Modified Newton solver for yield stress fluids

Saptarshi Mandal, Abderrahim Ouazzi, and Stefan Turek

Abstract. The aim of this contribution is to present a new Newton-type solver for yield stress fluids, for instance for viscoplastic Bingham fluids. In contrast to standard globally defined ('outer') damping strategies, we apply weighting strategies for the different parts inside of the resulting Jacobian matrices (after discretizing with FEM), taking into account the special properties of the partial operators which arise due to the differentiation of the corresponding nonlinear viscosity function. Moreover, we shortly discuss the corresponding extension to fluids with a pressure-dependent yield stress which are quite common for modelling granular material. From a numerical point of view, the presented method can be seen as a generalized Newton approach for non-smooth problems.

Key words: Newton method, Bingham flow, granular material, yield stress fluids

1 Introduction

Continuum theory for slow viscoplastic fluids based on corresponding flow rules typically relates the shear stress and the strain rate in a plastic frictional system via Bingham-like constitutive laws

$$\begin{cases} \boldsymbol{\tau} = 2\nu \mathbf{D}(\boldsymbol{u}) + \tau_s \dfrac{\mathbf{D}(\boldsymbol{u})}{\|\mathbf{D}(\boldsymbol{u})\|} & \text{if } \|\mathbf{D}(\boldsymbol{u})\| \neq 0 \\ \|\boldsymbol{\tau}\| \leq \tau_s & \text{if } \|\mathbf{D}(\boldsymbol{u})\| = 0 \end{cases} \tag{1}$$

where $\mathbf{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$ denotes the strain rate tensor, and τ_s denotes the yield stress. The shear stress has two contributions: a viscous part, and a strain rate independent part. Furthermore, for the deformation of dense granular material, the stress and strain rate tensors are always coaxial. So, for unequal stresses, Schaeffer [5] postulated that the stresses contract in the directions of greater stress and expand in directions of smaller stress. As a consequence, the deviatoric part of the related Schaeffer model for flow of dry powder in the quasi-static regime [5] is

$$\boldsymbol{\tau} = \sin(\phi)\, p \frac{\mathbf{D}(\boldsymbol{u})}{\|\mathbf{D}(\boldsymbol{u})\|} \tag{2}$$

where ϕ denotes the angle of internal friction: Hence, this model can be interpreted as pressure-dependent yield stress fluid. Moreover, the interesting transition from

S. Mandal · A. Ouazzi · S. Turek
Institut für Angewandte Mathematik, LSIII, TU Dortmund, Vogelpothsweg 87, D-44227 Dortmund, Germany
e-mail: saptarshi.mandal@math.tu-dortmund.de; abderrahim.ouazzi@math.tu-dortmund.de; ture@featflow.de

solid-like to fluid-like behavior of granular material was investigated experimentally and numerically in [3]. Here, the unified constitutive model for the static and intermediate regimes is given by the following constitutive law (with an appropriate $n > 0$ and $b \in \mathbb{R}^+$, see [3]):

$$\boldsymbol{\tau} = p \left\{ \sin(\phi) + b\cos(\phi)\|\mathbf{D}(\boldsymbol{u})\|^n \right\} \frac{\mathbf{D}(\boldsymbol{u})}{\|\mathbf{D}(\boldsymbol{u})\|} \tag{3}$$

Similarly, Pouliquen et al. [2] proposed an extended constitutive model for dense granular material, where the stress tensor is given as a function of the inertia number $\mathbf{I} = \mathbf{D}(\boldsymbol{u})d/\sqrt{p\rho_p}$ (again with appropriate values of ρ_p and d, see [2])

$$\boldsymbol{\tau} = p\mu(\mathbf{I}) \frac{\mathbf{D}(\boldsymbol{u})}{\|\mathbf{D}(\boldsymbol{u})\|} \tag{4}$$

where $\mu(\cdot)$ is an empirical friction law:

$$\mu(\mathbf{I}) = \mu_1 + \frac{\mu_2 - \mu_1}{\mathbf{I}_0/\mathbf{I} + 1} \tag{5}$$

