Towards a monolithic multiphase flow solver via surface stress-based formulations

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Abstract. In this note we discuss the combination of the classical CSS and CSF approaches with special FEM techniques towards a fully implicit distance-free level set method. Based on a new surface stress formulation, neither normals nor curvature have to be explicitly calculated, and also the explicit redistancing can be avoided so that a monolithic formulation of the corresponding multiphase problem gets feasible. Prototypical numerical tests of benchmarking character for a rising 2D bubble are provided for validating the accuracy of this new approach.

Key words: Level set, FEM, Multiphase flow, Continuum surface force, Continuum surface stress, Navier-Stokes equations

1 Introduction

Interface capturing methods, as for instance level set formulations, are widely used in fluid dynamics due to their flexibility to handle two phase flow [4] and free surfaces [5]. In this paper we discuss ideas for a monolithic treatment of multiphase flow problems: The surface tension is treated without calculating normals and curvature and the explicit localization of the surface force or the surface stress on the interface is not required. Moreover, we discuss a variant avoiding the explicit redistancing procedure which is an essential part in most level set approaches.

From a methodological point of view, we combine the classical CSS [11] and CSF [2] approaches with conservative level set techniques [13] to introduce an improved variant of a fully implicit level set method. In the first step, inspired by the fictitious boundary methodology [12], the CSF force term in the level set approach is calculated as a volume force using a cutoff function generated by the distance based level set function so that, as a consequence, the localization of the force on the surface using the Dirac delta function can be avoided. Then, the resulting vectorial force is upgraded as a tensor field, similar to the CSS approach, in which neither normals nor curvature evaluations are required. First numerical tests based on quantitative benchmark results for a rising bubble [9] in 2D are provided which show the high accuracy of this approach in the context of level set FEM.

Furthermore, an implicit algorithm for preserving the signed distance property is introduced, based on the ideas in [13]. There are several potential advantages of this approach: First, it leads to a more implicit algorithm where no capillary time step restriction remains and makes it possible to use a monolithic treatment of multiphase flow problems. Second, corresponding multiphase flow formulations can be simulated with a standard Navier-Stokes solver with homogeneous force terms, that means omitting the surface tension force as right hand side term. Third, special FEM spaces for the introduced surface stress can be used which might lead to improved approximation properties.

The paper is organized as follows: The next section introduces the level set methodology for multiphase flow problems, where the continuous surface force is added to the momentum equation with the help of the Dirac delta function. In the third section, the Dirac delta function is removed using the gradient of a material function for both approaches, the classical CSS [11] and CSF approach [2], then we analyze the corresponding behaviour of the different level set variants w.r.t. accuracy, based on the well established benchmark of a rising 2D bubble [9]. To derive a fully implicit level set variant, evolution equations for the material function and a signed distance level set function which do not require any reinitialization are introduced in the fourth section. A summary with several future research directions is given in the last section.

2 Level set method for multiphase problems

For multiphase flow problems, defined on a domain Ω, the materials are separated by an interface, Γ, which is assumed to be a lower-dimensional manifold. In what follows all functions are defined on $\Omega \times (0,t)$.
The equations of motion for each phase are presented as follows

$$\begin{aligned} \rho(\Gamma)\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) - \operatorname{div}\boldsymbol{\tau} + \nabla p = 0, \\ \operatorname{div}\boldsymbol{u} = 0 \end{aligned} \tag{1}$$

where $\boldsymbol{u}$ is the velocity, ρ the density, p the pressure, $\boldsymbol{\tau} = \boldsymbol{\tau}_s = 2\mu(\Gamma)\mathbf{D}(\boldsymbol{u})$ the (standard) stress tensor. The only external force is the gravity, nevertheless the momentum equation is presented with homogeneous force terms for the sake of the clarity of the main objective of the paper.

At the interface, the usual jump conditions need to be imposed

$$[u]_{|\Gamma} = 0 \quad , \quad -[\boldsymbol{\tau}\cdot\boldsymbol{n}]_{|\Gamma} = \kappa\sigma\boldsymbol{n} \tag{2}$$

where κ is the curvature, σ is the surface force constant, and $\boldsymbol{n}$ is the normal to the interface.

