

Anisotropic slope limiting for discontinuous Galerkin methods

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SUMMARY

In this paper, we present an anisotropic version of a vertex-based slope limiter for discontinuous Galerkin (DG) methods. The limiting procedure is carried out locally on each mesh element utilizing the bounds defined at each vertex by the largest and smallest mean value from all elements containing the vertex. The application of this slope limiter guarantees the preservation of monotonicity. Unnecessary limiting of smooth directional derivatives is prevented by constraining the x - and y -components of the gradient separately. As an inexpensive alternative to optimization-based methods based on solving small linear programming (LP) problems, we propose a simple operator splitting technique for calculating the correction factors for the x - and y -derivatives. We also provide the necessary generalizations for using the anisotropic limiting strategy in an arbitrary rotated frame of reference and in the vicinity of exterior boundaries with no Dirichlet information. The limiting procedure works for elements of arbitrary polygonal shape and can be extended to three dimensions in a straightforward fashion. The performance of the new anisotropic slope limiter is illustrated by two-dimensional numerical examples.

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1. INTRODUCTION

Slope limiters are widely used to enforce geometric maximum principles in finite volume and discontinuous Galerkin (DG) methods for conservation laws. A well-designed limiter should be able to detect and eliminate nonphysical under-/overshoots, preserve the approximation order in regions of smoothness, and keep the numerical solution free of spurious effects.

Starting with the classical *minmod* limiter of Cockburn and Shu [5, 6] a number of variations and improvements were introduced over the course of the years [9, 10, 20]. A typical limiting technique constrains the derivatives of a piecewise-linear or high-order approximation so as to impose some inequality constraints on the solution values at certain control points. The accuracy of the slope-limited solution depends on the location of the control points, definition of the bounds at these control points, and the algorithm used to enforce these bounds. A class of generalized limiters was developed to constrain higher-order moments reconstructed using the solution on a patch including the element itself and some neighbors [15, 16, 17, 21].

In the case of modal discontinuous Galerkin methods, vertex-based slope limiters [2, 11] impose less restrictive constraints than algorithms in which the control points are located at edge/face midpoints. In the ‘standard’ version, the limiter is applied to the coefficients of a Taylor polynomial,

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and all derivatives of the same order are multiplied by the same correction factor, which may be inappropriate in the presence of strong directional derivatives. Moreover, for elements adjoining exterior domain boundaries with no Dirichlet information, any solution variation in the direction normal to the exterior boundary immediately results in a complete cancellation of gradients.

In this paper, we discuss the cause and cure of the above effects in the context of a time-explicit piecewise linear DG approximation. We introduce an anisotropic version of the vertex-based slope limiter developed in [11] and evaluate it using test problems that illustrate the need for an anisotropic limiting strategy. The proposed algorithm extends the optimization approach presented in [1] and is based on the same set of inequality constraints as optimization-based anisotropic limiters [8, 4, 14]; however, it avoids solving explicitly the optimization problems that compute the correction factors for gradient components or directional derivatives of the solution. Our limiter relies on simple operator splitting techniques instead. Furthermore, we evaluate modifications allowing to improve limiter efficiency for boundary points. The proposed anisotropic limiting scheme is further generalized to support arbitrary, locally defined frames of reference; by making a specific choice of local limiting directions the limiting can be reduced to the one-dimensional case. The performance of new limiters is illustrated by numerical studies in 2D.

2. MODEL PROBLEM

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $T > 0$. We set $Q_T = \Omega \times (0, T)$ and by Γ we denote the boundary of Ω . As a standard model problem, we consider the linear convection equation

$$\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v}u) = 0 \quad \text{in } Q_T, \quad (1)$$

where $u : Q_T \rightarrow \mathbb{R}$ is a conserved scalar quantity, and $\mathbf{v} : Q_T \rightarrow \mathbb{R}^2$ is a continuous velocity field. The initial condition is given by

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega, \quad (2)$$

where $u_0 : \Omega \rightarrow \mathbb{R}$. On the inflow boundary $\Gamma_{\text{in}} = \{\mathbf{x} \in \Gamma \mid \mathbf{v}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0\}$ ($\mathbf{n}$ denotes the exterior unit normal to the boundary), the convective flux is given by the weakly imposed Dirichlet boundary condition

$$(\mathbf{v}u) \cdot \mathbf{n} = (\mathbf{v}u_{\text{in}}) \cdot \mathbf{n} \quad \text{on } \Gamma_{\text{in}} \times (0, T), \quad (3)$$

where $u_{\text{in}} : \Gamma_{\text{in}} \times (0, T) \rightarrow \mathbb{R}$ is a given function. Due to hyperbolicity, no boundary conditions are prescribed on $\Gamma \setminus \Gamma_{\text{in}}$.

To introduce the discontinuous Galerkin method, we consider an arbitrary element $K \in \mathcal{T}_h$ of a (possibly unstructured) computational mesh $\mathcal{T}_h$. Multiplying (1) by a suitable test function w , integrating over K , and using Green's formula, one obtains the local variational formulation

$$\int_K \left(w \frac{\partial u}{\partial t} - \nabla w \cdot \mathbf{v}u \right) dx + \int_{\partial K} w \hat{u} \mathbf{v} \cdot \mathbf{n} ds = 0, \quad \forall w \in V, \quad (4)$$

where V is the broken Sobolev space of trial/test functions, and $\hat{u}$ is the upwind-sided trace of the (generally discontinuous) function $u \in V$.

Selecting the upwind-sided value, we define the convective fluxes using

$$\hat{u}(\mathbf{x}, t) = \begin{cases} \lim_{\varepsilon \rightarrow +0} u(\mathbf{x} + \varepsilon \mathbf{n}, t) & \text{if } \mathbf{x} \in \partial K_{\text{in}} \setminus \Gamma_{\text{in}}, \\ u_{\text{in}}(\mathbf{x}, t) & \text{if } \mathbf{x} \in \partial K_{\text{in}} \cap \Gamma_{\text{in}}, \\ \lim_{\varepsilon \rightarrow -0} u(\mathbf{x} + \varepsilon \mathbf{n}, t) & \text{otherwise,} \end{cases} \quad (5)$$

where $\partial K_{\text{in}} = \{\mathbf{x} \in \partial K \mid \mathbf{v}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0\}$ denotes the inflow boundary of K .