All models show the relationship between granular and Bingham fluids. In order to incorporate friction into viscoplasticity in mixing wet granular materials, El Khouja et al. [1] introduced the dependency of the pressure in yield stress flow model, i.e. the yield stress $\tau_s(\cdot)$ is a function of the pressure, namely let $\tau_{\min}, \tau_{\max} \in \mathbb{R}^+$, so that $\tau_s(\cdot)$ can be defined as:

$$\tau_s(p) = \min\{\max\{p, \tau_{min}\}, \tau_{\max}\} \tag{6}$$

In what follows, we consider steady problems of (slow) Bingham flow with pressure dependent yield stress that satisfies

$$\begin{cases} -\nabla \cdot \boldsymbol{\tau} + \nabla p = 0 & \text{in } \Omega \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{g}_D & \text{on } \Gamma_D \end{cases} \tag{7}$$

and proceed within the framework of generalized Stokes problems. So, we introduce the second invariant of the strain rate tensor $\gamma_{\rm II} = \frac{1}{2}(2\mathbf{D} : 2\mathbf{D})$, resp., $\|\mathbf{D}\| = \frac{1}{\sqrt{2}}\gamma_{\rm II}^{\frac{1}{2}}$, and define a generalized viscosity $\eta(\cdot,\cdot)$ which depends on the pressure and the shear rate:

$$\eta(\gamma_{\rm II}, p) = \nu + \frac{\sqrt{2}}{2} \frac{\tau_s(p)}{\gamma_{\rm II}^{\frac{1}{2}}} \tag{8}$$

To define the viscosity everywhere, we introduce the classical regularization:

$$\eta(\gamma_{\rm II}, p) = \nu + \frac{\sqrt{2}}{2} \frac{\tau_s(p)}{(\gamma_{\rm II} + \epsilon^2)^{\frac{1}{2}}} \tag{9}$$

As a consequence, Bingham flow with pressure dependent yield stress is the limit case, $\epsilon = 0$, of the regularized problem. However, it is well known that the accuracy of the solution is strongly dependent on this parameter ϵ. Summarizing the previous considerations, the considered system of equations in the primitive variables $\boldsymbol{u}$ and p is given as follows:

$$\begin{cases} -\nabla \cdot (2\eta(\gamma_{\rm II}, p)\mathbf{D}(\boldsymbol{u})) + \nabla p = 0 & \text{in } \Omega \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{g}_D & \text{on } \Gamma_D \end{cases} \tag{10}$$

2 Non-standard saddle point problem formulation

After discretization, for instance with standard Q2P1 finite elements, let $\tilde{\boldsymbol{u}} = (\boldsymbol{u}, p)$ and $\mathcal{R}_{\tilde{\boldsymbol{u}}}$ denote the discrete residuals for the system (10). We use the Newton method which means that the nonlinear iteration is updated with the correction $\delta\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{u}}^{n+1} = \tilde{\boldsymbol{u}}^n + \delta\tilde{\boldsymbol{u}}$. Then, the Newton linearization provides the following approximation for the residuals:

$$\begin{aligned}\mathcal{R}(\tilde{\boldsymbol{u}}^{n+1}) =& \mathcal{R}(\tilde{\boldsymbol{u}}^n + \delta\tilde{\boldsymbol{u}}) \\ \simeq& \mathcal{R}(\tilde{\boldsymbol{u}}^n) + \left[\frac{\partial \mathcal{R}(\tilde{\boldsymbol{u}}^n)}{\partial \tilde{\boldsymbol{u}}}\right] \delta\tilde{\boldsymbol{u}}\end{aligned} \tag{11}$$

Hence, one iteration of the Newton method can be written as follows:

$$\begin{bmatrix} \boldsymbol{u}^{n+1} \\ \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}^{n} \\ \\ p^{n} \end{bmatrix} - \omega_n \begin{bmatrix} \dfrac{\partial \mathcal{R}_{\boldsymbol{u}}(\boldsymbol{u}^n, p^n)}{\partial \boldsymbol{u}} & \dfrac{\partial \mathcal{R}_{\boldsymbol{u}}(\boldsymbol{u}^n, p^n)}{\partial p} \\ \dfrac{\partial \mathcal{R}_{p}(\boldsymbol{u}^n, p^n)}{\partial \boldsymbol{u}} & \dfrac{\partial \mathcal{R}_{p}(\boldsymbol{u}^n, p^n)}{\partial p} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{R}_{\boldsymbol{u}}(\boldsymbol{u}^n, p^n) \\ \\ \mathcal{R}_{p}(\boldsymbol{u}^n, p^n) \end{bmatrix} \tag{12}$$

