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Abstract

We analyze a model of strategic network formation prior to a Manea (2011) bargaining game: ex-ante homogeneous players form costly undirected links, anticipating expected equilibrium payoffs from the subsequent network bargaining. Assuming patient players, we provide a complete characterization of generically pairwise stable networks: specific unions of separated pairs, odd circles, and isolated players constitute this class. We also show that many other structures, such as larger trees or unbalanced bipartite networks, cannot be pairwise stable at all. As an important implication, this reveals that the diversity of possible bargaining outcomes is substantially narrowed down, provided that the underlying network is (generically) pairwise stable.

Keywords: Bargaining, Network Formation, Noncooperative Games

JEL-Classification: C72, C78, D85

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1 Introduction

People often engage in bi- and multilateral bargaining: firms bargain with workers' unions over contracts, firms with other firms over prices or collaborations, politicians over environmental or trade agreements, or even friends and family members over household duties or other arrangements. However, in most of the situations that come to mind, not everyone will be able or willing to bargain with anyone else. This idea can be expressed by means of a network. One's bargaining power in negotiations then commonly depends on the number and types of alternative partners since they present outside options. Rational strategic agents typically intend to maximize their expected profit from bargaining, which suggests that they might want to influence and optimize their network of potential bargaining partners. In such situations, the underlying network should not be regarded as being exogenously given but as the outcome of strategic interaction among agents. However, forming a link to someone else usually costs some time and effort, which should be taken into account as well. This gives rise to a nontrivial trade-off between the costs of forming links and potential benefits from it. This consideration is the topic of this paper. Which networks can one expect to form at which level of linking costs? And which bargaining outcomes do these networks induce? These are the main research questions we address.

We set up and analyze a sequential model of strategic network formation prior to a Manea (2011) infinite-horizon network bargaining game. We consider ex ante homogeneous players who in the first stage strategically form undirected, costly links. In this context, one might think of one-time initiation or communication costs that players have to bear. In the second stage, we take the resulting network as given and players sequentially bargain with a neighbor for the division of a mutually generated unit surplus. According to Manea (2011), all subgame perfect equilibria of the bargaining game are payoff equivalent. Players are supposed to anticipate these outcomes during the preceding network formation game and to choose their actions accordingly. For the case in which players are infinitely patient, we examine their strategic behavior regarding network formation, characterize the stable network structures, and determine the induced bargaining outcomes.

After giving a description of the model, including a summary of the underlying Manea (2011) bargaining game and his decisive results, we consider the seminal concept of pairwise stability established by Jackson and Wolinsky (1996). As the first of two main results, we establish a complete characterization of generically pairwise stable networks. To this end, we first state and prove sufficient conditions for the structure of a network to be pairwise stable (Theorem 1). When costs are relatively high, these are

specific unions of separated pairs and isolated players. When costs decrease, odd circles of increasing size can additionally emerge. This result also establishes the existence of pairwise stable networks at each level of linking costs. For each combination of the above subnetworks, we specify for which cost range it is pairwise stable and for which it is not (Corollary 1). In considering the necessary conditions for pairwise stability, we differentiate between "equitable" networks and those in which at least two players in one component receive different payoffs. We provide a complete characterization of equitable pairwise stable networks by demonstrating that, in any such network, any nonisolated player has to be contained in a separated pair or in an odd circle (Theorem 2). Complementary to this, we show that any network that induces heterogeneous payoffs within a component can at be pairwise stable at no more than a single cost level (Theorem 3). This concludes the complete characterization of generically pairwise stable networks (Corollary 3).
As our second main result, we conclude that pairwise stability substantially narrows down the diversity of induced bargaining outcomes among players. However, though players are ex ante homogeneous, they do not have to be completely equal in this respect (Corollary 4). Finally, we reveal that networks containing a tree (with more than three players), a certain kind of "cut-player", or a certain class of bipartite subnetworks cannot even be nongenerically pairwise stable (Proposition 2).

For a concrete economic application captured by our model which might contribute to a better understanding of the framework, consider a number of similar firms that begin operation at the same time. This might be, for instance, in a new industry which just got established, thanks to a recent innovation. These firms can jointly generate an (additional) surplus within bilateral projects by exploiting synergy potentials. This possibility could arise from capacity constraints or cost-saving opportunities. However, since no prior cooperation network exists at this early stage, it will have to be created through a strategic interaction between the firms. Those taking charge of that interaction will be project managers who receive bonus payments proportional to their employer's profit from the project. Here, one-time initiation costs might arise to prepare each two firms for mutual projects (for instance, it might be necessary to adjust the IT or to organize joint trainings for workers, etc.). We assume that each project manager keeps her position only until she reaches an agreement with a counterpart. Such two project managers will then leave their positions (to carry out the project, for instance) or get promoted and, in either case, are each replaced by a successor. This means that the network remains unchanged after it has initially been established by the first project managers.

In such a context, it makes good sense to take as a starting point the suitable framework and convenient results established in Manea (2011). To my best knowledge, that is the only work which focuses purely on the impact of explicit network structures on players' bargaining power and outcomes in a setting of decentralized bilateral bargaining without imposing any restrictions ex ante to the class of networks considered. Thus, none of the distorting effects are present that might otherwise arise from additional incentives to add or delete links, from ex-ante heterogeneity among players, or in buyer-seller scenarios (which impose bipartite network structures). Also, stochastic effects do not play a role. Moreover, Manea's network bargaining game is analytically tractable and has some important properties. For any level of time discount, all subgame perfect equilibria are payoff equivalent. Beyond that, Manea develops an equally convenient and sophisticated algorithm which determines the limit equilibrium payoffs for a given network of infinitely patient players. We make extensive use of this algorithm and contribute to a deeper understanding of its features throughout this paper.

1.1 Additional Related Literature

The analysis of bargaining problems has a long tradition in the economic literature and dates back to the work of Nash (1950, 1953). A Nash bargaining solution is based on factors like players' bargaining power and outside options, whereas their origin is not part of the theory. This also applies to Rubinstein (1982), who analyzes perfect equilibrium partitions in a two-player framework of sequential bargaining in discrete time with an infinite horizon; and Rubinstein and Wolinsky (1985), who set up a model of bargaining in stationary markets with two populations. The work of Manea (2011), to which we add a preceding stage of strategic interaction, can be regarded as an extension of or microfoundation for these four seminal papers. Here, bargaining power is endogenized in a natural and well-defined manner as an outcome of the given network structure and the respective player's position in it. Other important contributions to the literature on decentralized bilateral bargaining in exogenously given networks have been made by Abreu and Manea (2012) and Corominas-Bosch (2004); the latter considers the special case of buyer-seller networks.

In addition, this paper contributes to the more recently emerging literature on strategic network formation which has been inspired mainly by the seminal paper of Jackson and Wolinsky (1996). Other prominent works which have been carried out since then (although not in a bargaining framework) are those by Bala and Goyal (2000), Calvó-Armengol (2004), Galeotti et al. (2006), Goyal and Joshi (2003, 2006), and Watts (2001), to name a few. Some effort has also been dedicated to gaining rather general insights regarding the existence, uniqueness, and structure of stable networks; see, for example, Hellmann (2013) and Hellmann and Landwehr (2014).[1]

Thus far, only a few papers have combined these two fields of research. Calvó-Armengol (2003) studies a bargaining framework like that in Rubinstein (1982), embedded in a network context, and considers stability and efficiency issues. However, the mechanism that determines bargaining partners is different from that in Manea (2011) and the network bargaining game ends after the first agreement has been found. As a consequence, in Calvó-Armengol's (2003) model a player's network position does not affect her bargaining power as such, but only the probability that she is selected as proposer or responder. This leads to a characterization of pairwise stable networks in which the players' neighborhood size is the only relevant feature of the network structure. It therefore differs substantially from our results though we both have in common the assumption that links are costly. In contrast, Manea (2011, Online Appendix) abstracts from explicit linking costs when approaching the issue of network formation as an extension of his model. He shows that for zero linking costs, a network is pairwise stable if and only if it is equitable. Though results differ and get more complex for positive linking costs, we will see that the present work is in line with this finding in such a way that both works complement one another.[2] Most other papers studying strategic network formation in a bargaining context focus on buyer-seller networks, which is also complementary to our more general approach; examples of such papers include Kranton and Minehart (2001) and Polanski and Vega-Redondo (2013). Again, the latter does not involve explicit linking costs.

The rest of the paper is organized as follows. In Section 2 we introduce the model, including the decisive results of Manea (2011). The main results on the structure of stable networks and induced bargaining outcomes are developed in Section 3. Finally, Section 4 concludes. The rather complex and lengthy proofs are presented in the appendix.

2 The Model

Let time be discrete and denoted by $t = 0, 1, 2, ...$ For the initial period $t = 0$ consider a finite set of players $N = \{1, 2, ..., n\}$. A *connection* or *(undirected) link* between two players $i, j \in N$, $i \neq j$, is denoted by $\{i, j\}$ which we abbreviate for simplicity

[1]Note, however, that the results of Hellmann (2013) and Hellmann and Landwehr (2014) are in general not applicable to our framework, because our model does not include certain crucial conditions that would permit such an application. For details, see Gauer (2016, Appendix 3.B).

[2]In fact, we show that only "skeletons" of equitable networks (that is, certain unions of separated pairs and odd circles) survive if costs are positive. However, nonequitable networks, such as unions of odd circles and an isolated player, can also be pairwise stable in our setting.

by $ij = ji := \{i, j\}$. A collection of such links is an *undirected graph* or *network* $g \subseteq g^N := \{ij \mid i, j \in N, i \neq j\}$ where g^N is called the *complete network*. Let $N_i(g) := \{j \in N \mid ij \in g\}$ denote the set of player i's *neighbors* in g and let $\eta_i(g) := |N_i(g)|$ be its cardinality which is also referred to as the *degree* of player i.

Furthermore, for a network g, a set $C \subseteq N$ is said to be a *component* if there exists a path between any two players in C and it is $N_j(g) \cap C = \emptyset$ for all $j \notin C$.[3,4] The set of all components of g then gives a partition of the player set N. Moreover, a *subnetwork* $g' \subseteq g$ is said to be *component-induced* if there exists a component C of g such that $g' = g|_C$. In general, for any set $K \subseteq N$, we denote $g|_K := \{ij \in g \mid i, j \in K\}$ and we commonly consider such a subnetwork as being defined on the player set K instead of N (thus, disregarding isolated players in $K^{\complement}$). In addition, for two networks $g, g' \subseteq g^N$ let $g - g' := g \backslash g'$ ($g + g' := g \cup g'$, respectively) denote the network obtained by deleting the set of links $g' \cap g$ from (adding the set of links $g' \backslash g$ to) the network g.

In our model, ex ante, i.e. apart from their potentially differing network positions, players are assumed to be identical.[5] These players are then assumed to strategically form links in period $t = 0$. The outcome of this network formation game is a network g. The interpretation of a link $ij \in g$ is that players $i, j \in N$ are able to mutually generate a unit surplus. On the contrary, each link causes costs of link formation $c > 0$ for both players involved. Thus, player i has to bear total costs of $\eta_i(g)c$ in $t = 0$.

We take this as a starting point for an infinite-horizon network bargaining game à la Manea (2011). In each period $t = 0, 1, 2, ...$ nature randomly chooses one link $ij \in g$ which means that i and j are matched to bargain for a mutually generated unit surplus. One of the two players is randomly assigned the role of the proposer while the other one is selected as responder. Then the proposer makes an offer how to distribute the unit surplus and the responder has the choice: If she rejects, then both receive a payoff of zero and stay in the game whereas both leave with the shares agreed on if she accepts. In the latter case, both players get replaced one-to-one in the next period such that the initially formed network remains unchanged.[6,7] This implies that each

[3]We say that there exists a path between two players $i', i'' \in N$ in g if there exist players $i_1, i_2, ..., i_{\bar{m}} \in N$, $\bar{m} \in \mathbb{N}$, such that $i_1 = i', i_{\bar{m}} = i''$ and $i_m i_{m+1} \in g$ for $m = 1, 2, ..., \bar{m} - 1$.

[4]One can alternatively define the component $C_i(g) \subseteq N$ of player $i \in N$ in g as the minimal set of players such that both $i \in C_i(g)$ and $N_{i'}(g) \subseteq C_i(g)$ for all $i' \in C_i(g)$.

[5]In the literature, this is sometimes referred to as a "homogeneous society" (see e.g. Hellmann and Landwehr, 2014).

[6]This replacement is primarily due to technical reasons. The implication that the network structure does not change over time makes the model analytically tractable. However, recalling the motivating example on bilateral project cooperation from Section 1 gives a hint that there are real-world situations for which this is a good approximation.

[7]This is why Manea carefully distinguishes between network positions and (potentially) different players being in one and the same position in different periods. However, as we examine solely the stage of network formation at time $t = 0$ here, we can neglect this distinction.

of the (initial) players $1, 2, ..., n$ will bargain successfully one time at most. A player's strategy in this setting pins down the offer she makes as proposer and the answer she gives as responder after each possible history of the game. Based on this, a player's payoff is then specified as her discounted expected agreement gains. A strategy profile is said to be a "subgame perfect equilibrium" of the bargaining game if it induces Nash equilibria in subgames following every history (see Manea, 2011). Players are assumed to discount time by a uniform discount factor $\delta \in (0, 1)$.[8]

The key result from Manea is that all subgame perfect equilibria are payoff equivalent and that each player's equilibrium payoff exclusively depends on her network position and the discount factor δ (see Manea, 2011, Theorem 1). Moreover, the *equilibrium payoff vector*, which we denote as $v^{*\delta}(g) = (v_i^{*\delta}(g))_{i \in N}$, is the unique solution to the equation system

$$v_i = \left(1 - \sum_{j \in N_i(g)} \frac{1}{2d^{\#}(g)}\right) \delta v_i + \sum_{j \in N_i(g)} \frac{1}{2d^{\#}(g)} \max\{1 - \delta v_j, \delta v_i\}, \quad i \in N, \qquad (1)$$

where $d^{\#}(g)$ denotes the total number of links in the network g. If we have $\delta\Big(v_i^{*\delta}(g) + v_j^{*\delta}(g)\Big) < 1$ for $ij \in g$, then this means that player i and j find an agreement when their mutual link is selected whereas $\delta\Big(v_i^{*\delta}(g) + v_j^{*\delta}(g)\Big) > 1$ means that each of them prefers to wait for a potentially better deal with a weaker partner.[9] This gives rise to the definition of the so called *equilibrium agreement network* $g^{*\delta} := \Big\{ij \in g \mid \delta\Big(v_i^{*\delta}(g) + v_j^{*\delta}(g)\Big) \leq 1\Big\}$. For the bargaining game, we assume players to play a subgame perfect equilibrium-strategy profile.

