

Zak transforms and Gabor frames of totally positive functions and exponential B-splines[★]

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Abstract

We study totally positive (TP) functions of finite type and exponential B-splines as window functions for Gabor frames. We establish the connection of the Zak transform of these two classes of functions and prove that the Zak transforms have only one zero in their fundamental domain of quasi-periodicity. Our proof is based on the variation-diminishing property of shifts of exponential B-splines. For the exponential B-spline B_m of order m , we determine a large set of lattice parameters $\alpha, \beta > 0$ such that the Gabor family $\mathcal{G}(B_m, \alpha, \beta)$ of time-frequency shifts $e^{2\pi i l \beta} B_m(\cdot - k\alpha)$, $k, l \in \mathbb{Z}$, is a frame for $L^2(\mathbb{R})$. By the connection of its Zak transform to the Zak transform of TP functions of finite type, our result provides an alternative proof that TP functions of finite type provide Gabor frames for all lattice parameters with $\alpha\beta < 1$. For even two-sided exponentials $g(x) = \frac{\lambda}{2}e^{-\lambda|x|}$ and the related exponential B-spline of order 2, we find lower frame-bounds A , which show the asymptotically linear decay $A \sim (1 - \alpha\beta)$ as the density $\alpha\beta$ of the time-frequency lattice tends to the critical density $\alpha\beta = 1$.

Keywords: Gabor frame, total positivity, exponential B-spline, Zak transform

Introduction

The Gabor transform provides an important tool for the analysis of a given signal $f : \mathbb{R} \rightarrow \mathbb{C}$ in time and frequency. It is a fundamental tool for time-frequency analysis and serves various purposes, such as signal denoising or compression. A window function $g \in L^2(\mathbb{R})$ has time-frequency shifts

$$M_\xi T_y g(x) = e^{2\pi i \xi x} g(x - y), \quad \xi, y \in \mathbb{R}.$$

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The Gabor transform of a square-integrable signal f is defined as

$$\mathcal{S}_g f(k\alpha, l\beta) = \langle f, M_{l\beta} T_{k\alpha} g \rangle, \quad k, l \in \mathbb{Z},$$

where the parameters $(k\alpha, l\beta) \in \alpha\mathbb{Z} \times \beta\mathbb{Z}$ of the time-frequency shifts of g form a lattice in $\mathbb{R}^2$, with lattice parameters $\alpha, \beta > 0$. The family

$$\mathcal{G}(g, \alpha, \beta) := \{M_{l\beta} T_{k\alpha} g \mid k, l \in \mathbb{Z}\}$$

is called a Gabor family. An important problem in Gabor analysis is to determine, for a given window function $g \in L^2(\mathbb{R})$ and lattice parameters $\alpha, \beta > 0$, if the Gabor family $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R})$. This means that there exist constants $A, B > 0$, which depend on g, α, β , such that for every $f \in L^2(\mathbb{R})$ we have

$$A\|f\|^2 \leq \sum_{k,l \in \mathbb{Z}} |\langle f, M_{l\beta} T_{k\alpha} g \rangle|^2 \leq B\|f\|^2. \quad (1)$$

Moreover, it is interesting for the selection of sampling and modulation rates in signal analysis to know the whole *frameset*

$$\mathcal{F}_g := \{(\alpha, \beta) \in \mathbb{R}_+^2 \mid \mathcal{G}(g, \alpha, \beta) \text{ is a frame}\}$$

of a given function $g \in L^2(\mathbb{R})$. Until 2012, the framesets of only a few functions, like the one- and two-sided exponential, the Gaussian and the hyperbolic secant, were known. Then it was proved in [12, Theorem 1] that the frameset of a large class of functions, the *totally positive functions of finite type $m \geq 2$* , is given by

$$\mathcal{F}_g = \mathbb{H} := \{(\alpha, \beta) \in \mathbb{R}_+^2 \mid 0 < \alpha\beta < 1\}.$$

This result complements the known fact, that for continuous window functions of the *Wiener space*,

$$W(\mathbb{R}) := \{g \in L^\infty(\mathbb{R}) \mid \|g\|_W = \sum_{n \in \mathbb{Z}} \operatorname{ess\,sup}_{x \in [0,1]} |g(x+n)| < \infty\},$$

the frameset is a subset of $\mathbb{H}$.

In order to describe Gabor families of a window function g , a very helpful tool is the Zak transform

$$Z_\alpha g(x, \omega) := \sum_{k \in \mathbb{Z}} g(x - k\alpha) e^{2\pi i k \alpha \omega}, \quad (x, \omega) \in \mathbb{R}^2.$$

In Approximation Theory, the Zak transform was used by Schoenberg [26] in connection with cardinal spline interpolation. For a polynomial B-spline B_m of degree $m - 1$, Schoenberg called $Z_1 B$ the *exponential Euler spline*. The Zak transform $Z_\alpha g$ has the properties

$$Z_\alpha g(x, \omega + \tfrac{1}{\alpha}) = Z_\alpha g(x, \omega), \quad Z_\alpha g(x + \alpha, \omega) = e^{2\pi i \alpha \omega} Z_\alpha g(x, \omega). \quad (2)$$

Therefore, its values in the lattice cell $[0, \alpha) \times [0, \frac{1}{\alpha})$ define $Z_\alpha g$ completely. A well-known result for Gabor families $\mathcal{G}(g, \alpha, \beta)$, with $\alpha = 1$, $\beta = 1/N$, and $N \in \mathbb{N}$, states that the values

$$A_{\text{opt}} = \operatorname{ess\,inf}_{x, \omega \in [0, 1)} \sum_{j=0}^{N-1} |Z_1 g(x, \omega + \frac{j}{N})|^2, \quad B_{\text{opt}} = \operatorname{ess\,sup}_{x, \omega \in [0, 1)} \sum_{j=0}^{N-1} |Z_1 g(x, \omega + \frac{j}{N})|^2,$$

are the optimal constants A, B in the inequality (1), see [7], [16]. For rational values of $\alpha\beta$, a connection of the Zak transform $Z_\alpha g$ with the frame-bounds of the Gabor family $\mathcal{G}(g, \alpha, \beta)$ was given by Zibulsky and Zeevi [31]. Therefore, the presence and the location of zeros of $Z_\alpha g$ is relevant for the existence of lower frame-bounds in (1). For example, the celebrated Balian-Low theorem [7] states that a Gabor family at the critical density $\alpha\beta = 1$ cannot be a frame, if the window function g or its Fourier transform

$$\hat{g}(\omega) = \int_{\mathbb{R}} g(x) e^{-2\pi i x \omega} dx$$

is continuous and in $W(\mathbb{R})$. The proof in [13] uses the topological argument, that every continuous function with the property (2) must have a zero in every lattice cell.

Motivated by the results in [12], we study three problems in Gabor analysis. First, in Section 3, we investigate zero properties of the Zak transform $Z_\alpha g$, where g is a TP function of finite type. Our main tools are methods from the theory of Tschebycheffian splines which is elaborated in [29] and [22]. We establish the connection of $Z_\alpha g$ with the Zak transform $Z_1 B_m$ of an exponential B-spline in Theorem 3.4. For exponential B-splines B_m of order $m \geq 2$ (and the slightly larger class of periodic exponential B-splines defined in [22]), we show in Theorem 3.2 that the Zak transform $Z_1 B_m$ has exactly one zero $(x, \omega) \in [0, 1)^2$, and this zero is located somewhere on the line $\omega = \frac{1}{2}$. The main argument for the proof is the variation-dimishing property of Tschebycheffian B-splines. As a corollary, we obtain that $Z_\alpha g$ has only one zero $(x, \omega) \in [0, \alpha) \times [0, \frac{1}{\alpha})$, and this zero is located on the line $\omega = \frac{\alpha}{2}$. These results add two new families of examples to the study of Zak transforms with few zeros by Janssen [16].

In Section 4, we prove that the Gabor families $\mathcal{G}(B_m, \alpha, \beta)$ are frames for $L^2(\mathbb{R})$ for certain values $\alpha, \beta > 0$. Another important problem in Gabor analysis is the asymptotic behaviour of the lower frame-bound A in (1) near the critical density $\alpha\beta \approx 1$. This information is required in practical situations where only a minimal rate of oversampling is allowed. It is known that for every continuous window function in $W(\mathbb{R})$, the lower frame-bound tends to 0 as $\alpha\beta$ tends to 1. The only window functions (and scaled versions thereof) in the literature, where the asymptotic behaviour of the lower frame-bound was specified near the critical density, are the Gaussian $g(x) = e^{-\pi x^2}$ and the hyperbolic secant $g(x) = (\cosh \pi x)^{-1}$. It was proved in [3] that constants $c_1, c_2 > 0$ exist such that the optimal lower frame-bound A_{opt} of $\mathcal{G}(g, \alpha, \beta)$ satisfies

$$c_1(1 - \alpha\beta) \leq A_{\text{opt}} \leq c_2(1 - \alpha\beta) \quad \text{for} \quad \frac{1}{2} < \alpha\beta < 1.$$

In Section 5, we consider all symmetric exponential B-splines of order 2 and give explicit lower frame-bounds that exhibit the same linear asymptotic decay when $\alpha = 1$ and β tends to 1. The main ingredient in the proofs of the results in sections 4 and 5 is the fact that collocation matrices of Tschebycheffian B-splines are almost strict totally positive (ASTP) matrices, in the terminology of Gasca et al. [9]. This property is equivalent to the Schoenberg-Whitney conditions [29].

The connection of the Zak transforms of a TP function g of finite type m and an exponential B-spline B_m of order m in Theorem 3.4 opens a new corridor for further study of Gabor frames $\mathcal{G}(g, \alpha, \beta)$. In combination with the results for B_m in section 4, we obtain an alternative proof for [12, Theorem 1] that TP functions of finite type provide Gabor frames for all lattice parameters with $\alpha\beta < 1$. Moreover it allows us to give lower frame-bounds for $\mathcal{G}(g, \alpha, \beta)$ as well, where g is the two-sided exponential $g(x) = \frac{\lambda}{2}e^{-\lambda|x|}$ which is TP of finite type 2.

In the first two sections, we give a brief introduction and provide enough background on total positivity and Tschebycheffian B-splines for our purpose. For non-specialists in Approximation Theory, this may be helpful for reading the remaining sections. The results on Gabor frames which are needed in this article are contained in sections 3-5. For a detailed exposition on Gabor frames, we recommend the monographs [11], [5], a very short account is contained in the introduction in [12].

1. Totally positive functions, matrices and sequences

In this section, we give a short review on total positivity. We refer to [19] for a comprehensive exposition and include some more recent developments which will be helpful in our context.

Totally positive functions were introduced by Schoenberg in 1947 [24]. Schoenberg and Whitney [27] laid an important foundation for their applications in Approximation Theory, e.g. in spline interpolation.

Definition 1.1 (Totally positive (TP) function, [24]). A measurable, non-constant function $g : \mathbb{R} \rightarrow \mathbb{R}$ is called totally positive (TP), if for every $N \in \mathbb{N}$ and two sets of real numbers

$$x_1 < x_2 < \dots < x_N, \quad y_1 < y_2 < \dots < y_N,$$

the corresponding matrix $A = (g(x_j - y_k))_{j,k=1}^N$ has a non-negative determinant. If the determinant is always strictly positive, g is called strictly totally positive (STP).

An example of a TP function is the two-sided exponential function, given by $g(x) = e^{-|x|}$. The Gaussian $g(x) = e^{-\pi x^2}$ is an STP function. Schoenberg characterized TP functions by their Laplace transform [24] and gave a characterization of all integrable TP functions [25].