The damping parameter $\omega_n \in (0, 1]$ is typically chosen such that:

$$\begin{bmatrix} \mathcal{R}_{\boldsymbol{u}}(\boldsymbol{u}^{n+1}, p^{n+1}) \\ \mathcal{R}_{p}(\boldsymbol{u}^{n+1}, p^{n+1}) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{u}^{n+1} \\ p^{n+1} \end{bmatrix} \leq \begin{bmatrix} \mathcal{R}_{\boldsymbol{u}}(\boldsymbol{u}^{n}, p^{n}) \\ \mathcal{R}_{p}(\boldsymbol{u}^{n}, p^{n}) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{u}^{n} \\ p^{n} \end{bmatrix} \tag{13}$$

As we will demonstrate for the considered yield stress fluids, this damping parameter is not enough to ensure robust convergence. In what follows, we derive explicitly the Jacobian in order to segregate it into "bad" and "good" terms to get a robust nonlinear solver. The block matrices of the Jacobian are given as follows:

$$\begin{aligned}\left[\frac{\partial \mathcal{R}_{\boldsymbol{u}}(\boldsymbol{u}^n, p^n)}{\partial \boldsymbol{u}}\right] \boldsymbol{v} =& -\nabla \cdot \Big(2\eta(\gamma_{\mathrm{II}}^n, p^n) \mathbf{D}(\boldsymbol{v}) \\ &+ 8\eta_1'(\gamma_{\mathrm{II}}^n, p^n) \left[\mathbf{D}(\boldsymbol{u}^n) : \mathbf{D}(\boldsymbol{v})\right] \mathbf{D}(\boldsymbol{u}^n) \Big)\end{aligned} \tag{14}$$

where $\eta_1'(\gamma_{\mathrm{II}}, p) = \frac{\partial \eta(\gamma_{\mathrm{II}}, p)}{\partial \gamma_{\mathrm{II}}}$, the last term in the equation (14), is due to the shear dependent viscosity models. Furthermore, there holds

$$\left[\frac{\partial \mathcal{R}_{\boldsymbol{u}}(\boldsymbol{u}^n, p^n)}{\partial p}\right] q = \Big(\mathbf{I} - 2\eta_2'(\gamma_{\mathrm{II}}^n, p^n) \mathbf{D}(\boldsymbol{u}^n) \Big) \nabla q \tag{15}$$

where $\eta_2'(\gamma_{\mathrm{II}}, p) = \frac{\partial \eta(\gamma_{\mathrm{II}}, p)}{\partial p}$, the second term in the equation (15), is relevant for pressure-dependent viscosity models. Moreover, the incompressibility condition leads to

$$\left[\frac{\partial \mathcal{R}_{p}(\boldsymbol{u}^n, p^n)}{\partial \boldsymbol{u}}\right] \boldsymbol{v} = -\nabla \cdot \boldsymbol{v} \tag{16}$$

and additionally we obtain:

$$\left[\frac{\partial \mathcal{R}_{p}(\boldsymbol{u}^n, p^n)}{\partial p}\right] q = 0 \tag{17}$$

Let $\mathbf{V} := (\boldsymbol{H}_0^1(\Omega))^2$ and $\mathbf{Q} := L_0^2(\Omega)$, the weak formulation reads:

$$\int_\Omega \left[\frac{\partial \mathcal{R}_{\boldsymbol{u}}(\boldsymbol{u}^n, p^n)}{\partial \boldsymbol{u}}\right] \boldsymbol{u}\cdot\boldsymbol{v} dx = \int_\Omega 2\eta(\gamma_{\scriptscriptstyle \rm II}^n, p^n)\left[\mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{v})\right] dx + \int_\Omega 8\eta_1'(\gamma_{\scriptscriptstyle \rm II}^n, p^n)\left[\mathbf{D}(\boldsymbol{u}^n)\otimes\mathbf{D}(\boldsymbol{u})\right] : \left[\mathbf{D}(\boldsymbol{u}^n)\otimes\mathbf{D}(\boldsymbol{v})\right] dx \tag{18}$$

Next, let us introduce the following linear forms defined on $\mathbf{V} \longrightarrow \mathbf{V}'$

$$\begin{aligned} \langle \mathbf{A}_1\boldsymbol{u}, \boldsymbol{v}\rangle &:= \int_\Omega 2\eta(\gamma_{\scriptscriptstyle \rm II}^n, p^n)\left[\mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{v})\right] dx \\ \langle \mathbf{A}_2\boldsymbol{u}, \boldsymbol{v}\rangle &:= \int_\Omega 8\eta_1'(\gamma_{\scriptscriptstyle \rm II}^n, p^n)\left[\mathbf{D}(\boldsymbol{u}^n)\otimes\mathbf{D}(\boldsymbol{u})\right] : \left[\mathbf{D}(\boldsymbol{u}^n)\otimes\mathbf{D}(\boldsymbol{v})\right] dx \\ \langle \mathbf{A}\boldsymbol{u}, \boldsymbol{v}\rangle &:= \langle \mathbf{A}_1\boldsymbol{u}, \boldsymbol{v}\rangle + \langle \mathbf{A}_2\boldsymbol{u}, \boldsymbol{v}\rangle \end{aligned} \tag{19}$$