Throughout this paper, we focus on the limit case of $\delta \to 1$, meaning that players are infinitely patient. For this case, Manea (2011, Theorem 2) finds that, for all δ being greater than some bound, the corresponding equilibrium agreement networks are equal. This network $g^* \subseteq g$ is then called the *limit equilibrium agreement network*. Moreover, we again take from Manea (2011, Theorem 2) that the *limit equilibrium payoff vector* $v^*(g) := \lim_{\delta \to 1} v^{*\delta}(g)$ is well-defined, i.e. it always exists. For simplicity, we also refer to this as player i's *payoff* in this paper. It is important to precisely distinguish it from a player's *profit* which we define as her payoff net of linking costs. Thus, in period $t = 0$, each player $i \in N$ intends to maximize her profit

$$u_i^*(g) := v_i^*(g) - \eta_i(g)c.$$

[8]One might argue that players should be allowed to form (or delete) links in periods $t = 1, 2, ...$ as well. However, as the game has an infinite horizon, any player faces just the same situation in any period as (the player who was in her network position) in the previous period. Therefore, there do not arise additional or altered incentives regarding link formation over time.

[9]In the case $\delta\big(v_i^{*\delta}(g) + v_j^{*\delta}(g)\big) = 1$ both players are indifferent.

Here, we assume that players know the whole structure of the network g such that they are able to anticipate their profits.

Manea develops a smart algorithm to determine the limit equilibrium payoff vector $v^*(g)$ and we make heavily use of this computation method. To prepare for the implementation of the algorithm we need to introduce some additional notation. For any set of players $M \subseteq N$ and any network g let $L^g(M) := \{j \in N \mid ij \in g,\ i \in M\}$ be the corresponding *partner set* in g, that is the set of players having a link in g to a player in M.[10] Further, a set $M \subseteq N$ is called *g-independent* if we have $g|_M := \{ij \in g \mid i, j \in M\} = \emptyset$, i.e. if no two players contained in M are linked in g. Moreover, let $\mathcal{I}(g) \subseteq \mathcal{P}(N)$ denote the set of all nonempty g-independent subsets of N. Then the algorithm determining the payoff vector $v^*(g)$ is the following.

Definition 1 (Manea (2011)). *For a given network g on the player set N, the algorithm $\mathcal{A}(g)$ provides a sequence $(r_s, x_s, M_s, L_s, N_s, g_s)_{s=1,\dots,\bar{s}}$ which is defined recursively as follows. Let $N_1 := N$ and $g_1 := g$. For $s \geq 1$, if $N_s = \emptyset$ then stop and set $\bar{s} = s$. Otherwise, let*

$$r_s = \min_{M \subseteq N_s, M \in \mathcal{I}(g)} \frac{|L^{g_s}(M)|}{|M|}. \tag{2}$$

If $r_s \geq 1$ then stop and set $\bar{s} = s$. Otherwise, set $x_s = \frac{r_s}{1+r_s}$. Let M_s be the union of all minimizers M in (2). Denote $L_s := L^{g_s}(M_s)$. Let $N_{s+1} := N_s \backslash (M_s \cup L_s)$ and $g_{s+1} := g|_{N_{s+1}}$.

Given such a sequence $(r_s, x_s, M_s, L_s, N_s, g_s)_{s=1,\dots,\bar{s}}$ being the outcome of the described algorithm $\mathcal{A}(g)$, the limit equilibrium payoff vector for this network can be determined by applying a simple rule. Note that this rather sophisticated result of Manea (2011, Theorem 4) is absolutely fundamental for our work.

Payoff Computation (Manea (2011)). *Let $(r_s, x_s, M_s, L_s, N_s, g_s)_{s=1,\dots,\bar{s}}$ be the outcome of $\mathcal{A}(g)$ for a given network g. Then the limit equilibrium payoffs are given by*

$$\begin{aligned} v_i^*(g) &= x_s && \text{for all } i \in M_s, s < \bar{s}, \\ v_j^*(g) &= 1 - x_s && \text{for all } j \in L_s, s < \bar{s}, \\ v_k^*(g) &= \tfrac{1}{2} && \text{for all } k \in N_{\bar{s}}. \end{aligned} \tag{3}$$

Let us figure out what the algorithm $\mathcal{A}(g)$ in combination with the payoff calculation rule actually does. Starting with the network g and player set N, at each step s

[10] Although it does not make a big difference, do not confuse with the notation of Manea who refers to $L^{g^*}(M)$ instead.

it identifies the so called *minimal shortage ratio* r_s among the remaining players N_s in the network $g_s = g|_{N_s}$. There is a largest g-independent set M_s which minimizes this shortage ratio such that

$$r_s = \frac{|L_s|}{|M_s|},$$

where L_s is the partner set of M_s. The limit equilibrium payoff of the players in M_s is then given by $x_s = \frac{r_s}{1+r_s} = \frac{|L_s|}{|M_s|+|L_s|} < \frac{1}{2}$ while their partners in L_s receive $1 - x_s = \frac{|M_s|}{|M_s|+|L_s|} > \frac{1}{2}$. These players are then deleted from the player set and their links from the network and the algorithm moves forward to the next step. It stops as soon as there are either no more players left or if the minimal shortage ratio is greater than or equal to one. In the latter case, the limit equilibrium payoff of all remaining players is $\frac{1}{2}$.

For the considered setting, the algorithm quantifies the general idea that, when it comes to bilateral bargaining, players usually benefit from having multiple potential partners they can choose from. This improves their bargaining position as they are not too dependent on others in this case. Conversely, it is advantageous for players to have others being dependent on them. Hence, they will themselves benefit the most from partners with only few links. It becomes clear how these main forces affect players' payoffs here if, as convenient examples, one applies the algorithm to the networks sketched in Figure 1 (see Section 3) or in Manea (2011, Figure 2). Obviously, these forces lead to conflicting interests among strategic players who are to form a network of potential bargaining partners. This is particularly true if each player can only find an agreement once. If, in addition, forming links is costly, then this results in an interesting problem of network formation, the one we examine in this paper.

The described algorithm $\mathcal{A}(g)$ together with the previous considerations then also pins down the profit $u_i^*(g)$ of each player $i \in N$. It is important to note that the profile of payoffs and therefore also the profile of profits $u^* = (u_i^*)_{i\in N}$ is *component-decomposable*, meaning that $u_i^*(g) = u_i^*(g|_{C_i(g)})$ for all players $i \in N$ and networks g. Here, $C_i(g) \subseteq N$ denotes the component of player i in g. Thus, a player's profit is not affected by other subnetworks which are induced by components she is not contained in.

Beyond that, note that Manea develops the algorithm $\mathcal{A}(g)$ under the assumption that there are no isolated players in the underlying network g. However, it is easy to see that equations (3) are still fulfilled if one relaxes this restriction. It is clear that isolated players have a limit equilibrium payoff of zero since they have no bargaining partner they could generate a unit surplus with. At the same time, the algorithm $\mathcal{A}(g)$ provides $r_1 = 0$ such that $x_1 = 0$. In this case, M_1 is the set of all isolated players in the network and we have $L_1 = \emptyset$. Then, according to (3) and as required, all players in M_1 are assigned a limit equilibrium payoff of $x_1 = 0$.

In what follows, we assume that each player can influence the network structure by altering own links in $t = 0$, i.e. before the bargaining game starts.[11] This means that the network is no longer exogenously given as in the work of Manea but the outcome of strategic interaction between players. This gives rise to our analysis of stability issues, which is based on the seminal concept of pairwise stability as it was introduced by Jackson and Wolinsky (1996).

Definition 2 (Pairwise Stability, Jackson and Wolinsky (1996))**.** *Consider the player set N and a profile of network utility or profit functions $(u_i)_{i \in N}$. Then a network g is said to be* pairwise stable *if both*

(i) for all $ij \in g$: $u_i(g) \geq u_i(g - ij)$ and

(ii) for all $ij \notin g$: if $u_i(g + ij) > u_i(g)$, then $u_j(g + ij) < u_j(g)$.

So, according to this definition, a network is pairwise stable if no player can improve by deleting a single link and also no two players can both individually benefit from adding a mutual link. The analysis of our model demands to distinguish between pairwise stable networks for which the above conditions hold on a cost interval of positive length and those for which this is not the case.

Definition 3 (Generic and Nongeneric Pairwise Stability)**.** *In the considered framework with network profit function $u = u^*$ and linking costs $c > 0$, a network g is called*

- generically pairwise stable *if g is pairwise stable for all $c' \in (c - \epsilon, c + \epsilon)$ for some $\epsilon > 0$,*

- nongenerically pairwise stable *if g is pairwise stable but not generically pairwise stable.*[12]

The results we deduce in Section 3 will even reveal that networks can be pairwise stable at no more than a single cost level if they are not generically pairwise stable for any cost level. Thus, the notion of nongeneric pairwise stability is not robust at all with respect to changes of linking costs. One might say that it is even a singularity for such a network to encounter precisely the parametrization where it is pairwise stable.

[11] See again Footnote 8.

[12] See e.g. Baetz (2015) who refers to a "generic equilibrium" in a different setting but considering a similar definition.

We are therefore predominantly interested in generically pairwise stable networks and, in what follows, establish a complete characterization thereof.[13]

3 Characterization of Stable Networks

In addition to deriving a complete characterization of generically pairwise stable networks, we examine the implications for possible bargaining outcomes (see Subsection 3.1) as well as the possibility of networks to be nongenerically pairwise stable (see Subsection 3.2).

We consider period $t = 0$ and suppose that players, who anticipate the infinite-horizon network bargaining game, individually intend to maximize their expected profits. To establish our characterization result, we identify pairwise stable structures for all levels of linking costs $c > 0$ as a first step. Second, we gradually rule out the possibility to be pairwise stable for a broad range of networks until we arrive at a complete characterization of generically pairwise stable networks.[14]

To get a first impression of the problem, let us have a look at the situation for three players, i.e. for $N = \{1, 2, 3\}$. It turns out that this case already covers many important aspects of the network formation game. Figure 1 illustrates the four types of networks which might appear including induced profits u_i^* for each player $i \in N$.[15] To comprehend these profits, consider the algorithm introduced in Definition 1 and the subsequent payoff computation rule.

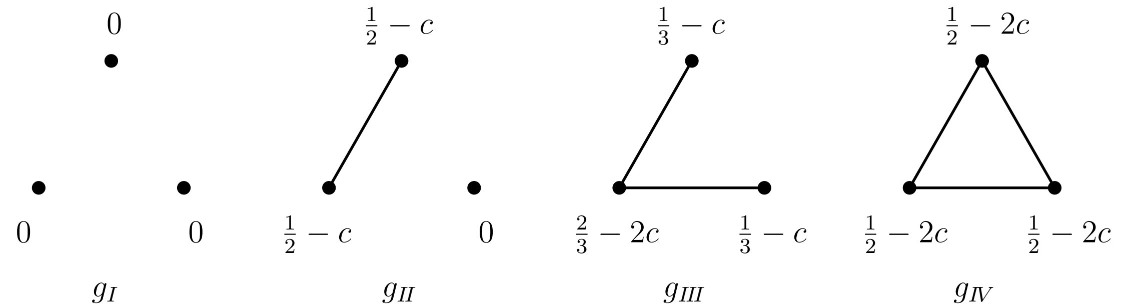


Figure 1: A sketch of the four network structures which can arise in the case $n = 3$ with induced profits

Let us consider these networks in detail. We see immediately that the network g_I is pairwise stable if and only if the linking costs c are greater than or equal to $\frac{1}{2}$.

[13]In fact, our results even reach beyond that and yield a complete characterization of networks being pairwise stable for more than one level of linking costs.

[14]See again Footnote 13.

[15]Note that all other possible networks can be derived by permuting players. This would not provide additional insights as players are assumed to be ex ante homogeneous.

Otherwise any two players could increase their profit from zero to $\frac{1}{2} - c > 0$ by creating a mutual link. However, for $c = \frac{1}{2}$ also no player wants to delete this link and indeed, the cost range $c \in (\frac{1}{6}, \frac{1}{2}]$ is the one for which g_{II} is pairwise stable. Here, link deletion is obviously not beneficial and if one of the two connected players creates a link to the third player, then she would end up with a profit of $\frac{2}{3} - 2c$ which is strictly smaller than $\frac{1}{2} - c$ for this cost range. These latter two terms are equal for $c = \frac{1}{6}$ but the third player would improve from zero to $\frac{1}{6}$ in this case. Therefore, at this or an even smaller cost level, g_{II} cannot be pairwise stable. But so is g_{III} for $c = \frac{1}{6}$. This is because here no player has incentives to delete a link and the two players who are not connected are indifferent between creating a mutual link and not creating it as for this cost level we have $\frac{1}{3} - c = \frac{1}{6} = \frac{1}{2} - 2c$. However, if linking costs are even smaller, then both would profit from this link. Thus, g_{III} is pairwise stable if and only if $c = \frac{1}{6}$. Finally, the network g_{IV} is pairwise stable for $c \in (0, \frac{1}{6}]$ but obviously not at higher cost levels.

It turns out that the observed mechanisms being crucial in the three-player case hold similarly also in general. Our first theorem reveals sufficient conditions for networks to be pairwise stable. More precisely, it identifies, for all cost levels, concrete network structures being pairwise stable.

Theorem 1 (Sufficient Conditions for Networks to Be Pairwise Stable)**.** *In our model, the following holds:*

(i) The empty network is pairwise stable if $c \geq \frac{1}{2}$.

(ii) A network which is a union of separated pairs and at most one isolated player is pairwise stable if $c \in (\frac{1}{6}, \frac{1}{2}]$.[16] *Additionally, if $c = \frac{1}{2}$, then several isolated players can coexist in a pairwise stable network.*

(iii) A network which is a union of odd circles with at most $\frac{1}{2c}$ players and either separated pairs or at most one isolated player is pairwise stable if $c \in (0, \frac{1}{6}]$.[17] *Additionally, if $c = \frac{1}{6}$ and given that there is no isolated player, then there can also exist lines of length three in a pairwise stable network.*[18]

The formal proof of this theorem is, like all other more complex or lengthy proofs, provided in Appendix A. It is important to note that, when considering the above mentioned unions of subnetworks, we do not mean that the respective network necessarily has to be composed of all of the stated subnetworks to be pairwise stable.

[16] A separated pair denotes a subnetwork induced by a two-player component.

[17] A circle denotes a component-induced subnetwork which is regular of degree two. A circle with m players or a m-player circle is induced by a component with cardinality $m \geq 3$ and it is called odd if this cardinality is an odd number.

[18] A line of length $m \geq 3$ denotes a subnetwork induced by a m-player component which can be transformed to a m-player circle by adding one link.

For instance, if we consider costs $c \in (0, \frac{1}{6}]$, then a network consisting only of separated pairs or only of (some of the permissible) odd circles is pairwise stable as well. Furthermore, note that all of these subnetworks are component-induced which implies that unions are disjoint with respect to contained links and players.

A byproduct of Theorem 1 is that it guarantees existence of a pairwise stable network at each level of linking costs. Furthermore, we have given a characterization of at least some pairwise stable networks for each level of costs. However, it is not clear at all that the types of networks mentioned in the theorem are in each case the only pairwise stable ones. Anyway, we can already state some consequences from our observations in the three-player case considered in Figure 1 and the proof of Theorem 1. This is done in the following corollary.

