^0| = |\hat{M}^0|$. Notice that $\hat{M}^0$ and $\hat{L}^0$ satisfy the conditions of implication A.5.
Based on this, we can construct a sequence of players $(j_1, j_2, j_3, ...)$ in a certain way such that according to the previous considerations, the underlying procedure which sequentially adds players to the sequence will never break up. As in the proofs of Lemma 2 and 3, this leads to a contradiction to the finiteness of the player set N. For this purpose, let $\hat{M}^m := \{j_l \ : \ 1 \leq l \leq m \text{ odd}\} \cup \hat{M}^0$ and $\hat{L}^m := \{j_l \ : \ 1 \leq l \leq m \text{ even}\} \cup \hat{L}^0$ for $m \in \mathbb{N}$. Now consider some even number $m \in \mathbb{N} \cup \{0\}$. Assume that $\hat{L}^m \subseteq L_1 \cap C_k(g|_{N'_{\tilde{s}'} \cap K}) \backslash \{k\}, N_k(g) \cap K \subseteq \hat{M}^m \subseteq M_1 \cap C_k(g|_{N'_{\tilde{s}'} \cap K})$ and $|\hat{L}^m| = |\hat{M}^m| \geq 1$. We then have:

- By implication (A.5) there exists $j_{m+1} \in L^{g'_{\tilde{s}'}}(\hat{L}^m) \backslash \hat{M}^m$. For this player it must hold that $j_{m+1} \in M_1 \cap C_k(g|_{N'_{\tilde{s}'} \cap K}) \backslash \hat{M}^m$ since $\hat{L}^m \subseteq L_1 \cap C_k(g|_{N'_{\tilde{s}'} \cap K}) \backslash \{k\}$.
- There then must exist $j_{m+2} \in L_1 \cap C_k(g|_{N'_{\tilde{s}'} \cap K}) \backslash (\hat{L}^{m+1} \dot\cup \{k\})$ since otherwise we would have

$$1 < \frac{|L_1 \cap C_k(g|_{N'_{\tilde{s}'} \cap K})|}{|M_1 \cap C_k(g|_{N'_{\tilde{s}'} \cap K})|} \leq \frac{|\hat{L}^{m+1} \dot\cup \{k\}|}{|\hat{M}^{m+1}|} = 1.$$

Thus it is $\hat{L}^{m+2} \subseteq L_1 \cap C_k(g|_{N'_{\tilde{s}'} \cap K}) \backslash \{k\}$, $N_k(g) \cap K \subseteq \hat{M}^{m+2} \subseteq M_1 \cap C_k(g|_{N'_{\tilde{s}'} \cap K})$ and $|\hat{L}^{m+2}| = |\hat{M}^{m+2}| = |\hat{L}^m| + 1 \geq 1$. By induction this leads to a contradiction to the finiteness of the player set N. This proves equation (A.4).

During the second step we will now use the previous result to construct a conclusive contradiction of similar kind arising from the assumption that it is $v_k^*(g') = \frac{1}{2}$. Additionally, we will need the following implication:

$$|\tilde{L}| = |\tilde{M}| \geq 1 \text{ for } \tilde{L} \subseteq L_1 \cap N'_{\tilde{s}'} \backslash K,\ \tilde{M} \subseteq M_1 \cap N'_{\tilde{s}'} \backslash K \quad \Rightarrow \quad L^{g'_{\tilde{s}'}}(\tilde{L}) \backslash (\tilde{M} \dot\cup K) \neq \emptyset \tag{A.6}$$

Its verification is as well postponed to the end of this proof. Moreover, notice that by definition it is $\frac{|L^{g'_{\tilde{s}'}}(\bar{M})|}{|\bar{M}|} \geq 1$ for all g'-independent sets $\bar{M} \subseteq N'_{\tilde{s}'}$. Based on this we can again construct a sequence of players $(i_1, i_2, i_3, ...)$ such that according to the previous considerations, the sequential adding of new players will never break up. Thus, we will again get a contradiction to the finiteness of the player set N. For this purpose, we define the sets $\tilde{M}^m = \{i_l \,:\, 1 \leq l \leq m \text{ odd}\}$ and $\tilde{L}^m = \{i_l \,:\, 1 \leq l \leq m \text{ even}\}$ for $m \in \mathbb{N}$.