Let $\varphi_1, \dots, \varphi_N$ be basis functions spanning a finite element space V_h . Using (4) with $w \in \{\varphi_1, \dots, \varphi_N\}$, one obtains a system of semi-discrete equations for the time-dependent coefficients

of the numerical solution

$$u_h(\mathbf{x}, t) = \sum_{j=1}^N u_j(t) \varphi_j(\mathbf{x}). \quad (6)$$

The time derivative can be discretized, e.g., using a strong stability preserving (SSP) explicit Runge-Kutta method [7]. For a more detailed description of the DG discretization procedure, we refer the interested reader to [11].

3. LIMITING THE GRADIENT COMPONENTS IN 2D

The piecewise-constant DG approximation is equivalent to the first-order finite volume method which is known to be monotonicity preserving. Hence, any nonphysical under-/overshoots are caused by higher-order terms.

In the case of a piecewise-linear 2D approximation, the restriction of u_h to a single element $K \in \mathcal{T}_h$ can be written as the Taylor polynomial

$$u_h(\mathbf{x}, t) = u_h(\mathbf{x}_0, t) + \nabla u_h \cdot (\mathbf{x} - \mathbf{x}_0), \quad \mathbf{x} = (x, y) \in K, \quad (7)$$

where $\nabla u_h = (u_x, u_y)^T$ and $\mathbf{x}_0 = (x_0, y_0)$ is the center of mass of K

$$\mathbf{x}_0 = \frac{1}{|K|} \int_K \mathbf{x} \, dx. \quad (8)$$

By linearity, we have

$$u_h(\mathbf{x}_0, t) = \frac{1}{|K|} \int_K u_h(\mathbf{x}, t) \, dx. \quad (9)$$

That is, the value of u_h at the point $\mathbf{x}_0$ coincides with the mean value in K .

Holding the time t fixed, we write the linear solution in the form

$$u_h(x, y) = u_0 + u_x(x - x_0) + u_y(y - y_0), \quad (10)$$

where $u_0 = u_h(x_0, y_0)$. Note that the partial derivatives u_x, u_y are constant on K and can be adjusted without changing the mean value of $u_h|_K$.

3.1. Inequality constraints

Since the linear components of u_h can cause the jumps of the DG solution across inter-element boundaries to violate the maximum principle, they may need to be limited. Denoting the limited solution by $\bar{u}_h$, we consider

$$\bar{u}_h(x, y) = u_0 + \alpha_x u_x(x - x_0) + \alpha_y u_y(y - y_0). \quad (11)$$

Given a set of control points $(x_1, y_1), \dots, (x_M, y_M)$ which may be placed at the vertices or edge midpoints of K , the correction factors $\alpha_x, \alpha_y \in [0, 1]$ are chosen so as to enforce inequality constraints of the form

$$u_i^{\min} \leq \bar{u}_h(x_i, y_i) \leq u_i^{\max}, \quad i = 1, \dots, M. \quad (12)$$

The bounds $u_i^{\max}$ and $u_i^{\min}$ are defined as the largest/smallest mean values in elements containing the point (x_i, y_i) . In this paper, we favor a vertex-based limiting strategy [11], i.e., the control points are the vertices of K .

3.2. Isotropic limiting

A slope limiter not accounting for anisotropy constrains the linear part of $u_h|_K$ using the same correction factor $\alpha_x = \alpha_y = \alpha$ for the x - and y -components of the gradient. For accuracy reasons, this correction factor should be chosen as close to 1 as possible without violating (12). Then α is calculated using the Barth-Jespersen formula [2, 11]

$$\alpha = \min_{1 \leq i \leq M} \begin{cases} \min \left\{ 1, \frac{u_i^{\max} - u_0}{u_i - u_0} \right\}, & \text{if } u_i - u_0 > 0, \\ 1, & \text{if } u_i - u_0 = 0, \\ \min \left\{ 1, \frac{u_i^{\min} - u_0}{u_i - u_0} \right\}, & \text{if } u_i - u_0 < 0, \end{cases} \quad (13)$$

where u_i denotes the unconstrained value of $u_h|_K$ at the control point (x_i, y_i) .

The use of a common correction factor for both components of the gradient corresponds to adjusting its magnitude while leaving its direction unchanged. As we will see later, this approach to slope limiting may give rise to unnecessary cancellation of smooth directional derivatives.

3.3. Anisotropic limiting based on operator splitting

Berger et al. [4, 14] proposed directional limiting formulations for the finite volume method, in which the optimal correction factors for the x - and y -components of the gradient are determined by solving inequality-constrained minimization problems. While their results indicate that anisotropic slope limiting is a better way to enforce inequality constraints of the form (12), the cost of an optimization-based algorithm is clearly higher than that of a closed-form expression as in (13).