Theorem 1.2 ([24], [25]). *A function $g : \mathbb{R} \rightarrow \mathbb{R}$, which is not an exponential $g(x) = Ce^{ax}$ with $C, a \in \mathbb{R}$, is a TP function, if and only if its two-sided Laplace transform exists in a strip $S = \{s \in \mathbb{C} \mid \alpha < \operatorname{Re} s < \beta\}$ with $-\infty \leq \alpha < \beta \leq \infty$ and is given by*

$$(\mathcal{L}g)(s) = \int_{-\infty}^{\infty} g(t)e^{-st} dt = Cs^{-n}e^{\gamma s^2 - \delta s} \prod_{\nu=1}^{\infty} \frac{e^{a_{\nu}^{-1}s}}{1 + a_{\nu}^{-1}s},$$

where $n \in \mathbb{N}_0$ and $C, \gamma, \delta, a_{\nu}$ are real parameters with

$$C > 0, \quad \gamma \geq 0, \quad a_{\nu} \neq 0, \quad 0 < \gamma + \sum_{\nu=1}^{\infty} \left(\frac{1}{a_{\nu}}\right)^2 < \infty.$$

Moreover, g is integrable and TP, if and only if its Fourier transform is given by

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} g(t)e^{-2\pi i t \omega} dt = Ce^{-\gamma \omega^2} e^{-2\pi i \delta \omega} \prod_{\nu=1}^{\infty} \frac{e^{2\pi i a_{\nu}^{-1} \omega}}{1 + 2\pi i a_{\nu}^{-1} \omega}$$

with the same conditions on $C, \gamma, \delta, a_{\nu}$ as above.

The subset of TP functions of finite type $m \in \mathbb{N}$ is given by their Fourier transform

$$\hat{g}(\omega) = C \prod_{\nu=1}^m (1 + 2\pi i \omega a_{\nu}^{-1})^{-1}, \quad a_{\nu} \in \mathbb{R} \setminus \{0\}, \quad C > 0. \quad (3)$$

In [12], these are considered being window functions of Gabor frames $\mathcal{G}(g, \alpha, \beta)$. For example, the one- and two-sided exponentials

$$g_1(x) = e^{-x} \chi_{[0, \infty)}(x), \quad g_2(x) = e^{-|x|}$$

are TP functions of finite type 1 and 2, respectively. The main result in [12] states that $\mathcal{G}(g, \alpha, \beta)$ is a frame for all lattice parameters $\alpha, \beta > 0$ with $\alpha\beta < 1$. Its proof is based on the following property of TP functions of finite type, which was characterized by Schoenberg and Whitney.

Theorem 1.3 ([27]). *Let $g \in L^1(\mathbb{R})$ be a TP function of finite type $m = m_1 + m_2 \in \mathbb{N}$, where $m_1 \in \mathbb{N}_0$ is the number of positive and $m_2 \in \mathbb{N}_0$ is the number of negative a_{ν} in (3). Then for every $N \in \mathbb{N}$ and two sets of real numbers*

$$x_1 < x_2 < \dots < x_N, \quad y_1 < y_2 < \dots < y_N,$$

the determinant of the matrix $A = (g(x_j - y_k))_{j,k=1}^N$ is strictly positive, if and only if

$$x_{j-m_1} < y_j < x_{j+m_2} \quad (4)$$

for all $j = 1, \dots, N$. Here, we let $x_k = -\infty$, if $k < 1$, and $x_k = \infty$, if $k > N$.

The property (4) is called Schoenberg-Whitney condition, or in terms of interpolation, interlacing property.

The notion of total positivity is also used in matrix linear algebra, see [19] and [21]. A rectangular matrix $A = (a_{j,k}) \in \mathbb{R}^{m \times n}$ is called totally positive (TP), if all its minors are non-negative, that is

$$\det((a_{i_k, j_l})_{k,l=1,\dots,s}) \geq 0 \quad \text{for all} \\ 1 \leq s \leq \min\{m, n\}, \quad 1 \leq i_1 < \dots < i_s \leq m, \quad 1 \leq j_1 < \dots < j_s \leq n.$$

It is called strictly totally positive (STP), if all its minors are strictly positive. Examples of TP matrices are the collocation matrices of polynomial B-splines or Tschebycheffian B-splines, see [29, Theorem 9.34] or Theorem 2.2 below. The following generalization of STP matrices is useful in our context.

Definition 1.4 (Almost strict totally positive (ASTP) matrix, [9]). An $m \times n$ TP matrix A is said to be almost strict totally positive (ASTP), if it satisfies the following two conditions:

- (1) Any minor of A with consecutive rows and columns of A is positive, and only if the diagonal entries of the minor are positive.
- (2) If A has a zero row or column, then the subsequent rows or columns of A are also zero.

It was proved in [9] that an arbitrary minor of an ASTP matrix A is positive, if and only if the diagonal entries of the corresponding submatrix are positive.

Yet another notion of total positivity is used for sequences of functions $(u_1, \dots, u_n)$ defined on a set $K \subset \mathbb{R}$. For pairwise distinct points $t_1, \dots, t_m \in K$ (not necessarily ordered), the matrix

$$M \begin{pmatrix} u_1, \dots, u_n \\ t_1, \dots, t_m \end{pmatrix} := (u_k(t_j))_{j=1,\dots,m; k=1,\dots,n}$$

denotes the collocation matrix of $(u_1, \dots, u_n)$ at the points $t_1, \dots, t_m$.

Definition 1.5 (TP and ASTP sequence, [4]). The sequence of functions $(u_1, \dots, u_n)$ on $K \subset \mathbb{R}$ is called totally positive (TP) (resp. almost strict totally positive (ASTP)), if all collocation matrices $M \begin{pmatrix} u_1, \dots, u_n \\ t_1, \dots, t_n \end{pmatrix}$ with ordered points $t_1 < \dots < t_n$ in K are TP (resp. ASTP).

An example of an ASTP sequence are Tschebycheffian B-splines, which we describe in the next section. The notion of ASTP matrices is useful in order to describe interlacing properties of interpolation nodes and spline knots, see Theorem 2.2.

2. Tschebycheffian B-splines and PEB-splines

Next we provide some background from [29] and [22] on a class of Tschebycheffian B-splines which will be used as window functions for Gabor frames in the rest of the article. We start from positive weight functions $w_j \in C^{m-j}[a, b]$ on an interval $[a, b] \subset \mathbb{R}$. Then the functions

$$\begin{aligned} u_1(x) &= w_1(x) \\ u_2(x) &= w_1(x) \int_a^x w_2(s_2) ds_2 \\ &\vdots \\ u_m(x) &= w_1(x) \int_a^x w_2(s_2) \int_a^{s_2} \cdots \int_a^{s_{m-1}} w_m(s_m) ds_m \cdots ds_2 \end{aligned}$$

are functions in $C^{m-1}[a, b]$, which form an *extended complete Tschebycheff* (ECT) system on $[a, b]$. This means that

$$\det \left(M \begin{pmatrix} u_1, \dots, u_s \\ t_1, \dots, t_s \end{pmatrix} \right) > 0$$

for all $1 \leq s \leq m$ and $t_1 \leq \dots \leq t_s \in [a, b]$. Here the matrix

$$M \begin{pmatrix} u_1, \dots, u_s \\ t_1, \dots, t_s \end{pmatrix}$$

is the collocation matrix of Hermite interpolation, if some nodes t_j coincide. The vector space

$$\mathcal{U}_m := \text{span}(u_1, \dots, u_m)$$

is called ECT-space.

An important result in spline theory is the existence of Tschebycheffian B-splines (TB-splines) $B_m^k \in C^{m-2}(\mathbb{R})$ of order m , associated with knots $a \leq y_k < \dots < y_{k+m} \leq b$, such that

$$\begin{aligned} \text{supp } B_m^k &= [y_k, y_{k+m}], \\ B_m^k|_{(y_j, y_{j+1})} &\in \mathcal{U}_m, \quad k \leq j \leq k+m-1. \end{aligned}$$

Up to normalization, B_m^k is uniquely determined by these properties. A detailed description of TB-splines of order m is given in [29, Ch. 9]. For our purpose, we define the differential operators

$$D_j f = \frac{d}{dx} \left(\frac{f}{w_j} \right), \quad L_j = D_j \cdots D_1, \quad j = 1, \dots, m, \quad (5)$$

as in [29, page 365]. Then the following results are true.

Lemma 2.1. (i) $L_j u_k = 0$ for $1 \leq k \leq j \leq m$, and $\mathcal{U}_m$ is the kernel of L_m .

(ii) $(L_j u_{j+1}, \dots, L_j u_m)$ is the ECT-system on $[a, b]$ with weight functions $w_{j+1}, \dots, w_m$, called the j -th reduced ECT-system.

- (iii) Let B_{m-1}^k denote the TB-spline of order $m-1$ with knots $y_k, \dots, y_{k+m-1}$ and with respect to the first reduced ECT-system. Then for every $k \in \mathbb{Z}$ there are constants $a_k, b_k > 0$ such that

$$L_1 B_m^k = a_k B_{m-1}^k - b_k B_{m-1}^{k+1}. \quad (6)$$

The assertions (i) and (ii) of the lemma are given in [29, page 365f], assertion (iii) follows from Theorems 9.23 and 9.28 in [29].

An important result of [29, Theorem 9.34] characterizes the uniqueness of spline interpolation. It requires an interlacing property of interpolation nodes and spline knots, in a similar way as the Schoenberg-Whitney conditions in Theorem 1.3. Moreover, it covers the case where an ordered subsequence of TB-splines is selected, with natural ordering of the support intervals on the real line. We present this result for the purpose of finding Gabor frames $\mathcal{G}(B, \alpha, \beta)$ in the next section, where we let B be a special TB-spline in a shift-invariant spline space.

Theorem 2.2 ([29]). *Let $\{B_m^k \mid k = 1, \dots, N\}$ be the TB-splines of order m with respect to the knot sequence $y_1 < \dots < y_{m+N}$ and weight functions $w_1, \dots, w_m$. Then for any selection $1 \leq k_1 < \dots < k_s \leq N$ and any $t_1 < \dots < t_s$,*

$$\det \left(M \begin{pmatrix} B_m^{k_1}, \dots, B_m^{k_s} \\ t_1, \dots, t_s \end{pmatrix} \right) \geq 0,$$

and strict positivity holds if and only if

$$t_j \in (y_{k_j}, y_{k_j+m}), \quad j = 1, \dots, s.$$

In other words, an ordered sequence of TB-splines constitutes an ASTP sequence, and therefore their collocation matrices with ordered points are ASTP matrices.

Another property of TB-splines is closely related with total positivity of the collocation matrices. It is called the *variation-diminishing property* of TB-splines. It is important for applications of TB-splines in Computer Aided Design, especially in the digital design of curves. Following [2], we say that a function $f : [a, b] \rightarrow \mathbb{R}$ has at least p strong sign changes, if there exists a nondecreasing sequence $(\tau_j)_{0 \leq j \leq p}$ in $[a, b]$ with $f(\tau_0) \neq 0$ and, in case $p \geq 1$, $f(\tau_{j-1})f(\tau_j) < 0$ for all $j = 1, \dots, p$. The supremum of the number of strong sign changes of f is denoted by $S^-(f)$. Similarly, we define the total number of sign changes $S^-(c)$ of a sequence of real numbers $c = (c_k)_{0 \leq k \leq N}$.