Initially, select a player $i_1 \in M_1 \cap N'_{\bar{s}'} \backslash K$. Notice that such a player must exist since $k \in L_1 \cap N'_{\bar{s}'}$ is part of a cycle in $g|_{N \backslash K \cup \{k\}}$ and according to Lemma 3 $M_1 \cap L'_s = \emptyset$ for all $s < \bar{s}'$. Now consider some odd number $m \in \mathbb{N}$. Assume that $\tilde{M}^m \subseteq M_1 \cap N'_{\bar{s}'} \backslash K$, $\tilde{L}^m \subseteq L_1 \cap N'_{\bar{s}'} \backslash K$ and that $|\tilde{M}^m| = \frac{m+1}{2} > \frac{m-1}{2} = |\tilde{L}^m|$. We then have:

- $\tilde{M}^m \,\dot{\cup}\, (M_1 \cap C_k(g|_{N'_{\bar{s}'} \cap K})) \subseteq N'_{\bar{s}'}$ is g'-independent and

  $$\frac{|\tilde{L}^m \,\dot{\cup}\, (L_1 \cap C_k(g|_{N'_{\bar{s}'} \cap K}))|}{|\tilde{M}^m \,\dot{\cup}\, (M_1 \cap C_k(g|_{N'_{\bar{s}'} \cap K}))|} < 1$$

  since it is $|L^{g'_{\bar{s}'}}(M_1 \cap C_k(g|_{N'_{\bar{s}'} \cap K}))| \leq |L_1 \cap C_k(g|_{N'_{\bar{s}'} \cap K})| = |M_1 \cap C_k(g|_{N'_{\bar{s}'} \cap K})|$ as we know from (A.4). Therefore, there must exist a player $i_{m+1} \in L^{g'_{\bar{s}'}}(\tilde{M}^m) \backslash (\tilde{L}^m \,\dot{\cup}\, K)$. Since $\tilde{M}^m \subseteq M_1$, it is $i_{m+1} \in L_1 \cap N'_{\bar{s}'} \backslash (\tilde{L}^m \,\dot{\cup}\, K)$.

- We then have $|\tilde{L}^{m+1}| = |\tilde{M}^{m+1}| = \frac{m+1}{2} \geq 1$ and $\tilde{L}^{m+1} \subseteq L_1 \cap N'_{\bar{s}'} \backslash K$, $\tilde{M}^{m+1} \subseteq M_1 \cap N'_{\bar{s}'} \backslash K$. Therefore by implication (A.6) there exists $i_{m+2} \in L^{g'_{\bar{s}'}}(\tilde{L}^{m+1}) \backslash (\tilde{M}^{m+1} \,\dot{\cup}\, K)$. Since $\tilde{L}^{m+1} \subseteq L_1$, it is $i_{m+2} \in M_1 \cap N'_{\bar{s}'} \backslash (\tilde{M}^{m+1} \,\dot{\cup}\, K)$.

Thus, it is $\tilde{M}^{m+2} \subseteq M_1 \cap N'_{\bar{s}'} \backslash K$, $\tilde{L}^{m+2} \subseteq L_1 \cap N'_{\bar{s}'} \backslash K$ and $|\tilde{M}^{m+2}| = \frac{(m+2)+1}{2} > \frac{(m+2)-1}{2} = |\tilde{L}^{m+2}|$. By induction this leads again to a contradiction to the finiteness of the player set N. So we have proven that player k's payoff $v_k^*(g')$ must indeed be strictly greater than $\frac{1}{2}$. The corresponding stability condition then leads to

$$u_k^*(g) \geq u_k^*(g - ki') \quad \Leftrightarrow \quad v_k^*(g) - c \geq v_k^*(g') \quad \Rightarrow \quad 1 - x_1 - c > \frac{1}{2} \quad \Leftrightarrow \quad x_1 + c < \frac{1}{2},$$

which is a contradiction to Theorem 3. Therefore, the network g cannot be pairwise stable.

It remains to prove implications (A.5) and (A.6). We start with the first one. Given the two sets $\hat{L} \subseteq L_1 \cap C_k(g|_{N'_{\bar{s}'} \cap K}) \backslash \{k\}$ and $\hat{M} \subseteq M_1 \cap C_k(g|_{N'_{\bar{s}'} \cap K})$ with $N_k(g) \cap K \subseteq \hat{M}$ and $|\hat{L}| = |\hat{M}| \geq 1$ assume for contradiction that $L^{g'_{\bar{s}'}}(\hat{L}) \subseteq \hat{M}$. Notice that it must be $N_j(g'_{\bar{s}'}) = N_j(g)$ for all $j \in \hat{L}$ since it is $\hat{L} \subseteq L_1 \cap N'_{\bar{s}'} \backslash \{k\}$ and according to Lemma 3 $M_1 \cap L'_s = \emptyset$ for all $s < \bar{s}'$. So we have that $L^g(M_1 \cap K \backslash \hat{M}) \subseteq L_1 \cap K \backslash \hat{L}$. Moreover, since $N_k(g) \cap K \subseteq \hat{M}$, it even is $L^g(M_1 \cap K \backslash \hat{M}) \subseteq L_1 \cap K \backslash (\hat{L} \,\dot{\cup}\, \{k\})$.
Further, we will establish and make use of the following inequalities:

$$\frac{|L_1 \cap K| - 1}{|M_1 \cap K|} \leq r_1 \leq \frac{|L_1 \cap K|}{|M_1 \cap K|} \leq 1 \tag{A.7}$$

