ANALYSIS OF A VISCOUS TWO-FIELD GRADIENT DAMAGE MODEL
PART II: PENALIZATION LIMIT

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Abstract. The paper analyses the behaviour for penalty parameter tending to infinity of a damage model which features two damage variables coupled through a penalty term. It turns out that in the limit both damage variables coincide and satisfy a classical viscous damage model.

Key words. Viscous damage evolution, energy identity, penalization

1. Introduction. This paper is concerned with a two-field damage model involving two different damage variables which are connected through a penalty term in the stored energy functional. While the well-posedness of the model was investigated in the companion paper, this work addresses the limit analysis for penalization parameter approaching ∞.

The penalized damage model analyzed in the companion paper [17] features two damage variables and describes the evolution in time of the local damage variable d in an elastic body, when applying a time-dependent force ℓ. The thereby induced displacement is denoted by $\boldsymbol{u}$, while the non-local damage variable is denoted by φ. The mathematical model reads

$$\left.\begin{aligned} (\boldsymbol{u}(t), \varphi(t)) &\in \operatorname*{arg\,min}_{(\boldsymbol{u},\varphi)\in V\times H^1(\Omega)} \mathcal{E}(t, \boldsymbol{u}, \varphi, d(t)), \\ -\partial_d \mathcal{E}(t, \boldsymbol{u}(t), \varphi(t), d(t)) &\in \partial \mathcal{R}_\delta(\dot{d}(t)), \quad d(0) = d_0 \text{ a.e. in } \Omega \end{aligned}\right\} \tag{P}$$

for almost all $t \in (0, T)$. The stored energy $\mathcal{E} : [0, T] \times V \times H^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ is given by

$$\mathcal{E}(t, \boldsymbol{u}, \varphi, d) := \frac{1}{2} \int_\Omega g(\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u})\, dx - \langle \ell(t), \boldsymbol{u}\rangle_V + \frac{\alpha}{2}\|\nabla\varphi\|_2^2 + \frac{\beta}{2}\|\varphi - d\|_2^2, \tag{1.1}$$

where $\alpha > 0$ denotes the gradient regularization and $\beta > 0$ stands for the penalization parameter. The viscous dissipation functional $\mathcal{R}_\delta : L^2(\Omega) \to [0, \infty]$ is defined as

$$\mathcal{R}_\delta(\eta) := \begin{cases} r \int_\Omega \eta\, dx + \frac{\delta}{2}\|\eta\|_2^2, & \text{if } \eta \geq 0 \text{ a.e. in } \Omega, \\ \infty, & \text{otherwise,} \end{cases} \tag{1.2}$$

where $r > 0$ stands for the fracture toughness of the material and $\delta > 0$ is the viscosity parameter.

In the present paper the viability of the penalty approach is established. This is twofold. Firstly, it turns out that working with two damage variables coupled through a penalty term makes sense from a mathematical point of view, since it turns out that they both become equal in the limit. Secondly, the resulting one-field gradient damage model falls into the category of classical partial damage models introduced in [7]. The limit behaviour $\beta \to \infty$ of the solutions of (P) is studied by means of an equivalent

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reformulation of (P) in terms of an energy identity. The latter one plays an essential role in the present work, as it not only ensures the convergence of the penalized solutions, but it is also crucial for deriving the energy inequality which characterizes the limit damage variable. In combination with well known convex analysis results, this allows for deriving a one field gradient damage model. This features one damage variable less, which is the limit of the sequence of (penalized) local as well as nonlocal damage variables. Moreover, the new damage variable possesses more space regularity. Nevertheless, the one field gradient damage model can be transformed into a classical partial damage model analyzed in [13].

Let us put our work into perspective. Numerous damage models have been addressed by many authors under different aspects. In [1–3, 6] various viscous damage models have been analyzed with regard to existence and regularity of solutions. The concept of viscosity also plays an important role in the mathematical treatment of rate-independent damage models, as the vanishing viscosity approach is a prominent method to establish solutions for rate-independent problems. We only refer to [5, 12–15, 19–21], and the references therein. An important tool in the context of damage modeling is the energy inequality. This can take many forms, depending on the properties of the energy- and dissipation functionals, see [18]. Therein an overview of various notions of solutions for rate-independent damage models is given. We only mention here the global energetic solutions and balanced viscosity (BV) solutions. The energy inequality is a powerful tool when it comes to limit analysis, e.g. it allows the construction of BV solutions for rate-independent damage models. This has been demonstrated in [13] for a gradient damage model in the spirit of [7]. However, to the best of our knowledge, a damage model containing two damage variables has never been investigated so far with regard to a rigorous mathematical analysis, although these models are frequently used for numerical simulations, cf. e.g. [16, 22–24, 26]. This concerns the existence and regularity of solutions, let alone the behavior of the damage variables and the displacement field, as the penalty vanishes. The mathematical model (P) was inspired by the one presented in [4]. The latter one was slightly modified because of mathematical reasons, by replacing the less regular variable, d, by the more regular, φ, in the balance of momentum equation. As expected, the deviation between (P) and the original model vanishes in the limit $\beta \to \infty$. In [13] and preprint, the existence of viscous solutions is obtained via time-discretization and regularization, respectively. Our final result shows that the existence thereof may also be obtained via penalization.

The paper is organized as follows. Section 2 collects the notations and standing assumptions, as well as known results from companion paper [17] which are needed in the actual paper. In Section 3 we derive an equivalent formulation of the evolution in (P), namely the energy identity. Section 4 is devoted entirely to the limit analysis $\beta \to \infty$ of (P). This is done in two steps. We first prove that the variables in (P) are bounded in suitable spaces, such that the existence of the limit variables is ensured. We then pass to the limit in the elliptic system which characterizes the minimization problem in (P), as well as in the energy identity. The latter one results in an energy inequality, which describes the evolution of the limit damage variable. Based on this results, Section 5 deals with deriving a one field gradient damage model in terms of an evolutionary equation and addresses the unique solvability thereof. In Section 6 it is established that the one field gradient damage model is equivalent to a viscous damage model analyzed in [13].

2. Notation, Standing Assumptions, and Known Results. Throughout the paper, C denotes a generic positive constant. If X and Y are two linear normed spaces, the space of linear and bounded operators from X to Y is denoted by $\mathcal{L}(X, Y)$. The dual of a linear normed space X will be denoted by X^*. For the dual pairing between X and X^* we write $\langle ., . \rangle_X$ and, if it is clear from the context, which dual pairing is meant, we just write $\langle ., . \rangle$. By $\|\cdot\|_p$ we denote the $L^p(\Omega)-$norm for $p \in [1, \infty]$ and by $(\cdot, \cdot)_2$ the $L^2(\Omega)-$scalar product. If X is compactly embedded in Y, we write $X \hookrightarrow\hookrightarrow Y$. In the rest of the paper $N \in \{2, 3\}$ denotes the spatial dimension. By bold-face case letters we denote vector valued variables and vector valued spaces.

Definition 2.1. *For $p \in [1, \infty]$ we define the following subspace of $\boldsymbol{W}^{1,p}(\Omega)$:*

$$\boldsymbol{W}^{1,p}_D(\Omega) := \{v \in \boldsymbol{W}^{1,p}(\Omega) : v|_{\Gamma_D} = 0\},$$

where Γ_D is a part of the boundary of the domain Ω, see Assumption 2.2 below. The dual space of $\boldsymbol{W}^{1,p'}_D(\Omega)$ is denoted by $\boldsymbol{W}^{-1,p}_D(\Omega)$, where p' is the conjugate exponent of p. If $p = 2$, we abbreviate $V := \boldsymbol{W}^{1,2}_D(\Omega)$.

Before we turn to our assumptions on the data, we summarize the often used symbols in Table 2.1 for convenience of the reader.

Table 2.1
Functionals, operators and variables

Symbol	Meaning	Definition
$\mathcal{I}$	Reduced energy functional	Definition 3.1
$\mathcal{R}_\delta$	Viscous dissipation functional	(1.2)
$\boldsymbol{u}$	Displacement	
φ	Nonlocal damage	
d	Local damage	
$\widetilde{\mathcal{E}}$	Energy functional without penalty	(??)
$\widetilde{\mathcal{I}}$	Reduced energy functional without penalty	Definition **??**
$\widetilde{\mathcal{R}}_1$	Dissipation functional after passing to the limit	(5.10)
$\widetilde{\mathcal{R}}_\delta$	Viscous dissipation functional after passing to the limit	Definition 4.13

Let us now state our standing assumptions. We begin with the smoothness of the computational domain.