and the associated bilinear forms defined on $\mathbf{V}\times\mathbf{V} \longrightarrow \mathbb{R}$

$$a(\boldsymbol{u}, \boldsymbol{v}) = \langle \mathbf{A}\boldsymbol{u}, \boldsymbol{v}\rangle, \quad a_1(\boldsymbol{u}, \boldsymbol{v}) = \langle \mathbf{A}_1\boldsymbol{u}, \boldsymbol{v}\rangle, \quad a_2(\boldsymbol{u}, \boldsymbol{v}) = \langle \mathbf{A}_2\boldsymbol{u}, \boldsymbol{v}\rangle \tag{20}$$

and the linear forms defined on $\mathbf{V} \longrightarrow \mathbf{Q}'$:

$$\langle \mathbf{B}\boldsymbol{u}, p\rangle := -\int_\Omega \nabla\cdot\boldsymbol{u}\; p\, dx \tag{21}$$

the new additional linear forms $\tilde{\mathbf{B}}$ and $\mathbf{C}$ are given as follows

$$\langle \tilde{\mathbf{B}}\boldsymbol{u}, p\rangle = \int_\Omega \nabla\cdot\left(2\eta_2'(\gamma_{\scriptscriptstyle \rm II}^n, p^n)\mathbf{D}(\boldsymbol{u}^n)\boldsymbol{u}\right) p\, dx \tag{22}$$

$$\langle \mathbf{C}\boldsymbol{u}, p\rangle = -\int_\Omega \nabla\cdot\left[\left(\mathbf{I} - 2\eta_2'(\gamma_{\scriptscriptstyle \rm II}^n, p^n)\mathbf{D}(\boldsymbol{u}^n)\right)\boldsymbol{u}\right] p\, dx \tag{23}$$

with the associated bilinear forms $b(\cdot,\cdot)$, $\tilde{b}(\cdot,\cdot)$, and $c(\cdot,\cdot)$ defined on $\mathbf{V}\times\mathbf{Q} \longrightarrow \mathbb{R}$ read:

$$b(\boldsymbol{v}, q) = \langle \mathbf{B}\boldsymbol{v}, q\rangle, \quad \tilde{b}(\boldsymbol{v}, q) = \langle \tilde{\mathbf{B}}\boldsymbol{v}, q\rangle, \quad c(\boldsymbol{v}, q) = b(\boldsymbol{v}, q) + \tilde{b}(\boldsymbol{v}, q) \tag{24}$$

So, the corresponding Newton iteration (12) after discretization becomes:

$$\begin{bmatrix} \boldsymbol{u}^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}^n \\ p^n \end{bmatrix} - \omega_n \begin{bmatrix} \mathbf{A} & \mathbf{C}^{\mathrm{T}} \\ \mathbf{B} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{R}_{\boldsymbol{u}}(\boldsymbol{u}^n, p^n) \\ \mathcal{R}_p(\boldsymbol{u}^n, p^n) \end{bmatrix} \tag{25}$$

In the case of pressure-dependent yields stress, the Jacobian has a nonsymmetric saddle point structure (if not, then $\mathbf{C}^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}$):

$$\mathbf{J} = \begin{bmatrix} \mathbf{A} & \mathbf{C}^{\mathrm{T}} \\ \mathbf{B} & 0 \end{bmatrix} \tag{26}$$

The Jacobian $\mathbf{J}$ can be decomposed, based on the block operators $\mathbf{A}$, into

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 \tag{27}$$

respectively, $\mathbf{C}$ into

$$\mathbf{C} = \mathbf{B} + \tilde{\mathbf{B}}. \tag{28}$$

In what follows, we mainly concentrate, first of all, onto the model (1) for yield stress only and take $\mathbf{C} = \mathbf{B}$ (at the end, we provide a preliminary result for pressure-dependent viscosity, too). Therefore, our studies focus on the discussed decomposition of the operator $\mathbf{A}$ due to (27).