First notice that it is $L^g(M_1 \cap K) \subseteq L_1 \cap K$ and similarly $L^g(M_1 \backslash K) \subseteq L_1 \backslash K \dot\cup \{k\}$. So it must be $r_1 \leq \frac{|L_1 \cap K|}{|M_1 \cap K|}$ and $r_1 \leq \frac{|L_1 \backslash K|+1}{|M_1 \backslash K|}$ since otherwise one would get a contradiction to the minimality of r_1. Moreover, it is $r_1 = \frac{|L_1|}{|M_1|} < 1$, $M_1 = (M_1 \cap K) \dot\cup (M_1 \backslash K)$ and $L_1 = (L_1 \cap K) \dot\cup (L_1 \backslash K)$. Therefore it must be $\frac{|L_1 \cap K|-1}{|M_1 \cap K|} = \frac{|L_1|-(|L_1 \backslash K|+1)}{|M_1|-|M_1 \backslash K|} \leq r_1$. And finally, if it was $\frac{|L_1 \cap K|}{|M_1 \cap K|} > 1$, then this would mean that also $\frac{|L_1 \cap K|-1}{|M_1 \cap K|} \geq 1$, which is a contradiction to the above.
The third inequality in (A.7) implies that $M_1 \cap K \backslash \hat{M} \neq \emptyset$ since otherwise it would be $|M_1 \cap K| = |\hat{M}| = |\hat{L}| < |\hat{L} \dot\cup \{k\}| \leq |L_1 \cap K|$. Taken together, this leads to the following contradiction:

$$
\begin{aligned}
r_1 \leq \frac{|L^g(M_1 \cap K \backslash \hat{M})|}{|M_1 \cap K \backslash \hat{M}|} \leq \frac{|L_1 \cap K \backslash (\hat{L} \dot\cup \{k\})|}{|M_1 \cap K \backslash \hat{M}|} = \frac{|L_1 \cap K| - |\hat{L}| - 1}{|M_1 \cap K| - |\hat{M}|} &= \frac{|L_1 \cap K| - 1 - |\hat{L}|}{|M_1 \cap K| - |\hat{L}|} \\
&< \frac{|L_1 \cap K| - 1}{|M_1 \cap K|} \\
&\leq r_1,
\end{aligned}
$$

where the last two inequalities are due to (A.7) and the fact that $r_1 < 1$.

Regarding implication (A.6), we start from two sets $\tilde{L} \subseteq L_1 \cap N'_{\bar{s}'} \backslash K$ and $\tilde{M} \subseteq M_1 \cap N'_{\bar{s}'} \backslash K$ with $|\tilde{L}| = |\tilde{M}| \geq 1$ and assume that it was $L^{g'_{\bar{s}'}}(\tilde{L}) \subseteq \tilde{M}$. Notice that it must be $N_j(g'_{\bar{s}'}) = N_j(g)$ for all $j \in \tilde{L}$ since according to Lemma 3 it is $M_1 \cap L'_s = \emptyset$ for all $s < \bar{s}'$. Therefore, we have that $L^g(M_1 \backslash \tilde{M}) \subseteq L_1 \backslash \tilde{L}$. Also, it is clear that $M_1 \backslash \tilde{M} \neq \emptyset$ since otherwise it would be $|M_1| = |\tilde{M}| = |\tilde{L}| \leq |L_1|$ contradicting $r_1 < 1$. Summing up, this implies

$$
r_1 \leq \frac{|L^g(M_1 \backslash \tilde{M})|}{|M_1 \backslash \tilde{M}|} \leq \frac{|L_1 \backslash \tilde{L}|}{|M_1 \backslash \tilde{M}|} = \frac{|L_1| - |\tilde{L}|}{|M_1| - |\tilde{M}|} = \frac{|L_1| - |\tilde{L}|}{|M_1| - |\tilde{L}|} < \frac{|L_1|}{|M_1|} = r_1,
$$

which is obviously again a contradiction. So we have that $L^{g'_{\bar{s}'}}(\tilde{L}) \backslash (\tilde{M} \dot\cup K) \neq \emptyset$ since it is $L^{g'_{\bar{s}'}}(\tilde{L}) \subseteq K^{\complement}$. This concludes the proof. □