Theorem 2.3 (Variation-diminishing property, see [29, Theorem 9.35]). *Let $\{B_m^k \mid k = 1, \dots, N\}$ be the TB-splines of order m with respect to the knot sequence $y_1 < \dots < y_{m+N}$ and weight functions $w_1, \dots, w_m$. Then $f = \sum_{k=0}^N c_k B_m^k$, with domain $[a, b] = [y_1, y_{m+N}]$, satisfies*

$$S^-(f) \leq S^-(c). \quad (7)$$

In order to approach our main concern of studying Zak transforms and Gabor frames whose window function is a TB-spline, we impose the special structure of shift-invariance with respect to the integer lattice on the sequence of TB-splines. This structure is obtained by

(S₁) letting the knots be integers $y_k = k$, $k \in \mathbb{Z}$, and

(S₂) requiring that the weight functions w_j , $1 \leq j \leq m$, have the form

$$w_j(x) = e^{\alpha_j x} r_j(x), \quad \alpha_j \in \mathbb{R}, \quad r_j(x+1) = r_j(x) \text{ for all } x \in \mathbb{R}.$$

Due to this form of the weight functions, B_m^k is called a *periodic exponential B-spline* (PEB-spline) in [22], and if $r_j \equiv 1$ for all j , then B_m^k is called *exponential B-spline* (EB-spline). As a consequence of (S₁) and (S₂), the PEB-splines of order m satisfy

$$B_m^k(x) = e^{\alpha_1 k} B_m^0(x - k), \quad k \in \mathbb{Z}. \quad (8)$$

Hence, the space of all spline functions $\sum_{k \in \mathbb{Z}} c_k B_m^k$ with complex coefficients c_k is shift-invariant. We let $B_m := B_m^0$, for short, and $B_{m-1} := B_{m-1}^0$ the PEB-spline of order $m-1$ with respect to the first reduced ECT-system and knots $0, 1, \dots, m-1$. Then the recursion formula (6) is specified as

$$L_1 B_m = a_{m-1}^{-1} (B_{m-1} - B_{m-1}(\cdot - 1)), \quad a_{m-1} > 0, \quad (9)$$

cf. [22, Proposition 3.2]. Moreover, the identity (8) allows us to write the variation-diminishing property in (7) as

$$S^-\left(\sum_{k=0}^N c_k B_m(\cdot - k)\right) \leq S^-(c), \quad (10)$$

where the coefficients c_k are real and the sum has the domain $[a, b] = [0, m+N]$.

Example 2.4. EB-splines of order $m \in \mathbb{N}$ (see [22, page 16f]).

Let $\Lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$, $\lambda_0 = 0$, and let exponential weight functions

$$w_j = e^{(\lambda_j - \lambda_{j-1})x}, \quad j = 1, \dots, m,$$

be given. Then, with proper normalization, the EB-spline $B_\Lambda := B_m$, with knots $0, 1, \dots, m$, is given by the convolution of the functions $e^{\lambda_j(\cdot)} \chi_{[0,1]}$,

$$B_\Lambda = e^{\lambda_1(\cdot)} \chi_{[0,1]} * e^{\lambda_2(\cdot)} \chi_{[0,1]} * \dots * e^{\lambda_m(\cdot)} \chi_{[0,1]}. \quad (11)$$

Its Fourier transform is

$$\hat{B}_\Lambda(\omega) = \prod_{j=1}^m \frac{e^{\lambda_j - 2\pi i \omega} - 1}{\lambda_j - 2\pi i \omega} \quad (12)$$

and the corresponding differential operators in (5) have the form

$$L_j = e^{-\lambda_j x} \prod_{k=1}^j \left(\frac{d}{dx} - \lambda_k \text{id} \right).$$

Note that the EB-spline B_{m-1} of the first reduced ECT-system with knots $0, \dots, m-1$ is, up to normalization, given by

$$B_{\{\lambda_2-\lambda_1, \dots, \lambda_m-\lambda_1\}} = e^{(\lambda_2-\lambda_1)(\cdot)} \chi_{[0,1)} * \dots * e^{(\lambda_m-\lambda_1)(\cdot)} \chi_{[0,1)}.$$

For our numerical computations, we use the explicit representation of B_Λ in each interval $[k-1, k)$, $1 \leq k \leq m$. If the numbers $\lambda_1, \dots, \lambda_m$ are pairwise distinct, then the ECT-space is

$$\mathcal{U}_m = \text{span}\{e^{\lambda_j x} \mid 1 \leq j \leq m\}.$$

For $\lambda_1 < \dots < \lambda_m$ and $m \geq 2$, Christensen and Massopust [6] give the closed form

$$B_\Lambda(x+k-1) = \sum_{j=1}^m \alpha_j^{(k)} e^{\lambda_j x}, \quad x \in [0, 1), \quad 1 \leq k \leq m,$$

with coefficients

$$\alpha_j^{(k)} = \begin{cases} \prod_{\substack{r=1, \\ r \neq j}}^m (\lambda_m - \lambda_r)^{-1}, & k = 1 \\ (-1)^{k-1} \frac{\sum_{\substack{1 \leq j_1 < \dots < j_{k-1} \leq m, \\ j_1, \dots, j_{k-1} \neq j}} e^{\lambda_{j_1} + \dots + \lambda_{j_{k-1}}}}{\prod_{\substack{r=1, \\ r \neq j}}^m (\lambda_m - \lambda_r)}, & k = 2, \dots, m. \end{cases}$$

For the general case

$$\Lambda = (\underbrace{\xi_1, \dots, \xi_1}_{s_1}, \dots, \underbrace{\xi_r, \dots, \xi_r}_{s_r}),$$

with pairwise distinct $\xi_1, \dots, \xi_r \in \mathbb{R}$, and each ξ_j repeated with multiplicity $s_j \in \mathbb{N}$, we have

$$\mathcal{U}_m = \text{span}\{e^{\xi_1 x}, x e^{\xi_1 x}, \dots, x^{s_1-1} e^{\xi_1 x}, \dots, e^{\xi_r x}, \dots, x^{s_r-1} e^{\xi_r x}\}$$

and

$$B_\Lambda(x+k-1) = \sum_{j=1}^r p_j^{(k)}(x) e^{\xi_j x}, \quad x \in [0, 1), \quad 1 \leq k \leq m, \quad (13)$$

with real polynomials $p_j^{(k)}$ of degree $s_j - 1$.

3. Zak transform of PEB-splines and TP functions of finite type

The Zak transform is an important tool for spline interpolation and Gabor frame analysis. For a given parameter $\alpha > 0$, the Zak transform $Z_\alpha f$ of a function $f : \mathbb{R} \rightarrow \mathbb{C}$ is defined by

$$Z_\alpha f(x, \omega) = \sum_{k \in \mathbb{Z}} f(x - k\alpha) e^{2\pi i k \alpha \omega},$$

whenever this series converges. Properties of the Zak transform and many more facts about this transform can be found in [11]. We present some of these properties which are needed for the results in the following sections.

Lemma 3.1. *Let f be an element of the Wiener space $W(\mathbb{R})$.*

a) *$Z_\alpha f(x, \omega)$ is bounded in $\mathbb{R}^2$, and if f is continuous, then $Z_\alpha f$ is continuous.*

b) *For every $n \in \mathbb{Z}$, we have the identities for periodicity*

$$Z_\alpha f(x, \omega + \frac{n}{\alpha}) = Z_\alpha f(x, \omega)$$

and quasi-periodicity

$$Z_\alpha f(x + n\alpha, \omega) = e^{2\pi i n \alpha \omega} Z_\alpha f(x, \omega).$$

c) *If $\hat{f} \in W(\mathbb{R})$ as well, then*

$$\alpha \cdot Z_\alpha f(x, \omega) = e^{2\pi i x \omega} Z_{1/\alpha} \hat{f}(\omega, -x).$$

d) *Let $f_\alpha = f(\alpha \cdot)$ be the scaled function of f . Then*

$$Z_\alpha f(x, \omega) = Z_1 f_\alpha(\frac{x}{\alpha}, \alpha \omega).$$

We study the Zak transform of PEB-splines B_m , with integer knots $0, 1, \dots, m$ and with respect to periodic exponential weights w_j , $j = 1, \dots, m$ on $[0, \infty)$. For the special case $m = 1$ (the only case where B_m is discontinuous), the PEB-spline is

$$B_1(x) = w_1(x) \chi_{[0,1)}(x) = e^{\alpha_1 x} r_1(x) \chi_{[0,1)}(x).$$

Its Zak transform $Z_1 B_1$ is

$$Z_1 B_1(x, \omega) = B_1(x - \lfloor x \rfloor) e^{2\pi i \omega \lfloor x \rfloor}, \quad (x, \omega) \in \mathbb{R}^2.$$

Since B_1 is positive on $[0, 1)$, the Zak transform $Z_1 B_1$ has no zeros in $\mathbb{R}^2$.

For $m \geq 2$, the PEB-spline B_m is continuous and belongs to the Wiener space $W(\mathbb{R})$. By Lemma 3.1 its Zak transform $Z_1 B_m$ is continuous. It was first pointed out in [30] that, if the Zak transform $Z_1 g$ is continuous, then it has a zero in $[0, 1) \times [0, 1)$. Later this fact was used in [7] to generalize the famous Balian-Low Theorem from orthogonal bases to frames. That Theorem leads to the fact that $\mathcal{G}(g, \alpha, \frac{1}{\alpha})$ cannot be a frame, if g is a continuous function in $W(\mathbb{R})$. Since B_m is real, we know from the results in [14] that there exists a zero $Z_1 B_m(\tilde{x}, \frac{1}{2}) = 0$ for some $\tilde{x} \in [0, 1)$.

The absence of more zeros of the Zak transform $Z_1 g$ was proved in the following special cases:

- $g = N_m$, the cardinal polynomial B-spline, which is the EB-spline B_Λ in Example 2.4 with $\Lambda = (0, \dots, 0)$:

A classical result in the theory of cardinal polynomial B-splines, dating back to Schoenberg's work in 1973 on cardinal spline interpolation [26], states that the only zero of $Z_1 N_m$ in $[0, 1) \times [0, 1)$ is located at $(\frac{1}{2}, \frac{1}{2})$ for even m and $(0, \frac{1}{2})$ for odd $m \geq 3$. Note that, in Schoenberg's terminology, the Zak transform $Z_1 N_m$ is called *exponential Euler spline*. See also [18] for a discussion of zeros of these functions.

- g is even and *super convex* on $[0, \infty)$:

Janssen proved in [16] that the Zak transform of an even, continuous function g of the form

$$g(t) = b(t) + b(t+1), \quad t \geq 0,$$

where b is integrable, non-negative and strictly convex on $[0, \infty)$, has only one zero in $[0, 1) \times [0, 1)$, located at $(\frac{1}{2}, \frac{1}{2})$.

PEB-splines, in general, do not satisfy any of these assumptions. Our next result proves that this property of having only one zero in $[0, 1) \times [0, 1)$ persists.

Theorem 3.2. *Let $m \geq 2$ and B_m a PEB-spline of order m . Then $Z_1 B_m$ has exactly one zero in $[0, 1)^2$. More precisely, there exists $\tilde{x} \in [0, 1)$, such that $Z_1 B_m(\tilde{x}, \frac{1}{2}) = 0$, and $Z_1 B_m(x, \omega) \neq 0$ for all $(x, \omega) \in [0, 1)^2 \setminus \{(\tilde{x}, \frac{1}{2})\}$.*

We will give two independent proofs of Theorem 3.2. The first proof makes use of the variation-diminishing property of shifts $B_m(\cdot - k)$ of a PEB-spline B_m , which was described in Theorem 2.3. The second proof is more elementary, but restricted to EB-splines of order m .

Proof. First proof of Theorem 3.2.

First, we let $\omega \in (-\frac{1}{2}, \frac{1}{2})$ and assume that there exists $\tilde{x} \in [0, 1)$ with $Z_1 B_m(\tilde{x}, \omega) = 0$. By quasi-periodicity of $Z_1 B_m$, the function

$$f(x) := \operatorname{Re}(Z_1 B_m(x, \omega)) = \sum_{k \in \mathbb{Z}} c_k B_m(x - k), \quad c_k = \cos(2\pi k \omega), \quad (14)$$

vanishes at all points $\tilde{x} + k$, $k \in \mathbb{Z}$. These points are isolated zeros of f by the following argument. By [28], the TB-splines B_m^k are locally linearly independent. This means that, if f vanishes on an interval $(a, b) \subset \mathbb{R}$, then all coefficients c_k with $\operatorname{supp} B_m(\cdot - k) \cap (a, b) \neq \emptyset$ vanish as well. Since $|\omega| < \frac{1}{2}$, no consecutive coefficients c_k, c_{k+1} of the function f in (14) vanish simultaneously. Therefore, due to $\operatorname{supp} B_m = [0, m]$ and $m \geq 2$, there is no non-empty interval (α, β) where f is identically zero. Moreover, since the components $f|_{[k, k+1]}$ are elements of an ECT-space of dimension m , there are at most $m - 1$ isolated zeros in each of these intervals.