In the framework of the level set method, as an example for interface capturing approaches, the interface is embedded in a higher dimensional smooth function, and the level set function satisfies the following equation

$$\frac{\partial\varphi}{\partial t} + \boldsymbol{u}\cdot\nabla\varphi = 0. \tag{3}$$

For two immiscible fluids, the (continuous) level set function is chosen to change the sign for the different fluid phases and the interface Γ corresponds to the zero level of the level set function, that means

$$\Gamma = \{\boldsymbol{x} \in \Omega, \varphi = 0\}\,. \tag{4}$$

The normals $\boldsymbol{n}$ to the interface and the curvature κ are given by (assuming that $|\nabla\varphi| \neq 0$)

$$\boldsymbol{n} = \frac{\nabla\varphi}{|\nabla\varphi|}, \quad \kappa = -\nabla \cdot \boldsymbol{n}. \tag{5}$$

The boundary condition (2) is imposed as surface force which leads to the well-known Continuum Surface Force (CSF) approach. The CSF approach [2] consists of adding the surface force term

$$\boldsymbol{f}_{\text{CSF}} = \kappa\sigma\boldsymbol{n}_{|\Gamma} \tag{6}$$

to the right hand side of the momentum equation and takes the form

$$\rho(\varphi)\left(\frac{\partial\boldsymbol{u}}{\partial t} + u \cdot \nabla\boldsymbol{u}\right) - \operatorname{div}\boldsymbol{\tau} + \nabla p = \boldsymbol{f}_{\text{CSF}}. \tag{7}$$

The surface force (6) can be rewritten as a volume force, $\boldsymbol{f}_{\text{CSF},1}$, using the Delta function, δ_Γ, localizing the interface

$$\boldsymbol{f}_{\text{CSF},1} = \kappa\sigma\boldsymbol{n}\delta_\Gamma. \tag{8}$$

For the numerical simulations the Delta function, δ_Γ, can be regularized as follows

$$\delta_\Gamma^\epsilon(x) = \begin{cases} \dfrac{1}{\epsilon}p\left(\dfrac{\text{dist}_\Gamma(x)}{\epsilon}\right) & \text{if } \text{dist}_\Gamma(x) \leq \epsilon \\ 0 & \text{if } \text{dist}_\Gamma(x) > \epsilon \end{cases} \tag{9}$$

where dist_Γ is the distance function to the interface Γ. The width of the regularization is denoted by ϵ which is usually related to the mesh size [8]. In our numerical simulations the function p is chosen as a polynomial [8]

$$p(z) = \frac{35}{32}(1 - 3z^2 + 3z^4 - z^6). \tag{10}$$

The distance to the interface is essential for the approximation of the Delta function. So, the straightforward choice for the level set function is the signed distance function in combination with a reinitialization procedure [16].

3 Curvature-free level set approach

For finite element methods, the surface evaluation of the force may suffer from mesh convergence [18]. Based on the numerical experience w.r.t. the evaluation of surface integrals of drag and lift coefficients particularly in flow simulations [1, 7, 10] in the context of fictitious boundary methods [12], it turns out that the evaluation of the surface integrals by integrals over the whole volume may enhance the accuracy.

The idea consists of introducing the cutoff function ψ:

$$\psi(x) = \begin{cases} 1 & \text{if } \varphi(x) \geq 0 \\ 0 & \text{if } \varphi(x) < 0 \end{cases} \tag{11}$$

Then, beside being a material characterization, the function (11) has the following important property regarding (8)

$$\boldsymbol{n}\delta_{\Gamma} = \boldsymbol{n}\delta_{\{\varphi=0\}} = \nabla\psi(\varphi). \tag{12}$$

Consequently, an alternative version of the surface tension is given as

$$\boldsymbol{f}_{\text{CSF},2} = \sigma\kappa\nabla\psi. \tag{13}$$

For the numerical simulations, the material cutoff function ψ can be regularized with the help of the signed distance function as for instance [13]

$$\psi^{\epsilon}(x) = \frac{-1}{1 + \exp\left(\frac{\text{sign}(\varphi(x))\text{dist}_{\Gamma}(x)}{\epsilon}\right)} + 0.5 \tag{14}$$

where

$$\text{sign}(z) = \begin{cases} +1 & \text{if } z > 0 \\ -1 & \text{if } z < 0 \end{cases} \tag{15}$$

Remark 1. Similar to the regularized Delta function in (9), the regularized cutoff function (14) requires the distance to the interface.