Proof of Example 2. There are several ways to easily check that g^N with $n \geq 4$ is not pairwise stable for any positive level of linking costs if players are infinitely patient. For any nonempty g^N-independent set M it is $|M| = 1$ and $|L^{g^N}(M)| = n-1$ and hence, according to the algorithm $\mathscr{A}(g^N)$ we have $v_i^*(g^N) = \frac{1}{2}$ for all $i \in N$. The situation does not change crucially if some link ij is deleted from g^N. In this case $\mathscr{A}(g^N - ij)$ provides $r_1 = \frac{n-2}{2} \geq 1$ and therefore it is again $v_i^*(g^N - ij) = \frac{1}{2}$ for all $i \in N$. Thus, it is

$$
u_i^*(g^N) = v_i^*(g^N) - \eta_i(g^N)c = \frac{1}{2} - (n-1)c < \frac{1}{2} - (n-2)c = u_i^*(g^N - ij).^{12}
$$

[12] An alternative way to verify this would be to apply Theorem 2 combined with Manea (2011, Theorem 5).

Now consider the case $\delta \in (0,1)$. As a starting point, we take the equation

$$v_{\bar{i}} = \left(1 - \frac{n-1}{n(n-1)}\right)\delta v_{\bar{i}} + \frac{n-1}{n(n-1)}(1 - \delta v_{\bar{i}})$$

for some fixed $\bar{i} \in N$. Solving this gives $v_{\bar{i}} = \frac{1}{(1-\delta)n+2\delta}$. Obviously, it is $v_{\bar{i}} \in \left(0, \frac{1}{2}\right)$, which implies $1 - 2\delta v_{\bar{i}} > 0$. Moreover, it is $d^{\#}(g^N) = \frac{n(n-1)}{2}$ and all players are in symmetric positions within the network g^N. This shortcut avoiding extensive calculations establishes that the n-tuple $(v_{\bar{i}}, v_{\bar{i}}, ..., v_{\bar{i}})$ solves the equation system (2.1). Therefore, it is for all $i \in N$

$$v_i^{*\delta}(g^N) = \frac{1}{(1-\delta)n + 2\delta}. \tag{A.8}$$

After that, consider the network $g^N - ij$ for some $i, j \in N$. For this purpose, let $\tilde{v} = (\tilde{v_1}, \tilde{v_2}, ..., \tilde{v_n})$ be given by

$$\begin{aligned}
\tilde{v_i} = \tilde{v_j} &= \frac{(1-\delta)n^2 + (2\delta-1)n - (\delta+2)}{(\delta^2 - 2\delta + 1)n^3 + (-3\delta^2 + 3\delta)n^2 + (2\delta^2 + 3\delta - 3)n - (2\delta^2 + 2\delta + 2)}, \\
\tilde{v_k} &= \frac{(1-\delta)n^2 + \delta n - (2\delta+1)}{(\delta^2 - 2\delta + 1)n^3 + (-3\delta^2 + 3\delta)n^2 + (2\delta^2 + 3\delta - 3)n - (2\delta^2 + 2\delta + 2)},
\end{aligned} \tag{A.9}$$

where $k \in N \backslash \{i, j\}$. By showing that the denominator in the terms (A.9) is in both cases greater than the nominator and that both nominators are greater than 0 we show first that $\tilde{v} \in (0,1)^n$. It is for $\delta \in (0,1)$ and $n \geq 4$:

$$\begin{aligned}
&\big((\delta^2 - 2\delta + 1)n^3 + (-3\delta^2 + 3\delta)n^2 + (2\delta^2 + 3\delta - 3)n - (2\delta^2 + 2\delta + 2)\big) \\
&- \big((1-\delta)n^2 + (2\delta - 1)n - (\delta + 2)\big) \\
=&(1-\delta)^2 n^3 + (1-\delta)(3\delta - 1)n^2 + (2\delta^2 + \delta - 2)n - \delta(2\delta + 1) \\
=&(1-\delta)n\big[(1-\delta)n^2 + (3\delta - 1)n - (2\delta + 3)\big] + n - \delta(2\delta + 1) \\
>&(1-\delta)n[2n - (2\delta + 3)] + n - \delta(2\delta + 1) \\
>&(1-\delta)n[2n - 5] + (n - 3) \\
>&0
\end{aligned}$$