Next, we will find a contradiction to the variation-diminishing property in (10) by counting strong sign changes. As some of the zeros of f may be double zeros, counting the strong sign changes of f may not be enough. Instead, we take

$$g(x) = L_1 f(x) = \frac{d}{dx} \left(\frac{f}{w_1} \right) = a_{m-1}^{-1} \sum_{k \in \mathbb{Z}} (c_k - c_{k+1}) B_{m-1}(x - k),$$

where we used (9) and the PEB-spline B_{m-1} of order $m - 1$. For every $N \in \mathbb{N}$, there are at least $N + 1$ zeros of f/w_1 in $[0, N + 1]$, and, by Rolle's theorem, we obtain

$$S^-(g) \geq N \quad \text{on} \quad [0, N + 1].$$

On the other hand, g is the sum of PEB-splines $B_{m-1}(\cdot - k)$ with real coefficients

$$d_k = a_{m-1}^{-1}(c_k - c_{k+1}) = a_{m-1}^{-1} \operatorname{Re} \left((1 - e^{2\pi i \omega}) e^{2\pi i k \omega} \right), \quad k \in \mathbb{Z}.$$

We notice that

$$|\arg \left((1 - e^{2\pi i \omega}) e^{2\pi i k \omega} \right) - \arg(1 - e^{2\pi i \omega})| = 2\pi |k\omega| \quad \text{for all } k \in \mathbb{Z}.$$

Therefore, the finite sequence $d = (d_k)_{-m+2 \leq k \leq N}$ satisfies

$$S^-(d) \leq 2(N + m - 2) |\omega|. \quad (15)$$

Therefore, for $|\omega| < \frac{1}{2}$ and large N we find

$$S^-(d) < N \leq S^-(g),$$

which is a contradiction to the variation-diminishing property of B_{m-1} in (10). This implies, that there is no zero of $Z_1 B_\Lambda(\cdot, \omega)$ with $|\omega| < \frac{1}{2}$.

For the case $\omega = \frac{1}{2}$, the assumption of having two distinct zeros (or a double zero) of $Z_1 B_\Lambda(\cdot, \frac{1}{2})$ in $[0, 1)$ leads to a contradiction by analogous arguments. $\square$

The second proof of Theorem 3.2 is more elementary, but it is restricted to EB-splines B_Λ in Example 2.4. We employ the notation

$$\Lambda = (\xi_1, \dots, \xi_1, \dots, \xi_r, \dots, \xi_r)$$

with multiplicities s_j of ξ_j in Λ as in Example 2.4.

Proof. For fixed $\omega \in \mathbb{R}$, we consider the complex function $h := Z_1 B_\Lambda(\cdot, \omega)$. By (13), we obtain for $x \in [0, 1)$

$$\begin{aligned} h(x) &= \sum_{k=0}^{m-1} B_\Lambda(x + k) e^{-2\pi i k \omega} = \sum_{k=1}^m \sum_{j=1}^r p_j^{(k)}(x) e^{\xi_j x} e^{-2\pi i k \omega} \\ &= \sum_{j=1}^r \underbrace{\sum_{k=1}^m p_j^{(k)}(x) e^{-2\pi i k \omega}}_{:= q_j(x)} e^{\xi_j x}, \end{aligned}$$

where q_j are complex polynomials of degree $\sigma_j - 1 \leq s_j - 1$. Some of the q_j are nonzero, as pointed out in the first proof, by local linear independence of the EB-splines. We let $\sigma_j = 0$ if $q_j = 0$ and define $\mu = \sigma_1 + \dots + \sigma_r$. Without loss of generality, we can assume $\sigma_1 \geq 1$, that is, the term $q_1(x) e^{\xi_1 x}$ in $h|_{[0,1)}$ is nonzero. We use the identity

$$e^{\xi_j x} \frac{d}{dx} (e^{-\xi_j x} h(x)) = q_j'(x) e^{\xi_j x} + \sum_{k \neq j} ((\xi_k - \xi_j) q_k(x) + q_k'(x)) e^{\xi_k x}.$$

Writing D_j for the differential operator on the left hand side and $\mathcal{D} := D_1^{\sigma_1-1} \prod_{j=2}^r D_j^{\sigma_j}$, we obtain

$$\mathcal{D}h(x) = be^{\xi_1 x}, \quad x \in (0, 1),$$

with a nonzero constant $b \in \mathbb{C}$. The quasi-periodicity of h leads directly to

$$\mathcal{D}h(x) = be^{\xi_1(x-k)} e^{2\pi i k \omega}, \quad x \in (k, k+1),$$

for all $k \in \mathbb{Z}$.

Now we let $\omega \in (-\frac{1}{2}, \frac{1}{2})$ and we assume that there exists $\tilde{x} \in [0, 1)$ with $Z_1 B_\Lambda(\tilde{x}, \omega) = 0$. By quasi-periodicity of $h = Z_1 B_\Lambda(\cdot, \omega)$, the function $f = \operatorname{Re} h$ vanishes at all points $\tilde{x} + k$, $k \in \mathbb{Z}$, and these points are isolated zeros of f by the same argument of local linear independence as before. This guarantees that f has at least $N \in \mathbb{N}$ isolated zeros in $[0, N]$. Note that $\mathcal{D}$ is a differential operator of order $\mu - 1 \leq m - 1$. Since $f \in C^{m-2}(\mathbb{R})$, with $f^{(m-2)}$ absolutely continuous, we obtain by Rolle's theorem that

$$S^-(\mathcal{D}f) \geq N - \mu + 1 \quad \text{on } [0, N]. \quad (16)$$

However, on each interval $[k, k+1)$ with $k \in \mathbb{Z}$, the sign of $\mathcal{D}f$ is fixed by

$$\operatorname{sign}(\mathcal{D}f)(x) = \operatorname{sign} \operatorname{Re}(b e^{2\pi i k \omega}), \quad x \in [k, k+1).$$

In the same way as in (15), this implies

$$S^-(\mathcal{D}f) \leq 2N|\omega| \quad \text{on } [0, N].$$

This is a contradiction to (16) for $|\omega| < \frac{1}{2}$ and large N .

An analogous argument leads to a contradiction, if we assume that $\omega = \frac{1}{2}$ and there are two distinct zeros of $Z_1 B_\Lambda$ with $x \in [0, 1)$. $\square$

Remark 3.3. The result of Theorem 3.2 can be slightly generalized. Let $m \geq 2$ and B_m a PEB-spline of order m . Consider the function $g = \sum_{l=0}^r a_l B_m(\cdot - l)$, where the coefficients a_l define the trigonometric polynomial $\hat{a}(\omega) = \sum_{l=0}^r a_l e^{-2\pi i l \omega}$ with no real zeros. Then the Zak transform $Z_1 g$ is

$$\begin{aligned} Z_1 g(x, \omega) &= \sum_{k \in \mathbb{Z}} \sum_{l=0}^r a_l B_m(x - k - l) e^{2\pi i k \omega} \\ &= \hat{a}(\omega) Z_1 B_m(x, \omega). \end{aligned}$$

Hence, the zeros of $Z_1 g$ are the same as the zeros of $Z_1 B_m$, and the result of Theorem 3.2 extends to the Zak transform $Z_1 g$.

Next we turn our attention to TP functions g of finite type $m \in \mathbb{N}$, given by the Fourier transform

$$\hat{g}(\omega) = \prod_{\nu=1}^m (1 + 2\pi i \omega a_\nu^{-1})^{-1},$$

where $a_1, \dots, a_m \in \mathbb{R} \setminus \{0\}$. We are able to express the Zak transform $Z_\alpha g$, with arbitrary $\alpha > 0$, in terms of the Zak transform $Z_1 B_\Lambda$ of an EB-spline of order m . This has several advantages:

1. Since an EB-spline B_Λ of order m has support $[0, m]$, its Zak transform can be computed accurately for given values of x and ω , without the need of truncation.
2. The property that $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for some pair (α, β) of lattice parameters is equivalent to $\mathcal{G}(B_\Lambda, 1, \alpha\beta)$ being a Gabor frame.
3. Finding frame-bounds of $\mathcal{G}(g, \alpha, \beta)$ is reduced to finding frame-bounds for $\mathcal{G}(B_\Lambda, 1, \alpha\beta)$.

More details on items 2. and 3. will be presented in the last two sections of this article. We mention that another explicit representation of $Z_\alpha g$, without the use of EB-splines, was recently found in [1].

Theorem 3.4. *Let $\alpha > 0$ and $g \in L^1(\mathbb{R})$ be a TP function of finite type, defined by its Fourier transform*

$$\hat{g}(\omega) = \prod_{\nu=1}^m (1 + 2\pi i \omega a_\nu^{-1})^{-1},$$

where $a_1, \dots, a_m \in \mathbb{R} \setminus \{0\}$. With $\lambda_\nu := -\alpha a_\nu$ and B_Λ defined in (11), we have

$$\alpha Z_\alpha g(x, \omega) = \prod_{\nu=1}^m \frac{\alpha a_\nu}{1 - e^{-\alpha(a_\nu + 2\pi i \omega)}} Z_1 B_\Lambda\left(\frac{x}{\alpha}, \alpha \omega\right), \quad (x, \omega) \in [0, \alpha) \times [0, \frac{1}{\alpha}).$$

Proof. The Fourier transform of B_Λ in (12) is

$$\begin{aligned} \hat{B}_\Lambda(\omega) &= \prod_{\nu=1}^m \frac{e^{\lambda_\nu - 2\pi i \omega} - 1}{\lambda_\nu (1 - 2\pi i \omega \lambda_\nu^{-1})} \\ &= \prod_{\nu=1}^m \frac{1 - e^{-(\alpha a_\nu + 2\pi i \omega)}}{\alpha a_\nu} \prod_{\nu=1}^m (1 + 2\pi i \omega (\alpha a_\nu)^{-1})^{-1} \\ &\quad \underbrace{\hspace{10em}}_{=:t(\omega)} \\ &= t(\omega) \alpha \hat{g}_\alpha(\omega), \end{aligned}$$

where t is a one-periodic function in ω and $g_\alpha = g(\alpha \cdot)$. This implies

$$Z_1 \hat{B}_\Lambda(\omega, x) = t(\omega) \alpha Z_1 \hat{g}_\alpha(\omega, x)$$

and finally, since $g, B_\Lambda \in W(\mathbb{R})$, with Lemma 3.1 we get

$$\begin{aligned} \alpha Z_\alpha g(x, \omega) &= \alpha Z_1 g_\alpha\left(\frac{x}{\alpha}, \alpha \omega\right) \\ &= \alpha Z_1 \hat{g}_\alpha\left(\alpha \omega, -\frac{x}{\alpha}\right) e^{2\pi i x \omega} \\ &= t(\alpha \omega)^{-1} Z_1 \hat{B}_\Lambda\left(\alpha \omega, -\frac{x}{\alpha}\right) e^{2\pi i x \omega} \\ &= t(\alpha \omega)^{-1} Z_1 B_\Lambda\left(\frac{x}{\alpha}, \alpha \omega\right), \end{aligned}$$

which completes the proof. □

Clearly, the factor $t(\alpha\omega)$ in the proof is nonzero and bounded. Hence, we obtain the following corollary of Theorem 3.2.

Corollary 3.5. *Let g be a totally positive function of finite type $m \geq 2$ and $\alpha > 0$. Then there exists $\tilde{x} \in [0, \alpha)$, such that $Z_\alpha g(\tilde{x}, \frac{1}{2\alpha}) = 0$, and $Z_\alpha g(x, \omega) \neq 0$ for all $(x, \omega) \in [0, \alpha) \times [0, \frac{1}{\alpha}) \setminus \{(\tilde{x}, \frac{1}{2\alpha})\}$.*

With reference to our paper, the result of Corollary 3.5 was already used in [1] for the construction of periodic and discrete Gabor frames $\mathcal{G}(g, \alpha, \frac{1}{\alpha})$ for the spaces $L^2([0, K])$ and $\mathbb{C}^K$, respectively, at the critical density. In other words, families of Riesz bases for $L^2([0, K])$ or bases for $\mathbb{C}^K$ are constructed in [1], which result from periodization resp. discretization of Gabor families with a TP window function of finite type at the critical density.