While these representations of the surface tension in (8), respectively in (13), are classical, our motivation to derive a curvature free formulation leads us to go further with the investigation of the curvature. In what follows we assume more regularity for ψ and express $\boldsymbol{f}_{\text{CSF},2}$ in terms of ψ, by using (5) so that we obtain

$$\begin{aligned} \boldsymbol{f}_{\text{CSF},2} &= -\sigma\nabla\cdot\left(\frac{\nabla\psi}{|\nabla\psi|}\right)\nabla\psi \\ &= -\sigma\left(\frac{\nabla\cdot\nabla\psi}{|\nabla\psi|} - \frac{\nabla|\nabla\psi|\nabla\psi}{|\nabla\psi|^2}\right)\nabla\psi \\ &= -\sigma\left(\frac{1}{|\nabla\psi|}\triangle\psi\nabla\psi - \nabla|\nabla\psi|\right). \end{aligned} \tag{16}$$

Moreover, we have

$$\triangle\psi\nabla\psi = \nabla\cdot(\nabla\psi\otimes\nabla\psi) - \frac{1}{2}\nabla|\nabla\psi|^2. \tag{17}$$

Then we obtain

$$\begin{aligned} \frac{1}{|\nabla\psi|}\triangle\psi\nabla\psi &= \frac{\nabla\cdot(\nabla\psi\otimes\nabla\psi)}{|\nabla\psi|} - \frac{1}{2}\frac{\nabla|\nabla\psi|^2}{|\nabla\psi|} \\ &= \nabla\cdot\left(\frac{\nabla\psi\otimes\nabla\psi}{|\nabla\psi|}\right) + \frac{(\nabla\psi\otimes\nabla\psi)\nabla|\nabla\psi|}{|\nabla\psi|^2} - \nabla|\nabla\psi| \\ &= \nabla\cdot\left(\frac{\nabla\psi\otimes\nabla\psi}{|\nabla\psi|}\right) - \left(\mathbf{I} - \frac{\nabla\psi}{|\nabla\psi|}\otimes\frac{\nabla\psi}{|\nabla\psi|}\right)\nabla|\nabla\psi| \\ &= \nabla\cdot\left(\frac{\nabla\psi\otimes\nabla\psi}{|\nabla\psi|}\right) - \nabla_s|\nabla\psi| \end{aligned} \tag{18}$$

where

$$\nabla_s = (\mathbf{I} - \boldsymbol{n} \otimes \boldsymbol{n})\nabla. \tag{19}$$

Since

$$\nabla_s |\nabla\psi| = 0, \tag{20}$$

we recover the CSS expression as in [15]

$$\boldsymbol{f}_{\mathrm{CSF},2} = -\sigma \left\{ \nabla \cdot \left(\frac{\nabla\psi \otimes \nabla\psi}{|\nabla\psi|} \right) - \nabla |\nabla\psi| \right\}. \tag{21}$$

It is important to notice that the gradient term $\sigma\nabla |\nabla\psi|$ can be absorbed in the pressure. So, we finally introduce $\boldsymbol{f}_{\mathrm{CSF},3}$ as follows

$$\boldsymbol{f}_{\mathrm{CSF},3} = -\sigma\nabla \cdot \left(\frac{\nabla\psi \otimes \nabla\psi}{|\nabla\psi|} \right), \tag{22}$$

and

$$\nabla p_{\mathrm{CSF},3} = \nabla p_{\mathrm{CSF},1/2} - \sigma\nabla |\nabla\psi|. \tag{23}$$

Remark 2. The splitting of the CSS stress into two parts in (21) allows a direct consistency distribution of the force via the choice of the corresponding finite element spaces similar to three field formulations of the Stokes problem, while the classical volumetric surface force representation via CSF or CSS needs special care to avoid spurious velocities [6, 14].

Remark 3. The formulation in divergence form in (22) is advantageous in the FEM framework since due to partial integration, weaker regularity requirements have to be imposed and, additionally, only information about the gradient of the indicator function are required.