and

$$\begin{aligned}
&\big((\delta^2 - 2\delta + 1)n^3 + (-3\delta^2 + 3\delta)n^2 + (2\delta^2 + 3\delta - 3)n - (2\delta^2 + 2\delta + 2)\big) \\
&- \big((1-\delta)n^2 + \delta n - (2\delta + 1)\big) \\
=&(1-\delta)^2 n^3 + (1-\delta)(3\delta - 1)n^2 + (2\delta^2 + 2\delta - 3)n - (2\delta^2 + 1) \\
=&(1-\delta)n\big[(1-\delta)n^2 + (3\delta - 1)n - (2\delta + 4)\big] + n - (2\delta^2 + 1) \\
>&(1-\delta)n[2n - (2\delta + 4)] + n - (2\delta^2 + 1) \\
>&(1-\delta)n[2n - 6] + (n - 3)
\end{aligned}$$

$>0,$

and moreover

$$(1-\delta)n^2+(2\delta-1)n-(\delta+2)>n-(\delta+2)>n-3>0,$$
$$(1-\delta)n^2+\delta n-(2\delta+1)>n-(2\delta+1)>n-3>0.$$

Next, we show that it is $1-\delta\tilde{v_i}-\delta\tilde{v_k}>0$ and $1-2\delta\tilde{v_k}>0$, which implies $\max\{1-\delta\tilde{v_i},\delta\tilde{v_k}\}=1-\delta\tilde{v_i}$, $\max\{1-\delta\tilde{v_k},\delta\tilde{v_i}\}=1-\delta\tilde{v_k}$ and $\max\{1-\delta\tilde{v_k},\delta\tilde{v_k}\}=1-\delta\tilde{v_k}$. We calculate

$$\begin{aligned}
1-\delta\tilde{v_i}-\delta\tilde{v_k} &= \frac{(\delta^2-2\delta+1)n^3+(-\delta^2+\delta)n^2+(-\delta^2+4\delta-3)n+(\delta^2+\delta-2)}{(\delta^2-2\delta+1)n^3+(-3\delta^2+3\delta)n^2+(2\delta^2+3\delta-3)n-(2\delta^2+2\delta+2)}\\
&= \frac{(1-\delta)\left[(1-\delta)n^3+\delta n^2+(\delta-3)n-(\delta+2)\right]}{(\delta^2-2\delta+1)n^3+(-3\delta^2+3\delta)n^2+(2\delta^2+3\delta-3)n-(2\delta^2+2\delta+2)}\\
&> \frac{(1-\delta)[n^2+(\delta-3)n-(\delta+2)]}{(\delta^2-2\delta+1)n^3+(-3\delta^2+3\delta)n^2+(2\delta^2+3\delta-3)n-(2\delta^2+2\delta+2)}\\
&> \frac{(1-\delta)[\overbrace{n^2-3n-3}^{>0}]}{(\delta^2-2\delta+1)n^3+(-3\delta^2+3\delta)n^2+(2\delta^2+3\delta-3)n-(2\delta^2+2\delta+2)}\\
&> 0
\end{aligned}$$

and

$$\begin{aligned}
1-2\delta\tilde{v_k} &= \frac{(\delta^2-2\delta+1)n^3+(-\delta^2+\delta)n^2+(3\delta-3)n+(2\delta^2-2)}{(\delta^2-2\delta+1)n^3+(-3\delta^2+3\delta)n^2+(2\delta^2+3\delta-3)n-(2\delta^2+2\delta+2)}\\
&= \frac{(1-\delta)\left[(1-\delta)n^3+\delta n^2-3n-2(\delta+1)\right]}{(\delta^2-2\delta+1)n^3+(-3\delta^2+3\delta)n^2+(2\delta^2+3\delta-3)n-(2\delta^2+2\delta+2)}\\
&> \frac{(1-\delta)[n^2-3n-2(\delta+1)]}{(\delta^2-2\delta+1)n^3+(-3\delta^2+3\delta)n^2+(2\delta^2+3\delta-3)n-(2\delta^2+2\delta+2)}\\
&> \frac{(1-\delta)[\overbrace{n^2-3n-4}^{\geq 0}]}{(\delta^2-2\delta+1)n^3+(-3\delta^2+3\delta)n^2+(2\delta^2+3\delta-3)n-(2\delta^2+2\delta+2)}\\
&\geq 0
\end{aligned}$$

Furthermore, notice that it is $d^{\#}(g^N-ij)=\frac{n(n-1)-2}{2}$ and hence, for the network g^N-ij the equation system (2.1) is equivalent to

$$\begin{aligned}
v_l &= \left(1-\frac{n-2}{n(n-1)-2}\right)\delta v_l+\frac{n-2}{n(n-1)-2}\max\{1-\delta v_k,\delta v_l\},\\
v_k &= \left(1-\frac{n-1}{n(n-1)-2}\right)\delta v_k+\frac{2}{n(n-1)-2}\max\{1-\delta v_l,\delta v_k\}+\frac{n-3}{n(n-1)-2}\max\{1-\delta v_k,\delta v_k\},
\end{aligned} \tag{A.10}$$

$l \in \{i, j\}$, $k \in N \backslash \{i, j\}$. Using our preparatory work, one can show by straightforward calculations that $\tilde{v}$ as given in (A.9) solves the system (A.10) and hence for the equilibrium payoff vector it is $v^{*\delta}(g^N - ij) = \tilde{v}$.