Corollary 1. *In our model, a network cannot be pairwise stable if it contains*

(i) *more than one isolated player while* $c < \frac{1}{2}$*,*

(ii) *a separated pair while* $c > \frac{1}{2}$*,*

(iii) *a line of length three while* $c \neq \frac{1}{6}$*,*

(iv) *an odd circle with more than* $\frac{1}{2c}$ *members,*[19]

(v) *an isolated player combined with a separated pair or a line of length three while* $c \leq \frac{1}{6}$*.*

Statements (i)–(iv) as well as the first part of Statement (v) of Corollary 1 follow immediately from what we learned in the three-player case and the proof of Theorem 1. To see that an isolated player and a line of length three cannot coexist in a network being pairwise stable for some $c \leq \frac{1}{6}$ is quite obvious as well. An isolated player's profit is always zero while each of the two players in a line of length three having one link receives $\frac{1}{3} - c$ as we know from the three-player case. If one of these players links to an isolated player, then the algorithm $\mathcal{A}(\cdot)$ yields that all players in the new component receive a payoff of $\frac{1}{2}$. Thus, it is beneficial for both players to build this mutual link as for $c \leq \frac{1}{6}$ we have $\frac{1}{2} - 2c \geq \frac{1}{3} - c$ and $\frac{1}{2} - c > 0$. One should perhaps mention that according to (iii) it is clear anyway that we cannot have a line of length three in a pairwise stable network if $c < \frac{1}{6}$. So the above additional consideration is actually only relevant for $c = \frac{1}{6}$.

Together with this corollary, the results we establish in the further course of this section reveal that, for generically pairwise stable networks, the corresponding conditions stated in Theorem 1 are not only sufficient but also necessary. Note, however,

[19] In particular, this means that there can be no odd circles at all in pairwise stable networks as long as $c > \frac{1}{6}$.

that a network which is composed of several isolated players and at least one separated pair is only nongenerically pairwise stable (only at $c = \frac{1}{2}$, see Theorem 1(ii)). Similarly, networks containing a line of length three can at most be nongenerically pairwise stable (only at $c = \frac{1}{6}$, see Theorem 1(iii)).

In general, it is clear that a network can only be pairwise stable if none of the contained links harms any of the respective two players involved. Therefore, intuition says that pairwise stable networks cannot have so called *disagreement links*, that is links which are not contained in the corresponding limit equilibrium agreement network. Such a link causes additional costs for both players connected through it whereas it seems to be irrelevant regarding payoffs. Roughly, this is what we establish in the following proposition.

Proposition 1. *If a network g is pairwise stable for some cost level $c > 0$ in our model, then the network itself and the corresponding limit equilibrium agreement network coincide, that is $g = g^*$, meaning that g does not contain disagreement links. In particular, this implies that we have $v_i^*(g) + v_j^*(g) = 1$ for all $ij \in g$.*

Proof of Proposition 1. Assume that there exists a disagreement link in the pairwise stable network, w.l.o.g. say between players 1 and 2, that is we have $12 \in g \backslash g^*$. Manea (2011, Theorem 2) then yields that there exists some $\underline{\delta} \in (0,1)$ such that $12 \notin g^{*\delta}$ for all $\delta \in (\underline{\delta}, 1)$. By definition of the equilibrium agreement network, this means that we have $\delta(v_1^{*\delta}(g) + v_2^{*\delta}(g)) > 1$, implying $\max\{1 - \delta v_2^{*\delta}(g), \delta v_1^{*\delta}(g)\} = \delta v_1^{*\delta}(g)$ and $\max\{1 - \delta v_1^{*\delta}(g), \delta v_2^{*\delta}(g)\} = \delta v_2^{*\delta}(g)$ for all $\delta \in (\underline{\delta}, 1)$.

Thus, for the equilibrium payoff vector, which is the unique solution to the equation system (1) (see Section 2), we have that for all $\epsilon > 0$ there exists $\delta' \in (\underline{\delta}, 1)$ such that $||v^{*\delta}(g) - v^{*\delta}(g - 12)||_\infty < \epsilon$ for all $\delta \in (\delta', 1)$.[20] Again by Manea (2011, Theorem 2),

[20] The only reason why we do not have equality here is that $d^\#(g - 12) \neq d^\#(g)$. However, the effect of switching from $d^\#(g)$ to $d^\#(g - 12)$ on the corresponding equilibrium payoff vector v^* gets arbitrarily small as soon as δ is sufficiently close to one. To see this, note that the equation system (1) is equivalent to

$$2(1-\delta)d^\#(g)v_i = -\sum_{j \in N_i(g^*)} \delta v_i + \sum_{j \in N_i(g^*)} (1 - \delta v_j), \quad i \in N.$$

for $\delta \in (\underline{\delta}, 1)$. If we replace $d^\#(g)$ by $d^\#(g - 12)$ here, then the altered equation system again has a unique solution, say $\tilde{v}^\delta(g - 12) \in [0,1]^n$ (cf. Manea, 2011, Proof of Proposition 1). Considering Cramer's rule, for instance, then reveals that for all $\epsilon > 0$ there exists $\delta' \in (\underline{\delta}, 1)$ such that $||v^{*\delta}(g) - \tilde{v}^\delta(g-12)||_\infty < \epsilon$ for all $\delta \in (\delta', 1)$. Thus, according to Manea (2011, Proposition 1), if δ' is sufficiently close to one, then for all $\delta \in (\delta', 1)$ and $ij \in g$ it holds that $\delta(\tilde{v}_i^\delta(g - 12) + \tilde{v}_j^\delta(g - 12)) \leq 1$ if and only if $ij \in g^*$. This implies that $\tilde{v}^\delta(g - 12)$ also solves the equation system (1) for the network $g - 12$, meaning that $v^{*\delta}(g - 12) = \tilde{v}^\delta(g - 12)$ for all $\delta \in (\delta', 1)$.

this gives

$$u_i^*(g) - u_i^*(g-12) = v_i^*(g) - v_i^*(g-12) - c = \lim_{\delta \to 1} \left(v_i^{*\delta}(g) - v_i^{*\delta}(g-12) \right) - c < 0$$

for both $i \in \{1,2\}$. Hence, g cannot be pairwise stable in the framework with $\delta \to 1$ as for both players 1 and 2 it is beneficial to delete their mutual link. Arriving at a contradiction, this proves that a pairwise stable network cannot contain a disagreement link.

Finally, together with Manea (2011, Proposition 2), this implies that we have $v_i^*(g) + v_j^*(g) = 1$ for all $ij \in g$ if g is pairwise stable. □

In achieving a complete characterization of generically pairwise stable networks, this insight will prove to be of great importance. As the first of two major steps towards establishing this complete characterization, we now consider networks inducing a homogeneous payoff structure. Here, in line with Manea (2011), we call a network *equitable* if every player receives a payoff of $\frac{1}{2}$. Moreover, for a given network g with player set N, we define the subset $\tilde{N}(g) := \{i \in N \mid v_i^*(g) = \frac{1}{2}\}$. We utilize this notation in the following theorem, which, in combination with Proposition 1, reveals that a network can only be pairwise stable if any player receiving a payoff of $\frac{1}{2}$ is contained in a component which either induces a separated pair or an odd circle.

Theorem 2 (Necessary Conditions for Networks to Be Pairwise Stable – Part 1/2)**.** *If a network g is pairwise stable for some cost level $c > 0$ in our model, then $g|_{\tilde{N}(g)}$ must be a union of separated pairs and odd circles.*[21]

The proof, which is again given in Appendix A, is by contradiction. The idea is to assume that g is pairwise stable but $g|_{\tilde{N}(g)}$ is not a union of separated pairs and odd circles. Note that by Proposition 1 a link from a player in $\tilde{N}(g)$ to a player outside this set cannot exist which implies that we have $v_i^*(g) = v_i^*(g|_{\tilde{N}(g)})$ for all $i \in \tilde{N}(g)$ as payoffs are component-decomposable. Further, we make use of both directions of Manea (2011, Theorem 5) who establishes that a network is equitable if and only if it has a so called "edge cover" g' composed of separated pairs and odd circles. A network g' is said to be an *edge cover* of $g|_{\tilde{N}(g)}$ if it fulfills $g' \subseteq g|_{\tilde{N}(g)}$ and no player in $\tilde{N}(g)$ is isolated in g'. This implies that any player in $\tilde{N}(g)$ has an incentive to delete each of her links not contained in g'.

Though statements differ, note that Theorem 2 is in line with Manea (2011, Theorem 1(ii) of the Online Appendix). The latter establishes that for zero linking costs a network is pairwise stable if and only if it is equitable. Of course, in this case no player

[21] As usual, $g|_{\tilde{N}(g)}$ here is considered as being defined on the player set $\tilde{N}(g)$ instead of N.

can gain anything from deleting redundant links from an equitable network. This then gives rise to a larger class of pairwise stable (equitable) networks. For instance, any even circle or line of even length is equitable and therefore pairwise stable as long as there are no linking costs whereas Theorem 2 rules out this possibility for $c > 0$. On the contrary, as we have seen in Figure 1 and Theorem 1, for positive linking costs there additionally exist nonequitable structures, such as networks composed of an isolated player combined with separated pairs or odd circles, which can be pairwise stable. Another example for this is the line of length three. However, as we already know, such a component-induced subnetwork can only occur in a nongenerically pairwise stable network and, to be more precise, only at the single cost level $c = \frac{1}{6}$. In what follows, this kind of singularity is central to our investigation.

Summing up our results so far, for all levels of positive linking costs, we achieved a complete characterization of networks which are pairwise stable and induce homogeneous payoffs within each of its components. In these networks, all payoffs must be equal to either zero (for isolated players) or $\frac{1}{2}$ by Proposition 1. According to Theorem 1, Corollary 1 and Theorem 2 certain unions of separated pairs, odd circles and isolated players constitute this class of networks.

Thus, it remains to consider structures which induce heterogeneous payoffs within a component. Most of the rest of the section is devoted to the examination of such networks and the question whether and in which cases they can potentially be pairwise stable. To begin with, let us make sure to be aware of the following property of pairwise stable nonequitable networks. Taking into account the payoff computation rule, this corollary is an immediate consequence of Proposition 1.

Corollary 2. *In our model, let $g \neq \emptyset$ be a nonequitable network having only one component and assume that it is pairwise stable for some cost level $c > 0$. Then there exists a unique partition $M \dot{\cup} L = N$ with $|M| > |L|$ and $g|_M = g|_L = \emptyset$, meaning that g is bipartite.*[22] *Payoffs are then given by*

$$v_i^*(g) = x \quad \text{for all } i \in M \quad \text{and}$$
$$v_j^*(g) = 1 - x \quad \text{for all } j \in L,$$

where $x = \frac{|L|}{|M|+|L|}$.

Note here that, according to Manea (2011, Proposition 3), the sequence of minimal shortage ratios provided by the algorithm in Definition 1 is strictly increasing for any network. Thus, Corollary 2 implies that, for any nonequitable pairwise stable network

[22] If we write $M \dot{\cup} L$, this simply denotes the union of two sets M and L being disjoint. We use this notation whenever disjointness is of importance.

g consisting of only one component, the algorithm $\mathcal{A}(g)$ has to stop after removing all players during the first step. This then leads to the heterogeneous payoff distribution with two different payoffs, one below and one above $\frac{1}{2}$.

Based on Corollary 2, the following theorem concludes the complete characterization of generically pairwise stable networks. In fact, it establishes that any remaining network, that is any network in which at least two players belonging to one component receive different payoffs, can be pairwise stable at no more than a single cost level. Thus, any network not mentioned in Theorem 1 cannot be generically pairwise stable.

Theorem 3 (Necessary Conditions for Networks to Be Pairwise Stable – Part 2/2)**.** *If a network is pairwise stable for some cost level $c > 0$ in our model and there is a component in which players receive heterogeneous payoffs, then in any such component there must occur exactly two different payoffs $x \in (0, \frac{1}{2})$ and $1 - x \in (\frac{1}{2}, 1)$ with*

$$x + c = \frac{1}{2}. \tag{4}$$

The proof rests on two lemmas which are of some independent interest. We shall now state these lemmas, one after the other, and then show how they combine to establish the theorem.

We first show that, if any two players whose payoffs in a pairwise stable network are strictly smaller than $\frac{1}{2}$, link to each other, then both receive a payoff of $\frac{1}{2}$ in the resulting network.

Lemma 1. *In our model, consider a pairwise stable network g for which the algorithm $\mathcal{A}(g)$ provides $(r_1, x_1, M_1, L_1, N_1, g_1)$, i.e. $\bar{s} = 1$, such that $r_1 \in (0, 1)$. Then for all $i, j \in M_1$ it is*

$$v_i^*(g + ij) = v_j^*(g + ij) = \frac{1}{2}.$$

Further, if the player set $N = \{1, ..., n\}$ is extended by a player $n+1$ while the network g remains unchanged, it similarly is $v_i^*(g + i(n+1)) = v_{n+1}^*(g + i(n+1)) = \frac{1}{2}$.

The second lemma, in contrast, considers link deletion and players who receive a payoff being strictly greater than $\frac{1}{2}$ in a pairwise stable network. It establishes that one link deletion cannot effect these players' payoffs to fall below $\frac{1}{2}$.

Lemma 2. *In our model, consider a pairwise stable network g for which the algorithm $\mathcal{A}(g)$ provides $(r_1, x_1, M_1, L_1, N_1, g_1)$, i.e. $\bar{s} = 1$, such that $r_1 \in (0, 1)$. Then for all $j \in L_1$, $i \in M_1$ and $kl \in g$ it is*

$$v_j^*(g - kl) \geq \frac{1}{2} \geq v_i^*(g - kl).$$

The proofs of these lemmas are somewhat lengthy and as usual provided in the appendix. In both cases we show that, if the respective statement were not true, then this would imply that the player set is infinite. To arrive at this contradiction, we make use of an additional, rather technical lemma which we also provide in the appendix (see Lemma 3). Based on these lemmas, the proof of the theorem is straightforward.

Proof of Theorem 3. Let g be a pairwise stable network inducing heterogeneous payoffs within a component $C \subseteq N$. Let $g' := g|_C$. According to Corollary 2, the algorithm $\mathcal{A}(g')$ (with $N_1 = C$) has to stop after the first step, i.e. $\bar{s}' = 1$.[23] Let $(r'_1, x, M'_1, L'_1, N'_1, g'_1)$ be the outcome of $\mathcal{A}(g')$ and $i \in M'_1$, $j \in L'_1$. Then any player in C must either receive a payoff of $x = \frac{|L'_1|}{|M'_1|+|L'_1|} \in (0, \frac{1}{2})$ or $1 - x = \frac{|M'_1|}{|M'_1|+|L'_1|} \in (\frac{1}{2}, 1)$. Then Lemma 1 provides the stability condition

$$x - \eta_i(g')c \geq \frac{1}{2} - (\eta_i(g') + 1)c \quad \Leftrightarrow \quad x + c \geq \frac{1}{2}.$$

Similarly, according to Lemma 2 we must have

$$(1 - x) - \eta_j(g')c \geq \frac{1}{2} - (\eta_j(g') - 1)c \quad \Leftrightarrow \quad x + c \leq \frac{1}{2}.$$

So payoffs must be $x = \frac{1}{2} - c$ and $1 - x = \frac{1}{2} + c$. Obviously, this has to hold for all components of g in which players receive heterogeneous payoffs. □

Notice, by considering the limit case $c \to 0$, that Theorem 3 is in line with Manea's (2011, Online Appendix) result that for zero linking costs any pairwise stable network must be equitable. As an immediate consequence of Theorem 3 and the previous findings, we arrive at the main result of this paper, which can now be stated as a corollary.