3.1 Numerical validation

For the numerical validation, we consider the rising bubble benchmark [9], see also (www.featflow.de/en/benchmarks.html) for the details of the benchmark configuration. A (gas) bubble is placed at the lower part of an 1×2 rectangular geometry with a radius of $r = 0.25$. Given a different density and viscosity between the two immiscible fluids, the bubble rises due to buoyancy force when solving the Navier-Stokes equations. The evolution of the bubbles has been tracked for 3 time units during which the defined benchmark quantities should be measured. We restrict our validation to the test case of a rising bubble with Reynolds number $\mathrm{R}e = 35$, Eötvös number $\mathrm{E}o = 10$, and both density and viscosity ratios equal to 10. Table 1 lists the fluid and physical parameters which specify the test cases.

We investigate quantitatively (see **Fig.1**) the new curvature-free level set FEM approach using the benchmark quantities, namely circularity, rising speed and center point of the bubble. Also, we provide the shape of the bubble at time $t = 3$.

Table 1. Physical parameters and dimensionless numbers defining the test case

ρ_1	ρ_2	μ_1	μ_2	g	σ	Re	Eo	ρ_1/ρ_1	μ_1/μ_2
1000	100	10	1	0.98	24.5	35	10	10	10

We used biquadratic (Q_2) finite elements for the approximation of the level set function on an equidistant mesh with mesh size $h = 1/64$: First, we consider the classical method, $\boldsymbol{f}_{\text{CSF},1}$, with the evaluation of normals and curvature using the volume integration with the help of the δ_Γ function. Second, the volume force method, $\boldsymbol{f}_{\text{CSF},2}$, with the evaluation of curvature only using the volume integration. Third, the curvature-free approach, $\boldsymbol{f}_{\text{CSF},3}$, without the evaluation of normals or curvature using the volume integration. The reference results (ref.) are taken from [9]. **Fig.2** shows the mesh convergence of the rising bubble benchmark quantities. Furthermore, the characteristic material function is derived directly from the signed distance level set function using (14) and the level set is reinitialized using the "brute force" approach [4].

Clearly, the numerical simulations confirm the very similar approximation properties of the three level set methods and the mesh convergence for the curvature-free level set method, which is the main focus of this paper. However, to have the full advantage of using the curvature-free level set FEM approach we will introduce a new multiphase stress and incorporate a reinitialization-free level set function as well. In the next section we discuss two possible ways for a fully implicit approach for multiphase flow problems.

4 Towards a fully implicit monolithic approach for multiphase flow problems

Let us introduce the new multiphase, resp., surface stress $\boldsymbol{\tau}_m$ from (22), which is well defined for $\psi \in H^1(\Omega)$, as

$$\boldsymbol{\tau}_m = -\sigma \left(\frac{\nabla\psi \otimes \nabla\psi}{|\nabla\psi|} \right). \tag{24}$$

So, in the first step the full stress is modified as follows

$$\boldsymbol{\tau} = \boldsymbol{\tau}_s + \boldsymbol{\tau}_m, \tag{25}$$

and the momentum equation reads after eliminating the CSF force term

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + u \cdot \nabla \boldsymbol{u} \right) - \operatorname{div} \boldsymbol{\tau} + \nabla p = 0 \tag{26}$$

which together with (3, 14) forms the complete system.

The distance to the interface is required for the regularized Delta function (9) in the first CSF approach as well as for the regularized cutoff function (14) in the second CSF approach. So, the natural choice for the level set equation is the signed distance function.