4. Gabor frames of PEB-splines and TP functions of finite type

In this section, we prove that every PEB-spline B_m of order m is the window function of a Gabor frame $\mathcal{G}(B_m, \alpha, \beta)$, for a large set of lattice parameters $\alpha, \beta > 0$. In combination with Theorem 3.4, this will provide an alternative proof of the result in [12], that every Gabor family $\mathcal{G}(g, \alpha, \beta)$ with TP function g of finite type and with lattice parameters $\alpha, \beta > 0$, $\alpha\beta < 1$, is a frame for $L^2(\mathbb{R})$, see Theorem 4.8 below.

Theorem 4.1. *Let $m \in \mathbb{N}$ and B_m be a PEB-spline with knots $0, \dots, m$. Then the set $\mathcal{G}(B_m, \alpha, \beta)$ constitutes a frame in the following cases:*

- (1) $0 < \alpha < m$ and $0 < \beta \leq m^{-1}$,
- (2) $\alpha \in \{1, 2, \dots, m-1\}$, $\beta > 0$ and $\alpha\beta < 1$,
- (3) $\alpha > 0$, $\beta \in \{1, 2^{-1}, \dots, (m-1)^{-1}\}$ and $\alpha\beta < 1$.

Proof. The upper frame-bound of the Gabor family $\mathcal{G}(B_m, \alpha, \beta)$ is finite, because B_m is piecewise continuous with compact support, so it is an element of the Wiener space $W(\mathbb{R})$. Finding a positive lower frame-bound is the more difficult part.

For case (1), we have $\beta^{-1} \geq \text{vol}(\text{supp } B_m)$. Hence, the assertion is a direct consequence of the more general result in [8], which states that the optimal lower frame-bound is given by

$$A_{\text{opt}} = \frac{1}{\beta} \text{ess inf}_{x \in [0, \alpha)} \sum_{k \in \mathbb{Z}} B_m(x + k\alpha)^2 > 0. \quad (17)$$

Let us now consider the cases (2) and (3), so α or β^{-1} are in $\{1, 2, \dots, m-1\}$. This excludes the case $m = 1$, so we only consider $m \geq 2$ from now on. In addition, by the result for case (1), we can assume that

$$0 < \alpha < \beta^{-1} < m.$$

The main idea of the proof that $\mathcal{G}(B_m, \alpha, \beta)$ has a lower frame-bound is taken from [12, Theorem 8]. The proof is constructive and produces a dual window function γ such that $\mathcal{G}(\gamma, \alpha, \beta)$ is a dual frame for $\mathcal{G}(B_m, \alpha, \beta)$. The construction is based on [12, Lemma 5], which we repeat here in the univariate setting for the reader's convenience.

Lemma 4.2 ([12]). *Let $g \in W(\mathbb{R})$ and $\alpha, \beta > 0$. Assume that there exists a Lebesgue measurable vector-valued function $\sigma(x)$ from $\mathbb{R}$ to $\ell^2(\mathbb{Z})$ with period α , such that*

$$\sum_{k \in \mathbb{Z}} \sigma_k(x) \bar{g}(x + \alpha k - \frac{l}{\beta}) = \delta_{l,0} \quad a.a. \ x \in \mathbb{R}. \quad (18)$$

If

$$\sum_{k \in \mathbb{Z}} \sup_{x \in [0, \alpha]} |\sigma_k(x)| < \infty, \quad (19)$$

then $\mathcal{G}(g, \alpha, \beta)$ is a frame. Moreover, with

$$\gamma(x) = \beta \sum_{k \in \mathbb{Z}} \sigma_k(x) \chi_{[0, \alpha)}(x - \alpha k), \quad x \in \mathbb{R},$$

the set $\mathcal{G}(\gamma, \alpha, \beta)$ is a dual frame of $\mathcal{G}(g, \alpha, \beta)$.

Now we make use of this lemma in our proof of Theorem 4.1. We define the pre-Gramian matrix

$$P(x) = \left(B_m(x + k\alpha - \frac{l}{\beta}) \right)_{k, l \in \mathbb{Z}}, \quad x \in \mathbb{R}.$$

Let

$$k_0 := \left\lceil \frac{m - \frac{1}{\beta}}{\frac{1}{\beta} - \alpha} \right\rceil - 1.$$

We select the compact interval

$$I = [m - \frac{1}{\beta}, m - \frac{1}{\beta} + \alpha]$$

of length α inside the support of B_m , and for every $x \in I$, we construct the row vector $\sigma(x)$ such that (18) holds, in other words

$$\sigma(x)P(x) = (\delta_{l,0})_{l \in \mathbb{Z}}. \quad (20)$$

For every $x \in I$, the inequalities

$$\begin{aligned} 0 &< m - \frac{1}{\beta} \leq x \leq m - \left(\frac{1}{\beta} - \alpha\right) < m, \\ 0 &< x - k_0 \left(\frac{1}{\beta} - \alpha\right) \leq \frac{1}{\beta} < m \end{aligned}$$

are readily verified. Therefore, the rows $k = 0, 1, \dots, k_0$ of $P(x)$ can be described as follows:

- all diagonal entries $B_m\left(x - k\left(\frac{1}{\beta} - \alpha\right)\right)$, $k = 0, 1, \dots, k_0$, are strictly positive,
- all entries in columns $l < 0$ vanish, because $x + k\alpha - \frac{l}{\beta} \geq m$,
- all entries in columns $l > k_0$ vanish, because $x + k\alpha - \frac{l}{\beta} \leq 0$.

Therefore, the only nonzero block of $P(x)$ in rows $0 \leq k \leq k_0$ is the square matrix

$$P_0(x) := \left(B_m\left(x + k\alpha - \frac{l}{\beta}\right) \right)_{k,l=0,\dots,k_0}$$

with strictly positive diagonal entries. For case (3), where $\beta^{-1} \in \{1, 2, \dots, m-1\}$, the matrix $P_0(x)$ is the collocation matrix

$$P_0(x) = M \begin{pmatrix} B_m & B_m(\cdot - \beta^{-1}) & \cdots & B_m(\cdot - k_0\beta^{-1}) \\ x & x + \alpha & \cdots & x + k_0\alpha \end{pmatrix}.$$

By Theorem 2.2, $P_0(x)$ is invertible. Likewise, for case (2), where $\alpha \in \{1, 2, \dots, m-1\}$, the transpose $P_0(x)^T$ is the collocation matrix

$$P_0(x)^T = M \begin{pmatrix} B_m & B_m(\cdot + \alpha) & \cdots & B_m(\cdot + k_0\alpha) \\ x & x - \beta^{-1} & \cdots & x - k_0\beta^{-1} \end{pmatrix},$$

with reverse ordering of the support of the B-splines and nodes as compared to Theorem 2.2. Again we conclude that $P_0(x)$ is invertible. Next we define

$$(\sigma_0(x), \dots, \sigma_{k_0}(x)), \quad x \in I, \quad (21)$$

to be the first row of the inverse of $P_0(x)$, and extend this vector with components $\sigma_k(x) = 0$ for $k < 0$ and $k > k_0$. Then the identity (20) follows immediately.

It remains to show that $\sigma(x)$ satisfies

$$\sum_{k \in \mathbb{Z}} \sup_{x \in I} |\sigma_k(x)| < \infty, \quad (22)$$

which is equivalent to (19) by periodicity. Recall that $m \geq 2$, so B_m is continuous. The determinant of $P_0(x)$ is strictly positive and continuous as a function of $x \in I$. Since I is compact, there is a constant $c > 0$ such that $\det P_0(x) \geq c$ for all $x \in I$. By Cramer's rule, there is a uniform bound for all entries of $P_0(x)^{-1}$, and this gives (22). $\square$

Remark 4.3. Gabor frames with window function B_m were also considered in [6]. In their work, the lattice parameters $0 < \alpha < m$, $0 < \beta \leq \frac{1}{2m}$ define highly redundant Gabor frames, and explicit formulas for dual windows γ with support $[0, m]$ are presented.

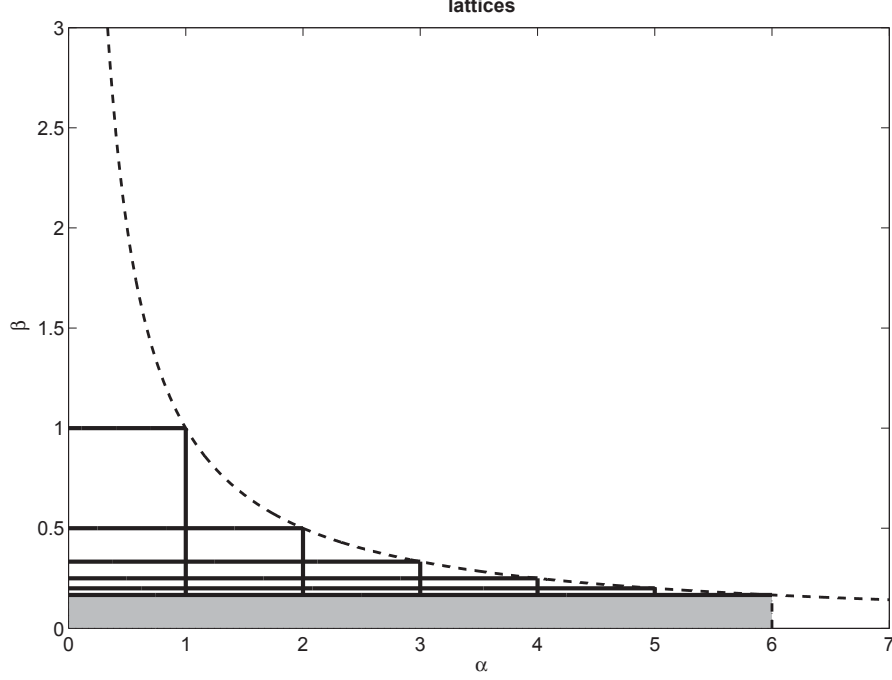


Figure 1: Lattice parameters (α, β) , where every PEB-spline of order $m = 6$ induces a Gabor frame, include the rectangle $0 < \alpha < 6$, $0 < \beta \leq 1/6$ and the lines $\alpha \in \{1, \dots, 5\}$ and $\beta^{-1} \in \{1, \dots, 5\}$ with $\alpha\beta < 1$.

Remark 4.4. Different dual windows for the Gabor frame $\mathcal{G}(B_m, \alpha, \beta)$ can be computed, if we choose larger rectangular submatrices $P_1(x)$ of $P(x)$ in the following way. We include more columns of $P(x)$ to the left and right of $P_0(x)$ and select the maximal set of rows of $P(x)$, such that only zeros appear on both sides of $P_1(x)$. Similar arguments as in the proof of Theorem 4.1 show that the matrix $P_1(x)$ has full column rank. We take its Moore-Penrose pseudo-inverse Γ_1 and define the central part of the vector $\sigma(x)$ in (21) by the row vector with index $k = 0$ of Γ_1 . The matrix $P_1(x)$ often has a better ℓ_2 -condition number than $P_0(x)$, and the corresponding dual window γ has smaller norm $\|\gamma\|_W$. For this case, the proof of (19) needs some small adaptations as in [20], where TP functions were considered. For comparison of this construction of the dual window γ , we mention that the canonical dual window $\tilde{\gamma}$ is defined by the row vector $\sigma(x)$ in row $k = 0$ of the Moore-Penrose pseudo-inverse

$$\Gamma(x) = (P(x)^T P(x))^{-1} P(x)^T$$

of the (full) pre-Gramian $P(x)$, in the same way that $\tilde{\gamma}(x + k\alpha) = \beta\sigma_k(x)$ for all $k \in \mathbb{Z}$.