In this paper, we propose an inexpensive alternative to optimization-based anisotropic limiting. It is based on a simple operator splitting technique. To enforce the inequality constraints

$$u_i^{\min} - u_0 \leq \alpha_x u_x(x_i - x_0) + \alpha_y u_y(y_i - y_0) \leq u_i^{\max} - u_0 \quad (14)$$

at each control point (x_i, y_i) , $i = 1, \dots, M$ of element K , we define

$$\alpha_x = \min_{1 \leq i \leq M} \alpha_{x,i}, \quad \alpha_y = \min_{1 \leq i \leq M} \alpha_{y,i}. \quad (15)$$

To avoid unnecessary limiting in the case when the x and y variations cancel out, we check if the inequality constraints

$$u_i^{\min} - u_0 \leq u_x(x_i - x_0) + u_y(y_i - y_0) \leq u_i^{\max} - u_0 \quad (16)$$

hold for each control point (x_i, y_i) , $i = 1, \dots, M$ of element K . If this is the case, then no limiting is required, so we set $\alpha_x = \alpha_y = 1$. Otherwise, we start with prelimiting the variation in a given direction. Without loss of generality let it be the x direction. Adding the limited x variation to the mean value, we construct

$$\hat{u}_h(x, y) = u_0 + \alpha_x u_x(x - x_0), \quad (17)$$

where the correction factor $\alpha_x \in [0, 1]$ is chosen so as to enforce the inequality constraints

$$u_i^{\min} \leq u_0 + \alpha_x u_x(x_i - x_0) \leq u_i^{\max}. \quad (18)$$

The corresponding nodal correction factors $\alpha_{x,i} \in [0, 1]$ are calculated using the following modification of the Barth-Jespersen formula (13) for limiting the variations in the x direction:

$$\alpha_{x,i} = \begin{cases} \min \left\{ 1, \frac{u_i^{\max} - u_0}{u_x(x_i - x_0)} \right\}, & \text{if } u_x(x_i - x_0) > 0, \\ 1, & \text{if } u_x(x_i - x_0) = 0, \\ \min \left\{ 1, \frac{u_i^{\min} - u_0}{u_x(x_i - x_0)} \right\}, & \text{if } u_x(x_i - x_0) < 0. \end{cases} \quad (19)$$

Given the prelimited split solution $\hat{u}_h$, we add the y variation multiplied by a correction factor $\alpha_y \in [0, 1]$ in order to enforce the inequality constraints

$$u_i^{\min} \leq \hat{u}_h(x_i, y_i) + \alpha_y u_y(y_i - y_0) \leq u_i^{\max}. \quad (20)$$

Again, we use a modification of the Barth-Jespersen formula to calculate the nodal correction factors

$$\alpha_{y,i} = \begin{cases} \min \left\{ 1, \frac{u_i^{\max} - \bar{u}_i(x_i, y_i)}{u_y(y_i - y_0)} \right\}, & \text{if } u_y(y_i - y_0) > 0, \\ 1, & \text{if } u_y(y_i - y_0) = 0, \\ \min \left\{ 1, \frac{u_i^{\min} - \bar{u}_i(x_i, y_i)}{u_y(y_i - y_0)} \right\}, & \text{if } u_y(y_i - y_0) < 0. \end{cases} \quad (21)$$

Note that the outcomes of the above sequential limiting procedure can be generally affected by the order in which the x - and y -components are constrained (prelimiting in the x direction followed by limiting in the y direction or otherwise).

3.4. Anisotropic limiting based on solving LP problems

An anisotropic slope limiter based on solving linear programming (LP) problems was introduced in [4]. In the present paper, we compare this approach to the proposed operator splitting strategy. Therefore, a brief description of the LP limiter is in order. Again, we consider the limited solution

$$\bar{u}_h(x, y) = u_0 + \alpha_x u_x(x - x_0) + \alpha_y u_y(y - y_0). \quad (22)$$

The correction factors $\alpha_x, \alpha_y \in [0, 1]$ are chosen so as to enforce inequality constraints of the form

$$u_i^{\min} \leq \bar{u}_h(x_i, y_i) \leq u_i^{\max}, \quad i = 1, \dots, M. \quad (23)$$

In contrast to [4], the control points are placed at the vertices of the element K . Following [4], the limiter is formulated as a constrained optimization method based on solving small LP problems with the goal of minimizing the l^1 -norm of the difference between limited and unlimited gradient. That is, the correction factors α_x and α_y are chosen so as to minimize the objective function

$$-\alpha_x |u_x| - \alpha_y |u_y| \quad (24)$$

subject to (23). For a detailed description of the LP minimization algorithm we refer to [4].

4. GENERALIZATION TO AN ARBITRARY FRAME OF REFERENCE

While the use of different correction factors for the x - and y -derivatives is a marked improvement compared to the 'isotropic' limiting strategy formulated in (13), the DG solutions may still exhibit anisotropies which are not aligned with the axes of the Cartesian coordinate system. In many cases, controlling another pair of directional derivatives $(u_\xi, u_\eta)^T$ associated with a rotated Cartesian reference frame is appropriate. The use of frame-invariant directions is to be preferred, especially in extensions of the anisotropic limiter to vector fields (cf. [12, 13, 18, 19]).

In elements adjoining the domain boundary, the local reference frame may need to be aligned with the normal and tangential directions (see below). The limiting techniques proposed here can also be easily adapted to skew coordinate systems.

The inverse coordinate transformation associated with a pair of orthonormal direction vectors $\xi = (\cos \theta, -\sin \theta)^T$ and $\eta = (\sin \theta, \cos \theta)^T$ is defined by

$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \hat{x} - x_0 \\ \hat{y} - y_0 \end{pmatrix}, \quad (25)$$

which corresponds to a rotation of the local coordinate system around the center of mass (x_0, y_0) . Thus, a piecewise linear solution can be written as

$$\begin{aligned} u_h(x, y) &= u_0 + u_x(x - x_0) + u_y(y - y_0) \\ &= u_0 + (u_x, u_y) \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \hat{x} - x_0 \\ \hat{y} - y_0 \end{pmatrix} \\ &= u_0 + u_\xi(\hat{x} - x_0) + u_\eta(\hat{y} - y_0), \end{aligned}$$

where u_ξ and u_η are the two components of the transformed gradient

$$\begin{pmatrix} u_\xi \\ u_\eta \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix}. \quad (26)$$

The directional derivatives u_ξ and u_η can be constrained using the same strategy as the x - and y -components in the algorithm presented in Section 3.3. After the multiplication by the correction factors α_ξ and α_η we obtain

$$\begin{aligned} \bar{u}_h(x, y) &= u_0 + \alpha_\xi u_\xi(\hat{x} - x_0) + \alpha_\eta u_\eta(\hat{y} - y_0) \\ &= u_0 + (u_\xi, u_\eta) \begin{pmatrix} \alpha_\xi & 0 \\ 0 & \alpha_\eta \end{pmatrix} \begin{pmatrix} \hat{x} - x_0 \\ \hat{y} - y_0 \end{pmatrix} \\ &= u_0 + (u_x, u_y) \Phi(\theta) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}, \end{aligned}$$

where Φ is a symmetric 2×2 matrix of correction factors

$$\Phi(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \alpha_\xi & 0 \\ 0 & \alpha_\eta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}. \quad (27)$$

Thus, the application of the anisotropic slope limiter to the components of $(u_\xi, u_\eta)^T$ is equivalent to replacing $(u_x, u_y)^T$ by $\Phi(\theta)(u_x, u_y)^T$.