Corollary 3 (Complete Characterization). *In our model, the class of generically pairwise stable networks is completely characterized by Theorem 1 for each level of linking costs $c > 0$.*[24] *Thus, specific unions of isolated players, separated pairs and odd circles constitute this class.*

To see this, recall first that, according to Theorem 2, any network g not mentioned in Theorem 1 can only be pairwise stable if it induces heterogeneous payoffs within at least one component. Each player contained in such a component must either receive a payoff of $x = \frac{1}{2} - c$ or $1 - x = \frac{1}{2} + c$ by Theorem 3.[25] Be aware that these equations

[23] Disregarding isolated players here by considering the restricted player set is w.l.o.g. as the profile of payoffs respectively profits is component-decomposable.

[24] Of course, nongenerically pairwise stable networks mentioned in Theorem 1 are to be ignored here.

[25] Recalling Corollary 2, the induced subnetwork must be bipartite.

do not represent calculation rules determining payoffs in g but necessary conditions for a network to (possibly) be pairwise stable. Recall that x is in fact determined by the algorithm $\mathcal{A}(g)$, meaning that it solely depends on the structure of g and that $c > 0$ is an independent parameter of the model. Therefore, such a network g can only be pairwise stable at the single cost level $c = \frac{1}{2} - x$. Together with Corollary 1 this establishes Corollary 3.

Beyond that, given this crucial cost level $c = \frac{1}{2} - x$, it is of course not at all clear that a network in which each player receives a payoff of $x \in (0, \frac{1}{2})$ or $1 - x$ is actually pairwise stable. Even if this is the case, then any two players with a payoff of x must be indifferent between leaving the network unchanged and adding a mutual link (see Lemma 1). Also, any player receiving a payoff of $1 - x$ must be indifferent between keeping all of her links and deleting any one of them (see Lemma 2). In this sense, it is a very special case that a network inducing heterogeneous payoffs within a component is pairwise stable and does indeed form. This is why, so far, we did not specifically examine possible structures of such networks. However, we are able to specify one such network (and variations respectively generalizations of it as a component-induced subnetwork), namely the line of length three. It induces payoffs $x = \frac{1}{3}$ and $1 - x = \frac{2}{3}$ and is pairwise stable if and only if $c = \frac{1}{6}$. As opposed to this, we even rule out the possibility to be pairwise stable at a single cost level for a broad range of network structures in the further course of this section (see Subsection 3.2).

3.1 Stability and Bargaining Outcomes

After characterizing (generically) pairwise stable networks, we turn first to see what our findings imply for outcomes of the infinite-horizon network bargaining game. As our second main result, we demonstrate that payoffs and profits induced by (generically) pairwise stable networks are in general highly but not completely homogeneous. Given the previous results of this section, this can be stated as a corollary.

Corollary 4 (Limited Outcome Diversity)**.** *In our model, consider a network g which is pairwise stable at a given level of linking costs $c > 0$. Then players' payoffs must be such that either $v_i^*(g) \in \{\frac{1}{2} - c, \frac{1}{2}, \frac{1}{2} + c\}$ with $c \in (0, \frac{1}{4}]$ or $v_i^*(g) \in \{0, \frac{1}{2}\}$ for all $i \in N$. Moreover, if g is generically pairwise stable, then only the latter of these two cases can occur and there exists a set $P(g) \subset \{0, \frac{1}{2} - 2c, \frac{1}{2} - c\}$ with $|P(g)| \leq 2$ such that for players' profits it holds that $u_i^*(g) \in P(g)$ for all $i \in N$.*

This result is basically a consequence of Theorems 2 and 3, Corollary 1, and Lemma 1. To see this, recall that, in pairwise stable networks, there can only occur four kinds of players in terms of payoffs. Namely, these are isolated players receiving zero, players belonging to a separated pair or an odd circle with a payoff of $\frac{1}{2}$, and players contained in a component with heterogeneous payoffs who receive $\frac{1}{2}+c$ or $\frac{1}{2}-c$. Note, however, that the former and the latter player type cannot coexist in a pairwise stable network by the second part of Lemma 1. Moreover, considering a component with heterogeneous payoffs, linking costs cannot be greater than $\frac{1}{4}$. This is because any player $i \in N$ receiving a payoff of $x = \frac{1}{2} - c$ can save costs of c when deleting a link while falling back to a payoff of zero in the worst case. Provided pairwise stability, this yields the stability condition

$$x - \eta_i(g)c \geq 0 - (\eta_i(g) - 1)c \;\Leftrightarrow\; x \geq c \;\Leftrightarrow\; \frac{1}{2} - c \geq c \;\Leftrightarrow\; c \leq \frac{1}{4}. \tag{5}$$

Further, one will only observe profits of zero, $\frac{1}{2} - 2c$, and $\frac{1}{2} - c$ in generically pairwise stable networks as, by our characterization result consolidated in Corollary 3, any such network must be a union of isolated players, odd circles, and separated pairs. For any cost level $c > 0$, even only two of these three kinds of component-induced subnetworks can coexist in a pairwise stable network according to Corollary 1(iv) and (v).

Taken together, we have that the diversity of possible bargaining outcomes gets narrowed down substantially compared to the work of Manea (2011) if one considers a preceding stage of strategic network formation. To this end, observe that in Manea's basic framework with $\delta \to 1$ one can obtain any rational number from the interval $[0, 1)$ as a payoff induced by an appropriate network on a sufficiently large player set.[26]

3.2 Nongeneric Pairwise Stability

As already announced, we now conclude this section by ruling out the possibility to be pairwise stable at all for a broad range of network structures not considered yet. Up to here, according to our previous results, any network not considered in Theorem 1 can at most be nongenerically pairwise stable. In fact, we even have that any remaining network can be pairwise stable at no more than a single cost level and, moreover, it must have a component in which players receive heterogeneous payoffs (recall Theorem 3).

In the following proposition, we consider specific classes of networks of that kind. The main idea of most of the proofs is to identify generic network positions in which the respective player receives a payoff strictly greater than $\frac{1}{2}$ and still does so after deleting a certain link. Applying the notation of Theorem 3, the resulting stability

[26]For the rational number $\frac{n'}{n''} \in [0, 1)$ with $n', n'' \in \mathbb{N}$, consider the player set N with $n = n''$ and the complete bipartite network with n' players on one side and $n'' - n'$ players on the other side. Then the algorithm $\mathcal{A}(\cdot)$ yields payoffs $\frac{n''-n'}{n''}$ and $\frac{n'}{n''}$.

condition then yields $x + c < \frac{1}{2}$. Thus, arriving at a contradiction, such a network cannot be pairwise stable. Also, we consider players who are in a weak bargaining position but whose loss in payoff from dropping a certain own link is too small to be compatible with the condition $x + c = \frac{1}{2}$.

To be more precise, we establish that networks containing a tree (apart from the ones considered in Theorem 1), a certain kind of cut-player, or a certain class of bipartite subnetworks cannot be pairwise stable at all.[27] For a given network g, a player $k \in N$ is called *cut-player* if $g|_{N\setminus\{k\}}$ has more components than g.[28]

Proposition 2. *If a network g is pairwise stable in our model, then*

(i) it cannot have a component of more than three players which induces a tree,

(ii) there cannot be a cut-player who is part of a cycle and receives a payoff strictly greater than $\frac{1}{2}$,

(iii) it cannot have a component which induces a bipartite subnetwork with $m \in \mathbb{N}$ players on one side and less than $\frac{m}{3}$ on the other.

Let us consider this proposition in detail. To start with, Part (i) further reduces the class of potentially pairwise stable networks extensively. It implies that any component of a pairwise stable network either contains at most three players or induces a subnetwork which has a cycle.[29] The former case has been analyzed exhaustively in Theorem 1 and Corollary 1. Thus, the only structures which are not captured by our analysis yet are networks which have a cycle and in which players receive heterogeneous payoffs.

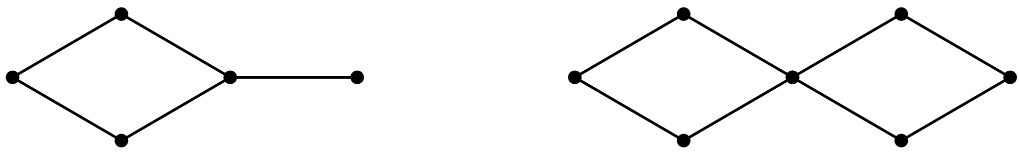
A significant subclass of such networks is captured by Part (ii). It rules out the possibility to be pairwise stable for several generic kinds of networks. For instance, many networks containing a component-induced subnetwork which has a cycle and a loose-end player, i.e. there is a player who has one link, are excluded. See Figure 2 for an illustration of exemplary subnetworks which cannot even be contained in a nongenerically pairwise stable network according to Proposition 2(ii).

Finally, Part (iii) further reduces the class of potentially pairwise stable networks by establishing that too unbalanced bipartite structures cannot occur.

[27] A tree denotes a component-induced subnetwork which is minimally connected.

[28] This notation comes from graph theory where vertices of that kind are typically called "cut-vertices" (see e.g. West, 2001). For instance, each player contained in a component which induces a tree and having more than one link is a cut-player.

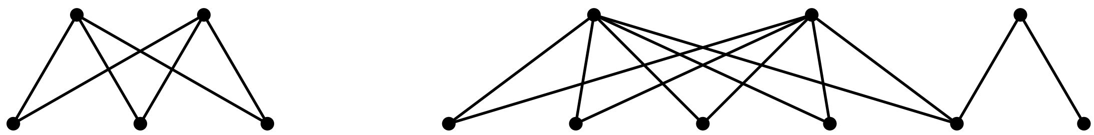
[29] A network g is said to have a cycle if there exist distinct players $i_1, i_2, ..., i_{\bar{m}} \in N$, $\bar{m} \geq 3$, such that $i_1 i_{\bar{m}} \in g$ and $i_m i_{m+1} \in g$ for $m = 1, 2, ..., \bar{m} - 1$. For instance, this implies that any network containing a circle has a cycle.



g_V g_{VI}

Figure 2: A sketch of networks which cannot be pairwise stable according to Proposition 2(ii)

However, there exist other networks not captured by our (explicitly stated) results which could potentially still be nongenerically pairwise stable. Two examples for this are given in Figure 3. Though a further generalization is not reached here, it is easy to check that the concrete networks illustrated in the figure cannot be pairwise stable.[30] It remains as a conjecture that, in our model, the only nongenerically pairwise stable networks inducing heterogeneous payoffs within a component are those containing a line of length three at cost level $c = \frac{1}{6}$.



g_{VII} g_{VIII}

Figure 3: A sketch of networks which, based on our general results in Section 3, might still be nongenerically pairwise stable

4 Discussion and Conclusion

In this paper, we develop an analytically tractable model of strategic network formation in the context of decentralized bilateral bargaining, involving ex ante homogeneous players and explicit linking costs. One reasonable application of our model can be seen in the stylized example of project collaboration between firms provided in the introduction.

Considering infinitely patient players, we derive a complete characterization of generically pairwise stable networks. This class of networks is constituted by specific unions of separated pairs, odd circles, and isolated players, where the concrete struc-

[30] In g_{VII}, it is obvious that one can delete any link without changing payoffs. Also, for instance as an immediate consequence of Manea (2011, Theorem 6), a network like g_{VIII} is not pairwise stable at any cost level either.

ture depends on the level of linking costs. For a sufficiently high cost level (for $c > \frac{1}{4}$ to be more precise), our result even yields a complete characterization of pairwise stable networks. The induced bargaining outcomes are mostly homogeneous but a certain level of diversity regarding players' payoffs and profits can still occur. In addition, we study the remaining networks which could possibly be nongenerically pairwise stable and succeed in ruling out this possibility for a broad range of structures. These results are complementary to Manea (2011, Online Appendix).

Beyond these core results, Gauer (2016, Corollary 3.5) demonstrates, in regard to alternative stability concepts, that all generically pairwise stable networks even prove to be pairwise Nash stable. Moreover, Gauer (2016, Theorem 3.4) provides a complete characterization of the networks which are efficient in terms of a utilitarian welfare criterion. As pointed out in Gauer (2016, Corollary 3.7), the efficient networks coincide with the pairwise stable ones as long as linking costs are sufficiently high. If costs are low, however, then the former networks constitute a proper subset of the latter ones. And there also exists an intermediate cost range which does not even yield such a subset relation. Finally, the assumption that players are infinitely patient is relaxed as a robustness check in Gauer (2016, Section 3.5).

Altogether, the present work contributes to a better understanding of the behavior of players in a noncooperative setting of decentralized bilateral bargaining when the underlying network is not exogenously given but is instead the outcome of prior strategic interaction. We gain insights concerning the structure of resulting networks, induced bargaining outcomes, and the effects that influence players when they are aiming to optimize their bargaining position in the network.

In regard to future research, a reasonable next step would be to develop a complete characterization of pairwise stable networks in general, that would go beyond the case of generic pairwise stability and would cover all levels of linking costs. This would call for a further discussion of networks which, according to our results, might be nongenerically pairwise stable for low costs. For this purpose, a promising approach might be to strive for a generalization of Manea (2011, Theorem 6) to the case where the buyer-seller ratio is not (necessarily) an integer. Beyond that, it could be valuable to fully examine the case of players who are less than infinitely patient. Further important insights could be generated by considering alternative stability concepts, such as pairwise stability with transfers, which seems quite natural in a bargaining context. Finally, it would surely be interesting to set up an analytically tractable model of network formation in a bargaining framework in which players do not get replaced one-to-one after dropping out. The resulting stochastic change of the network structure over time would certainly make this another challenging research topic.

A Proofs

A.1 Proof of Theorem 1

Consider a component $C \subseteq N$ of some network g which induces a circle or a separated pair. Then in both cases it is impossible to find a g-independent subset $M \subseteq C$ such that for the corresponding partner set we have $|L^g(M)| < |M|$. Hence, the algorithm $\mathcal{A}(g)$ yields a payoff of $\frac{1}{2}$ for each player contained in C in both cases (recall Definition 1 and the subsequent payoff computation rule). Now consider two players $i, j \in N$ with $ij \notin g$ where

(a) both are contained in the same component inducing an odd circle,

(b) they are contained in different components each inducing an odd circle,

(c) they are contained in different components each inducing a separated pair,

(d) one is contained in a component inducing an odd circle and the other one is contained in a component inducing a separated pair, or

(e) one is contained in a component inducing an odd circle and the other one is an isolated player.