Example 4.5. We consider the EB-spline B_{Λ_1} , with $\Lambda_1 = (-2, -1, 1, 2)$, drawn in the top left of Figure 2. Two different dual windows γ of the Gabor frame $\mathcal{G}(B_{\Lambda_1}, 1, 0.86)$ are computed by different selections of the matrix $P_1(x)$ with an increasing number of columns of the pre-Gramian $P(x)$. The dual window functions γ are drawn in the left half of Figure 2. They are discontinuous at the boundary points of the interval I of length 1 chosen for the computation

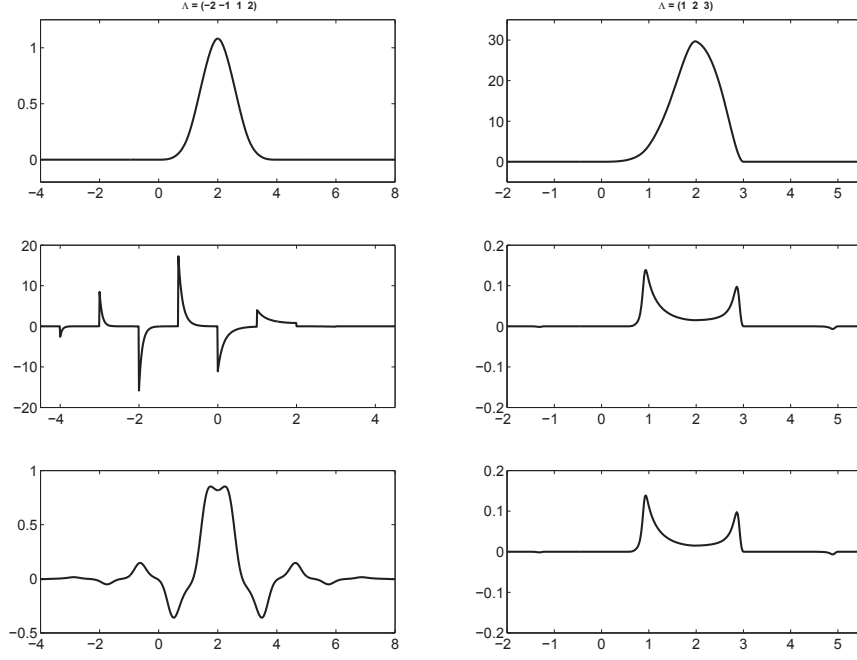


Figure 2: EB-spline B_Λ with $\Lambda = (-2, -1, 1, 2)$ and two different dual windows for $(\alpha, \beta) = (1, 0.86)$ (left side); EB-spline B_Λ with $\Lambda = (1, 2, 3)$ and two different dual windows for $(\alpha, \beta) = (2, 0.45)$ (right side).

of the row vector $\sigma(x)$ in the proof of Theorem 4.1, and the integer shifts of these points. The larger the submatrix $P_1(x)$, the smaller is the jump at these discontinuities. We also observe that the dual window γ approaches the canonical dual window $\tilde{\gamma}$, as we increase the number of columns of $P_1(x)$.

Similar computations are performed with the EB-spline B_{Λ_2} , $\Lambda_2 = (1, 2, 3)$, shown on the top right of Figure 2. Two different dual windows γ of the Gabor frame $\mathcal{G}(B_{\Lambda_2}, 2, 0.45)$ are drawn in the right half of Figure 2.

The relation between the Zak transform of TP functions and EB-splines in Theorem 3.4 can be used in order to compare frame-bounds of the different Gabor systems. The general method behind was first described by Janssen and Strohmer [17] and used for comparing the Gaussian and the hyperbolic secant as window functions for Gabor frames. This method is based on the following characterization of frame-bounds given by Ron and Shen [23].

Theorem 4.6. *Let $g \in L^2(\mathbb{R})$ be a window-function, $\alpha, \beta > 0$ some lattice-parameters, and $P(x) = (\bar{g}(x + k\alpha - \frac{l}{\beta}))_{k,l \in \mathbb{Z}}$ the pre-Gramian. Then $0 < A \leq B < \infty$ are frame-bounds for $\mathcal{G}(g, \alpha, \beta)$, if and only if*

$$\inf_{x \in [0, \alpha]} \|P(x) c\|_2^2 \geq \beta A \|c\|_2^2,$$

$$\sup_{x \in [0, \alpha]} \|P(x) c\|_2^2 \leq \beta B \|c\|_2^2, \quad \forall c \in \ell^2(\mathbb{Z}).$$

If $g \in W(\mathbb{R})$ and $c \in \ell_1(\mathbb{Z})$, an application of Parseval's identity reveals that

$$\begin{aligned} \|P(x)c\|_2^2 &= \sum_{k \in \mathbb{Z}} \left| \sum_{l \in \mathbb{Z}} c_l \bar{g}(x + \alpha k - \frac{l}{\beta}) \right|^2 \\ &= \alpha \int_0^{1/\alpha} \left| \sum_{k, l \in \mathbb{Z}} c_l \bar{g}(x + \alpha k - \frac{l}{\beta}) e^{-2\pi i k \alpha \omega} \right|^2 d\omega \\ &= \alpha \left\| \sum_{l \in \mathbb{Z}} c_l Z_\alpha \bar{g}(x - \frac{l}{\beta}, \cdot) \right\|_{L^2(0, \frac{1}{\alpha})}^2. \end{aligned}$$

Based on this identity, the observation by Janssen and Strohmer in [17] can be summarized as follows.

Proposition 4.7. *Let $\alpha, \beta, \kappa > 0$. If $\mathcal{G}(g, \alpha, \beta)$ is a frame with frame-bounds $0 < A \leq B < \infty$, and $h \in L^2(\mathbb{R})$ satisfies*

$$Z_{\kappa\alpha} h(\kappa x, \kappa^{-1}\omega) = D(\omega) \cdot Z_\alpha g(x, \omega), \quad (x, \omega) \in [0, \alpha) \times [0, \frac{1}{\alpha}),$$

where

$$0 < \text{ess inf } |D(\omega)|^2 \leq \text{ess sup } |D(\omega)|^2 < \infty,$$

then $\mathcal{G}(h, \kappa\alpha, \kappa^{-1}\beta)$ is also a frame with frame-bounds $\kappa A \cdot \text{ess inf } |D(\omega)|^2$ and $\kappa B \cdot \text{ess sup } |D(\omega)|^2$.

If g is a TP function of finite type, we found the factorization of $Z_\alpha g$ in Theorem 3.4. Moreover, together with Theorem 4.1 this leads to an alternative proof of [12, Theorem 1].

Theorem 4.8. *Let g be a totally positive function of finite type $m \geq 2$. Then $\mathcal{G}(g, \alpha, \beta)$ is a frame for every $\alpha, \beta > 0$ with $\alpha\beta < 1$.*

5. Lower frame-bounds for Gabor frames of even two-sided exponentials

The pre-Gramian matrices $P(x) = (B_m(x + k\alpha - \frac{l}{\beta}))_{k, l \in \mathbb{Z}}$ were already used in the proof of Theorem 4.1. We observed that, for $\alpha = 1$, the finite submatrices are collocation matrices of PEB-splines, hence they are ASTP matrices. In this section, we apply methods for the decomposition of ASTP matrices in order to compute lower frame-bounds for Gabor frames in a special case, namely for the symmetric EB-spline of order 2, given by

$$B(x) = (e^{\lambda(\cdot)} \chi_{[0,1]} * e^{-\lambda(\cdot)} \chi_{[0,1]})(x) = \begin{cases} \frac{\sinh(\lambda x)}{\lambda}, & 0 \leq x \leq 1, \\ \frac{\sinh(\lambda(2-x))}{\lambda}, & 1 < x \leq 2, \end{cases} \quad (23)$$

with $\lambda \in \mathbb{R}_+$ and $\text{supp } B = [0, 2]$. Then we use Theorem 3.4 once more for deriving the corresponding lower frame-bound of the Gabor frame $\mathcal{G}(g, \alpha, \beta)$, where g is the even two-sided exponential function

$$g(x) = \frac{\lambda}{2} e^{-\lambda|x|}.$$

Note that g is a TP function of finite type 2. It was shown by Janssen [15] that $\mathcal{G}(g, \alpha, \beta)$ is a frame for all $\alpha, \beta > 0$ with $\alpha\beta < 1$, see also Theorem 4.8. It is also known that the lower frame-bound A of these frames approaches 0 when $\alpha\beta$ tends to 1. Our goal of this section is to find a quantitative result which describes these properties.

Remark 5.1. There are only two window functions (and scaled versions thereof) in the literature, where the asymptotic behaviour of the lower frame-bound was specified near the critical density $\alpha\beta \approx 1$. These are the Gabor frames $\mathcal{G}(g, \alpha, \alpha)$ with g the Gaussian window $g(x) = e^{-\pi x^2}$ and the hyperbolic secant $g(x) = (\cosh \pi x)^{-1}$. It was proved in [3] that constants $c_1, c_2 > 0$ exist such that the optimal lower frame-bound A_{opt} of $\mathcal{G}(g, \alpha, \alpha)$ satisfies

$$c_1(1 - \alpha) \leq A_{opt} \leq c_2(1 - \alpha) \quad \text{for} \quad \frac{1}{2} < \alpha < 1.$$

In this section, we give explicit lower bounds of the same linear asymptotic decay as $\alpha\beta$ tends to 1, if the window function is the symmetric EB-spline (23) of order 2 and $\alpha = 1$ (Theorem 5.3), or the TP function $g(x) = \frac{\lambda}{2}e^{-\lambda|x|}$ of finite type 2 and general $\alpha, \beta > 0$ (Theorem 5.6).

Example 5.2. The left-hand side of Figure 3 depicts lower frame-bounds for the Gabor frame $\mathcal{G}(B, 1, \beta)$ with the window function B in (23) and $\lambda = 1$. The points mark the optimal lower frame-bound A_{opt} for rational values $\beta > 1/2$, taken at $\beta = k/61$, $31 \leq k \leq 60$. The computation of the optimal lower frame-bound follows the method of Zibulski and Zeevi, based on a matrix-valued Zak transform of B , see [31] or [11, Theorem 8.3.3]. The lower frame-bound A in Theorem 5.3 is drawn as a solid line. Note that, by the distinction of three different cases for $\frac{1}{2} < \beta < 1$, the lower bound has a jump at $\beta = \frac{5}{6}$. The asymptotically linear decay of A near $\beta = 1$ is expressed by

$$A^{-1} = \left(2 \sinh^2\left(\frac{1}{2}\right)(1 - \beta)\right)^{-1} + \mathcal{O}\left((1 - \beta)^{-1/2}\right).$$

The right half of Figure 3 gives the same type of information for the Gabor frame $\mathcal{G}(g, 1, \beta)$, where $g(x) = \frac{1}{2}e^{-|x|}$. The solid line shows the lower frame-bound A in Theorem 5.6.

We start with the discussion of lower frame-bounds for $\mathcal{G}(B, 1, \beta)$, where B is the EB-spline in (23). For this purpose, we inspect the pre-Gramian matrices

$$P(x) = \left(B\left(x + k - \frac{l}{\beta}\right)\right)_{k, l \in \mathbb{Z}}, \quad x \in \mathbb{R},$$

in more detail.