Assumption 2.2. *The domain $\Omega \subset \mathbb{R}^N$, $N \in \{2, 3\}$, is bounded with Lipschitz boundary Γ. The boundary consists of two disjoint measurable parts Γ_N and Γ_D such that $\Gamma = \Gamma_N \cup \Gamma_D$. While Γ_N is a relatively open subset, Γ_D is a relatively closed subset of Γ with positive measure.*

In addition, the set $\Omega \cup \Gamma_N$ is regular in the sense of Gröger, cf. [9]. That is, for every point $x \in \Gamma$, there exists an open neighborhood $\mathcal{U}_x \subset \mathbb{R}^N$ of x and a bi-Lipschitz map (a Lipschitz continuous and bijective map with Lipschitz continuous inverse) $\Psi_x : \mathcal{U}_x \to \mathbb{R}^N$ such that $\Psi_x(x) = 0 \in \mathbb{R}^N$ and $\Psi_x\big(\mathcal{U}_x \cap (\Omega \cup \Gamma_N)\big)$ equals one of the following sets:

$$E_1 := \left\{y \in \mathbb{R}^N : |y| < 1,\ y_N < 0\right\},$$
$$E_2 := \left\{y \in \mathbb{R}^N : |y| < 1,\ y_N \leq 0\right\},$$
$$E_3 := \{y \in E_2 : y_N < 0 \text{ or } y_1 > 0\}.$$

A detailed characterization of Gröger-regular sets in two and three spatial dimensions is given in [10].

Assumption 2.3. *The function* $g : \mathbb{R} \to [\epsilon, 1]$ *satisfies* $g \in C^2(\mathbb{R})$ *and* $g', g'' \in L^\infty(\mathbb{R})$ *with some* $\epsilon > 0$. *With a little abuse of notation the Nemystkii-operators associated with* g *and* g'*, considered with different domains and ranges, will be denoted by the same symbol.*

Assumption 2.4. *The fourth-order tensor* $\mathbb{C} \in L^\infty(\Omega; \mathcal{L}(\mathbb{R}^{N\times N}_{\text{sym}}))$ *is symmetric and uniformly coercive, i.e., there is a constant* $\gamma_{\mathbb{C}} > 0$ *such that*

$$\mathbb{C}(x)\boldsymbol{\sigma} : \boldsymbol{\sigma} \geq \gamma_{\mathbb{C}} |\boldsymbol{\sigma}|^2 \quad \forall \boldsymbol{\sigma} \in \mathbb{R}^{N\times N}_{\text{sym}} \text{ and f.a.a. } x \in \Omega, \tag{2.1}$$

where $|\cdot|$ *denotes the Frobenius norm on* $\mathbb{R}^{N\times N}$ *and* $(\cdot : \cdot)$ *the scalar product inducing this norm.*

Assumption 2.5. *For the applied volume and boundary load we require*

$$\ell \in C^1([0,T]; \boldsymbol{W}^{-1,p}_D(\Omega)),$$

where $p > N$ *is specified below, see Assumption 2.7.1 and Assumption 5.4.*

Moreover, the initial damage is supposed to satisfy $d_0 \in L^2(\Omega)$.

Our last assumption concerns the balance of momentum associated with the energy functional in (1.1). While the aforementioned assumptions are comparatively mild, this condition is rather restrictive, at least in three spatial dimensions, see Remark 2.8 below. For its precise statement we need the following

Definition 2.6. *For given* $\varphi \in L^1(\Omega)$ *we define the linear form* $A_\varphi : V \to V^*$ *as*

$$\langle A_\varphi \boldsymbol{u}, v\rangle_V := \int_\Omega g(\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v})\, dx.$$

The operator A_φ *considered with different domains and ranges will be denoted by the same symbol for the sake of convenience.*

Assumption 2.7. *For the rest of the paper we require the following:*

1. *There exists* $p > N$ *such that, for all* $\overline{p} \in [2, p]$ *and all* $\varphi \in L^1(\Omega)$*, the operator* $A_\varphi : \boldsymbol{W}^{1,\overline{p}}_D(\Omega) \to \boldsymbol{W}^{-1,\overline{p}}_D(\Omega)$ *is continuously invertible. Moreover, there exists a constant* $c > 0$*, independent of* φ *and* $\overline{p}$*, such that*

$$\|A_\varphi^{-1}\|_{\mathcal{L}(\boldsymbol{W}^{-1,\overline{p}}_D(\Omega), \boldsymbol{W}^{1,\overline{p}}_D(\Omega))} \leq c$$

   holds for all $\overline{p} \in [2, p]$ *and all* $\varphi \in L^1(\Omega)$.
2. *The penalization parameter* β *is sufficiently large, depending only on* α, p*, and* N.

Remark 2.8. *The critical assumption is Assumption 2.7.1. If* $N = 2$*, then this condition is automatically fulfilled, see [17, Lemma 3.2]. The situation chances however, if one turns to* $N = 3$*. In this case this assumption can be guaranteed by imposing additional and rather restrictive conditions on the data, in particular on the ellipticity and boundedness constants associated with* $\mathbb{C}$ *and* g*, see [17, Remark 3.21] for more details. However, as explained in [17, Remark 3.22], one could alternatively modify the energy functional in* (1.1) *by replacing* $\|\nabla\varphi\|_2^2$ *with the* $H^{3/2}$*-seminorm. This* *would allow to drop Assumption 2.7.1 in the three dimensional case, too. However, we chose not to work with the $H^{3/2}$-seminorm, as the associated bilinear form is difficult to realize in numerical computations.*

As an immediate consequence of Assumption 2.7.1 and the regularity of ℓ in Assumption 2.5 one can introduce the following

Definition 2.9. *We define the operator $\mathcal{U} : [0,T] \times H^1(\Omega) \to \boldsymbol{W}^{1,p}_D(\Omega)$ by $\mathcal{U}(t,\varphi) := A_\varphi^{-1}\ell(t)$. We will frequently consider $\mathcal{U}$ with different range and domain (w.r.t. the second variable), but denote it by the same symbol.*

Thanks to Assumption 2.7.1 there exists a constant $c > 0$, independent of t and φ, such that

$$\|\mathcal{U}(t,\varphi)\|_{\boldsymbol{W}^{1,p}_D(\Omega)} \le c \quad \forall\,(t,\varphi) \in [0,T] \times H^1(\Omega), \tag{2.2}$$

which will be frequently used in the sequel.

In the rest of this section we recall some known results and definitions from the companion paper [17] that will be used in the upcoming analysis.

Lemma 2.10 (Lipschitz continuity of $\mathcal{U}$, [17, Proposition 3.7]). *Let $p > 2$ and $r \in \big[2p/(p-2), \infty\big]$ be given. Then there exists $L > 0$ such that for all $\varphi_1, \varphi_2 \in H^1(\Omega) \cap L^r(\Omega)$ and all $t_1, t_2 \in [0,T]$ it holds*

$$\|\mathcal{U}(t_1,\varphi_1) - \mathcal{U}(t_2,\varphi_2)\|_{\boldsymbol{W}^{1,\pi}_D(\Omega)} \le L(|t_1 - t_2| + \|\varphi_1 - \varphi_2\|_r), \tag{2.3}$$

where $1/\pi = 1/p + 1/r$.

Lemma 2.11 (Fréchet differentiability of $\mathcal{U}$, [17, Proposition 5.6]). *It holds $\mathcal{U} \in C^1([0,T] \times H^1(\Omega); V)$ and at all $t \in [0,T]$ and $\varphi, \delta\varphi \in H^1(\Omega)$ we have*

$$\partial_t \mathcal{U}(t,\varphi) = A_\varphi^{-1}\dot{\ell}(t) \in \boldsymbol{W}^{1,p}_D(\Omega), \tag{2.4a}$$

$$A_\varphi\big(\partial_\varphi \mathcal{U}(t,\varphi)(\delta\varphi)\big) = \operatorname{div}\big(g'(\varphi)(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi)\big) \quad \text{in } V^*, \tag{2.4b}$$

where $\operatorname{div} : L^2(\Omega; \mathbb{R}^{n\times n}_{sym}) \to V^*$ denotes the distributional divergence. Moreover, there exists a constant $c > 0$, independent of t and φ, such that

$$\|\partial_t \mathcal{U}(t,\varphi)\|_{\boldsymbol{W}^{1,p}_D(\Omega)} \le c \quad \forall\,(t,\varphi) \in [0,T] \times H^1(\Omega). \tag{2.5}$$

In order to state the Euler-Lagrange equations associated with the energy minimization in (P) let us further define the mappings $B : H^1(\Omega) \to H^1(\Omega)^*$ and $F : [0,T] \times H^1(\Omega) \to H^1(\Omega)^*$ by

$$\langle B\varphi, \psi\rangle_{H^1(\Omega)} := \int_\Omega \alpha \nabla\varphi \cdot \nabla\psi + \beta\varphi\psi \, dx, \qquad \varphi, \psi \in H^1(\Omega), \tag{2.6}$$

$$\langle F(t,\varphi), \psi\rangle_{H^1(\Omega)} := \frac{1}{2}\int_\Omega g'(\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi)) : \varepsilon(\mathcal{U}(t,\varphi))\psi \, dx, \quad \varphi, \psi \in H^1(\Omega). \tag{2.7}$$

Note that F is well defined because of the Sobolev embedding $H^1(\Omega) \in L^s(\Omega)$ with $s = 6$ for $N = 3$ and $s < \infty$ for $N = 2$ in combination with Assumption 2.7.1.