Then in each of these cases the algorithm $\mathcal{A}(g + ij)$ again yields a payoff of $\frac{1}{2}$ for all players contained in the new component $C_i(g + ij) = C_j(g + ij)$. The best way to see this is again to consider Definition 1 and the subsequent payoff computation rule.[31] Therefore, at least one of the two players i and j (in Cases (a)–(d) even both) will receive an unchanged payoff after having established this link. Regarding profits this means, however, that this player is worse off as she has to bear additional linking costs of $c > 0$. Thus, the respective link will never be formed.

Next, recall the three-player case. From this it is straightforward to see that Part (i) of the theorem is indeed true. Also, we can deduce that a pairwise stable network can contain both an isolated player and a separated pair if we have linking costs $c \in (\frac{1}{6}, \frac{1}{2}]$. Together with the above Case (c) this establishes Part (ii) of the theorem.

Consider again a network g and now two players $i', j' \in N$ with $i'j' \in g$. Moreover, assume that these players are contained in a component C which induces an odd circle with $m \geq 3$ players. We already know that g induces a payoff of $\frac{1}{2}$ for both players. Now consider the network $g' := g - i'j'$ which is obviously a line of length m. Let

[31] However, a shortcut would be to consider Manea (2011, Theorem 5) which we make use of when proving our Theorem 2.

$(r'_s, x'_s, M'_s, L'_s, N'_s, g'_s)_{s=1,\dots,\bar{s}'}$ be the outcome of $\mathcal{A}(g')$ (with $N_1 = C$).[32] As m is an odd number, we have that $\bar{s}' = 1$ and $i', j' \in M'_1$. Further, it is $|M'_1| = \frac{m+1}{2}$ and $|L'_1| = \frac{m-1}{2}$ which implies that $v^*_{i'}(g') = v^*_{j'}(g') = x'_1 = \frac{m-1}{2m}$. As a stability condition this gives

$$u^*_{i'}(g) - u^*_{i'}(g') = \frac{1}{2} - 2c - \Big(\frac{m-1}{2m} - c\Big) \geq 0 \;\Leftrightarrow\; \frac{1}{2m} \geq c \;\Leftrightarrow\; m \leq \frac{1}{2c}.$$

Of course, the same holds for player j'. Together with the above Cases (a) and (b) this means that a network which is composed of odd circles is pairwise stable if and only if each circle has at most $\frac{1}{2c}$ members. Note that a pairwise stable network can therefore contain odd circles only if we have $c \leq \frac{1}{6}$ since a circle must have at least three members by definition.

Furthermore, observe that for the cost range $c \in (0, \frac{1}{6}]$ we have $\frac{1}{2} - c > 0$ which means that no player contained in a component inducing a separated pair has incentives to delete her link. This together with the above Cases (c)–(e) gives that, potentially besides one or several odd circles with a permissible number of players, a network being pairwise stable at $c \in (0, \frac{1}{6}]$ can contain separated pairs or one isolated player. As we know from the three-player case, however, an isolated player and a separated pair cannot coexist in a pairwise stable network at these levels of linking costs. This proves the first statement in Part (iii).

Finally, consider the transition point $c = \frac{1}{6}$. In what follows, let the network g be composed of two lines of length three, an odd circle, and a separated pair as sketched in Figure 4.

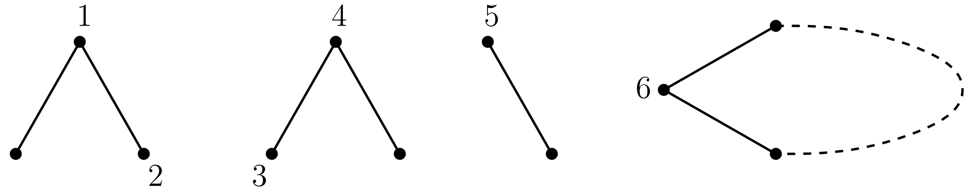


Figure 4: A sketch of a network g containing lines of length three

W.l.o.g. we focus on the labeled players $1, 2, ..., 6$. At $c = \frac{1}{6}$ the algorithm $\mathcal{A}(g)$ yields the following profits:

$$u^*_1(g) = \frac{2}{3} - 2c = \frac{1}{3}, \quad u^*_2(g) = u^*_3(g) = \frac{1}{3} - c = \frac{1}{6}, \quad u^*_6(g) = \frac{1}{2} - 2c = \frac{1}{6}$$

Based on this, we are able to establish that link addition either leads to a worsening for at least one of the two players or both are indifferent. Specifically, applying the

[32]Disregarding players in $C^{\complement}$ is w.l.o.g. as the profile of payoffs respectively profits is component-decomposable.

respective algorithm gives

$$
\begin{aligned}
&u_2^*(g+23) = u_3^*(g+23) = \frac{1}{2} - 2c = \frac{1}{6} = u_2^*(g) = u_3^*(g),\\
&u_1^*(g+13) = u_1^*(g+14) = u_1^*(g+15) = u_1^*(g+16) = \frac{2}{3} - 3c = \frac{1}{6} < \frac{1}{3} = u_1^*(g),\\
&u_2^*(g+25) = \frac{2}{5} - 2c = \frac{1}{15} < \frac{1}{6} = u_2^*(g),\\
&u_6^*(g+26) = \frac{1}{2} - 3c = 0 < \frac{1}{6} = u_6^*(g).
\end{aligned}
$$

Since we know from the three-player case that within the component of a line of length three there are no incentives to add or delete a link at this cost level, this concludes the proof of Part (iii) and of the whole theorem. □

A.2 Proof of Theorem 2

Consider a pairwise stable network g and assume that $g|_{\tilde{N}(g)}$ is not a union of separated pairs and odd circles. Notice that due to Proposition 1 for any component $C \subseteq N$ of g it must either be $C \subseteq \tilde{N}(g)$ or $C \subseteq \tilde{N}(g)^{\complement}$. Furthermore, recall that the profile of payoffs is component-decomposable, meaning that $v_i^*(g) = v_i^*(g|_{\tilde{N}(g)})$ for all $i \in \tilde{N}(g)$. Thus, the network $g|_{\tilde{N}(g)}$ is equitable such that by Manea (2011, Theorem 5) respectively by Berge (1981) it has a so called edge cover composed of separated pairs and odd circles. This means that there exists a union of separated pairs and odd circles $g' \subseteq g|_{\tilde{N}(g)}$ such that no player $i \in \tilde{N}(g)$ is isolated in g'. By assumption there must exist a link $ij \in g|_{\tilde{N}(g)} \backslash g'$. Obviously, the network g' is also an edge cover of the network $g|_{\tilde{N}(g)} - ij$. Again from Manea (2011, Theorem 5) respectively from Berge (1981) it then follows that $g|_{\tilde{N}(g)} - ij$ is still equitable. Hence, recalling the implication of Proposition 1 mentioned above, this gives

$$
\begin{aligned}
u_i^*(g) = v_i^*(g|_{\tilde{N}(g)}) - \eta_i(g|_{\tilde{N}(g)})c = \frac{1}{2} - \eta_i(g|_{\tilde{N}(g)})c &< \frac{1}{2} - \Big(\eta_i(g|_{\tilde{N}(g)}) - 1\Big)c\\
&= v_i^*\Big(g|_{\tilde{N}(g)} - ij\Big) - \eta_i\Big(g|_{\tilde{N}(g)} - ij\Big)c\\
&= u_i^*(g - ij).
\end{aligned}
$$

Thus, arriving at a contradiction, this concludes the proof. □

A.3 Proof of Lemma 1 and Lemma 2

As announced in Section 3, the proofs of both lemmas rest on another rather technical lemma which be provide and prove first.

Lemma 3. *Let $\tilde{g}$ be a network with $\mathcal{A}(\tilde{g})$ providing $(\tilde{r}_s, \tilde{x}_s, \tilde{M}_s, \tilde{L}_s, \tilde{N}_s, \tilde{g}_s)_s$. For any step $s < \bar{s}$ and any set $I \subseteq N$ the following implications must apply:*

$$\begin{aligned} &\textit{(i)} \quad 1 \leq |\tilde{M}_s \cap I| \leq |\tilde{L}_s \cap I| \quad \Rightarrow \quad |L^{\tilde{g}_s}(\tilde{L}_s \cap I) \cap \tilde{M}_s \cap I^{\complement}| \geq 1, \\ &\textit{(ii)} \quad 1 \leq |\tilde{M}_s \cap I| < |\tilde{L}_s \cap I| \quad \Rightarrow \quad |L^{\tilde{g}_s}(\tilde{L}_s \cap I) \cap \tilde{M}_s \cap I^{\complement}| \geq 2. \end{aligned}$$

Proof of Lemma 3. We prove the two parts of the lemma one after the other.

Part (i):
Assume that we have $1 \leq |\tilde{M}_s \cap I| \leq |\tilde{L}_s \cap I|$ and $L^{\tilde{g}_s}(\tilde{L}_s \cap I) \cap \tilde{M}_s \cap I^{\complement} = \emptyset$ for some step $s < \bar{s}$ and some set $I \subseteq N$. Recalling Definition 1, this obviously implies

$$\frac{|\tilde{L}_s|}{|\tilde{M}_s|} = \tilde{r}_s < 1 \leq \frac{|\tilde{L}_s \cap I|}{|\tilde{M}_s \cap I|}.$$

Additionally, we have that $\tilde{M}_s = (\tilde{M}_s \cap I) \,\dot{\cup}\, (\tilde{M}_s \backslash I)$ and $\tilde{L}_s = (\tilde{L}_s \cap I) \,\dot{\cup}\, (\tilde{L}_s \backslash I)$. This induces that $\tilde{M}_s \backslash I \neq \emptyset$ since it is $|\tilde{M}_s \cap I| \leq |\tilde{L}_s \cap I| \leq |\tilde{L}_s|$ but $|\tilde{M}_s| > |\tilde{L}_s|$. It follows that

$$\frac{|\tilde{L}_s \backslash I|}{|\tilde{M}_s \backslash I|} < \frac{|\tilde{L}_s|}{|\tilde{M}_s|}.$$

Moreover, it is $L^{\tilde{g}_s}(\tilde{M}_s \backslash I) \subseteq \tilde{L}_s \backslash I$ since by assumption $L^{\tilde{g}_s}(\tilde{L}_s \cap I) \cap \tilde{M}_s \subseteq I$. Taken together, this then gives

$$\frac{|L^{\tilde{g}_s}(\tilde{M}_s \backslash I)|}{|\tilde{M}_s \backslash I|} \leq \frac{|\tilde{L}_s \backslash I|}{|\tilde{M}_s \backslash I|} < \frac{|\tilde{L}_s|}{|\tilde{M}_s|} = \tilde{r}_s,$$

which contradicts the minimality of $\tilde{r}_s$.

Part (ii):
It remains to show that having $1 \leq |\tilde{M}_s \cap I| < |\tilde{L}_s \cap I|$ and $|L^{\tilde{g}_s}(\tilde{L}_s \cap I) \cap \tilde{M}_s \cap I^{\complement}| = 1$ in some step $s < \bar{s}$ and for some set $I \subseteq N$ leads to a contradiction as well. In such a situation, slightly different from Part (i), we have

$$\frac{|\tilde{L}_s|}{|\tilde{M}_s|} = \tilde{r}_s < 1 \leq \frac{|\tilde{L}_s \cap I|}{|\tilde{M}_s \cap I| + 1}.$$

Again, it holds that $\tilde{M}_s = (\tilde{M}_s \cap I) \,\dot{\cup}\, (\tilde{M}_s \backslash I)$ and $\tilde{L}_s = (\tilde{L}_s \cap I) \,\dot{\cup}\, (\tilde{L}_s \backslash I)$, which in this case even guarantees that $|\tilde{M}_s \backslash I| \geq 2$ since it is $|\tilde{M}_s \cap I| < |\tilde{L}_s \cap I| \leq |\tilde{L}_s|$, but $|\tilde{M}_s| > |\tilde{L}_s|$. This gives

$$\frac{|\tilde{L}_s \backslash I|}{|\tilde{M}_s \backslash I| - 1} < \frac{|\tilde{L}_s|}{|\tilde{M}_s|}.$$

Moreover, we have that there exists exactly one player $\tilde{i} \in L^{\tilde{g}_s}(\tilde{L}_s \cap I) \cap \tilde{M}_s \cap I^{\complement}$. Similarly to Part (i) this implies that it is $L^{\tilde{g}_s}(\tilde{M}_s \backslash (I \dot{\cup} \{\tilde{i}\})) \subseteq \tilde{L}_s \backslash I$, which combined with the above leads to

$$\frac{|L^{\tilde{g}_s}(\tilde{M}_s \backslash (I \dot{\cup} \{\tilde{i}\}))|}{|\tilde{M}_s \backslash (I \dot{\cup} \{\tilde{i}\})|} \leq \frac{|\tilde{L}_s \backslash I|}{|\tilde{M}_s \backslash I| - 1} < \frac{|\tilde{L}_s|}{|\tilde{M}_s|} = \tilde{r}_s,$$

which again contradicts the minimality of $\tilde{r}_s$. □

Now, we can turn to the proof of the first of the two lemmas which are stated in Section 3.

Proof of Lemma 1. For $i, j \in M_1$ consider the network $g' := g + ij$. Let $(r'_s, x'_s, M'_s, L'_s, N'_s, g'_s)_{s=1,...,\bar{s}'}$ be the outcome of $\mathcal{A}(g')$. Assume for contradiction that there exists a step $\hat{s} \in \{1, ..., \bar{s}' - 1\}$ such that $L_1 \cap M'_s = M_1 \cap L'_s = \emptyset$ for all $s \in \{1, ..., \hat{s}-1\}$ but $M_1 \cap L'_{\hat{s}} \neq \emptyset$. Note that $L_1 \cap M'_{\hat{s}} \neq \emptyset$ would also entail $M_1 \cap L'_{\hat{s}} \neq \emptyset$. This is because any player contained in $L_1 \cap M'_{\hat{s}}$ must have a neighbor $k \in M_1$ in g due to the minimality of $r_1 < 1$ and it can obviously neither be $k \in L'_s$ nor $k \in M'_s$ for any $s \in \{1, ..., \hat{s} - 1\}$. In what follows, we construct a sequence of players $(i_0, i_1, i_2, ...)$ and show by induction that the underlying procedure which sequentially adds players to it can never break up so that we get a contradiction to the finiteness of the player set N. For $m \in \mathbb{N}$ let $I_m := \{i_0, i_1, ..., i_m\} \subseteq N$ denote the players of the sequence up to the mth one. We need to distinguish two cases.