For example one may align the axes of the rotated coordinate system with the direction

$$\xi = \frac{\nabla u_h}{\|\nabla u_h\|} = (\xi_x, \xi_y)^T \quad (28)$$

parallel to the unconstrained gradient $\nabla u_h|_K$ and the orthogonal direction

$$\eta = (-\xi_y, \xi_x)^T. \quad (29)$$

By (26), this gives

$$\begin{pmatrix} u_\xi \\ u_\eta \end{pmatrix} = \begin{pmatrix} \frac{u_x}{\|\nabla u_h\|} & \frac{u_y}{\|\nabla u_h\|} \\ -\frac{u_y}{\|\nabla u_h\|} & \frac{u_x}{\|\nabla u_h\|} \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \|\nabla u_h\| \\ 0 \end{pmatrix}, \quad (30)$$

whence

$$u_h(x, y) = u_0 + u_\xi(\hat{x} - x_0). \quad (31)$$

For this particular choice of the rotated coordinate system, the anisotropic limiters introduced in Sections 3.3 and 3.4 lead to the same correction factors, as shown below.

The choice of the correction factor α_η has no effect on the value of $u_h(x, y)$ due to the fact that $u_\eta = 0$. Thus, we set

$$\alpha_\eta = 1. \quad (32)$$

In the limiter based on solving LP problems, the optimal correction factor $\alpha_\xi \in [0, 1]$ is chosen so as to enforce inequality constraints of the form

$$u_i^{\min} \leq u_0 + \alpha_\xi u_\xi(\hat{x}_i - x_0) \leq u_i^{\max}, \quad i = 1, \dots, M, \quad (33)$$

and the objective function is now given by

$$- \alpha_\xi |u_\xi|. \quad (34)$$

It follows that the solution of the minimization problem is

$$\alpha_\xi = \min_{1 \leq i \leq M} \alpha_{\xi, i},$$

where the nodal correction factors

$$\alpha_{\xi,i} = \begin{cases} \min \left\{ 1, \frac{u_i^{\max} - u_0}{u_{\xi}(\hat{x}_i - x_0)} \right\}, & \text{if } u_{\xi}(\hat{x}_i - x_0) > 0, \\ 1, & \text{if } u_{\xi}(\hat{x}_i - x_0) = 0, \\ \min \left\{ 1, \frac{u_i^{\min} - u_0}{u_{\xi}(\hat{x}_i - x_0)} \right\}, & \text{if } u_{\xi}(\hat{x}_i - x_0) < 0 \end{cases} \quad (35)$$

coincide with those obtained at the prelimiting step of the limiter based on operator splitting.

5. TREATMENT OF BOUNDARY POINTS

For the calculation of the correction factors for control points located on the boundary, the knowledge of the cell mean values in elements containing these points is insufficient to construct usable bounds $u_i^{\max}$ and $u_i^{\min}$. The unnecessary cancellation of normal derivatives in boundary elements can be avoided by using the Dirichlet boundary conditions and/or the mean values on the boundary edges to construct better bounds.

We estimate the solution bounds at the boundary vertices with no specified Dirichlet values using the mean solution values not only from elements containing the vertex, but also from all boundary edges containing the concerned vertex. Furthermore, for elements having a face on the exterior domain boundary, the normal and the tangential directions with respect to this boundary are considered as the frame-invariant directions.

Alternatively, the directional correction factors $\alpha_{n,i}$ associated with the normal derivatives may be set equal to 1 at boundary points. In other words, no inequality constraints should be imposed on the normal derivative u_n at these points, while the correction factors for the tangential derivative u_{τ} may be calculated as follows:

$$\alpha_{\tau,i} = \begin{cases} \min \left\{ 1, \frac{u_i^{\max} - u_0}{u_{\tau}(\mathbf{x}_i - \mathbf{x}_0) \cdot \boldsymbol{\tau}} \right\}, & \text{if } u_{\tau}(\mathbf{x}_i - \mathbf{x}_0) \cdot \boldsymbol{\tau} > 0, \\ 1, & \text{if } u_{\tau}(\mathbf{x}_i - \mathbf{x}_0) \cdot \boldsymbol{\tau} = 0, \\ \min \left\{ 1, \frac{u_i^{\min} - u_0}{u_{\tau}(\mathbf{x}_i - \mathbf{x}_0) \cdot \boldsymbol{\tau}} \right\}, & \text{if } u_{\tau}(\mathbf{x}_i - \mathbf{x}_0) \cdot \boldsymbol{\tau} < 0, \end{cases} \quad (36)$$

where $\boldsymbol{\tau}$ is a unit vector pointing in the tangential direction.

6. NUMERICAL EXAMPLES

In this section, we consider two-dimensional examples which illustrate the performance of the anisotropic slope limiting. The numerical study to be presented includes a comparison of the proposed algorithm to the ‘standard’ vertex-based slope limiter [11] and optimization-based techniques [4, 14]. In addition to a grid convergence study on uniform meshes, we study the performance of selected limiting techniques on nonuniform meshes and discuss the treatment of normal derivatives in boundary elements. All numerical experiments are performed using the open-source finite element library DEAL.II (<https://www.dealii.org>) [3].