Lemma 2.12. *The mapping F possesses the following properties:*

- *[17, Eq. (3.30)] It is Lipschitzian in the following sense: For all $t_1, t_2 \in [0,T]$ and all $\varphi_1, \varphi_2, \psi \in H^1(\Omega)$ there holds*

$$|\langle F(t_1,\varphi_1) - F(t_2,\varphi_2), \psi\rangle_{H^1(\Omega)}| \leq C\big(\|\varphi_1 - \varphi_2\|_{\frac{2p}{p-2}} + |t_1 - t_2|\big)\|\psi\|_{\frac{2p}{p-2}}, \quad (2.8)$$

  with a constant $C > 0$ independent of $(t_i, \varphi_i)_{i=1,2}$.
- *[17, Lemma 5.9] It is continuously Fréchet differentiable from $(0,T) \times H^1(\Omega)$ to $H^1(\Omega)^*$, and for all $(t,\varphi) \in [0,T] \times H^1(\Omega)$ and all $(\delta t, \delta\varphi) \in \mathbb{R} \times H^1(\Omega)$ we have*

$$\begin{aligned}\langle F'(t,\varphi)(\delta t, \delta\varphi), z\rangle_{H^1(\Omega)} = \frac{1}{2}\int_\Omega g''(\varphi)(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi)) : \varepsilon(\mathcal{U}(t,\varphi))z\,dx \\ + \int_\Omega g'(\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi)) : \varepsilon\Big(\mathcal{U}'(t,\varphi)(\delta t,\delta\varphi)\Big)z\,dx\end{aligned} \quad (2.9)$$

  for all $z \in H^1(\Omega)$.
- *[17, Eq. (5.28)] For its partial derivative w.r.t. φ there holds*

$$|\langle \partial_\varphi F(t,\varphi)z, z\rangle_{H^1(\Omega)}| \leq k\|z\|_2^2 + \widetilde{c}(k)\|z\|^2_{H^1(\Omega)} \quad (2.10)$$

  for all $z \in H^1(\Omega)$ and all $k > 0$, where $\widetilde{c} : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotonically decreasing function, which tends to 0 as $k \to \infty$.

With the mappings B and F at hand we can characterize the solution to the energy minimization in (P) as follows:

Lemma 2.13 (Energy minimizer, [17, Prop. 3.13, Thm. 3.18]). *For every $(t,d) \in [0,T] \times L^2(\Omega)$, the optimization problem*

$$\min_{(\boldsymbol{u},\varphi)\in V\times H^1(\Omega)} \mathcal{E}(t,\boldsymbol{u},\varphi,d)$$

admits a unique minimizer $(\boldsymbol{u},\varphi) \in \boldsymbol{W}^{1,p}_D(\Omega) \times H^1(\Omega)$ characterized by $\boldsymbol{u} = \mathcal{U}(t,\varphi)$ and $\varphi = \Phi(t,d)$, where $\Phi : [0,T] \times L^2(\Omega) \to H^1(\Omega)$ is defined by $\Phi(t,d) := (B + F(t,\cdot))^{-1}(\beta d)$.

Lemma 2.14 (Fréchet differentiability of Φ, [17, Prop. 5.12]). *The solution operator Φ is continuously Fréchet differentiable from $(0,T) \times L^2(\Omega)$ to $H^1(\Omega)$. Moreover, for all $(t,d) \in [0,T]\times L^2(\Omega)$ and all $(\delta t, \delta d) \in \mathbb{R}\times L^2(\Omega)$ its derivative solves the following linearized equation*

$$B\Phi'(t,d)(\delta t,\delta d) + F'(t,\varphi)\big(\delta t, \Phi'(t,d)(\delta t,\delta d)\big) = \beta\delta d, \quad (2.11)$$

where we use the abbreviation $\varphi := \Phi(t,d)$.

Finally we turn our attention to the differential inclusion in (P). First note that the functional $\mathcal{E}$ is partially Fréchet differentiable w.r.t. d on $[0,T] \times V \times H^1(\Omega) \times L^2(\Omega)$, and its partial derivative is given by

$$\partial_d\mathcal{E}(t,\boldsymbol{u},\varphi,d) = \beta(d-\varphi). \quad (2.12)$$

Therefore, in view Lemma 2.13, (P) reduces to the following evolutionary equation

$$-\beta\big(d(t) - \Phi(t,d(t))\big) \in \partial\mathcal{R}_\delta(\dot{d}(t)) \quad \forall\, t \in [0,T], \quad d(0) = d_0. \quad (2.13)$$

As shown in [17, Lemma 3.23], this equation is equivalent to the following non-smooth operator differential equation:

$$\dot{d}(t) = \frac{1}{\delta} \max\{-\beta\big(d(t) - \Phi(t, d(t))\big) - r, 0\} \quad \forall\, t \in [0, T], \quad d(0) = d_0. \tag{2.14}$$

This handy reformulation of (P) and (2.13), respectively, is a main advantage of the penalty-type regularization of partial damage models and provides a useful starting point for a numerical solution of (P), see **hier ref???!!!** We end this section by recalling the main result of the companion paper:

Theorem 2.15 (Existence and uniqueness for the penalized damage model, [17, Thm. 5.13]). *There exists a unique solution $(\boldsymbol{u}, \varphi, d)$ of the problem (P), satisfying $\boldsymbol{u} \in C^1([0,T]; V)$, $\varphi \in C^1([0,T]; H^1(\Omega))$, $d \in C^{1,1}([0,T]; L^2(\Omega))$ and the following system of differential equations:*

$$-\operatorname{div} g(\varphi(t))\varepsilon(\boldsymbol{u}(t)) = \ell(t) \qquad \text{in } \boldsymbol{W}_D^{-1,p}(\Omega) \tag{2.15a}$$

$$-\alpha\Delta\varphi(t) + \beta\,\varphi(t) + \frac{1}{2}\,g'(\varphi(t))\mathbb{C}\,\varepsilon(\boldsymbol{u}(t)) : \varepsilon(\boldsymbol{u}(t)) = \beta d(t) \quad \text{in } H^1(\Omega)^* \tag{2.15b}$$

$$\dot{d}(t) - \frac{1}{\delta}\max\{-\beta(d(t) - \varphi(t)) - r, 0\} = 0, \qquad d(0) = d_0. \tag{2.15c}$$

for every $t \in [0, T]$.

Note that, thanks to the definition of B and F, the equations in (2.15a) and (2.15b) are just equivalent to $\boldsymbol{u}(t) = \mathcal{U}(t, \varphi(t))$ and $\varphi(t) = \Phi(t, d(t))$, respectively. Note further that, thanks to the uniqueness of the solution, (2.15a)–(2.15c) is uniquely solvable, too, and therefore equivalent to (P).

3. Energy Identity. As seen in Theorem 2.15, for a given $\beta > 0$ sufficiently large, there exists a unique local damage variable, which we denote by d_β to indicate its dependency on the parameter β. The other variables are uniquely determined by d_β through $\varphi_\beta = \Phi(\cdot, d_\beta(\cdot))$ and $\boldsymbol{u}_\beta = \mathcal{U}(\cdot, \varphi_\beta(\cdot))$. The purpose of this section is to derive a characterization of the local damage d_β, which allows to find an estimate of the form $\|d_\beta\|_X \leq C$ for all $\beta > 0$, where X is a suitable reflexive Banach space and $C > 0$ is a constant independent of the penalty parameter β. Such an estimate will then allow to pass to the limit in the penalized damage model as $\beta \to \infty$, see Section 4 below. As seen in (2.13) and (2.14) above, there are various ways to describe the evolution of the local damage. However, all these descriptions have the disadvantage of containing the term $\beta(d_\beta - \varphi_\beta)$, which is not necessarily uniformly bounded w.r.t. β in suitable spaces that allow a passage to the limit. Our aim is therefore to find an alternative description of the evolution of the local damage, which only contains expressions that are bounded w.r.t. β. Such a description is given by the *energy identity* in Proposition 3.5 below.