<u>Case 1:</u> $i \in L'_{\hat{s}}$

Set $i_0 = i$. It then must be $|N_{i_0}(g'_{\hat{s}}) \cap M'_{\hat{s}}| \geq 2$ since otherwise one could reduce $r'_{\hat{s}}$ by not including i_0 and possibly her one contact belonging to $M'_{\hat{s}}$. This guarantees that there exists $i_1 \in N_{i_0}(g'_{\hat{s}}) \cap M'_{\hat{s}} \backslash \{j\}$. So it is $i_0 \in M_1 \cap L'_{\hat{s}}$ and $i_1 \in L_1 \cap M'_{\hat{s}}$. Let $I_1 = \{i_0, i_1\}$. Now consider some odd number $m \in \mathbb{N}$. Assume that $L_1 \cap I_m = M'_{\hat{s}} \cap I_m$, $M_1 \cap I_m = L'_{\hat{s}} \cap I_m$ and that the cardinalities of these two sets are equal. We then have:

- It is $1 \leq |M_1 \cap I_m| = |L_1 \cap I_m|$ and therefore by Lemma 3(i) there exists a player $i_{m+1} \in L^g(L_1 \cap I_m) \cap M_1 \cap I_m^{\complement}$. For this player it must hold that $i_{m+1} \in M_1 \cap L'_{\hat{s}} \backslash I_m$ since $L_1 \cap I_m \subseteq M'_{\hat{s}}$ and $M_1 \cap L'_s = \emptyset$ for all $s \in \{1, ..., \hat{s} - 1\}$.
- It then is $1 \leq |M'_{\hat{s}} \cap I_{m+1}| < |L'_{\hat{s}} \cap I_{m+1}|$ and therefore by Lemma 3(ii) there exists a player $i_{m+2} \in L^{g'_{\hat{s}}}(L'_{\hat{s}} \cap I_{m+1}) \cap M'_{\hat{s}} \cap I_{m+1}^{\complement} \backslash \{j\}$. For this player it must hold that $i_{m+2} \in L_1 \cap M'_{\hat{s}} \backslash I_{m+1}$ since $L'_{\hat{s}} \cap I_{m+1} \subseteq M_1$ and $i_{m+2} \neq j$.

Thus, it is $L_1 \cap I_{m+2} = M'_{\hat{s}} \cap I_{m+2}$, $M_1 \cap I_{m+2} = L'_{\hat{s}} \cap I_{m+2}$ and also the cardinalities of these two sets are equal. Moreover, it is $|I_{m+2}| = |I_m| + 2$. By induction it follows that the player set N must be infinitely large. Thus, we arrive at a contradiction.

Case 2: $i \notin L'_{\hat{s}}$
In this case we must have $j \notin M'_{\hat{s}}$ since by assumption $M_1 \cap L'_s = \emptyset$ for all $s \in \{1, ..., \hat{s} - 1\}$. For the same reason, $i \in M'_{\hat{s}}$ would imply $j \in L'_{\hat{s}}$ which is equivalent to Case 1. This is also true for $i \notin M'_{\hat{s}}$ and $j \in L'_{\hat{s}}$. So it remains to consider the case that $i, j \notin (M'_{\hat{s}} \cup L'_{\hat{s}})$. However, by assumption there must be a player $i_0 \in M_1 \cap L'_{\hat{s}}$. As in the previous case, existence of another player $i_1 \in N_{i_0}(g'_{\hat{s}}) \cap M'_{\hat{s}}$ is then guaranteed and it must be $i_1 \notin \{i, j\}$ since $i, j \notin M'_{\hat{s}}$. Therefore it is $i_1 \in L_1 \cap M'_{\hat{s}}$. Let again $I_1 = \{i_0, i_1\}$ and assume for some odd number $m \in \mathbb{N}$ that $L_1 \cap I_m = M'_{\hat{s}} \cap I_m$, $M_1 \cap I_m = L'_{\hat{s}} \cap I_m$ and that the cardinalities of these two sets are equal. Furthermore, assume that $i, j \notin I_m$. Similarly to the first case we have:

- There exists $i_{m+1} \in M_1 \cap L'_{\hat{s}} \backslash I_m$ for the stated reasons.
- By Lemma 3(ii) there then exists a player $i_{m+2} \in L^{g'_{\hat{s}}}(L'_{\hat{s}} \cap I_{m+1}) \cap M'_{\hat{s}} \cap I^{\complement}_{m+1}$. For this player it must hold that $i_{m+2} \in L_1 \cap M'_{\hat{s}} \backslash I_{m+1}$ since $L'_{\hat{s}} \cap I_{m+1} \subseteq M_1 \backslash \{i, j\}$.

Thus, it is again $L_1 \cap I_{m+2} = M'_{\hat{s}} \cap I_{m+2}$, $M_1 \cap I_{m+2} = L'_{\hat{s}} \cap I_{m+2}$ and also the cardinalities of these two sets are equal. Beyond that, we have $i, j \notin I_{m+2}$. By induction this leads again to a contradiction to the finiteness of the player set N.

Summing up, we have that $L_1 \cap M'_s = M_1 \cap L'_s = \emptyset$ for all $s < \bar{s}'$. Therefore, it must be $v^*_i(g'), v^*_j(g') \leq \frac{1}{2}$. On the contrary, we know by Manea (2011, Proposition 2) that $v^*_i(g') + v^*_j(g') \geq 1$. Taken together, this implies $v^*_i(g') = v^*_j(g') = \frac{1}{2}$.

With regard to the second part of the lemma consider the network $g' := g + i(n+1)$ and let $(r'_s, x'_s, M'_s, L'_s, N'_s, g'_s)_{s=1,...,\bar{s}'}$ be the outcome of $\mathcal{A}(g')$. It is clear that $n+1 \notin L'_s$ for all $s < \bar{s}'$ since otherwise one could simply reduce r'_s by deleting $n+1$ from L'_s and possibly her one neighbor i from M'_s. The possibility that $i \in L'_s$ for some $s < \bar{s}'$ can be ruled out by a line of argumentation which is equivalent to the proof of the first part if one substitutes $n+1$ for j, M_2 for M_1 and L_2 for L_1 (while taking into account that $\mathcal{A}(g)$ provides $M_1 = \{n+1\}$ and $L_1 = \emptyset$ in this case). □

And finally we establish the second of the two lemmas.

Proof of Lemma 2. W.l.o.g. assume that g has only one component.[33] Beside g consider the network $g' := g - kl$ and let $(r'_s, x'_s, M'_s, L'_s, N'_s, g'_s)_{s=1,...,\bar{s}'}$ be the outcome of $\mathcal{A}(g')$. Similarly to the proof of Lemma 1 assume for contradiction that there exists

[33]Again, this is w.l.o.g. as the profile of payoffs respectively profits is component-decomposable.

a step $\hat{s} \in \{1, ..., \bar{s}' - 1\}$ such that $L_1 \cap M'_s = M_1 \cap L'_s = \emptyset$ for all $s \in \{1, ..., \hat{s} - 1\}$ but $L_1 \cap M'_{\hat{s}} \neq \emptyset$. Observe that $M_1 \cap L'_{\hat{s}} \neq \emptyset$ would also entail $L_1 \cap M'_{\hat{s}} \neq \emptyset$ since due to the minimality of $r'_{\hat{s}}$ any player in $M_1 \cap L'_{\hat{s}}$ needs to have a g'-neighbor in $M'_{\hat{s}}$ who then must have been a neighbor in g as well. We again construct a sequence of players $(i_0, i_1, i_2, ...)$ and show by induction that the underlying procedure which sequentially adds players to it can never break up, meaning that we get a contradiction to the finiteness of the player set N. For $m \in \mathbb{N}$ let $I_m := \{i_0, i_1, ..., i_m\} \subseteq N$ denote the players of the sequence up to the mth one.

Initially, select some player $i_0 \in L_1 \cap M'_{\hat{s}}$. i_0 cannot be isolated or a loose-end player, i.e. she must have more than one link in g, since otherwise one could reduce r_1 by not including i_0 in L_1 and possibly her one contact in M_1. This guarantees that there exists $i_1 \in N_{i_0}(g')$. It must be $i_1 \in M_1 \cap L'_{\hat{s}}$ since by assumption $M_1 \cap L'_s = \emptyset$ for all $s \in \{1, ..., \hat{s} - 1\}$. Let $I_1 = \{i_0, i_1\}$. Now consider some odd number $m \in \mathbb{N}$. Assume that $L_1 \cap I_m = M'_{\hat{s}} \cap I_m$, $M_1 \cap I_m = L'_{\hat{s}} \cap I_m$ and that the cardinalities of these two sets are equal. We then have:

- It is $1 \leq |M'_{\hat{s}} \cap I_m| = |L'_{\hat{s}} \cap I_m|$ and therefore by Lemma 3(i) there exists a player $i_{m+1} \in L^{g'_{\hat{s}}}(L'_{\hat{s}} \cap I_m) \cap M'_{\hat{s}} \cap I_m^{\complement}$. For this player it must hold that $i_{m+1} \in L_1 \cap M'_{\hat{s}} \backslash I_m$ since it is $L'_{\hat{s}} \cap I_m \subseteq M_1$.

- Then it is $1 \leq |M_1 \cap I_{m+1}| < |L_1 \cap I_{m+1}|$ and therefore by Lemma 3(ii) there exists a player $i_{m+2} \in L^g(L_1 \cap I_{m+1}) \cap M_1 \cap I_{m+1}^{\complement} \cap L^{g'_{\hat{s}}}(L_1 \cap I_{m+1})$ since g' arose from g by a single link deletion and, additionally, $M_1 \cap L'_s = \emptyset$ for all $s \in \{1, ..., \hat{s} - 1\}$ and $L_1 \cap I_{m+1} \subseteq M'_{\hat{s}}$. This reasoning then also implies that $i_{m+2} \in M_1 \cap L'_{\hat{s}} \backslash I_{m+1}$.

Thus it is $L_1 \cap I_{m+2} = M'_{\hat{s}} \cap I_{m+2}$, $M_1 \cap I_{m+2} = L'_{\hat{s}} \cap I_{m+2}$ and also the cardinalities of these two sets are equal. Moreover, it is $|I_{m+2}| = |I_m| + 2$. Again, by induction this leads to a contradiction to the finiteness of the player set N. Consequently, it must be $L_1 \cap M'_s = M_1 \cap L'_s = \emptyset$ for all $s < \bar{s}'$. □

A.4 Proof of Proposition 2

In what follows, we consider the proposition's three parts separately.

Proof of Proposition 2(i). Consider a network g which is a tree with $n > 3$ players and assume that it is pairwise stable.[34] By Theorem 2 it cannot be the case that all players receive a payoff of $\frac{1}{2}$ in g. According to Proposition 1 and Corollary 2, the algorithm $\mathcal{A}(g)$ therefore has to stop after the first step providing an outcome

[34]Again, it is w.l.o.g. to assume that g consists of only one component as the profile of payoffs respectively profits is component-decomposable.

$(r_1, x_1, M_1, L_1, N_1, g_1)$ with $M_1 \dot{\cup} L_1 = N$, $|M_1| > |L_1|$ and $g|_{M_1} = g|_{L_1} = \emptyset$. So we have $r_1 \in (0,1)$ and $v_i^*(g) = 1 - v_j^*(g) = x_1 \in (0, \frac{1}{2})$ for all $i \in M_1,\ j \in L_1$. Theorem 3 then implies that

$$x_1 + c = \frac{1}{2}. \tag{6}$$

The class of tree networks we consider here can be divided into the following subclasses:

(a) No player has more than two links in g, meaning that g is a line network.

(b) There is a player who has more than two links in g such that at least two of her neighbors are loose-end players.[35]

(c) There is a player who has more than two links in g but no player has more than one loose-end contact.

In the following, we distinguish between these three subclasses and show separately that there arises a contradiction to pairwise stability.

Subclass (a):

W.l.o.g. let $g := \{12, 23, ..., (n-1)n\}$. Here n must be odd since otherwise it would obviously be $\frac{|L^g(M)|}{|M|} \geq 1$ for all g-independent sets $M \subseteq N$ inducing a payoff of $\frac{1}{2}$ for every player. So by assumption it must be $n \geq 5$. Considering the algorithm $\mathcal{A}(g)$, we find that the shortage ratio is minimized by the g-independent set which contains the players $1, 3, ..., n-2, n$. Therefore, it is $r_1 = \frac{n-1}{n+1}$ and $x = \frac{n-1}{2n}$. Hence, here equation (6) is equivalent to

$$c = \frac{1}{2n}. \tag{7}$$

Now, if player 3 deletes her link to player 2, then she becomes a loose-end player. Moreover, in the network $g - 23$ she is contained in a component with an odd number of players which induces a line of length $n-2$. Hence, it is $v_3^*(g-23) = \frac{n-3}{2(n-2)}$. Taking into account equation (7), the corresponding stability condition yields

$$\begin{aligned} u_3^*(g) - u_3^*(g-23) \geq 0 \quad &\Leftrightarrow \quad v_3^*(g) - v_3^*(g-23) - c \geq 0 \\ &\Leftrightarrow \quad \frac{n-1}{2n} - \frac{n-3}{2(n-2)} - \frac{1}{2n} \geq 0 \end{aligned}$$

[35] Recall that some $i \in N$ is said to be a loose-end player if it is $\eta_i(g) = 1$, that is if she has exactly one link in g.

$$\Leftrightarrow \quad \frac{4-n}{2n(n-2)} \geq 0.$$

Obviously, this is never fulfilled for $n \geq 5$, meaning that a line network cannot be pairwise stable.

Subclass (b):

Let $k \in N$ be a player with at least three neighbors including two or more loose-end players. Then Manea (2011, Theorem 3) implies that it is $v_k^*(g) \geq \frac{2}{3}$. So it must be $k \in L_1$. Select a player $i \in N_k(g)$ such that $\eta_i(g) \geq \eta_{i'}(g)$ for all $i' \in N_k(g)$. Note that in the network $g - ki$, player k still has at least two loose-end contacts such that again according to Manea (2011, Theorem 3) we have $v_k^*(g - ki) \geq \frac{2}{3}$. The corresponding stability condition then gives

$$u_k^*(g) \geq u_k^*(g-ki) \;\Leftrightarrow\; v_k^*(g) - c \geq v_k^*(g-ki) \;\Rightarrow\; 1 - x_1 - c \geq \frac{2}{3} \;\Leftrightarrow\; x_1 + c \leq \frac{1}{3}.$$

This obviously contradicts equation (6). Thus, a network g belonging to Subclass (b) cannot be pairwise stable.

Subclass (c):

First deliberate the following: For any tree network $\tilde{g}$ and any player $k \in N$ there exists a unique partition $\left(Br_\nu^k\right)_{\nu \in N_k(\tilde{g})}$ of $N \backslash \{k\}$ such that for all $\nu \in N_k(\tilde{g})$ it is $\nu \in Br_\nu^k$ and $\tilde{g}|_{Br_\nu^k}$ is connected, i.e. $\tilde{g}|_{Br_\nu^k}$ has only one component (if one restricts the player set to Br_ν^k). Based on this observation, we define the subnetworks $\left(\tilde{g}|_{Br_\nu^k}\right)_{\nu \in N_k(g)}$ to be the *branches* of player k in $\tilde{g}$ and $\nu \in N_k(\tilde{g})$ is said to be the *fork player* of $\tilde{g}|_{Br_\nu^k}$.

Note that if g belongs to Subclass (c), then there exists a player $k \in N$ who has more than two links such that for at least all but one of her branches, all players contained in these have at most two links in g. If this would not be the case, the following procedure would never stop, meaning that there would have to be infinitely many players in N: Initially, select a player k_0 having more than two links and one of her branches containing another player k_1 with more than two links. Then by assumption player k_1 must have a branch in g which does not contain player k_0 but a player k_2 who also has more than two links. For this player k_2 there must again be a branch in g not containing k_0 and k_1 but a player k_3 having more than two links. Continuing this way, for any $m \in \mathbb{N}$ there is a player $k_{m+1} \in N \backslash \{k_0, ..., k_m\}$, which then gives a contradiction by induction. Thus, a player k as mentioned above must indeed exist.