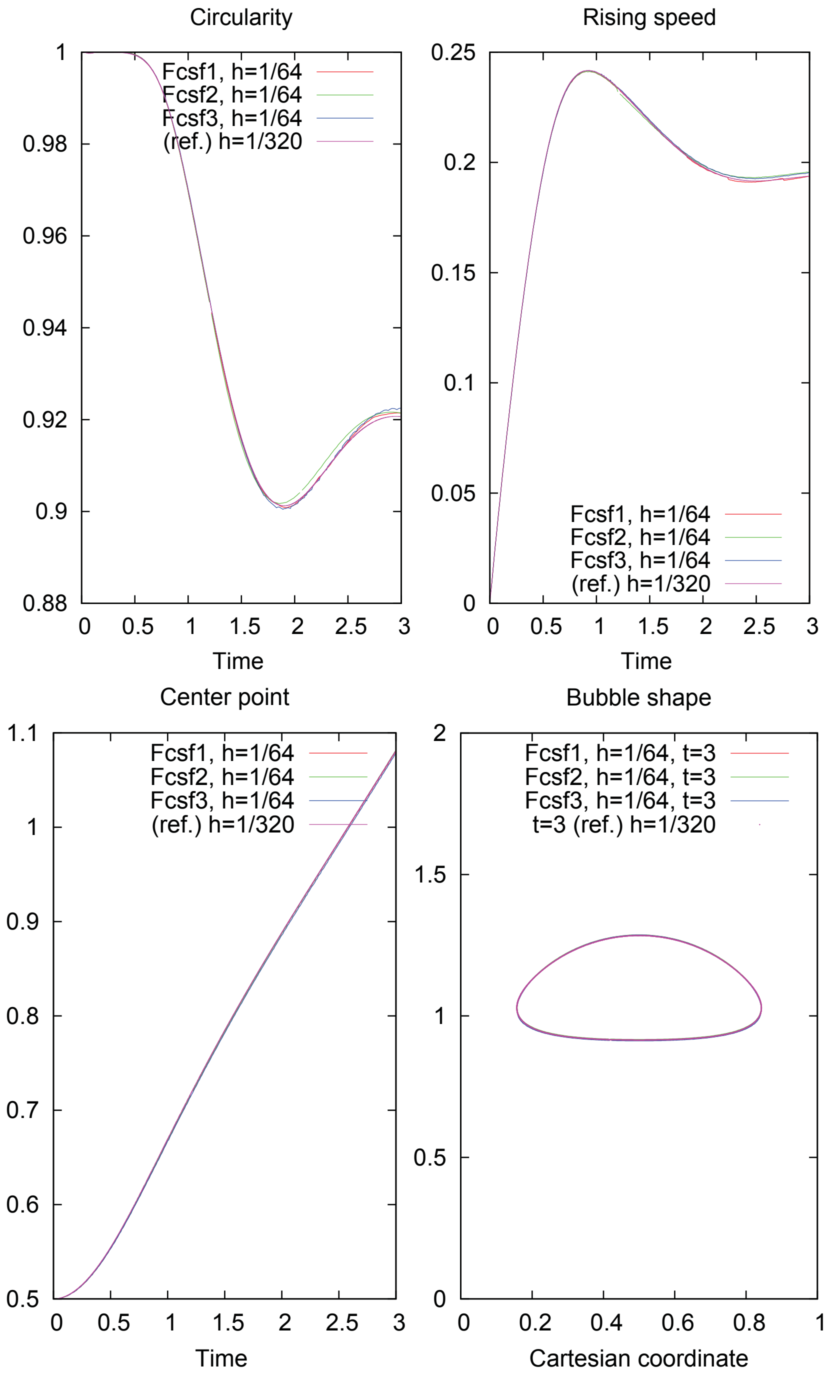

Fig. 1. Numerical validation of the curvature-free level set method: The rising bubble benchmark quantities with the three methods of calculating the CSF force in level set FEM approaches using the Q_2 finite element for the approximation of the level set function on equidistant mesh with mesh size $h = 1/64$. The classical method with the evaluation of normals and curvature using the volume integration with the help of δ_Γ function (F_{csf1}), the volume force method with the evaluation of curvature only using the volume integration (F_{csf2}), and the curvature-free approach without the evaluation of neither normals nor curvature using the volume integration (F_{csf3}). The reference results (ref.) are taken from [9].