For the rest of this section we drop the index β to shorten the notation. As already indicated above, the displacement $\boldsymbol{u}$ and the nonlocal damage φ are uniquely determined by the local damage d so that it is reasonable to reduce the whole system to the variable d only. For this purpose we define the following:

Definition 3.1. *The reduced energy functional $\mathcal{I} : [0,T] \times L^2(\Omega) \to \mathbb{R}$ is given by*

$$\mathcal{I}(t, d) := \mathcal{E}(t, \mathcal{U}(t, \Phi(t, d)), \Phi(t, d), d).$$

The reduced energy functional will be a key ingredient for deriving the energy identity. On account of (1.1) and Definitions 2.6 and 2.9 it can be rewritten as

$$
\begin{aligned}
\mathcal{I}(t,d) &= \frac{1}{2}\langle A_{\Phi(t,d)}\big(\mathcal{U}(t,\Phi(t,d))\big), \mathcal{U}(t,\Phi(t,d))\rangle_V - \langle \ell(t), \mathcal{U}(t,\Phi(t,d))\rangle_V \\
&\quad + \frac{\alpha}{2}\|\nabla\Phi(t,d)\|_2^2 + \frac{\beta}{2}\|\Phi(t,d)-d\|_2^2 \\
&= -\frac{1}{2}\langle \ell(t), \mathcal{U}(t,\Phi(t,d))\rangle_V + \frac{\alpha}{2}\|\nabla\Phi(t,d)\|_2^2 + \frac{\beta}{2}\|\Phi(t,d)-d\|_2^2.
\end{aligned}
\tag{3.1}
$$

This reformulation of the reduced energy allows to show the following

Lemma 3.2 (Fréchet differentiability of $\mathcal{I}$). *It holds $\mathcal{I} \in C^1([0,T]\times L^2(\Omega))$ and, at all $(t,d) \in [0,T]\times L^2(\Omega)$, we have*

$$
\partial_t \mathcal{I}(t,d) = -\langle \dot\ell(t), \mathcal{U}(t,\Phi(t,d))\rangle_V, \quad \partial_d \mathcal{I}(t,d) = \beta(d-\Phi(t,d)). \tag{3.2}
$$

Proof. First note that the mapping

$$
f : [0,T]\times L^2(\Omega) \to \mathbb{R}, \quad f(t,d) := \langle \ell(t), \mathcal{U}(t,\Phi(t,d))\rangle_V
$$

can be seen as product of the functions ℓ and $[0,T]\times L^2(\Omega) \ni (t,d) \mapsto \mathcal{U}(t,\Phi(t,d)) \in V$. The latter one is continuously Fréchet differentiable, thanks to Lemmas 2.11 and 2.14. Together with Assumption 2.5 the product rule yields $f \in C^1([0,T]\times L^2(\Omega))$. Thus, thanks to Lemma 2.14, we deduce from (3.1) that $\mathcal{I} \in C^1([0,T]\times L^2(\Omega))$ and, for given $(t,d) \in [0,T]\times L^2(\Omega)$ and $(\delta t, \delta d) \in \mathbb{R}\times L^2(\Omega)$, it holds

$$
\begin{aligned}
\mathcal{I}'(t,d)(\delta t,\delta d) &= -\frac{1}{2}\langle \dot\ell(t)\delta t, \mathcal{U}(t,\Phi(t,d))\rangle_V - \frac{1}{2}\langle \ell(t), \mathcal{U}'(t,\Phi(t,d))(\delta t,\delta\varphi)\rangle_V \\
&\quad + \alpha(\nabla\Phi(t,d),\nabla\delta\varphi)_2 + \beta(\Phi(t,d)-d, \delta\varphi-\delta d)_2,
\end{aligned}
\tag{3.3}
$$

where we abbreviate $\delta\varphi = \Phi'(t,d)(\delta t,\delta d)$. To derive the formulas for the partial derivatives, first observe that (2.4a) tested with $\mathcal{U}(t,\varphi)$, Definitions 2.6 and 2.9, and the symmetry of $\mathbb{C}$ imply

$$
\begin{aligned}
\langle \dot\ell(t), \mathcal{U}(t,\Phi(t,d))\rangle_V &= \langle A_{\Phi(t,d)}\partial_t\mathcal{U}(t,\Phi(t,d)), \mathcal{U}(t,\Phi(t,d))\rangle_V \\
&= \langle \ell(t), \partial_t\mathcal{U}(t,\Phi(t,d))\rangle_V.
\end{aligned}
\tag{3.4}
$$

If one tests (2.4b) tested with $\mathcal{U}(t,\Phi(t,d)) \in V$, one further obtains

$$
\begin{aligned}
&-\frac{1}{2}\langle \ell(t), \partial_\varphi\mathcal{U}(t,\Phi(t,d))\delta\varphi\rangle_V + \alpha(\nabla\Phi(t,d),\nabla\delta\varphi)_2 + \beta(\Phi(t,d)-d,\delta\varphi-\delta d)_2 \\
&\quad = -\frac{1}{2}\langle \operatorname{div}\big(g'(\Phi(t,d))(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\Phi(t,d))\big), \mathcal{U}(t,\Phi(t,d))\rangle_V \\
&\qquad + \alpha(\nabla\Phi(t,d),\nabla\delta\varphi)_2 + \beta(\Phi(t,d)-d,\delta\varphi-\delta d)_2 \\
&\quad = \langle F\big(t,\Phi(t,d)\big) + B\Phi(t,d), \delta\varphi\rangle_{H^1(\Omega)} - \beta(d,\delta\varphi)_2 + \beta(d-\Phi(t,d),\delta d)_2 \\
&\quad = \beta(d-\Phi(t,d),\delta d)_2 \quad \forall\, \delta d \in L^2(\Omega),
\end{aligned}
\tag{3.5}
$$

where $\operatorname{div} : L^2(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}}) \to V^*$ denotes the distributional divergence. Note that the last two equalities follow from (2.6), (2.7), and the definition of Φ, respectively. Inserting (3.4) and (3.5) in (3.3) leads to (3.2). ∎

As an immediate consequence of Lemma 3.2 and the chain rule, one obtains the following

COROLLARY 3.3 (Total derivative of $\mathcal{I}(\cdot, d(\cdot))$). *Let $d \in C^1([0,T], L^2(\Omega))$ be given. Then the map $[0,T] \ni t \mapsto \mathcal{I}(t, d(t))$ is continuously differentiable with*

$$\frac{d}{dt}\mathcal{I}(t,d(t)) = \partial_t \mathcal{I}(t,d(t)) + (\partial_d \mathcal{I}(t,d(t)), \dot{d}(t))_2 \quad \forall\, t \in [0,T].$$

With the help of the reduced energy $\mathcal{I}$ we will deduce the energy identity from the evolutionary equation in (2.13). To this end note first that, due to the second equation in (3.2), the evolutionary equation (2.13) or equivalently (2.14) can also be written as

$$-\partial_d \mathcal{I}(t,d(t)) \in \partial \mathcal{R}_\delta(\dot{d}(t)) \quad \forall\, t \in [0,T], \quad d(0) = d_0. \tag{3.6}$$

Since $\mathcal{R}_\delta$ is proper and convex, this is in turn equivalent to

$$\mathcal{R}_\delta(\dot{d}(t)) + \mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(t,d(t))\big) = (-\partial_d \mathcal{I}(t,d(t)), \dot{d}(t))_2 \;\; \forall\, t \in [0,T], \quad d(0) = d_0, \tag{3.7}$$

which will be the starting point for proving the energy identity in Proposition 3.5 below. To summarize, we obtained the following four alternative, but yet equivalent formulations:

- the subdifferential formulations in (2.13) and (3.6), respectively,
- the nonsmooth operator differential equation in (2.14),
- Young's equation in (3.7).

In all what follows, we refer to these equivalent formulations simply as *penalized damage evolution*. Note that, since (2.13) and (2.14), respectively, are uniquely solvable by Theorem 2.15, the same holds for (3.6) and (3.7).