After we have successfully calculated the payoffs in both networks g^N and $g^N - ij$, it remains to show that for all $\delta \in (0,1)$, $n \geq 4$, there exists $c > 0$ such that for all $i, j \in N$ it is

$$v_i^{*\delta}(g^N) - v_i^{*\delta}(g^N - ij) \geq c. \tag{A.11}$$

For this purpose let

$$c = \frac{2(1-\delta)(n-1)}{\big((1-\delta)n + 2\delta\big)\big((\delta^2 - 2\delta + 1)n^3 + (-3\delta^2 + 3\delta)n^2 + (2\delta^2 + 3\delta - 3)n - (2\delta^2 + 2\delta + 2)\big)},$$

where the denominator is the product of the denominators of the terms in (A.8) and (A.9) and therefore positive as well. Hence, it is $c > 0$ and we calculate

$$\begin{aligned}
& v_i^{*\delta}(g^N) - v_i^{*\delta}(g^N - ij) \\
=& \frac{1}{(1-\delta)n + 2\delta} - \frac{(1-\delta)n^2 + (2\delta - 1)n - (\delta + 2)}{(\delta^2 - 2\delta + 1)n^3 + (-3\delta^2 + 3\delta)n^2 + (2\delta^2 + 3\delta - 3)n - (2\delta^2 + 2\delta + 2)} \\
=& \frac{(\delta^2 - 2\delta + 1)n^2 + (-3\delta^2 + 4\delta - 1)n + (2\delta - 2)}{\big((1-\delta)n + 2\delta\big)\big((\delta^2 - 2\delta + 1)n^3 + (-3\delta^2 + 3\delta)n^2 + (2\delta^2 + 3\delta - 3)n - (2\delta^2 + 2\delta + 2)\big)} \\
=& \frac{(1-\delta)\big[(1-\delta)n^2 + (3\delta - 1)n - 2\big]}{\big((1-\delta)n + 2\delta\big)\big((\delta^2 - 2\delta + 1)n^3 + (-3\delta^2 + 3\delta)n^2 + (2\delta^2 + 3\delta - 3)n - (2\delta^2 + 2\delta + 2)\big)} \\
>& \frac{(1-\delta)[2n - 2]}{\big((1-\delta)n + 2\delta\big)\big((\delta^2 - 2\delta + 1)n^3 + (-3\delta^2 + 3\delta)n^2 + (2\delta^2 + 3\delta - 3)n - (2\delta^2 + 2\delta + 2)\big)} \\
=& c.
\end{aligned}$$

□

Proof of Theorem 4 (n odd). It remains to prove the statements for the case $n = |N|$ odd. Let g denote such a network. We adopt the notation we introduced for the case that n is even. Based on the player set N we consider again the corresponding set $N'(g) \subseteq N$ excluding isolated players and $g' = g|_{N'(g)}$ on this player set. Let $g^{SPL}_{N'(g)}$ denote a representative of the networks on $N'(g)$ consisting of $\frac{|N'(g)|-3}{2}$ separated pairs and one line of length three. Similarly, let $g^{SPC}_{N'(g)}$ be a network consisting of $\frac{|N'(g)|-3}{2}$ separated pairs and one three players circle. Since we did not use that $|N|$ was even to derive inequality (5.1), we again have

$$U^*(g) \leq \frac{1}{2}|N'(g)| - 2d^{\#}(g')c.$$

Again, since $\eta_i(g') \geq 1$ for all $i \in N'(g)$, it must be $d^{\#}(g') \geq \frac{1}{2}|N'(g)|$. We distinguish now between

$|N'(g)|$ even and odd.