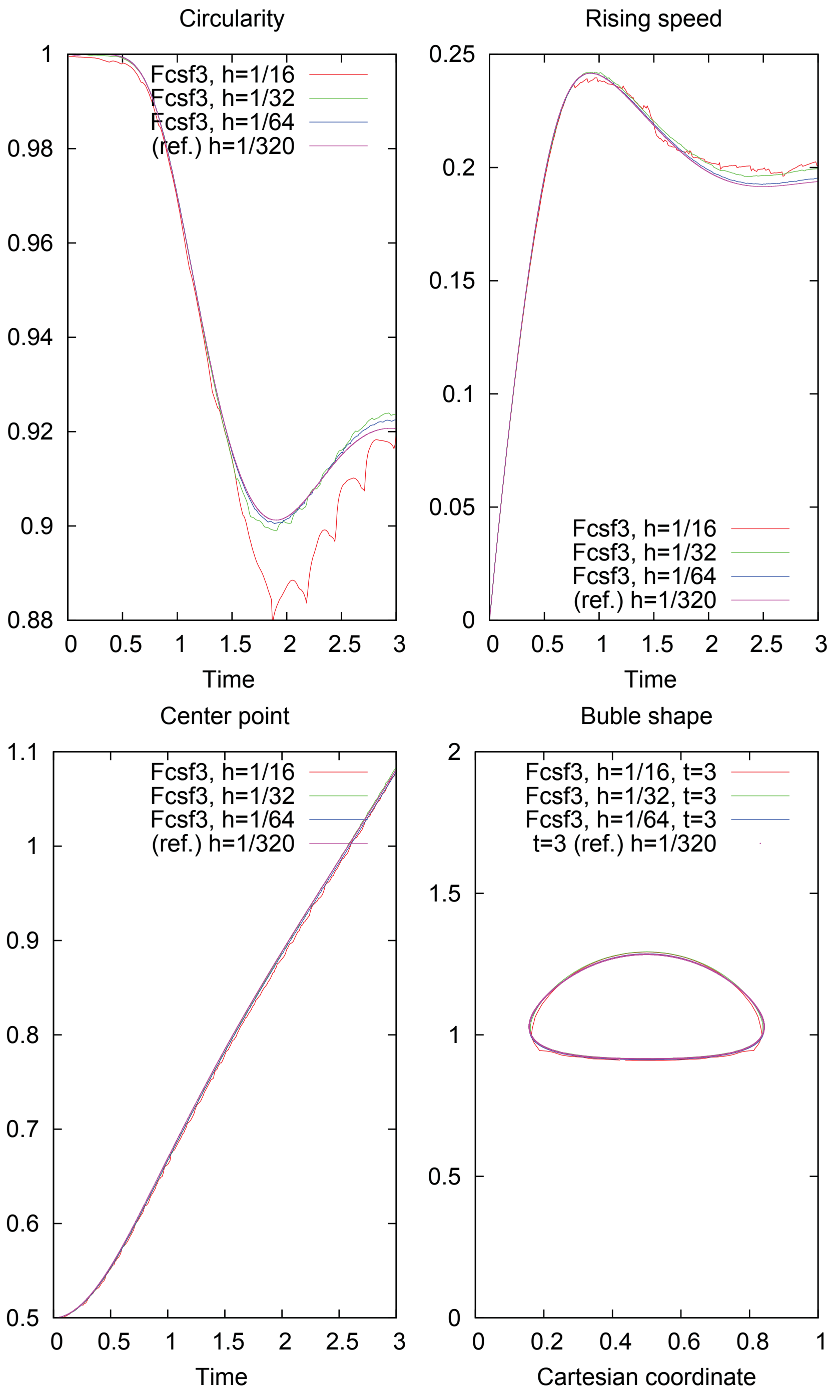

Fig. 2. Mesh convergence of the curvature-free level set method: The rising bubble benchmark quantities for the curvature-free level set method (F_{csf3}) with respect to mesh refinement. The circularity, rising speed, central point, and the bubble shape for three mesh sizes h set to 1/16, 1/32, and 1/32. The reference results (ref.) are taken from [9].

In order to derive a monolithic approach for multiphase flow simulations, two approaches will be presented. First, a reinitialization-free signed distance level set method while the cutoff function is derived via a postprocessing step. Second, a reinitialization-free cutoff function approach will be introduced.