6.1. Anisotropic convection in a unit square

In the first numerical example, we solve the 2D convection equation (1) with $\mathbf{v}(x, y) = (0, 1)$ in the unit square $\Omega = (0, 1) \times (0, 1)$. The initial profile displayed in Fig. 1 is defined by the formula

$$u_0(x, y) = w(x)4y(1 - y), \quad (37)$$

where

$$w(x) = \begin{cases} 2, & \text{if } 0.2 \leq x \leq 0.4, \\ 1, & \text{otherwise.} \end{cases} \quad (38)$$

The numerical experiments are performed on a sequence of successively refined rectangular meshes with spacing $h = 2^{-i}$, $i = 3, \dots, 9$. The time step is adapted to the value of h so as to maintain the fixed CFL number of 0.8. The main challenge of this test problem is to limit the discontinuities present in $w(x)$ without canceling the smooth gradient in the y direction.

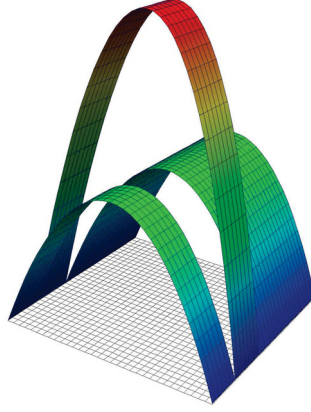


Figure 1. Initial solution profile (L^2 projection) for the anisotropic convection in a unit square.

We compare the following limiting techniques: the isotropic, vertex-based Barth-Jespersen (BJ) limiter, the limiter based on solving LP problems (LP limiter), the anisotropic limiter based on operator splitting (OX limiter) with prelimiting in the x direction followed by limiting in the y direction, the anisotropic limiter based on operator splitting (OY limiter) with prelimiting in the y direction followed by limiting in the x direction, and the limiter using the generalization to an arbitrary frame of reference (AR limiter). In the AR version we use the direction aligned with the unconstrained gradient $\nabla u_h|_K$.

The numerical solutions produced by the methods under investigation are presented in Fig. 3. For a better comparison, the solution profiles in the cross-sections $y = 0.5$ and $x = 0.8$ are plotted in Fig. 4. The numerical errors w.r.t. the L^2 -norm are listed in Tables I and II. One can see that the derivatives in the y -direction are unreasonably diminished by the isotropic BJ limiter. On the other hand, the solution obtained using the OX limiter does not suffer from this effect and is comparable to the one obtained using the LP limiter. A comparison of anisotropic limiters for different directional splittings does show some sensitivity, in particular, the limiter first treating the y -direction (OY limiter) appears to produce somewhat more accurate results. Another interesting phenomenon to note is the excellent performance of the limiter aligned with the gradient of the numerical solution (AR limiter). The convergence rates for all limiter versions are approximately the same. Furthermore, this test illustrates the difference due to the use of the mean values from adjacent boundary edges for constructing the bounds in boundary elements. The same kind of boundary treatment can be used in the isotropic BJ limiter with similar results.

6.2. Anisotropic convection in a quarter-annulus

This numerical example is a modification of the first one. The domain is represented by a quarter annulus with the inner radius of 1.0, and the outer radius of 1.43. We solve once again the 2D convection equation (1) with $\mathbf{v}(x, y) = (-y, x)$, which corresponds to a counterclockwise rotation about the origin. The initial profile displayed in Fig. 2 is defined by the formula

$$u_0(x, y) = \frac{4}{0.43^2} w(x, y) (d(x, y) - 1) (1.43 - d(x, y)), \quad (39)$$

where

$$w(x, y) = \begin{cases} 2, & \text{if } 0.1 \leq \frac{y}{d(x, y)} \leq 0.3, \\ 1, & \text{otherwise,} \end{cases} \quad (40)$$

h	$\ u - u_h\ _{L^2}$	EOC	h	$\ u - u_h\ _{L^2}$	EOC
1/8	0.2553700	-	1/8	0.2402730	-
1/16	0.1429410	0.8371693	1/16	0.1436230	0.7423877
1/32	0.0941314	0.6026718	1/32	0.0936814	0.6164523
1/64	0.0717202	0.3922965	1/64	0.0724032	0.3717092
1/128	0.0532687	0.4290914	1/128	0.0532026	0.4445567
1/256	0.0399040	0.4167547	1/256	0.0398477	0.4170003
1/512	0.0306948	0.3785391	1/512	0.0307252	0.3750741

Table I. Unit square, L^2 -norm of the error and experimental order of convergence of the solution at time 0.5, isotropic BJ (left) and OX (right) versions of the vertex-based limiter.

h	$\ u - u_h\ _{L^2}$	EOC	h	$\ u - u_h\ _{L^2}$	EOC
1/8	0.2205020	-	1/8	0.2220550	-
1/16	0.1291970	0.7712192	1/16	0.1324320	0.7456653
1/32	0.0937267	0.4630406	1/32	0.0946501	0.4845758
1/64	0.0723795	0.3728789	1/64	0.0728858	0.3769663
1/128	0.0540595	0.4210330	1/128	0.0543733	0.4227394
1/256	0.0405578	0.4145688	1/256	0.0407117	0.4174549
1/512	0.0312260	0.3772316	1/512	0.0311288	0.3871935

Table II. Unit square, L^2 -norm of the error and experimental order of convergence of the solution at time 0.5, LP (left) and AR (right) anisotropic versions of the vertex-based limiter.

and $d(x, y)$ is the distance from the origin.

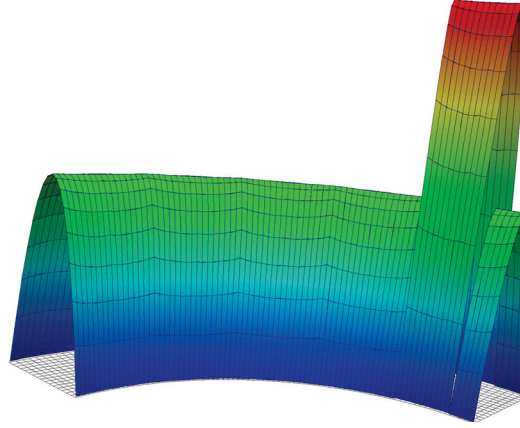


Figure 2. Anisotropic convection in a quarter-annulus, initial profile.