In the former case, if it is $d^{\#}(g') = \frac{1}{2}|N'(g)|$, then this implies again $g' = g^{SP}_{N'(g)}$. So conversely, for a network g with $g' \neq g^{SP}_{N'(g)}$ this means that it is $d^{\#}(g') > \frac{1}{2}|N'(g)|$ and therefore according to (5.1)

$$U^*(g) < \frac{1}{2}|N'(g)| - |N'(g)|c = \sum_{i \in N'(g)} \left(\frac{1}{2} - c\right) = U^*(g^{SP}_{N'(g)}).$$

In the case that $|N'(g)|$ is odd, notice first that it must be $d^{\#}(g') \geq \frac{1}{2}(|N'(g)|+1)$ since the interval $\left[\frac{1}{2}|N'(g)|, \frac{1}{2}(|N'(g)|+1)\right)$ does not contain a natural number. Consider the following subcases:

- If it is $d^{\#}(g') = \frac{1}{2}(|N'(g)|+1)$, then it follows $g' = g^{SPL}_{N'(g)}$. That is because otherwise we would either have $\eta_j(g') \geq 3$ for at least one player $j \in N'(g)$ or $\eta_k(g'), \eta_l(g') \geq 2$ for $k \neq l \in N'(g)$, which both implies $d^{\#}(g') \geq \frac{1}{2}(4 + (|N'(g)|-2)) = \frac{1}{2}(|N'(g)|+2) > \frac{1}{2}(|N'(g)|+1)$.
- If it is $d^{\#}(g') = \frac{1}{2}(|N'(g)|+3)$, then g' must be of one of the following types:
  - A network with three players having two links each and $|N'(g)|-3$ players with one link,
  - a network consisting of one player with three links, one player with two links and $|N'(g)|-2$ players with one link or
  - a network with one player having four links and $|N'(g)|-1$ players with one link.

  Notice that the network $g^{SPC}_{N'(g)}$ is included here. On closer examination, one realizes that for any other $g' \neq g^{SPC}_{N'(g)}$ being of one of the above types, the algorithm $\mathscr{A}(g')$ yields $|M'_1| > |L'_1|$. This implies a strict inequality in (5.1). Hence, for g inducing $g' \neq g^{SPC}_{N'(g)}$ with $d^{\#}(g') = \frac{1}{2}(|N'(g)|+3)$ it is

  $$U^*(g) < \frac{1}{2}|N'(g)| - 2d^{\#}(g')c = \frac{1}{2}|N'(g)| - (|N'(g)|+3)c = |N'(g)|\left(\frac{1}{2} - c\right) - 3c = U^*(g^{SPC}_{N'(g)}).$$

- Finally, for g with $d^{\#}(g') > \frac{1}{2}(|N'(g)|+3)$ it is again according to (5.1)

  $$U^*(g) \leq \frac{1}{2}|N'(g)| - 2d^{\#}(g')c < \frac{1}{2}|N'(g)| - (|N'(g)|+3)c = |N'(g)|\left(\frac{1}{2} - c\right) - 3c = U^*(g^{SPC}_{N'(g)}).$$

Summarizing this, we have shown that a network g inducing $g' \notin \left\{g^{SP}_{N'(g)}, g^{SPL}_{N'(g)}, g^{SPC}_{N'(g)}\right\}$ cannot be efficient. To conclude the proof, we have to examine, which of the remaining candidates is efficient depending on the level of linking costs. Notice that the set $N'(g)$ always satisfies $0 \leq |N'(g)| \leq |N|$ and $|N'(g)| \neq 1$. Moreover, recall that $g^{SP}_{N'(g)}$ is only well-defined for $|N'(g)|$ even and $g^{SPL}_{N'(g)}, g^{SPC}_{N'(g)}$

only for $|N'(g)|$ odd. Hence, it is

$$\max_{N'(g)\ feasible} U^*(g^{SP}_{N'(g)}) = \max_{N'(g)\ feasible} |N'(g)|\Big(\frac{1}{2}-c\Big) = \begin{cases} 0, & \text{for} \quad c \geq \frac{1}{2} \\ (|N|-1)\big(\frac{1}{2}-c\big), & \text{for} \quad c \in \big(0,\frac{1}{2}\big) \end{cases},$$

$$\max_{N'(g)\ feasible} U^*(g^{SPL}_{N'(g)}) = \max_{N'(g)\ feasible} |N'(g)|\Big(\frac{1}{2}-c\Big) - \Big(c+\frac{1}{6}\Big) = \begin{cases} \frac{4}{3}-4c, & \text{for} \quad c \geq \frac{1}{2} \\ |N|\big(\frac{1}{2}-c\big)-\big(c+\frac{1}{6}\big), & \text{for} \quad c \in \big(0,\frac{1}{2}\big) \end{cases},$$

$$\max_{N'(g)\ feasible} U^*(g^{SPC}_{N'(g)}) = \max_{N'(g)\ feasible} |N'(g)|\Big(\frac{1}{2}-c\Big) - 3c = \begin{cases} \frac{3}{2}-6c, & \text{for} \quad c \geq \frac{1}{2} \\ |N|\big(\frac{1}{2}-c\big)-3c, & \text{for} \quad c \in \big(0,\frac{1}{2}\big) \end{cases}.$$