4.1 Reinitialization-free signed distance level set method

The signed distance function needs to satisfy the constraint

$$|\nabla \varphi| = 1, \tag{27}$$

which can be expressed equivalently as

$$\boldsymbol{n} \cdot \nabla \varphi = 1. \tag{28}$$

For the implicit approach, the constraint (28) is imposed in a variational sense through a least squares formulation. Then, φ is the solution of the following variational problem

$$\int_{\Omega} \left(\frac{\partial \varphi}{\partial t} + \boldsymbol{u} \cdot \nabla \varphi \right) \phi \, dx + \gamma_{nd} \int_{\Omega} (\boldsymbol{n} \cdot \nabla \varphi) \ (\boldsymbol{n} \cdot \nabla \phi) \ dx = \gamma_{nd} \int_{\Omega} \boldsymbol{n} \cdot \nabla \phi \, dx \tag{29}$$

where γ_{nd} is a penalty parameter. The equivalent strong form is given as follows

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{u} \cdot \nabla \varphi - \gamma_{nd} \nabla \cdot ((\boldsymbol{n} \cdot \nabla \varphi - 1)\boldsymbol{n}) = 0. \tag{30}$$

The level set equation (3) is extended to a single PDE equation with additional normal diffusion term. Then, the complete system reads:

$$\left\{ \begin{aligned} \rho(\psi) \left(\frac{\partial \boldsymbol{u}}{\partial t} + u \cdot \nabla \boldsymbol{u} \right) - \operatorname{div} \boldsymbol{\tau} + \nabla p &= 0 \\ \nabla \cdot \boldsymbol{u} &= 0 \\ \boldsymbol{\tau} - 2\mu(\psi)\mathbf{D}(\boldsymbol{u}) + \sigma \left(\frac{\nabla \psi \otimes \nabla \psi}{|\nabla \psi|} \right) &= 0 \\ \frac{\partial \varphi}{\partial t} + \boldsymbol{u} \cdot \nabla \varphi - \gamma_{nd} \nabla \cdot \left(\left(\frac{\nabla \varphi}{|\nabla \varphi|} \cdot \nabla \varphi - 1 \right) \frac{\nabla \varphi}{|\nabla \varphi|} \right) &= 0 \\ \psi = \frac{-1}{1 + \exp(\varphi/\epsilon)} &+ 0.5 \end{aligned} \right. \tag{31}$$

4.2 Reinitialization-free cutoff level set method

The material cutoff function ψ can be derived directly from the signed distance function in (14) via postprocessing. However, a direct PDE approach for ψ can be adopted, too. The following equation for ψ has been introduced in [13] as

$$\frac{\partial \psi}{\partial t} + \nabla \cdot (\gamma_{nc} \psi (1 - \psi) \boldsymbol{n}) - \nabla \cdot (\gamma_{nd} (\nabla \psi \cdot \boldsymbol{n}) \boldsymbol{n}) = 0 \tag{32}$$

with γ_{nc} and γ_{nd} denoting the relaxation parameters for the nonlinear convection term in the normal direction and the normal diffusion. The nonlinear convection in the direction of the normal tends to build the Heaviside step function independent of the convective parameter, and the normal diffusion controls the sharpness of the interface.

Finally, the full set of equations for the fully implicit curvature-free level set approach for multiphase flow problems is given as follows:

$$\left\{\begin{aligned} \rho(\psi)\left(\frac{\partial \boldsymbol{u}}{\partial t} + u \cdot \nabla \boldsymbol{u}\right) - \operatorname{div} \boldsymbol{\tau} + \nabla p = 0 \\ \nabla \cdot \boldsymbol{u} = 0 \\ \boldsymbol{\tau} - 2\mu(\psi)\mathbf{D}(\boldsymbol{u}) + \sigma\left(\frac{\nabla\psi \otimes \nabla\psi}{|\nabla\psi|}\right) = 0 \\ \frac{\partial \psi}{\partial t} + \boldsymbol{u} \cdot \nabla\psi + \nabla \cdot \left(\gamma_{nc}\psi(1-\psi)\frac{\nabla\psi}{|\nabla\psi|}\right) \\ -\nabla \cdot \left(\gamma_{nd}\left(\nabla\psi \cdot \frac{\nabla\psi}{|\nabla\psi|}\right)\frac{\nabla\psi}{|\nabla\psi|}\right) = 0 \end{aligned}\right. \tag{33}$$

The system of equations (33) is a four field system with the unknowns $(\boldsymbol{u}, \tau, \psi, p)^{\mathrm{T}}$. So, the choice of finite element spaces is subject to compatibility conditions (i.e. LBB constraints). The regularity requirement for the material function is reduced. Moreover, the momentum equation is given with homogeneous force terms, so that as a consequence standard Navier-Stokes solver can be used for the simulation of multiphase flow problems (particularly, if solvers based on operator splitting, resp., pressure correction approaches are used). Furthermore the system of equations (33) can be seen as an hybrid phase field and level set approach for multiphase flow problems: The system does not require the free energy function, as in phase field methodology, neither the curvature evaluations, as in level set methodology.