In this example, the gradient-aligned reference frame varies locally. Snapshots of the numerical solutions at the final time 1.0 are presented in Fig. 5. The comparison of the solution profiles in two cross-sections is presented in Fig. 6. Once again the AR limiter – together with the LP limiter – produces results with the least amount of numerical diffusion. One has to note that the apparent undershoots in the right panel graphs in Fig. 6 are due to the orientation of the cross-section with respect to the curved solution graph: Since some of the mean solution values from elements surrounding the location of the cross-section lie below the level $z=1$, the limiting also does allow such values. The anisotropic OX and OY limiters are not quite as successful in this example as the LP and AR limiters, thus supporting our claim that certain types of anisotropy need limiting procedures that are based on the arbitrary frame of reference.

7. CONCLUSIONS

The presented numerical study illustrates the potential of anisotropic vertex-based slope limiting in the context of time-explicit discontinuous Galerkin methods. The proposed algorithm constitutes an attractive alternative to optimization-based techniques and produces numerical results of comparable quality. Of particular interest is the generalization of the proposed method to arbitrary local frames of reference that extends the anisotropic limiting scheme to a much larger class of domains. Applications of this algorithm to vector fields, higher-order discretizations, and three-dimensional problems are feasible and subject of ongoing work.

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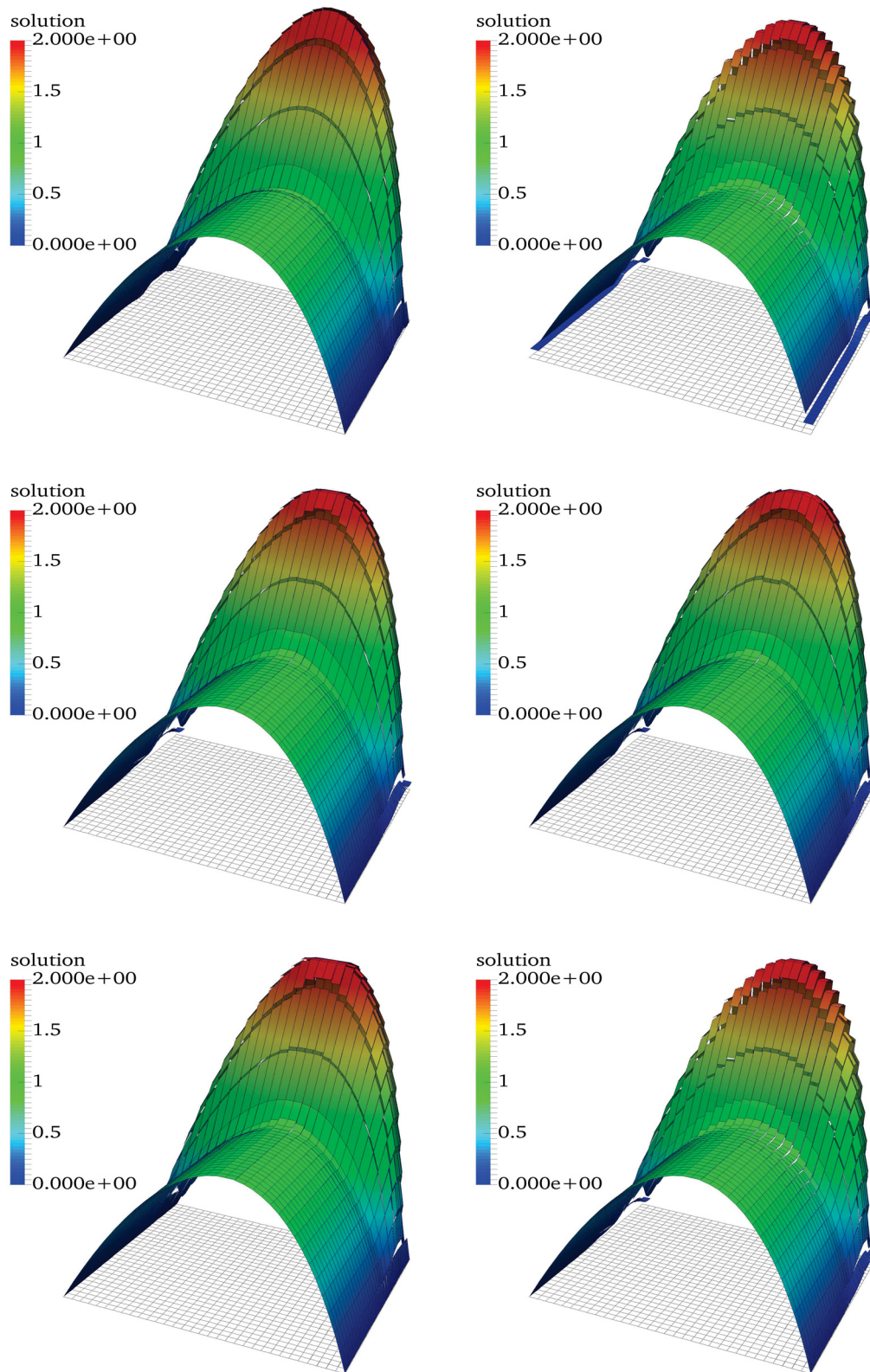


Figure 3. Unit square, solutions at time 0.5, $h = 1/32$, without limiter (top left), isotropic vertex-based BJ (top right), anisotropic OX (middle left), anisotropic OY (middle right), LP (bottom left), AR (bottom right).