LEMMA 3.4. *If d satisfies the penalized damage evolution, then, for every $t \in [0,T]$, there holds $\mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(t,d(t))\big) = \frac{\delta}{2}\|\dot{d}(t)\|_2^2$.*

Proof. Let d satisfy the penalized damage evolution. Then it follows from (3.6) that $\partial \mathcal{R}_\delta(\dot{d}(t)) \neq \emptyset$ so that $\dot{d} \geq 0$. Hence, inserting (1.2) and (3.2) in (3.7) leads to

$$\mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(t,d(t))\big) = (-\beta(d(t)-\varphi(t)), \dot{d}(t))_2 - r\|\dot{d}(t)\|_1 - \frac{\delta}{2}\|\dot{d}(t)\|_2^2 \quad \forall\, t \in [0,T], \tag{3.8}$$

where we again abbreviated $\varphi = \Phi(\cdot, d(\cdot))$. From the equivalent formulation (2.14) multiplied with $\dot{d}(t)$ and integrated over Ω, we deduce that $\delta\|\dot{d}(t)\|_2^2 = (-\beta(d(t) - \varphi(t)), \dot{d}(t))_2 - r\|\dot{d}(t)\|_1$. Inserting this into (3.8) gives the assertion. ☐

PROPOSITION 3.5 (The energy identity). *The unique solution $d \in C^1([0,T]; L^2(\Omega))$ of the penalized damage evolution fulfills for all $0 \leq s \leq t \leq T$ the energy identity*

$$\begin{aligned}\int_s^t \mathcal{R}_\delta(\dot{d}(\tau))d\tau + \int_s^t \mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(\tau,d(\tau))\big)d\tau + \mathcal{I}(t,d(t)) \\ = \mathcal{I}(s,d(s)) + \int_s^t \partial_t \mathcal{I}(\tau,d(\tau))d\tau.\end{aligned} \tag{3.9}$$

Proof. Corollary 3.3 combined with (3.7) yields at all $\tau \in [0,T]$ the identity

$$\mathcal{R}_\delta(\dot{d}(\tau)) + \mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(\tau,d(\tau))\big) = \partial_t \mathcal{I}(\tau,d(\tau)) - \frac{d}{dt}\mathcal{I}(\tau,d(\tau)). \tag{3.10}$$

Recall that $\dot{d} \geq 0$ as a result of (2.14). This implies in view of (1.2) and $\dot{d} \in C([0,T], L^2(\Omega))$ that the map $[0,T] \ni \tau \mapsto \mathcal{R}_\delta(\dot{d}(\tau)) \in \mathbb{R}$ is continuous. From Lemma 3.4 and Corollary 3.3 we deduce the continuity w.r.t. time of all terms in (3.10) and therefore, the integrability thereof. Integrating (3.10) w.r.t. time then yields (3.9). □

Remark 3.6. *One can show that the reverse statement of Proposition 3.5 is also true such that the energy identity is actually just another equivalent formulation of the penalized damage evolution. To do so, one combines the energy identity* (3.9) *with Corollary 3.3 and Young's inequality and in this way obtains* (3.7)*, which was one of the equivalent formulations of the penalized damage evolution. However, for the upcoming analysis we only need the implication stated in Proposition 3.5 so that we do not go into more details.*

4. Limit Analysis. This section proves the viability of the penalty approach in the sense that one can pass to the limit $\beta \to \infty$ and in this way obtain a one-field damage model. In Section 5 below, we will see that the limit system is equivalent to a classical viscous partial damage model.

In the first part of this section we focus on finding bounds independent of β in suitable spaces for the local and nonlocal damage, respectively. Note that, for the displacement, such a bound is already given in (2.2). This allows us to find weakly convergent subsequences. The limiting behaviour thereof, as $\beta \to \infty$, is studied in the second and third part of this section.

4.1. Uniform Boundedness. The starting point for the derivation of bounds independent of β is the energy identity in Proposition 3.5. For this purpose we require the following additional assumption, which is rather self-evident in many practical applications:

Assumption 4.1. *From now on we assume that at the beginning of the process the body is completely sound, i.e.* $d_0 \equiv 0$*, and that there is no load acting upon the body at initial time, i.e.* $\ell(0) \equiv 0$.

As a first consequence of Assumption 4.1, we obtain in view of (2.15) that

$$\boldsymbol{u}(0) = \varphi(0) = \dot{d}(0) \equiv 0. \tag{4.1}$$

Lemma 4.2 (Boundedness of the local damage). *Let Assumption 4.1 hold. Then there exists a constant* $C > 0$*, independent of* β*, such that* $\|d\|_{H^1(0,T;L^2(\Omega))} \leq C$.

Proof. The result follows mainly from the energy identity in Proposition 3.5. In order to see this, set $s := 0$ and $t = T$ in (3.9) and use (1.2), Lemma 3.4, (3.1), and (3.2), as well as Assumption 4.1, and (4.1) to obtain

$$\begin{aligned}
\delta \int_0^T \|\dot{d}(\tau)\|_2^2 \, d\tau &= \frac{1}{2} \langle \ell(T), \boldsymbol{u}(T) \rangle_V + \int_0^T \langle -\dot{\ell}(\tau), \boldsymbol{u}(\tau) \rangle_V \, d\tau \\
&\quad - \Big(r \int_0^T \|\dot{d}(\tau)\|_1 \, d\tau + \frac{\alpha}{2} \|\nabla\varphi(T)\|_2^2 + \frac{\beta}{2} \|\varphi(T) - d(T)\|_2^2 \Big) \\
&\leq \int_0^T \|\dot{\ell}(\tau)\|_{V^*} \|\boldsymbol{u}(\tau)\|_V \, d\tau + \frac{1}{2} \|\ell(T)\|_{V^*} \|\boldsymbol{u}(T)\|_V \leq C
\end{aligned}$$

with $C > 0$ independent of β. The assertion then follows from $d_0 = 0$ and Poincaré-Friedrich's inequality. □

Next let us turn to the uniform boundedness of φ. We will establish the existence of a constant C independent of β such that $\|\varphi\|_{H^1(0,T;H^1(\Omega))} \leq C$. In view of (4.1) and Poincaré-Friedrich's inequality, we only need to show that there is $C > 0$ independent of β such that

$$\|\dot{\varphi}\|^2_{L^2(0,T;H^1(\Omega))} = \int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau + \int_0^T \|\nabla\dot{\varphi}(\tau)\|_2^2 \, d\tau \leq C. \tag{4.2}$$

The starting point herefor is the equation characterizing the time derivative of the nonlocal damage. In view of Lemmata 2.12 and 2.14 this equation is given by

$$B\dot{\varphi}(t) + \partial_t F(t, \varphi(t)) + \partial_\varphi F(t, \varphi(t))\dot{\varphi}(t) = \beta\dot{d}(t) \quad \text{in } H^1(\Omega)^*. \tag{4.3}$$

Testing (4.3) with $\dot{\varphi}(t)$, integrating over $[0, T]$, and using (2.6) lead to

$$\begin{aligned}\int_0^T \alpha\|\nabla\dot{\varphi}(\tau)\|_2^2 \, d\tau = \beta \overbrace{\int_0^T (\dot{d}(\tau) - \dot{\varphi}(\tau), \dot{\varphi}(\tau))_2 \, d\tau}^{=:\, I_1} \\ - \underbrace{\int_0^T \langle \partial_t F(t, \varphi(t)) + \partial_\varphi F(t, \varphi(t))\dot{\varphi}(t), \dot{\varphi}(t)\rangle \, d\tau}_{=:\, I_2}.\end{aligned} \tag{4.4}$$

Lemma 4.3. *Under Assumption 4.1 it holds* $I_1 \leq 0$.

Proof. From Theorem 2.15 we recall that $\dot{d}$ and φ are Lipschitz continuous, and therefore $\dot{d} \in W^{1,\infty}(0, T; L^2(\Omega))$. Hence, by [25, Theorem 3.1.40], the mapping $f : [0, T] \to L^2(\Omega)$ defined through

$$f(t) := \delta\dot{d}(t) + \beta(d(t) - \varphi(t)) + r \tag{4.5}$$

is almost everywhere differentiable. Let now $t \in (0, T)$ be arbitrary, but fixed and $h > 0$ sufficiently small such that $t+h \in (0, T)$. From (2.14) it follows that $\dot{d}(\tau, x) \geq 0$, $f(\tau, x) \geq 0$, and $f(\tau, x)\, \dot{d}(\tau, x) = 0$ for all $\tau \in [0, T]$ and almost all $x \in \Omega$. Thus we arrive at

$$\Big(\frac{f(t \pm h) - f(t)}{h}, \dot{d}(t)\Big)_2 \geq 0$$

Passing to the limit $h \searrow 0$ and keeping in mind the fact that f is almost everywhere differentiable implies $(\dot{f}(t), \dot{d}(t))_2 = 0$ f.a.a. $t \in (0, T)$. Thanks to (4.5) this is equivalent to $\delta(\ddot{d}(t), \dot{d}(t))_2 + (\beta(\dot{d}(t) - \dot{\varphi}(t)), \dot{d}(t))_2 = 0$, which can be continued as

$$\frac{\delta}{2}\frac{d}{dt}\|\dot{d}(t)\|_2^2 + \beta\|\dot{d}(t) - \dot{\varphi}(t))\|_2^2 + \beta(\dot{d}(t) - \dot{\varphi}(t), \dot{\varphi}(t))_2 = 0 \tag{4.6}$$

for almost all $t \in (0, T)$. Due to Theorem 2.15, $\dot{\varphi}$ and $\dot{d}$ are both continuous with values in $L^2(\Omega)$ so that we can integrate (4.6) over $[0, T]$. This finally yields

$$\frac{\delta}{2}\|\dot{d}(T)\|_2^2 - \frac{\delta}{2}\|\dot{d}(0)\|_2^2 + \beta\int_0^T \|\dot{d}(\tau) - \dot{\varphi}(\tau))\|_2^2 \, d\tau + \beta\int_0^T (\dot{d}(\tau) - \dot{\varphi}(\tau), \dot{\varphi}(\tau))_2 \, d\tau = 0,$$

which on account of (4.1) gives the assertion. □

Lemma 4.4. *For all* $k > 0$ *it holds*

$$|I_2| \leq \widehat{c}(k)\int_0^T \|\dot{\varphi}(\tau)\|^2_{H^1(\Omega)} \, d\tau + k\int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau + C\, k,$$

where $\widehat{c} : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotonically decreasing function, independent of β, which tends to 0 as $k \to \infty$ and $C > 0$ is a constant independent of β.