4.3 Monolithic solver for multiphase flow problems

The system (33) allows already a fully implicit approach. In order to develop a monolithic solver for multiphase flow problems, we consider the Newton method as outer iteration. Let $\tilde{\boldsymbol{u}} = (\boldsymbol{u}, \tau, \psi)^{\mathrm{T}}$, so that $R_{\tilde{\boldsymbol{u}}} = (R_{\boldsymbol{u}}, R_\tau, R_\psi)^{\mathrm{T}}$ and R_p are the nonlinear residuals

$$\begin{bmatrix} \tilde{\boldsymbol{u}}^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{u}}^{n} \\ p^{n} \end{bmatrix} - \omega \begin{bmatrix} \dfrac{\partial R_{\tilde{\boldsymbol{u}}}(\tilde{\boldsymbol{u}}^n, p^n)}{\partial \tilde{\boldsymbol{u}}} & \dfrac{\partial R_{\tilde{\boldsymbol{u}}}(\tilde{\boldsymbol{u}}^n, p^n)}{\partial p} \\ \dfrac{\partial R_p(\tilde{\boldsymbol{u}}^n, p^n)}{\partial \tilde{\boldsymbol{u}}} & \dfrac{\partial R_p(\tilde{\boldsymbol{u}}^n, p^n)}{\partial p} \end{bmatrix}^{-1} \begin{bmatrix} R_{\tilde{\boldsymbol{u}}}(\tilde{\boldsymbol{u}}^n, p^n) \\ R_p(\tilde{\boldsymbol{u}}^n, p^n) \end{bmatrix} \tag{34}$$

where ω is a relaxation parameter and the Jacobian reads in an abstract form:

$$\begin{bmatrix} \dfrac{\partial R_{\tilde{\boldsymbol{u}}}(\tilde{\boldsymbol{u}}^n, p^n)}{\partial \tilde{\boldsymbol{u}}} & \dfrac{\partial R_{\tilde{\boldsymbol{u}}}(\tilde{\boldsymbol{u}}^n, p^n)}{\partial p} \\ \dfrac{\partial R_p(\tilde{\boldsymbol{u}}^n, p^n)}{\partial \tilde{\boldsymbol{u}}} & \dfrac{\partial R_p(\tilde{\boldsymbol{u}}^n, p^n)}{\partial p} \end{bmatrix} = \begin{bmatrix} A & B^{\mathrm{T}} \\ B & 0 \end{bmatrix} \tag{35}$$

The linear system to be solved has a 2×2 block saddle point structure. Thus, we may apply the concept of local Multilevel Pressure Schur Complement [17] approaches which solves exactly on fixed patches and performs an outer Gauß-Seidel iteration. This approach can be interpreted as generalization of block-Jacobi/Gauß-Seidel methods for saddle point problems which contains modifications of classical schemes like the Vanka smoother [3]. It is important to notice that the trade-off between the strong coupling and the fixed and smaller size matrix is the assembly process in a block-Jacobi/Gauß-Seidel approach. Therefore, reducing the linear system (35) while preserving the coupling is under investigation. So, a standard monolithic Newton-multigrid Navier-Stokes solver (see, for instance, [3]) can be used for the fully implicit treatment of multiphase flow problems.

5 Summary

We presented a new formulation for multiphase flow problems where neither normals nor curvature are explicitly calculated. The new method is (potentially) fully implicit, so that no capillary time restriction remains. Furthermore, standard Navier-Stokes solvers can be used which do not have to take into account inkomogeneous force terms due to prescribed surface tension, and explicit reinitialization steps are avoided in the context of level set approaches. To validate the introduced methodology, numerical investigations for a rising bubble benchmark [9] were performed which confirmed the equivalent approximation properties compared to standard level set approaches. Moreover, further benefits towards a monolithic solution approach have been discussed which is subject of our current work including special FEM approximations for the corresponding surface stress.

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