We find that for $c \geq \frac{1}{2}$ it is

$$\max_{N'(g)\ feasible} U^*(g^{SPL}_{N'(g)}),\ \max_{N'(g)\ feasible} U^*(g^{SPC}_{N'(g)}) < 0 = \max_{N'(g)\ feasible} U^*(g^{SP}_{N'(g)}),$$

so in this case a network g with $g' \in \big\{g^{SPL}_{N'(g)}, g^{SPC}_{N'(g)}\big\}$ cannot be efficient. For $c > \frac{1}{2}$ $N'(g) = \emptyset$ is the unique maximizer of $U^*(g^{SP}_{N'(g)})$ and therefore the empty network is uniquely efficient. For $c = \frac{1}{2}$ any feasible $N'(g)$ maximizes $U^*(g^{SP}_{N'(g)})$ meaning that a network g is efficient if and only if it is a disjoint union of a number of separated pairs and isolated players.

With regard to $c \in \big(0, \frac{1}{2}\big)$ we calculate

$$\max\Big\{-\Big(\frac{1}{2}-c\Big), -\Big(c+\frac{1}{6}\Big), -3c\Big\} = \begin{cases} -\big(\frac{1}{2}-c\big), & \text{for} \quad c \geq \frac{1}{6} \\ -\Big(c+\frac{1}{6}\Big), & \text{for} \quad c \in \big[\frac{1}{12},\frac{1}{6}\big] \\ -3c, & \text{for} \quad c \leq \frac{1}{12} \end{cases}$$

This means that for $n = |N|$ odd a network which is

- a disjoint union of $\frac{|N|-1}{2}$ separated pairs and one isolated player is efficient iff $c \in \big[\frac{1}{6}, \frac{1}{2}\big]$,
- a disjoint union of $\frac{|N|-3}{2}$ separated pairs and a line of length three is efficient iff $c \in \big[\frac{1}{12}, \frac{1}{6}\big]$,
- a disjoint union of $\frac{|N|-3}{2}$ separated pairs and a three player circle is efficient iff $c \in \big(0, \frac{1}{12}\big]$.

Since we have already ruled out any other network, this concludes the proof. □

References

Abreu, D. and Manea, M. (2012). Bargaining and efficiency in networks. *Journal of Economic Theory*, 147(1):43–70.

Bala, V. and Goyal, S. (2000). A noncooperative model of network formation. *Econometrica*, 68(5):1181–1229.

Berge, C. (1981). Some common properties for regularizable graphs, edge-critical graphs and b-graphs. In *Graph Theory and Algorithms*, pages 108–123. Springer.

Calvó-Armengol, A. (2003). Stable and efficient bargaining networks. *Review of Economic Design*, 7(4):411–428.

Condorelli, D. and Galeotti, A. (2012). Endogenous trading networks. *University of Essex Department of Economics Discussion Paper No. 705.*

Corominas-Bosch, M. (2004). Bargaining in a network of buyers and sellers. *Journal of Economic Theory*, 115(1):35–77.

Hellmann, T. (2013). On the existence and uniqueness of pairwise stable networks. *International Journal of Game Theory*, 42(1):211–237.

Hellmann, T. and Landwehr, J. (2014). Stable networks in homogeneous societies. *IMW Working Paper No. 517.*

Jackson, M. O. (2010). *Social and Economic Networks*. Princeton University Press.

Jackson, M. O. and Wolinsky, A. (1996). A strategic model of social and economic networks. *Journal of Economic Theory*, 71(1):44–74.

Kranton, R. and Minehart, D. F. (2001). A theory of buyer-seller networks. *American Economic Review*, 91(3):485–508.

Manea, M. (2011). Bargaining in stationary networks. *The American Economic Review*, 101(5):2042–2080.

Nash, J. F. (1950). The bargaining problem. *Econometrica*, 18(2):155–162.

Polanski, A. and Vega-Redondo, F. (2013). Markets, bargaining, and networks with heterogeneous agents. *Working Paper, available at SSRN 2218605.*

Rubinstein, A. (1982). Perfect equilibrium in a bargaining model. *Econometrica*, 50(1):97–109.

Rubinstein, A. and Wolinsky, A. (1985). Equilibrium in a market with sequential bargaining. *Econometrica*, 53(5):1133–1150.