Proof. Let $t \in [0, T]$ be arbitrary, but fixed. From (2.9) we deduce

$$\langle \partial_t F(t, \varphi(t)), \dot{\varphi}(t) \rangle = -\langle \operatorname{div} \big(g'(\varphi(t)) \dot{\varphi}(t) \mathbb{C} \varepsilon(\boldsymbol{u}(t)) \big), \partial_t \mathcal{U}(t, \varphi(t)) \rangle_V .$$

Due to $p > N$ we have $H^1(\Omega) \hookrightarrow L^{\frac{2p}{p-2}}(\Omega)$ and thus, Hölder's inequality with $(p - 2)/2p + 1/p + 1/2 = 1$ in combination with (2.2) and (2.5) yields

$$\begin{aligned} \big| \langle \partial_t F(t, \varphi(t)), \dot{\varphi}(t) \rangle_{H^1(\Omega)} \big| &\le \| g'(\varphi(t)) \|_\infty \| \dot{\varphi}(t) \|_{\frac{2p}{p-2}} \| \boldsymbol{u}(t) \|_{\boldsymbol{W}^{1,p}_D(\Omega)} \| \partial_t \mathcal{U}(t, \varphi(t)) \|_V \\ &\le C \| \dot{\varphi}(t) \|_{H^1(\Omega)}, \end{aligned}$$

with $C > 0$ independent of β. On account of the generalized Young inequality, this can be continued as follows

$$\big| \langle \partial_t F(t, \varphi(t)), \dot{\varphi}(t) \rangle_{H^1(\Omega)} \big| \le \frac{1}{4k} \| \dot{\varphi}(t) \|^2_{H^1(\Omega)} + Ck \quad \forall\, k > 0. \tag{4.7}$$

Together with (2.10) and the definition of I_2 in (4.4), this gives the assertion with $\widehat{c}(k) = \frac{1}{4k} + \widetilde{c}(k)$ so that $\widehat{c}$ is indeed independent of β, monotonically decreasing and tends to 0 as $k \to \infty$. □

LEMMA 4.5 (Boundedness of the gradient). *Let Assumption 4.1 hold. Then there exist constants $C_1, C_2 > 0$, independent of β, such that*

$$\int_0^T \| \nabla \dot{\varphi}(\tau) \|_2^2 \, d\tau \le C_1 \int_0^T \| \dot{\varphi}(\tau) \|_2^2 \, d\tau + C_2.$$

Proof. Applying Lemmata 4.3 and 4.4 to the right hand side in (4.4) yields

$$(\alpha - \widehat{c}(k)) \int_0^T \| \nabla \dot{\varphi}(\tau) \|_2^2 \, d\tau \le (\widehat{c}(k) + k) \int_0^T \| \dot{\varphi}(\tau) \|_2^2 \, d\tau + C\, k$$

for all $k > 0$. Since $\widehat{c}$ tends to zero as $k \to \infty$, there is a $K > 0$ such that $\widehat{c}(K) < \alpha$ holds. Choosing $k = K$ thus yields the assertion. Note that K does not depend on β, since $\widehat{c}$ is independent of β. □

LEMMA 4.6 (Boundedness of the L^2-component). *Let Assumption 4.1 hold. Then, for $\beta > 0$ sufficiently large, there holds $\| \dot{\varphi} \|_{L^2(0,T;L^2(\Omega))} \le C$ with a constant $C > 0$ independent of β.*

Proof. From (4.4) and Lemma 4.3 we deduce

$$\begin{aligned} \int_0^T \| \dot{\varphi}(\tau) \|_2^2 \, d\tau &\le \int_0^T (\dot{d}(\tau), \dot{\varphi}(\tau))_2 \, d\tau - \frac{1}{\beta} I_2 + I_1 - \frac{\alpha}{\beta} \int_0^T \| \nabla \varphi(\tau) \|_2^2 \, d\tau \\ &\le \int_0^T (\dot{d}(\tau), \dot{\varphi}(\tau))_2 \, d\tau - \frac{1}{\beta} I_2. \end{aligned} \tag{4.8}$$

Young inequality implies for the first term on the right hand side in (4.8) that

$$\int_0^T (\dot{d}(\tau), \dot{\varphi}(\tau))_2 \, d\tau \le \frac{1}{2} \int_0^T \| \dot{d}(\tau) \|_2^2 \, d\tau + \frac{1}{2} \int_0^T \| \dot{\varphi}(\tau) \|_2^2 \, d\tau. \tag{4.9}$$

In order to estimate the second term on the right hand side in (4.8), we apply Lemma 4.4 for some fixed $k > 0$, which thanks to Lemma 4.5 gives

$$\frac{1}{\beta}\,|I_2| \leq \frac{C_1}{\beta}\int_0^T \|\dot{\varphi}(\tau)\|_2^2\,d\tau + \frac{C_2}{\beta} \tag{4.10}$$

with constants C_1 and C_2 independent of β on account of Lemmata 4.4 and 4.5. Inserting (4.9) and (4.10) in (4.8) then implies

$$\int_0^T \|\dot{\varphi}(\tau)\|_2^2\,d\tau \leq \frac{1}{2}\int_0^T \|\dot{d}(\tau)\|_2^2\,d\tau + \Big(\frac{1}{2} + \frac{C_1}{\beta}\Big)\int_0^T \|\dot{\varphi}(\tau)\|_2^2\,d\tau + \frac{C_2}{\beta}. \tag{4.11}$$

Now, for β sufficiently large such that $C_1/\beta < 1/2$, the assertion follows from Lemma 4.2. □

As a consequence of Lemmata 4.5 and 4.6 and Poincaré-Friedrich's inequality together with (4.1) we can now state the main result of this section:

Corollary 4.7 (Boundedness of the nonlocal damage). *Under Assumption 4.1 there exists a $\beta_0 > 0$ and a constant $C > 0$ such that $\|\varphi\|_{H^1(0,T;H^1(\Omega))} \leq C$ for all $\beta \geq \beta_0$.*

4.2. Passing to the Limit in the Elliptic System. We start our limit analysis with the elliptic system in (2.15a) and (2.15b). In order to emphasize the dependency on the penalty parameter, we do not longer suppress the index β and denote the unique solution of (P) and (2.15a)–(2.15c), respectively, by $(\boldsymbol{u}_\beta, \varphi_\beta, d_\beta)$.

Proposition 4.8 (Passing to the limit in (2.15a)). *Let Assumption 4.1 hold. Then, for every sequence $\beta_n \to \infty$, there exist a (not relabeled) subsequence $\{\varphi_{\beta_n}\}_{n\in\mathbb{N}}$ such that*

$$\varphi_{\beta_n} \rightharpoonup \varphi \qquad \text{in } H^1(0,T;H^1(\Omega)), \tag{4.12}$$

$$\boldsymbol{u}_{\beta_n} = \mathcal{U}(\cdot, \varphi_{\beta_n}(\cdot)) \to \mathcal{U}(\cdot, \varphi(\cdot)) =: \boldsymbol{u} \quad \text{in } C([0,T];V) \tag{4.13}$$

as $n \to \infty$.

Proof. Since $H^1(0,T;H^1(\Omega))$ is a reflexive Banach space, Corollary 4.7 implies the existence of a (not relabeled) subsequence of $\{\varphi_{\beta_n}\}_{n\in\mathbb{N}}$ such that (4.12) holds.

To prove the second assertion, we apply Lemma 2.10 with $\pi = 2$. Then the number r in Lemma 2.10 is given by $r = 2p/(p-2)$ and (2.3) implies that the mapping $\mathcal{U}_c : C([0,T];L^r(\Omega) \ni \varphi \mapsto \mathcal{U}(\cdot,\varphi(\cdot)) \in C([0,T];V)$ is Lipschitz continuous with constant $L > 0$. Note that L is independent of β, since β does not appear in the elliptic equation (2.15a) associated with $\mathcal{U}$. Now, since $p > 2$ by Assumption 2.7.1, the embedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact, which implies that $H^1(0,T;H^1(\Omega)) \hookrightarrow\hookrightarrow C([0,T];L^r(\Omega))$ is a compact as well, cf. [25, Corollary 3.1.42]. Consequently (4.12) leads to $\varphi_{\beta_n} \to \varphi$ in $C([0,T];L^r(\Omega))$ and the Lipschitz continuity of $\mathcal{U}_c$ then gives (4.13). □

For the rest of this section we denote by $\{\beta_n\}_{n\in\mathbb{N}}$ a fixed sequence such that $\{\varphi_{\beta_n}\}$ converges weakly in $H^1(0,T;H^1(\Omega))$ and by φ the limit of this particular sequence. Proposition 4.8 guarantees the existence of such a sequence. Notice however that (at this point) φ depends on the chosen subsequence. Nevertheless, as we will see in Proposition 5.7 below, under a (rather restrictive) regularity condition on the elliptic operator A_φ, the weak limit is unique so that the whole sequence converges weakly.