3 Robust nonlinear solver

To develop a robust nonlinear solver we introduce a new control parameter δ_n in order to balance the operators $\mathbf{A}_1$ (corresponding to the typical fixed point approach) and $\mathbf{A}_2$, both being part of the complete Jacobian $\mathbf{A}$:

$$\mathbf{A} = \mathbf{A}_1 + \delta_n \mathbf{A}_2 \tag{29}$$

In the present note, we concentrate on the choice of the optimal parameter δ_n balancing the fixed point and the full Newton iteration. We take the classical flow around cylinder benchmark [4], and perform corresponding simulations for Bingham flow.

First, we take a very small yield stress parameter, $\tau_s = 10^{-4}$, and apply the fixed point ($\delta_n = 0$) and classical Newton ($\delta_n = 1$) methods. **Table 1** shows the resulting numbers of nonlinear iterations. Both methods, Newton and fixed point, are easily converging towards the solution, more or less independent of the mesh level. Moreover, the Newton method overcomes the fixed point method, as expected, due to the moderate nonlinearity. To highlight the insufficiency of the globally damped Newton (13) to simulate Bingham flow problems, we further increase the yield stress. Now, the Newton method can only converge with a strong damping parameter ω_n as the yield stress increases, for instance $\omega_n = 0.1$ for $\tau_s = 10^{-2}$, and no convergence at all can be obtained for higher yield stress, $\tau_s \geq 10^{-1}$. Instead, the fixed point method can converge for all cases, however being very slow and not being robust w.r.t. mesh level and/or yield stress.

Table 1. Globally damped Newton: The numbers of nonlinear iterations for Bingham flow with fixed point method and globally damped Newton for increasing yield stress

Level	τ_s=10^{-4}		τ_s=10^{-3}		τ_s=10^{-2}		τ_s=10^{-1}	τ_s=1
	Fixed point	Newton ω_n=1.0	Fixed point	Newton ω_n=0.2	Fixed point	Newton ω_n=0.1	Fixed point	Fixed point
2	21	3	67	99	212	210	490	1032
3	24	5	84	95	308	200	728	2135
4	20	5	98	90	408	190	1375	3444

Clearly, with increasing yield stress, it is hard if not impossible to solve the corresponding flow problems with the globally damped Newton. Therefore, in the next step, we take a static δ_n, i.e. $\delta_n = \delta_0$ for $n \geq 1$, which has been introduced in (29).

The balancing parameter δ_n is taken as a constant increasing from 0 to 1. **Table 2** presents the numbers of nonlinear iterations for Bingham flow with different values for the yield stress. From the results in **Table 2**, it is clear that increasing

Table 2. Statically balanced Newton: The numbers of nonlinear iterations for Bingham flow with two different yield stress values $\tau_s = 10^{-2}$ and $\tau_s = 10^{-1}$, with a statically balanced Jacobian, i.e. δ_n is kept constant

τ_s	Level	δ_n=0.1	δ_n=0.25	δ_n=0.5	δ_n=0.6
10^{-2}	2	236	198	135	110
	3	352	295	199	160
	4	455	380	256	206
10^{-1}	2	551	461	311	251
	3	848	708	475	382
	4	1455	1214	813	653

the contribution from the operator $\mathbf{A}_2$ improves the convergence behavior, but this contribution needs to remain under control. To do so, we go for a dynamic change of δ_n w.r.t. the residual changes. From the numerical experiment it can be noticed that the dynamic changes of the residual give a precious information about the singularity of the Jacobian. Indeed, the larger relative changes in the residual with the operator $\mathbf{A}_1$ reflect the 'singularity' of the operator $\mathbf{A}_2$. In this case, the parameter δ_n should have a small relative change and remain small. Moreover, when the relative changes in the residual are close to zero, this indicates that the operator $\mathbf{A}_2$ has the nicest properties and δ_n can be increased accordingly and maintained close to 1. We introduce the increment

$$\mathcal{Q}_n := \frac{\left\|\mathcal{R}(\gamma_{\text{II}}^n, p^n)\right\|}{\left\|\mathcal{R}(\gamma_{\text{II}}^{n-1}, p^{n-1})\right\|}, \tag{30}$$

and define the following continuous function for changes of δ_n w.r.t. the residual $\mathcal{R}_n$:

$$\frac{\delta_{n+1}}{\delta_n} = 0.2 + \frac{4}{0.7 + \exp(1.5\mathcal{Q}_n)} \tag{31}$$

It should be pointed out that the choice (31) of δ_n is derived so far based on simple and preliminary numerical experiments only. We check the robustness of the dynamic changes of δ_n in (31) for various values of yield stress. **Table 3** shows the numbers of nonlinear iterations for Bingham flow for a wide range of yield stress values and different starting weighting factors for the Jacobian, that means δ_0.