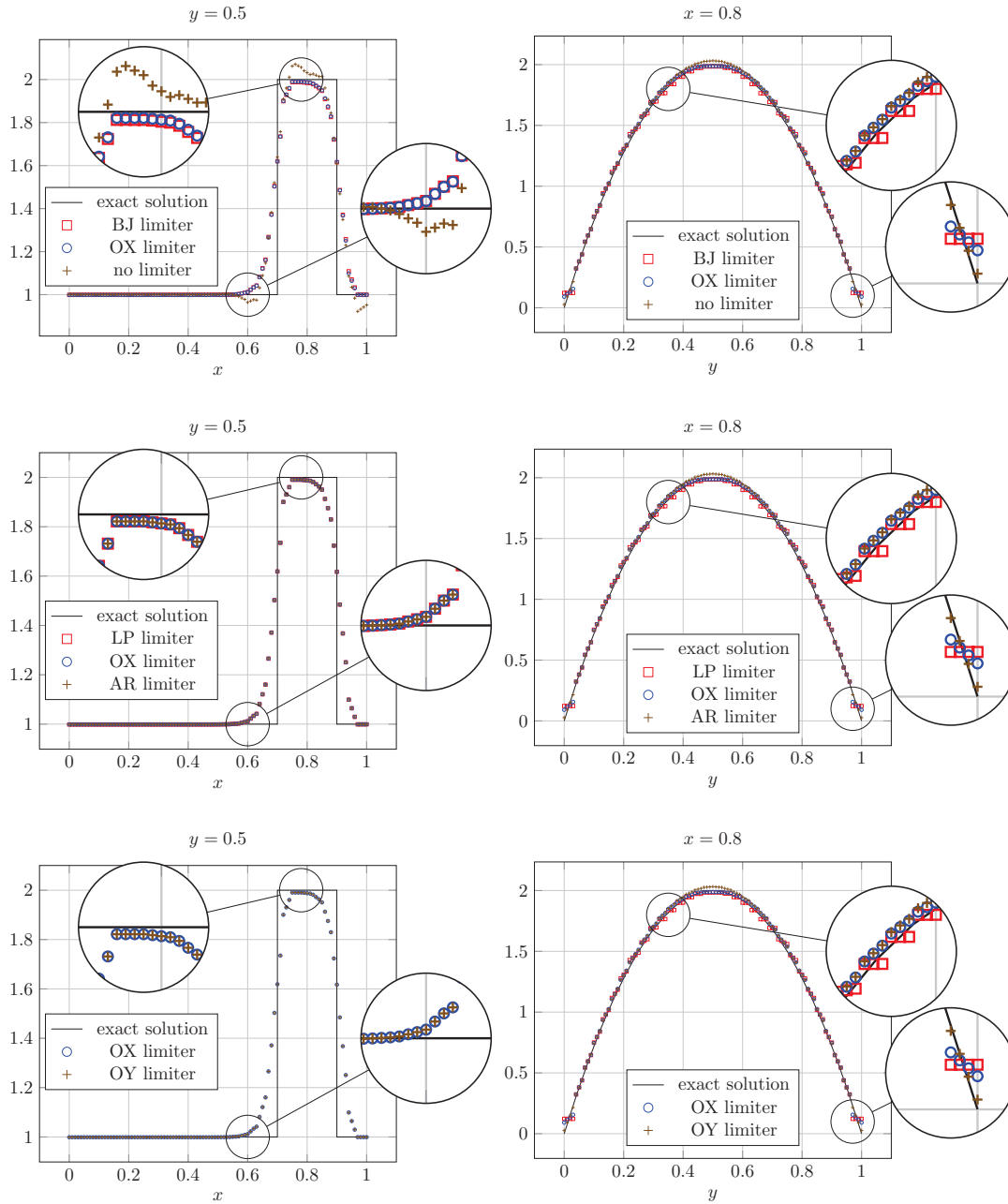


Figure 4. Unit square, solutions at time 0.5, $h = 1/32$, cross-sections at $y = 0.5$ (left) and $x = 0.8$ (right).