Proposition 4.9 (Passing to the limit in (2.15b)). *Let Assumption 4.1 hold and let $\{\beta_n\}_{n\in\mathbb{N}}$ be the subsequence from Proposition 4.8 and φ the corresponding limit.*

Then there holds

$$d_{\beta_n} \rightharpoonup \varphi \text{ in } H^1(0,T;L^2(\Omega)) \quad \text{as } n \to \infty,$$

which implies in particular that both damage variables coincide in the limit.

Proof. First of all, Lemma 4.2 yields the existence of a subsequence $\{\beta_{n_k}\}_{k\in\mathbb{N}}$ so that

$$d_{\beta_{n_k}} \rightharpoonup d \text{ in } H^1(0,T;L^2(\Omega)) \quad \text{as } k \to \infty. \tag{4.14}$$

Now, let $t \in [0,T]$ and $\psi \in H^1(\Omega)$ be arbitrary, but fixed. Testing (2.15b) with ψ gives the following estimate, where we use Corollary 4.7, the boundedness of g' from Assumption 2.3, and (2.2):

$$\begin{aligned}
&\int_\Omega \big(d_{\beta_n}(t) - \varphi_{\beta_n}(t)\big)\psi \, dx \\
&\quad\leq \frac{1}{\beta_n}\big(\alpha\|\nabla\varphi_{\beta_n}(t)\|_2 + \|g'(\varphi_{\beta_n}(t))\|_\infty \|\mathbb{C}\varepsilon(\boldsymbol{u}_{\beta_n}(t)) : \varepsilon(\boldsymbol{u}_{\beta_n}(t))\|_{\frac{p}{2}}\big)\|\psi\|_{H^1(\Omega)} \qquad (4.15)\\
&\quad\leq \frac{C}{\beta_n}\|\psi\|_{H^1(\Omega)} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Note that $H^1(0,T;H^1(\Omega)) \hookrightarrow C([0,T];H^1(\Omega))$, which yields the boundedness of $\|\nabla\varphi_{\beta_n}(t)\|_2$ uniformly in t. Of course the above estimate also holds for the subsequence $\{\beta_{n_k}\}_{k\in\mathbb{N}}$ and hence, (4.14) and the convergence of $\{\varphi_n\}$ by assumption imply

$$\int_\Omega (d(t) - \varphi(t))\psi \, dx \leq 0,$$

and, since t and ψ were arbitrary, this gives in turn $d(t) = \varphi(t)$ for all $t \in [0,T]$. Hence, we obtain $d_{\beta_{n_k}} \rightharpoonup \varphi$ in $H^1(0,T;L^2(\Omega))$ as $k \to \infty$. Thus the weak limit is unique and a well known argument implies the convergence of the whole sequence $\{d_{\beta_n}\}$ to φ. ☐

4.3. Passing to the Limit in the Energy Identity.

We no turn our attention to the passage to the limit in (2.15b). However, as already indicated at the beginning of Section 3, the term $\beta(d-\varphi)$ involved in (2.15b) is not bounded in suitable spaces that allow a passage to the limit. Passing to the limit therein will result in an energy inequality, which turns out to be equivalent to an evolutionary equation as shown in Section 5.

We begin by introducing the energy and dissipation functionals that will arise after passing to the limit. The energy without penalty term reads as follows

Definition 4.10 (Energy functionals without penalty). *We define the energy functional without penalty term by*

$$\begin{aligned}
&\widetilde{\mathcal{E}} : [0,T] \times V \times H^1(\Omega) \to \mathbb{R},\\
&\widetilde{\mathcal{E}}(t,\boldsymbol{u},\varphi) := \frac{1}{2}\int_\Omega g(\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u}) \, dx - \langle \ell(t), \boldsymbol{u}\rangle_V + \frac{\alpha}{2}\|\nabla\varphi\|_2^2.
\end{aligned}$$

The reduced energy functional without penalty is given by

$$\widetilde{\mathcal{I}} : [0,T] \times H^1(\Omega) \to \mathbb{R}, \quad \widetilde{\mathcal{I}}(t,\varphi) := \widetilde{\mathcal{E}}(t,\mathcal{U}(t,\varphi),\varphi).$$

Remark 4.11. *Since the expressions in the definition of $\widetilde{\mathcal{E}}$ involving the displacement are not affected by the penalty term, it can be shown completely analogously to [17, Proposition 3.4] that, for a given pair $(t,\varphi) \in [0,T] \times H^1(\Omega)$, $\boldsymbol{u}$ solves*

$$\min_{\boldsymbol{u}\in V} \widetilde{\mathcal{E}}(t,\boldsymbol{u},\varphi),$$

iff $\boldsymbol{u} = \mathcal{U}(t,\varphi)$ with $\mathcal{U}$ as defined in Definition 2.9. As a consequence we obtain $\widetilde{\mathcal{I}}(t,\varphi) = \min_{\boldsymbol{u}\in V} \widetilde{\mathcal{E}}(t,\boldsymbol{u},\varphi)$.

Completely analogous to (3.1), the definitions of $\widetilde{\mathcal{E}}$ and $\mathcal{U}$ allow to rewrite the reduced energy functional without penalty as

$$\widetilde{\mathcal{I}}(t,\varphi) = -\frac{1}{2}\langle \ell(t), \mathcal{U}(t,\varphi)\rangle_V + \frac{\alpha}{2}\|\nabla\varphi\|_2^2. \tag{4.16}$$

Lemma 4.12 (Fréchet differentiability of $\widetilde{\mathcal{I}}$). *It holds $\widetilde{\mathcal{I}} \in C^1([0,T]\times H^1(\Omega))$ and its partial derivatives read*

$$\partial_t\widetilde{\mathcal{I}}(t,\varphi) = -\langle \dot{\ell}(t), \mathcal{U}(t,\varphi)\rangle_V, \quad \partial_\varphi\widetilde{\mathcal{I}}(t,\varphi) = -\alpha\triangle\varphi + F(t,\varphi), \tag{4.17}$$

where $\triangle : H^1(\Omega) \to H^1(\Omega)^*$ denotes the distributional Laplace operator.

Proof. The proof is completely along the lines of the proof of Lemma 3.2 so that we shorten the depiction. By applying the product rule to (4.16), one obtains that $\mathcal{I}$ is indeed continuously Fréchet-differentiable with

$$\mathcal{I}'(t,\varphi)(\delta t,\delta\varphi) = -\frac{1}{2}\langle \dot{\ell}(t)\delta t, \mathcal{U}(t,\varphi)\rangle_V - \frac{1}{2}\langle \ell(t), \mathcal{U}'(t,\varphi)(\delta t,\delta\varphi)\rangle_V + \alpha(\nabla\varphi,\nabla\delta\varphi)_2.$$

Similarly to (3.4) and (3.5), the first to addends can be reformulated by using (2.4a), the symmetry of $\mathbb{C}$, (2.4b), and the definition of F to obtain

$$\frac{1}{2}\langle \dot{\ell}(t)\delta t, \mathcal{U}(t,\varphi)\rangle_V + \frac{1}{2}\langle \ell(t), \mathcal{U}'(t,\varphi)(\delta t,\delta\varphi)\rangle_V = \langle \dot{\ell}(t)\delta t, \mathcal{U}(t,\varphi)\rangle_V - \langle F(t,\varphi),\delta\varphi\rangle_{H^1(\Omega)},$$

which completes the proof. ◻

Next we introduce the viscous dissipation functional corresponding to the situation without penalty:

Definition 4.13 (Viscous dissipation functional without penalty). *We define the functional $\widetilde{\mathcal{R}}_\delta$ by*

$$\widetilde{\mathcal{R}}_\delta : H^1(\Omega) \to [0,\infty], \quad \widetilde{\mathcal{R}}_\delta(\eta) := \begin{cases} r\int_\Omega \eta\, dx + \frac{\delta}{2}\|\eta\|_2^2 & \text{if } \eta \geq 0 \text{ a.e. in } \Omega, \\ \infty & \text{otherwise.} \end{cases}$$

Note that $\widetilde{\mathcal{R}}_\delta$ coincides with $\mathcal{R}_\delta$ from (1.2) apart from its domain which is now $H^1(\Omega)$ instead of $L^2(\Omega)$.