Since the convergence typically gets harder with smaller values for the regularization parameter ϵ, we check the robustness of the dynamic changes of δ_n in (31) for decreasing ϵ and a wide range of yield stress values. **Table 4** shows the numbers of nonlinear iterations for Bingham flow using continuation strategies w.r.t. ϵ as well as w.r.t. τ_s.
Moreover, it should be pointed out that the parameters ϵ and τ_s can be seen as bounds for some physical quantities in models for granular material [2,3].

Finally, we want to perform some preliminary tests regarding the flexibility and robustness of the dynamically balanced Newton method for pressure-dependent

Table 3. Behavior of the weighted Newton w.r.t. starting parameter: The numbers of nonlinear iterations for the dynamically balanced Newton for Bingham flow for a wide range of yield stress values, varying from 10^{-3} to 5, for different initial values δ_0

δ_0	τ_s						
	0.001	0.01	0.1	0.5	1.0	2.0	5.0
0.0	10	15	20	19	19	20	20
0.3	10	16	20	19	19	20	20
0.7	18	18	22	22	20	18	18
1.0	46	14	19	21	21	22	22

Table 4. Convergence w.r.t. continuation strategies: The numbers of nonlinear iterations for the dynamically balanced Newton for Bingham flow for increasing yield stress values, from 10^{-3} to 5, and decreasing ϵ, from 10^{-2} to 10^{-5}

ϵ	τ_s						
	0.001	0.01	0.1	0.5	1.0	2.0	5.0
Continuation Newton w.r.t. ϵ							
10^{-2}	10	15	20	19	19	20	20
10^{-3}	11	11	12	17	16	15	15
10^{-4}	15	13	18	16	15	26	15
10^{-5}	16	10	22	22	17	15	17
Continuation Newton w.r.t. τ_s							
10^{-2}	10	14	19	12	8	7	7
10^{-3}	14	20	26	15	8	8	8
10^{-4}	21	26	34	23	10	17	8
10^{-5}	22	45	41	29	11	10	10

Table 5. Pressure dependent yield stress: The numbers of nonlinear iterations for Bingham flow with pressure dependent yield stress in (6) with fixed point method $\delta_n = 0.0$ and dynamically balanced Newton, varying the lower bound yield stress τ_{min} and fixed upper bound yield stress $\tau_{max} = 0.1$

Method	τ_{min}				
	0.0	0.0001	0.001	0.01	0.1
Fixed Point	356	356	356	356	356
Newton	79	68	68	56	27

yield stress. In a first step, the yield stress is taken as a function of the pressure as described in (6). We fix the upper bound of the yield stress and change the lower bound of the yield stress to allow significant changes in the pressure which should mainly influence the convergence behavior. However, due to the 'min-max' flow rule, we cannot differentiate w.r.t. the pressure so that we apply the described Newton modification for the velocity part only, while the pressure dependence is treated in a fixed point style only. Nevertheless, the comparison of the standard fixed point method and the newly dynamically balanced Newton, which is presented in **Table 5**, shows already a clearly improved behavior. In the next step, we will apply a flow model including pressure and shear rate which will allow differentiation w.r.t. both arguments (as demonstrated in the shown models for granular flow) so that an extension of the new Newton method to pressure-dependent yield stress fluids can be realized and numerically analyzed, too.

4 Summary

We shortly presented a new Newton-type method for flow problems with yield stress which are typical for viscoplastic Bingham models as well as granular flow models with pressure-dependent yield stress. The model is approximated with a regular approach to derive the Jacobian. Then, the partial contributions to the Jacobian are segregated in order to differ between 'good' and 'bad' parts (due to their expected numerical behavior). Firstly, we showed the insufficiency of the classical globally damped Newton. Secondly, we derived a statically balanced Newton approach, by taking different parts of the Jacobian in a static manner for different yield stress values. Thirdly, we went further with dynamic changes allowing the selection of the 'optimal' contributions inside of the Jacobian, here mainly based on the residual changes. The numerical results demonstrate the ability to simulate the Bingham viscoplastic model in the primitive variables for a small regularized parameter ϵ and pressure-dependent yield stress. Moreover, we pointed out how this approach can be extended to more complex (and more realistic) flow models which are typical for granular flow models.

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