In the following we distinguish two cases.

Case (c).1: $k \in L_1$

If there are other players having more than two links, then let $i \in N$ be the fork player of player k's branch which contains all of them. Otherwise, arbitrarily pick some $i \in N_k(g)$. In both cases consider the network $g - ki$ and the component $C \subset N$ which player k is contained in. In the network $g|_C$, there is only player k who might have more than two links. Furthermore, every branch of player k in $g|_C$ must be a line of odd length as Manea (2011, Theorem 3) implies that any loose-end player in g is contained in M_1. In turn, this implies that for any $g|_C$-independent set M with $\frac{|L^{g|_C}(M)|}{|M|} < 1$ it is $k \in L^{g|_C}(M)$. One example for such a set is $M_1 \cap C$ with partner set $L_1 \cap C$. Hence, it must be $v_k^*(g - ki) > \frac{1}{2}$. The corresponding stability condition then gives

$$u_k^*(g) \geq u_k^*(g - ki) \;\Leftrightarrow\; v_k^*(g) - c \geq v_k^*(g - ki) \;\Rightarrow\; 1 - x_1 - c > \frac{1}{2} \;\Leftrightarrow\; x_1 + c < \frac{1}{2}.$$

This obviously again contradicts equation (6). Consequently, a network g belonging to Subclass (c) with $k \in L_1$ cannot be pairwise stable.

Case (c).2:

We need to introduce some additional notation here. Identify a branch of player k which is a line network with minimal length among all of these line branches. We denote the set of players in this branch by $B^1 \subset N$. Note that any branch of player k which is a line must be of even length. Let $\hat{M}^1 := M_1 \cap B^1$ and $\hat{L}^1 := L_1 \cap B^1$. Then it is $|\hat{M}^1| = |\hat{L}^1|$. Let j denote the fork player of this branch. In addition, let $B^2 \subset N$ denote the set of all players contained in the other line branch(es) of player k. Let similarly $\hat{M}^2 := M_1 \cap B^2$ and $\hat{L}^2 := L_1 \cap B^2$. Then we have $|\hat{M}^2| = |\hat{L}^2| \geq |\hat{M}^1|$. Finally, let $B^3 := N \backslash (B^1 \,\dot{\cup}\, B^2 \,\dot{\cup}\, \{k\})$ and $\hat{M}^3 := M_1 \cap B^3$, $\hat{L}^3 := L_1 \cap B^3$. Then it must be $|\hat{M}^3| \geq |\hat{L}^3|$ as we have $|M_1| > |L_1|$.

Note that we must have $r_1 = \frac{|L_1|}{|M_1|} \leq \frac{|\hat{L}^3|}{|\hat{M}^3|}$ since r_1 is the minimal shortage ratio for g and obviously $L^g(\hat{M}^3) = \hat{L}^3$. Thus, applying the above notation gives

$$x_1 = \frac{|L_1|}{|M_1| + |L_1|} = \frac{|\hat{M}^1| + |\hat{M}^2| + |\hat{L}^3|}{2|\hat{M}^1| + 2|\hat{M}^2| + |\hat{M}^3| + |\hat{L}^3| + 1}. \tag{8}$$

Now consider the network $g' := g - kj$ and let $(r'_s, x'_s, M'_s, L'_s, N'_s, g'_s)_{s=1,\dots,\bar{s}'}$ be the outcome of the algorithm $\mathcal{A}(g')$. Notice first that the set $\hat{M}^2 \,\dot{\cup}\, \hat{M}^3 \,\dot{\cup}\, \{k\} \subset M_1$ is g'-independent and $\hat{L}^2 \,\dot{\cup}\, \hat{L}^3$ is the corresponding partner set in g'. Furthermore, we have

$$\frac{|\hat{L}^2| + |\hat{L}^3|}{|\hat{M}^2| + |\hat{M}^3| + 1} = \frac{|\hat{M}^2| + |\hat{L}^3|}{|\hat{M}^2| + |\hat{M}^3| + 1} < 1.$$

Assume for contradiction that there is another g'-independent set $M' \subseteq N$ with partner set $L' = L^{g'}(M') \subseteq N$ which is shortage ratio minimizing in step $s = 1$ of $\mathcal{A}(g')$. Since the set B^1 is a component of g' and it induces a line network of even length where every player receives a payoff of $\frac{1}{2}$, we must have $(M' \cup L') \cap B^1 = \emptyset$ and $\bar{s}' \geq 2$. Moreover, Lemma 2 yields that $M_1 \cap L'_s = L_1 \cap M'_s = \emptyset$ for all $s < \bar{s}'$. Hence, we must have $M' \subset \hat{M}^2 \,\dot{\cup}\, \hat{M}^3 \,\dot{\cup}\, \{k\}$ and $L' \subset \hat{L}^2 \,\dot{\cup}\, \hat{L}^3$ such that

$$\frac{|L'|}{|M'|} < \frac{|\hat{M}^2| + |\hat{L}^3|}{|\hat{M}^2| + |\hat{M}^3| + 1} < 1.$$

On the contrary, $M' \,\dot{\cup}\, \hat{M}^1 \subset M_1$ is g-independent and we have $L^g(M' \,\dot{\cup}\, \hat{M}^1) = L' \,\dot{\cup}\, \hat{L}^1$. The minimality of $r_1 = \frac{|L_1|}{|M_1|}$ in $\mathcal{A}(g)$ then implies

$$r_1 = \frac{|\hat{M}^2| + |\hat{L}^3| + |\hat{M}^1|}{|\hat{M}^2| + |\hat{M}^3| + 1 + |\hat{M}^1|} \leq \frac{|L'| + |\hat{M}^1|}{|M'| + |\hat{M}^1|} < 1 \quad \Rightarrow \quad \frac{|\hat{M}^2| + |\hat{L}^3|}{|\hat{M}^2| + |\hat{M}^3| + 1} \leq \frac{|L'|}{|M'|}.$$

Thus, arriving at a contradiction, this implies that

$$v_k^*(g') = \frac{|\hat{M}^2| + |\hat{L}^3|}{2|\hat{M}^2| + |\hat{M}^3| + |\hat{L}^3| + 1}. \tag{9}$$

Taking into account equation (7), the corresponding stability condition demands

$$\begin{aligned} u_k^*(g) \geq u_k^*(g - kj) \quad &\Leftrightarrow \quad v_k^*(g) - \eta_k(g)c \geq v_k^*(g') - \eta_k(g')c \\ &\Leftrightarrow \quad x_1 \geq v_k^*(g') + \frac{1}{2} - x_1 \\ &\Leftrightarrow \quad 2x_1 - v_k^*(g') \geq \frac{1}{2} \end{aligned} \tag{10}$$

However, we now establish that it must be $2x_1 - v_k^*(g') < \frac{1}{2}$. Recalling equations (8) and (9), some calculation yields

$$\begin{aligned} 2x_1 - v_k^*(g') &= \frac{2|\hat{M}^1| + 2(|\hat{M}^2| + |\hat{L}^3|)}{2|\hat{M}^1| + (2|\hat{M}^2| + |\hat{M}^3| + |\hat{L}^3| + 1)} - \frac{(|\hat{M}^2| + |\hat{L}^3|)}{(2|\hat{M}^2| + |\hat{M}^3| + |\hat{L}^3| + 1)} \\ &= \frac{2|\hat{M}^1|(|\hat{M}^2| + |\hat{M}^3| + 1) + (|\hat{M}^2| + |\hat{L}^3|)(2|\hat{M}^2| + |\hat{M}^3| + |\hat{L}^3| + 1)}{2|\hat{M}^1|(2|\hat{M}^2| + |\hat{M}^3| + |\hat{L}^3| + 1) + (2|\hat{M}^2| + |\hat{M}^3| + |\hat{L}^3| + 1)^2} \\ &= \frac{D - R}{2D}, \end{aligned}$$

where

$$D = 2|\hat{M}^1|(2|\hat{M}^2| + |\hat{M}^3| + |\hat{L}^3| + 1) + (2|\hat{M}^2| + |\hat{M}^3| + |\hat{L}^3| + 1)^2 > 0$$

and

$$\begin{aligned}
R = & -2|\hat{M}^1||\hat{M}^3| + 2|\hat{M}^1||\hat{L}^3| - 2|\hat{M}^1| + 2|\hat{M}^2||\hat{M}^3| - 2|\hat{M}^2||\hat{L}^3| + 2|\hat{M}^2| + |\hat{M}^3|^2 \\
& + 2|\hat{M}^3| - |\hat{L}^3|^2 + 1 \\
= & \, 2(\underbrace{|\hat{M}^2| - |\hat{M}^1|}_{\geq 0}) + 2(\underbrace{|\hat{M}^3| - |\hat{L}^3|}_{\geq 0})(\underbrace{|\hat{M}^2| - |\hat{M}^1|}_{\geq 0}) + (\underbrace{|\hat{M}^3|^2 - |\hat{L}^3|^2}_{\geq 0}) + 2|\hat{M}^3| + 1 \\
\geq & \, 2|\hat{M}^3| + 1 \\
> & \, 0.
\end{aligned}$$

Hence, we indeed have

$$2x_1 - v_k^*(g - kj) = \frac{D - R}{2D} < \frac{1}{2}.$$

This conclude the proof for Subclass (c) and of the whole proposition. □

Proof of Proposition 2(ii). Consider a pairwise stable network g and assume that there is a cut-player $k \in N$ who is part of a cycle and receives a payoff $v_k^*(g) > \frac{1}{2}$. Assume w.l.o.g. that g has only one component. According to Proposition 1 and Corollary 2, the algorithm $\mathcal{A}(g)$ must stop after the first step providing an outcome $(r_1, x_1, M_1, L_1, N_1, g_1)$ with $M_1 \dot{\cup} L_1 = N$, $|M_1| > |L_1|$ and $g|_{M_1} = g|_{L_1} = \emptyset$. So we have $r_1 = \frac{|L_1|}{|M_1|} \in (0,1)$, $k \in L_1$ and $v_k^*(g) = 1 - x_1$. Further, by Theorem 3 it is $x_1 + c = \frac{1}{2}$.

In what follows, we prove that player k can delete a certain link such that in the resulting network she still receives a payoff greater than $\frac{1}{2}$. To start with, note that by assumption there must be a set $K \subset N$ with $k \in K$ such that

- $L^g(K \backslash \{k\}) \subseteq K$,
- k is contained in a cycle in $g|_{K^\complement \cup \{k\}}$ and
- $g|_{K^\complement}$ has only one component (as usual, considering $K^\complement$ as player set).

As g has only one component, it must be $k \in L^g(K \backslash \{k\})$, meaning that $N_k(g) \cap K \neq \emptyset$. Moreover, there exists $i' \in N_k(g) \backslash K$ such that k and i' belong to the same cycle in g. Now consider the network $g' := g - ki'$ and let $(r'_s, x'_s, M'_s, L'_s, N'_s, g'_s)_{s=1,\ldots,\bar{s}'}$ be the outcome of $\mathcal{A}(g')$. Lemma 2 yields that $v_k^*(g') \geq \frac{1}{2}$. Assume for contradiction that we have $v_k^*(g') = \frac{1}{2}$, meaning that $k \in N'_{\bar{s}'}$.

Consider the set $C'_k := C_k(g'_{\bar{s}'}|_K) = C_k(g|_{N'_{\bar{s}'} \cap K})$, that is the component of player k in the network g restricted to the set $N'_{\bar{s}'} \cap K$. As a first step, we establish that it is

$$\frac{|L_1 \cap C'_k|}{|M_1 \cap C'_k|} = 1. \tag{11}$$

Note first that we have $N_k(g'|_K) \neq \emptyset$. Furthermore, it must be $N_k(g') \subseteq M_1 \cap N'_{\bar{s}'}$ as Lemma 2 yields $M_1 \cap L'_s = \emptyset$ for all $s < \bar{s}'$. This guarantees that $M_1 \cap C'_k \neq \emptyset$. Based on this, we can immediately rule out the possibility that the left-hand side of (11) is strictly smaller than one since $M_1 \cap C'_k$ is g'-independent and clearly $L^{g'_{\bar{s}'}}(M_1 \cap C'_k) \subseteq L_1 \cap C'_k$. So assume that the left-hand side of (11) is strictly greater than one. We make use of the following implication which we verify at the end of the proof:

$$|\hat{L}| = |\hat{M}| \geq 1 \text{ for } \hat{L} \subseteq L_1 \cap C'_k \backslash \{k\},\ N_k(g) \cap K \subseteq \hat{M} \subseteq M_1 \cap C'_k \\ \Rightarrow\ L^{g'_{\bar{s}'}}(\hat{L}) \backslash \hat{M} \neq \emptyset \quad (12)$$

We know that it is $\emptyset \neq N_k(g) \cap K \subseteq N_k(g') \subseteq N'_{\bar{s}'}$. Let $\hat{M}^0 := N_k(g) \cap K$ such that $\hat{M}^0 \subseteq M_1 \cap C'_k$. Hence, it must be $|L_1 \cap C'_k \backslash \{k\}| \geq |\hat{M}^0|$ since otherwise we would get

$$\frac{|L_1 \cap C'_k|}{|M_1 \cap C'_k|} \leq \frac{|L_1 \cap C'_k|}{|\hat{M}^0|} \leq 1,$$

that is a contradiction to our assumption. So select a set of players $\hat{L}^0 \subseteq L_1 \cap C'_k \backslash \{k\}$ with $|\hat{L}^0| = |\hat{M}^0|$. Note that $\hat{M}^0$ and $\hat{L}^0$ satisfy the conditions of implication (12).

Based on this, we can construct a sequence of players $(j_1, j_2, j_3, ...)$ in a certain way such that according to the previous considerations, the underlying procedure which sequentially adds players to the sequence can never break up. As in the proofs of Lemma 1 and 2, this leads to a contradiction to the finiteness of the player set N. Given such a sequence, let $\hat{M}^m := \{j_l \mid 1 \leq l \leq m,\ l \text{ odd}\} \cup \hat{M}^0$ and $\hat{L}^m := \{j_l \mid 1 \leq l \leq m,\ l \text{ even}\} \cup \hat{L}^0$ for $m \in \mathbb{N}$. Now consider some even number $m \in \mathbb{N} \cup \{0\}$. Assume that $\hat{L}^m \subseteq L_1 \cap C'_k \backslash \{k\}$, $N_k(g) \cap K \subseteq \hat{M}^m \subseteq M_1 \cap C'_k$ and $|\hat{L}^m| = |\hat{M}^m| \geq 1$. We then have:

- By implication (12) there exists $j_{m+1} \in L^{g'_{\bar{s}'}}(\hat{L}^m) \backslash \hat{M}^m$. For this player it must hold that $j_{m+1} \in M_1 \cap C'_k \backslash \hat{M}^m$ since $\hat{L}^m \subseteq L_1 \cap C'_k \backslash \{k\}$.
- Then there must exist $j_{m+2} \in L_1 \cap C'_k \backslash (\hat{L}^{m+1} \,\dot\cup\, \{k\})$ since otherwise we would have
$$1 < \frac{|L_1 \cap C'_k|}{|M_1 \cap C'_k|} \leq \frac{|\hat{L}^{m+1} \,\dot\cup\, \{k\}|}{|\hat{M}^{m+1}|} = 1.$$

Thus it is $\hat{L}^{m+2} \subseteq L_1 \cap C'_k \backslash \{k\}$, $N_k(g) \cap K \subseteq \hat{M}^{m+2} \subseteq M_1 \cap C'_k$ and $|\hat{L}^{m+2}| = |\hat{M}^{m+2}| = |\hat{L}^m| + 1 \geq 1$. By induction this leads to a contradiction to the finiteness of the player set N. This establishes equation (11), however, under the assumption of having $v^*_k(g') = \frac{1}{2}$.