Before we present our result in Theorem 5.3, let us illustrate our method for finding lower frame-bounds for $\mathcal{G}(B, 1, \beta)$ by first considering lattices with high redundancy. Let $1/\beta \geq 2 = \text{vol}(\text{supp}(B))$. As stated before in (17), the optimal lower frame-bound is given by

$$A_{opt} = \inf_{x \in [0, 1)} \frac{1}{\beta} \sum_{k \in \mathbb{Z}} |B(x + k)|^2.$$

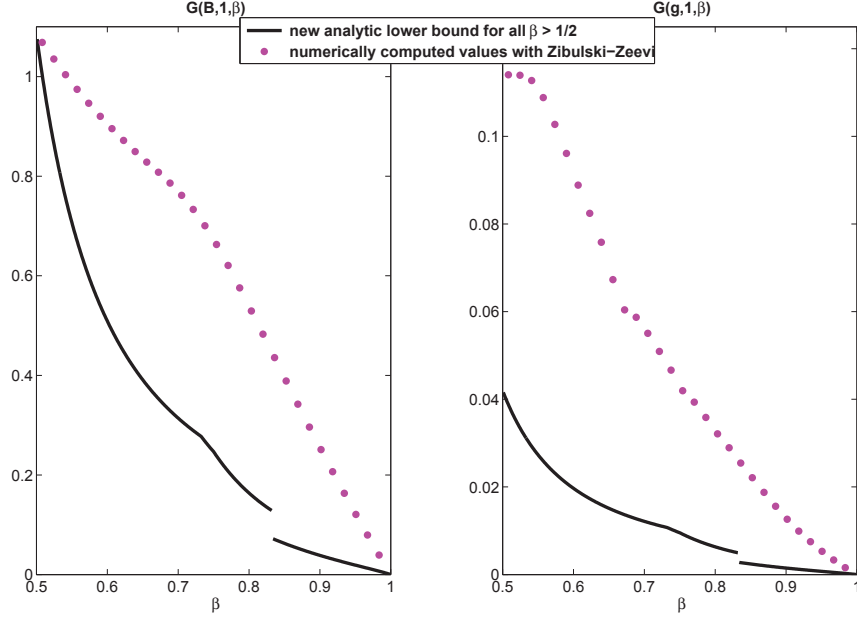


Figure 3: Points in the left figure show the optimal lower frame-bound of $\mathcal{G}(B, 1, \beta)$ for $\beta = k/61$, $31 \leq k \leq 60$, where B is the EB-spline (23) with $\lambda = 1$. The solid line in the left figure depicts the lower frame-bound A in Theorem 5.3. Points in the right figure show the optimal lower frame-bound of $\mathcal{G}(g, 1, \beta)$, where $g(x) = \frac{1}{2}e^{-|x|}$. The solid line in the right figure depicts the lower frame-bound in Theorem 5.6.

This was, for example, published in [8] and is also part of the Casazza-Christensen bound [5]. We next show how we can find this optimal lower frame-bound $A_\beta(B)$ from the pre-Gramian $P(x)$. For every $x \in [0, 1]$, there is at most one non-zero entry in each row of $P(x)$. Namely, if $\beta = 1/2$, we have

$$P(x) = \begin{pmatrix} \ddots & & & & \\ & B(x) & & & \\ & B(x+1) & & & \\ & & B(x) & & \\ & & B(x+1) & & \\ & & & B(x) & \\ & & & B(x+1) & \\ & & & & \ddots \end{pmatrix}, \quad x \in [0, 1]. \quad (24)$$

For $\beta < 1/2$, the entry of $P(x)$ in row k and column l is given by

$$p_{k,l}(x) = B(x + k - \frac{l}{\beta}).$$

This entry is nonzero, if and only if

$$k \in I_l := \mathbb{Z} \cap (\frac{l}{\beta} - x, \frac{l}{\beta} - x + 2).$$

The sets I_l are non-empty and, due to $\beta < 1/2$, they are pairwise disjoint. This implies that the matrix $P(x)$ has a similar form as (24), with the possibility

of some zero rows. The Moore-Penrose pseudo-inverse of $P(x)$ is

$$\Gamma(x) = (P(x)^T P(x))^{-1} P(x)^T,$$

and $P(x)^T P(x)$ is a diagonal matrix with diagonal entries

$$\sum_{j \in \mathbb{Z}} p_{j,l}(x)^2 = \sum_{j \in \mathbb{Z}} B(x + j\alpha - \frac{l}{\beta})^2, \quad l \in \mathbb{Z}.$$

Hence, the entry of $\Gamma(x)$ in row l and column k is

$$\gamma_{l,k}(x) = \frac{B(x + k - \frac{l}{\beta})}{\sum_{j \in \mathbb{Z}} B(x + j\alpha - \frac{l}{\beta})^2}.$$

Since nonzero entries in different rows of $\Gamma(x)$ appear in disjoint sets of columns, the operator norm is

$$\|\Gamma(x)\|_2 = \sup_{l \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} \gamma_{l,k}(x)^2 \right)^{1/2} = \sup_{l \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} B(x + j\alpha - \frac{l}{\beta})^2 \right)^{-1/2}.$$

Moreover, $\Gamma(x)$ is a left-inverse of $P(x)$ with the smallest ℓ_2 -operator norm. Therefore, with Theorem 4.6 the optimal lower frame-bound of the Gabor frame $\mathcal{G}(B, 1, \beta)$ with $0 < \beta \leq \frac{1}{2}$ is given by

$$A_{opt} = \beta^{-1} \left(\sup_{x \in [0,1)} \|\Gamma(x)\|_2 \right)^{-2} = \beta^{-1} \inf_{x \in [0,1)} (B(x)^2 + B(x+1)^2).$$

This infimum is obtained for $x = \frac{1}{2}$ and has the value

$$A_{opt} = \frac{2 \sinh^2(\frac{\lambda}{2})}{\beta \lambda^2}. \quad (25)$$

Our next theorem extends this construction to frames $\mathcal{G}(B, 1, \beta)$ with $\beta > 1/2$. It is no longer true that consecutive columns of $P(x)$ are supported in disjoint sets of rows. However, we can still find a block decomposition of the pre-Gramian $P(x)$ and a suitable left-inverse in order to prove the following result.

Theorem 5.3. *Let $\lambda > 0$, $0 < \beta < 1$, and B be the EB-spline in (23). Then the corresponding Gabor frame $\mathcal{G}(B, 1, \beta)$ has the lower frame-bound*

$$A = (\beta \lambda^2 c_\beta)^{-1} \min \left\{ 2 \sinh^2(\frac{\lambda}{2}), \sinh^2(\frac{\lambda}{2\beta}) \right\}, \quad (26)$$

where

$$c_\beta = \begin{cases} 1, & \text{if } 0 < \beta \leq \frac{1}{2}, \\ (3 - \beta^{-1})^2, & \text{if } \frac{1}{2} < \beta \leq \frac{3}{4}, \\ (3 - \beta^{-1})(11 - 11\beta^{-1} + 3\beta^{-2}), & \text{if } \frac{3}{4} < \beta \leq \frac{5}{6}, \\ \left(1 + \sqrt{\frac{\beta}{\pi(1-\beta)}}\right) \left(1 + \sqrt{\frac{\pi\beta}{4(1-\beta)}}\right), & \text{if } \frac{5}{6} < \beta < 1. \end{cases}$$

Remark 5.4. The parameter $\lambda = 0$ can be included. Then B in (23) is the linear polynomial B-spline

$$B(t) = (\chi_{[0,1]} * \chi_{[0,1]})(t) = \begin{cases} t, & 0 \leq t \leq 1, \\ (2-t), & 1 < t \leq 2, \end{cases}$$

and the lower frame-bound of $\mathcal{G}(B, 1, \beta)$ is

$$A = (\beta c_\beta)^{-1} \cdot \min \{1/2, 1/(2\beta)^2\}.$$

Proof. For $\beta \leq \frac{1}{2}$ the result was shown in (25).

Now let $1/2 < \beta < 1$ and $x \in [0, 1)$. As in our motivation, we construct a left-inverse $\Gamma(x)$ of the (full) pre-Gramian

$$P(x) = (p_{k,l}(x))_{k,l \in \mathbb{Z}} = \left(B(x + k - \frac{l}{\beta}) \right)_{k,l \in \mathbb{Z}}$$

with operator norm

$$\|\Gamma(x)\|_2 \leq c_\beta^{1/2} \lambda \left(\min \left\{ \sqrt{2} \sinh(\frac{\lambda}{2}), \sinh(\frac{\lambda}{2\beta}) \right\} \right)^{-1}.$$

Then the result follows from Theorem 4.6.

First, we take a close look at the form of $P(x)$ and detect a block structure which is more complicated than for $\beta = 1/2$ in (24). Since the support of B is $[0, 2]$, there are at most two non-zero entries in each row and in each column of $P(x)$. Moreover, every 2×2 submatrix of $P(x)$ contains at least one zero. The overall form of $P(x)$ is a staircase, similar to (24), with steps of height one if

$$x + k - \frac{l}{\beta}, x + k + 1 - \frac{l+1}{\beta} \in (0, 1],$$

and steps of height two if

$$x + k - \frac{l}{\beta} \in (0, 1], \quad x + k + 1 - \frac{l+1}{\beta} < 0.$$

This form is depicted in the following scheme, where a step of height two appears between the first and second column, followed by two steps of height one and so on,

$$P(x) = \begin{pmatrix} \ddots & * & 0 & 0 & 0 & 0 & 0 \\ & * & 0 & 0 & 0 & 0 & 0 \\ & & 0 & * & 0 & 0 & 0 \\ & & 0 & * & * & 0 & 0 \\ & & 0 & 0 & * & * & 0 \\ & & 0 & 0 & 0 & * & 0 \\ & & 0 & 0 & 0 & 0 & * & 0 \\ & & 0 & 0 & 0 & 0 & * & * \\ & & 0 & 0 & 0 & 0 & 0 & * & \ddots \end{pmatrix}.$$

We see that the pre-Gramian has the following block-structure.

1. If, for some $k, l \in \mathbb{Z}$, we have $x + k - \frac{l}{\beta} = 1$, then the (k, l) -entry $p_{k,l}(x) = B(1) = \frac{\sinh \lambda}{\lambda}$ defines a 1-by-1 block, with no other non-zero entries in row k and column l of $P(x)$.
2. All other blocks have the form

$$P_0 := \begin{pmatrix} a_1 & & & & \\ b_1 & a_2 & & & \\ & \ddots & \ddots & & \\ & & b_{s-1} & a_s & \\ & & & b_s & \end{pmatrix} \in \mathbb{R}^{(s+1) \times s} \quad (27)$$

with one more row than column and only zeros to the right and left, top and bottom of this block. The first entry $a_1 = B(x_1)$ is the entry in row k and column l of $P(x)$ with

$$x_1 := x + k - \frac{l}{\beta} \in (2 - \frac{1}{\beta}, 1),$$

such that only zeros appear to the left and right of a_1 in row k of $P(x)$ and also above in column l . Subsequent entries are

$$a_j = B(x_j), \quad b_j = B(x_j + 1) = B(1 - x_j), \quad 1 \leq j \leq s, \quad (28)$$

where

$$x_j := x_1 - (j - 1)(\frac{1}{\beta} - 1), \quad 1 \leq j \leq s, \quad (29)$$

and $s \in \mathbb{N}$ is defined by the condition that

$$x_s = x_1 - (s - 1)(\frac{1}{\beta} - 1) \in (0, \frac{1}{\beta} - 1].$$

Note that only zeros appear to the left and right of b_s in the corresponding row of $P(x)$ and also below in the corresponding column. Simple computations reveal that $s = \lceil \beta x_1 / (1 - \beta) \rceil$. So blocks have $s \approx \lceil \frac{\beta}{1 - \beta} \rceil$ columns.

A left-inverse $\Gamma(x)$ of $P(x)$ is constructed by considering each block separately. The operator norm of $\Gamma(x)$, as an operator on $\ell_2(\mathbb{Z})$, is the supremum of the operator norms of all blocks.

Now, let us consider a typical block P_0 of $P(x)$.

1. If P_0 is 1-by-1, then its inverse is $\Gamma_0 = \frac{\lambda}{\sinh \lambda}$.
2. If P_0 is $s+1$ -by- s , with s as described above, then we use a decomposition of P_0 which is similar to the Neville-elimination method in [10]. The following lemma provides an upper bound of the norm of a left-inverse Γ_0 and thus completes the proof of Theorem 5.3.

□

Lemma 5.5. *Let $\frac{1}{2} < \beta < 1$ and P_0 be the matrix in (27), with a_j, b_j in (28), $1 \leq j \leq s$. Then there exists a left-inverse Γ_0 of P_0 such that*

$$\|\Gamma_0\|_2 \leq c_\beta^{1/2} \lambda \left(\min \left\{ \sqrt{2} \sinh\left(\frac{\lambda}{2}\right), \sinh\left(\frac{\lambda}{2\beta}\right) \right\} \right)^{-1}$$

with the constant c_β in Theorem 5.3.