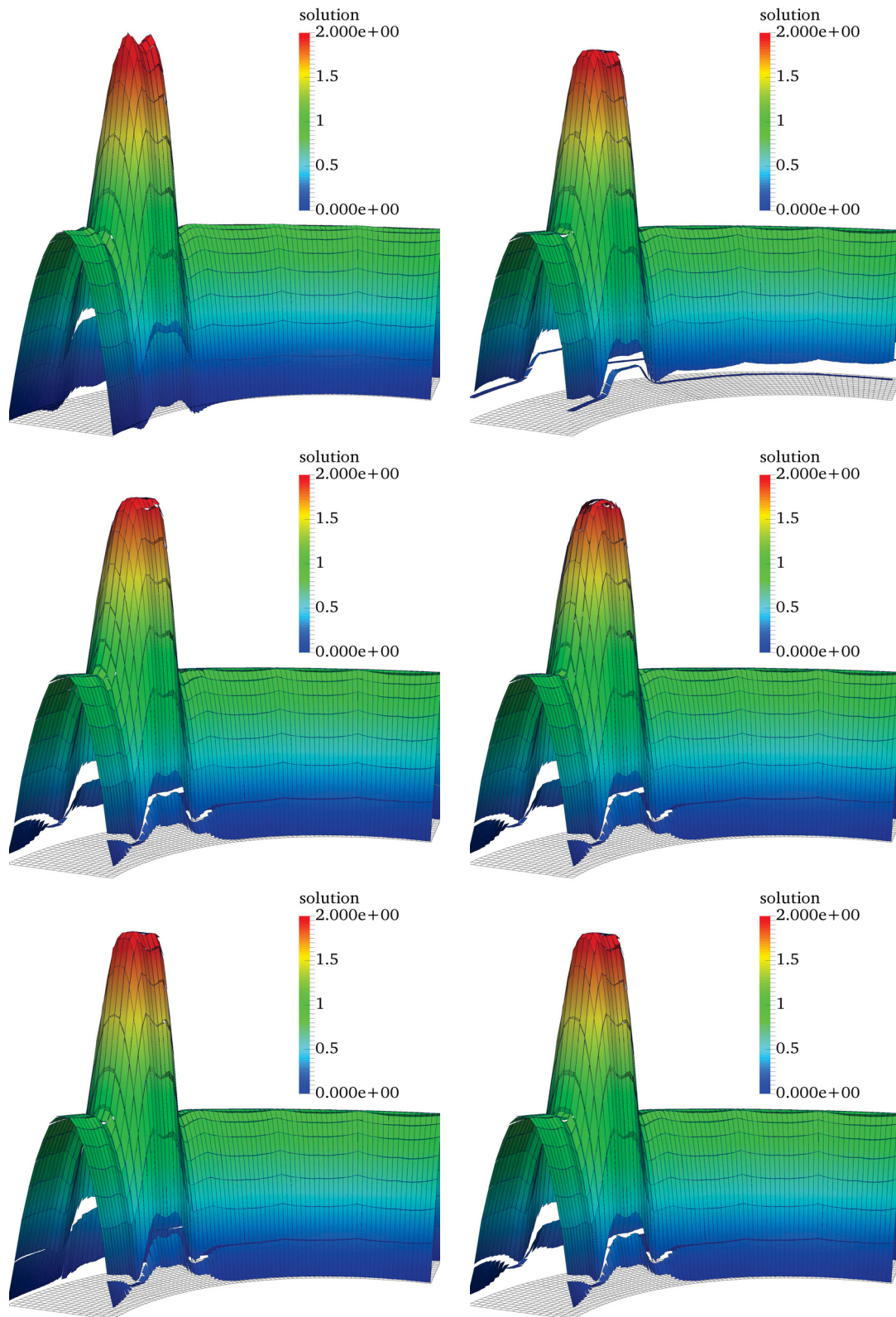


Figure 5. Quarter-annulus, solutions at time 1.0, $h = 0.01607$, without limiter (top left), isotropic vertex-based BJ (top right), anisotropic OX (middle left), anisotropic OY (middle right), LP (bottom left), AR (bottom right).

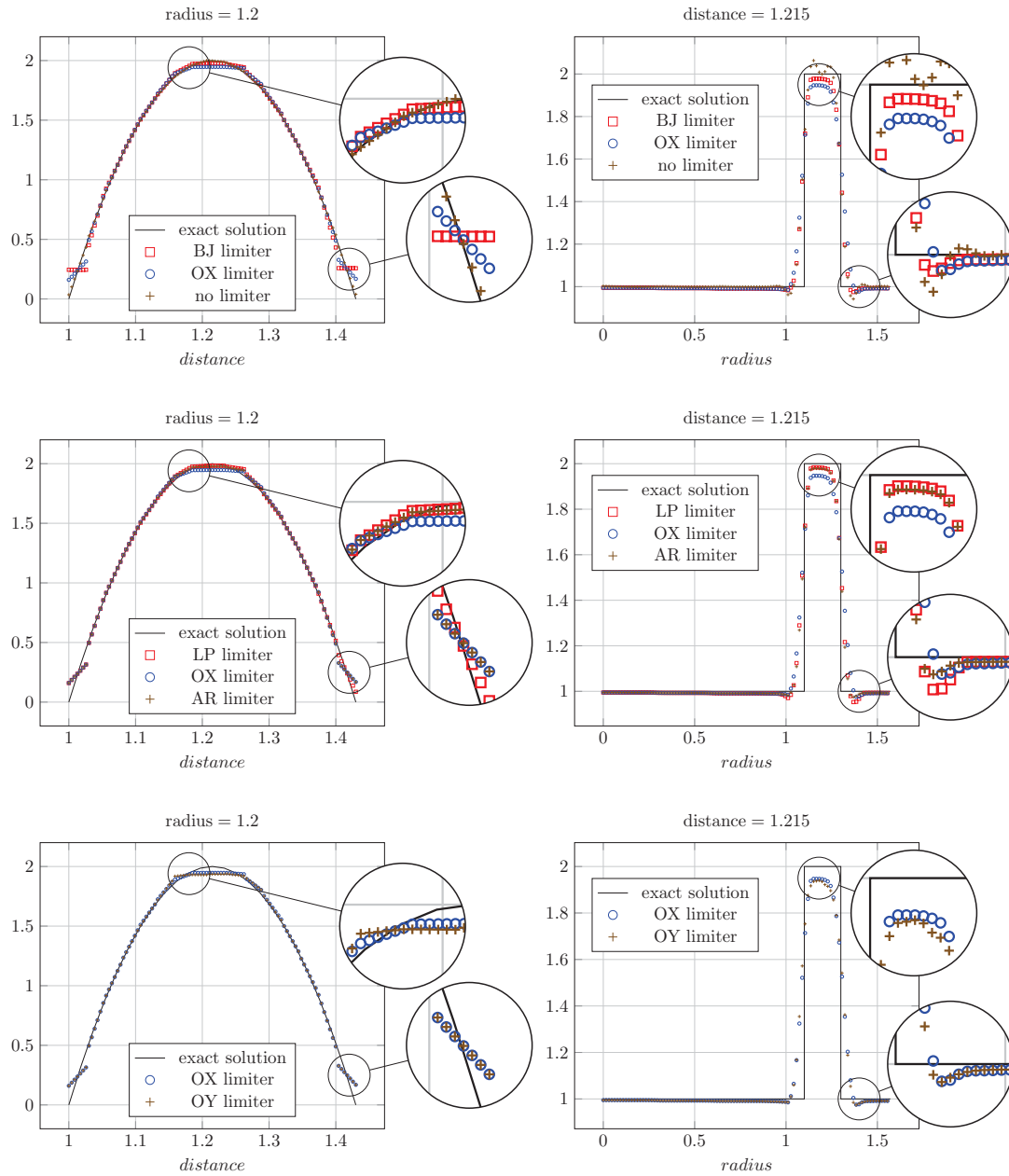


Figure 6. Quarter-annulus, solutions at time 1.0, $h = 0.01607$, cross-sections at radius = 1.2 (left) and distance = 1.215 (right).