During the second step we now use this to construct a concluding contradiction of similar kind arising from the assumption that $v_k^*(g') = \frac{1}{2}$. Here, we make use of the following implication:

$$|\tilde{L}| = |\tilde{M}| \geq 1 \text{ for } \tilde{L} \subseteq L_1 \cap N'_{\bar{s}'} \backslash K,\ \tilde{M} \subseteq M_1 \cap N'_{\bar{s}'} \backslash K \ \Rightarrow\ L^{g'_{\bar{s}'}}(\tilde{L}) \backslash (\tilde{M} \dot{\cup} K) \neq \emptyset \tag{13}$$

Its verification is postponed to the end of this proof as well. Note that by definition it is $\frac{|L^{g_{\bar{s}'}}(\bar{M})|}{|\bar{M}|} \geq 1$ for all g'-independent sets $\bar{M} \subseteq N'_{\bar{s}'}$. Based on this, we can again construct a sequence of players $(i_1, i_2, i_3, ...)$ such that, according to the previous considerations, the sequential adding of new players can never break up. Thus, we again get a contradiction to the finiteness of the player set N. For this purpose, we define the sets $\tilde{M}^m := \{i_l \mid 1 \leq l \leq m,\ l \text{ odd}\}$ and $\tilde{L}^m := \{i_l \mid 1 \leq l \leq m,\ l \text{ even}\}$ for $m \in \mathbb{N}$.

Initially, select a player $i_1 \in M_1 \cap N'_{\bar{s}'} \backslash K$. Such a player must exist as $k \in L_1 \cap N'_{\bar{s}'}$ is part of a cycle in $g|_{N \backslash K \cup \{k\}}$ and, according to Lemma 2, we have $M_1 \cap L'_s = \emptyset$ for all $s < \bar{s}'$. Now consider some odd number $m \in \mathbb{N}$. Assume that $\tilde{M}^m \subseteq M_1 \cap N'_{\bar{s}'} \backslash K$, $\tilde{L}^m \subseteq L_1 \cap N'_{\bar{s}'} \backslash K$ and that $|\tilde{M}^m| = \frac{m+1}{2} > \frac{m-1}{2} = |\tilde{L}^m|$. We then have:

- $\tilde{M}^m \dot{\cup} (M_1 \cap C'_k) \subseteq N'_{\bar{s}'}$ is g'-independent and

  $$\frac{|\tilde{L}^m \dot{\cup} (L_1 \cap C'_k)|}{|\tilde{M}^m \dot{\cup} (M_1 \cap C'_k)|} < 1$$

  since it is $|L_1 \cap C'_k| = |M_1 \cap C'_k|$ as we know from equation (11). As we have $k \in L^{g'_{\bar{s}'}}(M_1 \cap C'_k) \subseteq L_1 \cap C'_k$, this implies that there must exist a player $i_{m+1} \in L^{g'_{\bar{s}'}}(\tilde{M}^m) \backslash (\tilde{L}^m \dot{\cup} K)$. It is $i_{m+1} \in L_1 \cap N'_{\bar{s}'} \backslash (\tilde{L}^m \dot{\cup} K)$ since $\tilde{M}^m \subseteq M_1$.

- We then have $|\tilde{L}^{m+1}| = |\tilde{M}^{m+1}| = \frac{m+1}{2} \geq 1$ and $\tilde{L}^{m+1} \subseteq L_1 \cap N'_{\bar{s}'} \backslash K$, $\tilde{M}^{m+1} \subseteq M_1 \cap N'_{\bar{s}'} \backslash K$. Hence, by implication (13) there exists $i_{m+2} \in L^{g'_{\bar{s}'}}(\tilde{L}^{m+1}) \backslash (\tilde{M}^{m+1} \dot{\cup} K)$. It is $i_{m+2} \in M_1 \cap N'_{\bar{s}'} \backslash (\tilde{M}^{m+1} \dot{\cup} K)$ since $\tilde{L}^{m+1} \subseteq L_1$.

Thus, we have $\tilde{M}^{m+2} \subseteq M_1 \cap N'_{\bar{s}'} \backslash K$, $\tilde{L}^{m+2} \subseteq L_1 \cap N'_{\bar{s}'} \backslash K$ and $|\tilde{M}^{m+2}| = \frac{(m+2)+1}{2} > \frac{(m+2)-1}{2} = |\tilde{L}^{m+2}|$. Again, by induction this leads to a contradiction to the finiteness of the player set N. This proves that player k's payoff must indeed be strictly greater than $\frac{1}{2}$. The corresponding stability condition then yields

$$u_k^*(g) \geq u_k^*(g - ki') \ \Leftrightarrow\ v_k^*(g) - c \geq v_k^*(g') \ \Rightarrow\ 1 - x_1 - c > \frac{1}{2} \ \Leftrightarrow\ x_1 + c < \frac{1}{2},$$

which is a contradiction to Theorem 3. Hence, such a network g cannot be pairwise stable.

It remains to prove implications (12) and (13). We start with the first one. Given the two sets $\hat{L} \subseteq L_1 \cap C'_k \backslash \{k\}$ and $\hat{M} \subseteq M_1 \cap C'_k$ with $N_k(g) \cap K \subseteq \hat{M}$ and $|\hat{L}| = |\hat{M}| \geq 1$

assume for contradiction that $L^{g'_{\bar{s}'}}(\hat{L}) \subseteq \hat{M}$. Note that we have $N_j(g'_{\bar{s}'}) = N_j(g)$ for all $j \in \hat{L}$ since it is $\hat{L} \subseteq L_1 \cap N'_{\bar{s}'} \backslash \{k\}$ and, according to Lemma 2, $M_1 \cap L'_s = \emptyset$ for all $s < \bar{s}'$. Together with the assumption this implies that $L^g(M_1 \cap K \backslash \hat{M}) \subseteq L_1 \cap K \backslash \hat{L}$. Moreover, since $N_k(g) \cap K \subseteq \hat{M}$, it even is $L^g(M_1 \cap K \backslash \hat{M}) \subseteq L_1 \cap K \backslash (\hat{L} \,\dot{\cup}\, \{k\})$.

Additionally, we need the following inequalities:

$$\frac{|L_1 \cap K| - 1}{|M_1 \cap K|} \leq r_1 \leq \frac{|L_1 \cap K|}{|M_1 \cap K|} \leq 1 \tag{14}$$

To see that these are correct, note first that it is $L^g(M_1 \cap K) \subseteq L_1 \cap K$ and similarly $L^g(M_1 \backslash K) \subseteq L_1 \backslash K \,\dot{\cup}\, \{k\}$. So we must have $r_1 \leq \frac{|L_1 \cap K|}{|M_1 \cap K|}$ and $r_1 \leq \frac{|L_1 \backslash K|+1}{|M_1 \backslash K|}$ as r_1 is the minimal shortage ratio. Moreover, it is $r_1 = \frac{|L_1|}{|M_1|} < 1$, $M_1 = (M_1 \cap K) \,\dot{\cup}\, (M_1 \backslash K)$ and $L_1 = (L_1 \cap K) \,\dot{\cup}\, (L_1 \backslash K)$. Together this implies that $\frac{|L_1 \cap K|-1}{|M_1 \cap K|} = \frac{|L_1|-(|L_1 \backslash K|+1)}{|M_1|-|M_1 \backslash K|} \leq r_1$. In particular, this means that $|L_1 \cap K| - 1 < |M_1 \cap K|$ which in turn implies $\frac{|L_1 \cap K|}{|M_1 \cap K|} \leq 1$.

According to the third inequality in (14) we must have $M_1 \cap K \backslash \hat{M} \neq \emptyset$ since otherwise it would be $|L_1 \cap K| \leq |M_1 \cap K| = |\hat{M}| = |\hat{L}| < |\hat{L} \,\dot{\cup}\, \{k\}| \leq |L_1 \cap K|$. Taken together, this leads to the following contradiction:

$$\begin{aligned} r_1 \leq \frac{|L^g(M_1 \cap K \backslash \hat{M})|}{|M_1 \cap K \backslash \hat{M}|} &\leq \frac{|L_1 \cap K \backslash (\hat{L} \,\dot{\cup}\, \{k\})|}{|M_1 \cap K \backslash \hat{M}|} \\ &= \frac{|L_1 \cap K| - |\hat{L}| - 1}{|M_1 \cap K| - |\hat{M}|} \\ &= \frac{|L_1 \cap K| - 1 - |\hat{L}|}{|M_1 \cap K| - |\hat{L}|} < \frac{|L_1 \cap K| - 1}{|M_1 \cap K|} \leq r_1, \end{aligned}$$

where the last two inequalities are due to (14) and the fact that $r_1 < 1$.

Similarly, to prove implication (13), we consider the two sets $\tilde{L} \subseteq L_1 \cap N'_{\bar{s}'} \backslash K$ and $\tilde{M} \subseteq M_1 \cap N'_{\bar{s}'} \backslash K$ with $|\tilde{L}| = |\tilde{M}| \geq 1$ and assume for contradiction that we have $L^{g'_{\bar{s}'}}(\tilde{L}) \subseteq \tilde{M}$. Again according to Lemma 2, it must be $N_j(g'_{\bar{s}'}) = N_j(g)$ for all $j \in \tilde{L}$. Hence, we have that $L^g(M_1 \backslash \tilde{M}) \subseteq L_1 \backslash \tilde{L}$. Also, it is clear that $M_1 \backslash \tilde{M} \neq \emptyset$ since otherwise we would have $|L_1| < |M_1| = |\tilde{M}| = |\tilde{L}| \leq |L_1|$. Summing up, this implies

$$r_1 \leq \frac{|L^g(M_1 \backslash \tilde{M})|}{|M_1 \backslash \tilde{M}|} \leq \frac{|L_1 \backslash \tilde{L}|}{|M_1 \backslash \tilde{M}|} = \frac{|L_1| - |\tilde{L}|}{|M_1| - |\tilde{M}|} = \frac{|L_1| - |\tilde{L}|}{|M_1| - |\tilde{L}|} < \frac{|L_1|}{|M_1|} = r_1,$$

which is obviously again a contradiction. So we have that $L^{g'_{\bar{s}'}}(\tilde{L}) \backslash (\tilde{M} \,\dot{\cup}\, K) \neq \emptyset$ since it is $L^{g'_{\bar{s}'}}(\tilde{L}) \subseteq K^{\complement}$. This concludes the proof. □

Proof of Proposition 2(iii). Note first that, according to our characterization result, a component-induced subnetwork g' as mentioned in part (iii) could only be contained in a pairwise stable network if the algorithm $\mathcal{A}(g')$ stops after the first step.[36] Let $(r_1, x, M_1, L_1, N_1, g_1)$ denote its outcome. By assumption, we obviously have $|M_1| = m$ and $|L_1| < \frac{m}{3}$. On the contrary, recalling stability condition (5), we get

$$\frac{1}{4} \geq c = \frac{1}{2} - x = \frac{1}{2} - \frac{|L_1|}{|M_1| + |L_1|} = \frac{1}{2}\frac{|M_1| - |L_1|}{|M_1| + |L_1|} \Leftrightarrow 3|L_1| \geq |M_1|.$$

Arriving at a contradiction, this proves the proposition's part (iii). □

References

Abreu, D. and Manea, M. (2012). Bargaining and efficiency in networks. *Journal of Economic Theory*, 147(1):43–70.

Baetz, O. (2015). Social activity and network formation. *Theoretical Economics*, 10(2):315–340.

Bala, V. and Goyal, S. (2000). A noncooperative model of network formation. *Econometrica*, 68(5):1181–1229.

Berge, C. (1981). Some common properties for regularizable graphs, edge-critical graphs and b-graphs. In Saito, N. and Nishizeki, T., editors, *Graph Theory and Algorithms*, pages 108–123. Springer, Berlin Heidelberg.

Calvó-Armengol, A. (2003). Stable and efficient bargaining networks. *Review of Economic Design*, 7(4):411–428.

Calvó-Armengol, A. (2004). Job contact networks. *Journal of Economic Theory*, 115(1):191–206.

Corominas-Bosch, M. (2004). Bargaining in a network of buyers and sellers. *Journal of Economic Theory*, 115(1):35–77.

Galeotti, A., Goyal, S., and Kamphorst, J. (2006). Network formation with heterogeneous players. *Games and Economic Behavior*, 54(2):353–372.

Gauer, F. (2016). *Strategic Interaction and Socio-Economic Structure*. PhD thesis, Bielefeld University.

Goyal, S. and Joshi, S. (2003). Networks of collaboration in oligopoly. *Games and Economic Behavior*, 43(1):57–85.

[36] See again Footnote 23.

Goyal, S. and Joshi, S. (2006). Bilateralism and free trade. *International Economic Review*, 47(3):749–778.

Hellmann, T. (2013). On the existence and uniqueness of pairwise stable networks. *International Journal of Game Theory*, 42(1):211–237.

Hellmann, T. and Landwehr, J. (2014). Stable networks in homogeneous societies. Working Paper, Available at SSRN 2477452.

Jackson, M. O. and Wolinsky, A. (1996). A strategic model of social and economic networks. *Journal of Economic Theory*, 71(1):44–74.

Kranton, R. and Minehart, D. F. (2001). A theory of buyer-seller networks. *American Economic Review*, 91(3):485–508.

Manea, M. (2011). Bargaining in stationary networks. *American Economic Review*, 101(5):2042–2080.

Nash, J. F. (1950). The bargaining problem. *Econometrica*, 18(2):155–162.

Nash, J. F. (1953). Two-person cooperative games. *Econometrica*, 21(1):128–140.

Polanski, A. and Vega-Redondo, F. (2013). Markets, bargaining, and networks with heterogeneous agents. Working Paper, Available at SSRN 2218605.

Rubinstein, A. (1982). Perfect equilibrium in a bargaining model. *Econometrica*, 50(1):97–109.

Rubinstein, A. and Wolinsky, A. (1985). Equilibrium in a market with sequential bargaining. *Econometrica*, 53(5):1133–1150.

Watts, A. (2001). A dynamic model of network formation. *Games and Economic Behavior*, 34(2):331–341.

West, D. B. (2001). *Introduction to Graph Theory*, volume 2. Prentice-Hall, Englewood Cliffs, NJ.