Proof. Note that P_0 is an ASTP matrix and has full rank. Indeed, the s -by- s minor of the first s rows is positive as its diagonal entries are positive, see Theorem 2.2. We construct a decomposition of P_0 which is similar to the Neville elimination, as explained in [10], with slight modifications due to the rectangular form of P_0 .

For this purpose, let

$$\delta = \frac{1}{2}\left(\frac{1}{\beta} - 1\right), \quad \omega = \frac{1}{4\delta\beta} = \frac{1}{2 - 2\beta}.$$

Note that $0 < \delta < \frac{1}{2}$ and $\omega > 1$. We start from the observation that

$$1 > x_1 \geq 2 - \frac{1}{\beta} = 1 - 2\delta > \frac{1}{2} - \delta > 0, \quad 0 < x_s \leq 2\delta < \frac{1}{2} + \delta.$$

Therefore, we can choose $1 \leq r \leq s$ such that

$$|x_r - \frac{1}{2}| \leq \delta.$$

This is always possible, as the sequence of points x_j in (29) decreases with uniform stepsize 2δ , and we have

$$1 > x_1 > \cdots > x_{r-1} \geq \frac{1}{2} + \delta \geq x_r \geq \frac{1}{2} - \delta \geq x_{r+1} > \cdots > x_s > 0, \quad (30)$$

with the possibility of empty subsequences in these inequalities, if $r = 1$ or $r = s$. Due to $1 > x_1 \geq \frac{1}{2} + (2r - 3)\delta$, we find by simple computations that $r - 1 < \omega$.

Now we construct a left-inverse of P_0 . In contrast to Gauss elimination, the Neville elimination uses neighboring rows. In this manner, we obtain the factorization

$$P_0 = CS$$

where

$$C = \begin{pmatrix} 1 & & & & & & & \\ \frac{b_1}{a_1} & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \frac{b_{r-1}}{a_{r-1}} & 1 & & & & \\ & & & 1 & \frac{a_{r+1}}{b_{r+1}} & & & \\ & & & & \ddots & \ddots & & \\ & & & & & 1 & \frac{a_s}{b_s} \\ & & & & & & 1 \end{pmatrix},$$

$$S = \begin{pmatrix} a_1 & & & & & & & \\ & a_2 & & & & & & \\ & & \ddots & & & & & \\ & & & a_r & & & & \\ & & & b_r & & & & \\ & & & & \ddots & & & \\ & & & & & b_{s-1} & & \\ & & & & & & b_s \end{pmatrix}.$$

The Moore-Penrose pseudo-inverse $S^\dagger$ of S is readily computed and gives

$$\|S^\dagger\|_2 = \max \left\{ a_j^{-1} \text{ for } 1 \leq j \leq r-1, \quad b_j^{-1} \text{ for } r+1 \leq j \leq s, \quad (a_r^2 + b_r^2)^{-1/2} \right\}.$$

By (28) and (30) we obtain

$$\|S^\dagger\|_2 \leq \max \left\{ \frac{1}{B(\frac{1}{2} + \delta)}, \frac{1}{\sqrt{2}B(\frac{1}{2})} \right\} = \max \left\{ \frac{\lambda}{\sinh(\lambda(\frac{1}{2} + \delta))}, \frac{\lambda}{\sqrt{2} \sinh(\frac{\lambda}{2})} \right\}.$$

Next we find an upper bound for the norm of $F = C^{-1}$. Elementary linear algebra shows that F has the block form

$$F = \begin{pmatrix} L & \\ & U \end{pmatrix}$$

with lower triangular matrix L and upper triangular matrix U , and entries

$$f_{j,k} = (-1)^{j-k} \prod_{l=k}^{j-1} \frac{b_l}{a_l}, \quad 1 \leq k \leq j \leq r,$$

$$f_{s+2-j, s+2-k} = (-1)^{j-k} \prod_{l=k}^{j-1} \frac{a_{s+1-l}}{b_{s+1-l}}, \quad 1 \leq k \leq j \leq s-r+1.$$

Clearly, $\|F\|_2 = \max\{\|L\|_2, \|U\|_2\}$, by the block structure of F . We use Schur's test for finding an upper bound of $\|L\|_2$, and only mention here that the estimates for $\|U\|_2$ follow analogously.

First, we obtain from (29) and $x_{r-1} \geq \frac{1}{2} + \delta$, that

$$1 > x_{r-j} \geq \frac{1}{2} + (2j-1)\delta, \quad 1 \leq j \leq r-1.$$

Simple calculations give

$$\frac{\sinh \lambda x}{\sinh \lambda(1-x)} \leq 2x \quad \text{for all } x \in [0, \frac{1}{2}],$$

and thus

$$\frac{b_{r-j}}{a_{r-j}} = \frac{\sinh(\lambda(1-x_{r-j}))}{\sinh(\lambda x_{r-j})} \leq 2 - 2x_{r-j} \leq 1 - (4j-2)\delta = 1 - \frac{j - \frac{1}{2}}{\omega - \frac{1}{2}}.$$

By inserting these upper bounds, and by use of the basic inequality $\ln(1-x) \leq -x$ for $x \in [0, 1]$, we obtain

$$\begin{aligned} |f_{j,k}| &\leq \prod_{l=k}^{j-1} \left(1 - \frac{r-l-\frac{1}{2}}{\omega - \frac{1}{2}} \right) \\ &\leq \exp \left(- \sum_{l=k}^{j-1} \frac{r-l-\frac{1}{2}}{\omega - \frac{1}{2}} \right) \\ &= \exp \left(- \frac{(j-k)(2r-j-k)}{2\omega-1} \right) \end{aligned}$$

for all $1 \leq k \leq j \leq r$. Note that for $j=k$ we have the empty product $f_{j,j} = 1$.

Let us turn to the estimate of $\|L\|_\infty$. The sum of absolute values of all entries in row $1 \leq j \leq r$ of L is bounded by

$$\begin{aligned} \rho_j &= \sum_{k=1}^j |f_{j,k}| \leq \sum_{k=1}^j \exp \left(- \frac{(j-k)(2r-j-k)}{2\omega-1} \right) \\ &= \sum_{k=0}^{j-1} \exp \left(- \frac{k(2r-2j+k)}{2\omega-1} \right). \end{aligned}$$

The last sum is maximal for $j=r$, and the interpretation as a Riemann sum gives

$$\begin{aligned} \|L\|_\infty &\leq \sum_{k=0}^{r-1} \exp \left(- \frac{k^2}{2\omega-1} \right) \\ &\leq 1 + \int_0^\infty \exp \left(- \frac{x^2}{2\omega-1} \right) dx = 1 + \sqrt{\frac{(2\omega-1)\pi}{4}}. \end{aligned}$$

By the identity $2\omega-1 = \frac{\beta}{1-\beta}$, we obtain the upper bound

$$\|L\|_\infty \leq 1 + \sqrt{\frac{\pi\beta}{4(1-\beta)}}.$$

The same method provides a slightly smaller bound for the sums of absolute values in column k of L . More precisely, for $1 \leq k \leq r$, we have

$$\begin{aligned}
\kappa_k &= \sum_{j=k}^r |f_{j,k}| \leq \sum_{j=k}^r \exp\left(-\frac{(j-k)(2r-j-k)}{2\omega-1}\right) \\
&= \sum_{j=0}^{r-k} \exp\left(-\frac{j(2r-2k-j)}{2\omega-1}\right) \\
&\leq 1 + \int_0^{r-k} \exp\left(-\frac{x(2r-2k-x)}{2\omega-1}\right) dx \\
&= 1 + \sqrt{2\omega-1} \int_0^c e^{-x(2c-x)} dx,
\end{aligned}$$

where we let $c := (r-k)(2\omega-1)^{-1/2}$ in the last step. The function $h(c) = \int_0^c e^{-x(2c-x)} dx$ achieves its maximum over all positive values of c for $c_0 \approx 0.9241$, and $h(c_0) = 1/(2c_0) \approx 0.5410 < \pi^{-1/2}$. Therefore, we obtain

$$\|L\|_1 \leq 1 + \sqrt{\frac{\beta}{\pi(1-\beta)}}.$$

Schur's test gives

$$\|L\|_2^2 \leq \|L\|_1 \|L\|_\infty \leq \left(1 + \sqrt{\frac{\beta}{\pi(1-\beta)}}\right) \left(1 + \sqrt{\frac{\pi\beta}{4(1-\beta)}}\right).$$

More accurate bounds of $\|L\|_1$ and $\|L\|_\infty$ can be given for small values of r , e.g. for $r = 2$ (this is the case $\frac{1}{2} < \beta \leq \frac{3}{4}$), we have

$$\begin{aligned}
L &= \begin{pmatrix} 1 & 0 \\ f_{2,1} & 1 \end{pmatrix}, \quad |f_{2,1}| \leq 1 - \frac{\frac{1}{2}}{\omega - \frac{1}{2}} = 2 - \frac{1}{\beta}, \\
\|L\|_1 &= \|L\|_\infty \leq 3 - \frac{1}{\beta}.
\end{aligned}$$

Likewise, for $r = 3$ (this is the case $\frac{3}{4} < \beta \leq \frac{5}{6}$), we have

$$\begin{aligned}
L &= \begin{pmatrix} 1 & 0 & 0 \\ f_{2,1} & 1 & 0 \\ f_{3,1} & f_{3,2} & 1 \end{pmatrix}, \\
|f_{2,1}| &\leq 1 - \frac{\frac{3}{2}}{\omega - \frac{1}{2}} = 4 - \frac{3}{\beta}, \quad |f_{3,2}| \leq 1 - \frac{\frac{1}{2}}{\omega - \frac{1}{2}} = 2 - \frac{1}{\beta}, \quad |f_{3,1}| = |f_{2,1}f_{3,2}|, \\
\|L\|_1 &\leq 3 - \frac{1}{\beta}, \quad \|L\|_\infty \leq 11 - \frac{11}{\beta} + \frac{3}{\beta^2}.
\end{aligned}$$

These bounds lead to the particular upper bounds in Lemma 5.5. Therefore, the proof of the lemma is complete. $\square$

We can now add a similar result for Gabor frames $\mathcal{G}(g, \alpha, \beta)$, where g is the two-sided exponential

$$g(x) = \frac{\lambda}{2} e^{-\lambda|x|}$$

with $\lambda > 0$. The relation in Theorem 3.4 gives

$$Z_\alpha g(x, \omega) = \frac{1}{\alpha} C(\omega) Z_1 B\left(\frac{x}{\alpha}, \alpha\omega\right),$$

where B is the symmetric EB-spline of order 2 in (23) with parameter $\alpha\lambda$ instead of λ , and

$$C(\omega) = \frac{\alpha\lambda}{1 - e^{-(\alpha\lambda + 2\pi i\omega)}} \frac{-\alpha\lambda}{1 - e^{-(-\alpha\lambda + 2\pi i\omega)}}.$$

The minimum of $|C(\omega)|$ occurs at $\omega = \frac{1}{2}$,

$$|C(\omega)| \geq |C(\tfrac{1}{2})| = \frac{\alpha^2 \lambda^2}{4 \cosh^2(\frac{\alpha\lambda}{2})}.$$

With Proposition 4.7 and Theorem 5.3, where we replace λ by $\alpha\lambda$ and β by $\alpha\beta$ in (26), we obtain the following lower frame-bound.

Theorem 5.6. *Let g be the two-sided exponential with $g(x) = \frac{\lambda}{2} e^{-\lambda|x|}$, $\lambda > 0$, and $\alpha, \beta > 0$ some lattice parameters with $\alpha\beta < 1$. Then the Gabor frame $\mathcal{G}(g, \alpha, \beta)$ has the lower frame-bound*

$$A = \frac{\lambda^2 \min \left\{ 2 \sinh^2\left(\frac{\alpha\lambda}{2}\right), \sinh^2(\lambda/(2\beta)) \right\}}{16\beta c_{\alpha\beta} \cosh^4\left(\frac{\alpha\lambda}{2}\right)},$$

where $c_{\alpha\beta}$ is defined as in Theorem 5.3.

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