In order to pass to the limit in (3.9) we consider a sequence $\beta_n \to \infty$ such that

$$\varphi_{\beta_n} \rightharpoonup \varphi \quad \text{in } H^1(0,T;H^1(\Omega)), \tag{4.18}$$

$$d_{\beta_n} \rightharpoonup \varphi \quad \text{in } H^1(0,T;L^2(\Omega)), \tag{4.19}$$

$$\boldsymbol{u}_{\beta_n} \to \mathcal{U}(\cdot,\varphi(\cdot)) \quad \text{in } C([0,T];V). \tag{4.20}$$

Recall that such a sequence exists according to Propositions 4.8 and 4.9.

LEMMA 4.14. *Under Assumption 4.1 it holds for all $t \in [0,T]$ that*

$$\int_0^t \widetilde{\mathcal{R}}_\delta(\dot\varphi(\tau))\, d\tau \le \liminf_{n\to\infty} \int_0^t \mathcal{R}_\delta(\dot d_{\beta_n}(\tau))\, d\tau.$$

Proof. Let $t \in (0,T)$ be arbitrary, but fixed. From (4.19) it follows that

$$\dot d_{\beta_n} \rightharpoonup \dot\varphi \quad \text{in } L^2(0,t;L^2(\Omega)) \tag{4.21}$$

so that $\dot d_{\beta_n} \ge 0$ a.e. in $\Omega \times (0,t)$, see (2.15c), implies $\dot\varphi \ge 0$ a.e. in $\Omega \times (0,t)$ by the weak closedness of the set of non-negative functions in $L^2(0,t;L^2(\Omega))$. Thus Cavalieri's principle implies

$$\int_0^t \widetilde{\mathcal{R}}_\delta(\dot\varphi(\tau))\, d\tau = r\, \|\dot\varphi\|_{L^1(0,t;L^1(\Omega))} + \frac{\delta}{2}\, \|\dot\varphi\|^2_{L^2(0,t;L^2(\Omega))}$$

and the same obviously holds for $\mathcal{R}_\delta(\dot d_{\beta_n}(\tau))$, cf. (1.2). The result then follows from the weak lower semicontinuity of (squared) norms. ∎

LEMMA 4.15. *Let Assumption 4.1 hold. Then for all $t \in [0,T]$ we have*

$$\partial_d \mathcal{I}(t, d_{\beta_n}(t)) \rightharpoonup \partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t)) \ \textit{in } H^1(\Omega)^* \quad \textit{as } n \to \infty.$$

Proof. Let $t \in [0,T]$ be arbitrary, but fixed and set again $r = 2p/(p-2)$. As explained at the end of the proof of Proposition 4.8, Assumption 2.7.1 implies the compact embedding $H^1(0,T;H^1(\Omega)) \hookrightarrow\hookrightarrow C([0,T];L^r(\Omega))$ so that (4.18) results in

$$\varphi_{\beta_n}(t) \to \varphi(t) \text{ in } L^r(\Omega) \quad \text{for } n \to \infty. \tag{4.22}$$

Furthermore, since $H^1(0,T;H^1(\Omega)) \hookrightarrow C([0,T];H^1(\Omega))$, one finds for an arbitrary, but fixed $\omega \in L^2(\Omega;\mathbb{R}^d)$ that the functional

$$H^1(0,T;H^1(\Omega)) \ni v \mapsto \int_\Omega \omega \cdot \nabla v(t)\, dx \in \mathbb{R}$$

is linear and continuous so that (4.18) implies

$$\nabla\varphi_{\beta_n}(t) \rightharpoonup \nabla\varphi(t) \text{ in } L^2(\Omega) \quad \text{for } n \to \infty. \tag{4.23}$$

From (3.2) and (2.15b) we moreover deduce

$$\partial_d \mathcal{I}(t, d_{\beta_n}(t)) = \beta_n(d_{\beta_n}(t) - \varphi_{\beta_n}(t)) = -\alpha \triangle \varphi_{\beta_n}(t) + F(t,\varphi_{\beta_n}).$$

Together with (4.17) and (2.8) this yields for every $v \in H^1(\Omega)$ that

$$\begin{aligned} &|\langle \partial_d \mathcal{I}(t, d_{\beta_n}(t)) - \partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t)), v\rangle_{H^1(\Omega)}| \\ &\qquad \le \alpha |(\nabla\varphi_{\beta_n}(t) - \nabla\varphi(t), \nabla v)_2| + |\langle F(t,\varphi_{\beta_n}) - F(t,\varphi), v\rangle| \\ &\qquad \le \alpha |(\nabla\varphi_{\beta_n}(t) - \nabla\varphi(t), \nabla v)_2| + C\, \|\varphi_{\beta_n}(t) - \varphi(t)\|_r \|v\|_r. \end{aligned}$$

The result then follows from (4.22), (4.23), and $H^1(\Omega) \hookrightarrow L^r(\Omega)$ by Assumption 2.7.1. ☐

Lemma 4.16. *Under Assumption 4.1 it holds for all $t \in [0,T]$*

$$\int_0^t \widetilde{\mathcal{R}}_\delta^*\big(-\partial_\varphi \widetilde{\mathcal{I}}(\tau,\varphi(\tau))\big)\,d\tau \le \liminf_{n\to\infty} \int_0^t \mathcal{R}_\delta^*(-\partial_d \mathcal{I}(\tau, d_{\beta_n}(\tau)))\,d\tau.$$

Proof. Again, let $t \in [0,T]$ by arbitrary, but fixed. By definition of the Fenchel-conjugate, it holds for any $\xi \in L^2(\Omega)$ that

$$\widetilde{\mathcal{R}}_\delta^*(\xi) = \sup_{v\in H^1(\Omega)} \big((\xi,v)_2 - \widetilde{\mathcal{R}}_\delta(v)\big) \le \sup_{v\in L^2(\Omega)} \big((\xi,v)_2 - \mathcal{R}_\delta(v)\big) = \mathcal{R}_\delta^*(\xi). \tag{4.24}$$

Notice that we used in the above estimate that $\mathcal{R}_\delta$ and $\widetilde{\mathcal{R}}_\delta$ are defined with different domains, see (1.2) and Definition 4.13. Further, $\widetilde{\mathcal{R}}_\delta^* : H^1(\Omega)^* \to (-\infty,\infty]$ is convex and lower semicontinuous and thus weakly lower semicontinuous, which thanks to Lemma 4.15 leads to

$$\widetilde{\mathcal{R}}_\delta^*\big(-\partial_\varphi \widetilde{\mathcal{I}}(\tau,\varphi(\tau))\big) \le \liminf_{n\to\infty} \widetilde{\mathcal{R}}_\delta^*\big(-\partial_d \mathcal{I}(\tau, d_{\beta_n}(\tau))\big) \quad \forall\, \tau \in [0,t].$$

By setting $\xi := -\partial_d \mathcal{I}(\tau, d_{\beta_n}(\tau)) \in L^2(\Omega)$, see (3.2), in (4.24), the above estimate can be continued as

$$\widetilde{\mathcal{R}}_\delta^*\big(-\partial_\varphi \widetilde{\mathcal{I}}(\tau,\varphi(\tau))\big) \le \liminf_{n\to\infty} \mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(\tau, d_{\beta_n}(\tau))\big) = \liminf_{n\to\infty} \frac{\delta}{2}\,\|\dot{d}_{\beta_n}(\tau)\|_2^2 \tag{4.25}$$

for all $\tau \in [0,t]$, where the last equation follows from Lemma 3.4. Applying Fatou's lemma to the right hand side gives

$$\int_0^t \liminf_{n\to\infty} \mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(\tau, d_{\beta_n}(\tau))\big)\,d\tau \le \liminf_{n\to\infty} \int_0^t \mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(\tau, d_{\beta_n}(\tau))\big)\,d\tau. \tag{4.26}$$

Furthermore, arguing analogously to the derivation of (4.23), one sees that (4.19) implies $d_{\beta_n}(\tau) \rightharpoonup \varphi(\tau)$ in $L^2(\Omega)$ for every $\tau \in [0,T]$. Thus (4.25) shows that $\widetilde{\mathcal{R}}_\delta^*\big(-\partial_\varphi \widetilde{\mathcal{I}}(\tau,\varphi(\tau))\big)$ is finite for every τ. In addition, due to (4.17) and $H^1(0,T;H^1(\Omega)) \hookrightarrow C([0,T];H^1(\Omega))$, the map $[0,t] \ni \tau \mapsto \partial_\varphi \widetilde{\mathcal{I}}(\tau,\varphi(\tau)) \in H^1(\Omega)^*$ is continuous. Since $\widetilde{\mathcal{R}}_\delta^*$ is lower semicontinuous, it thus follows that the mapping

$$[0,t] \ni \tau \mapsto \widetilde{\mathcal{R}}_\delta^*(-\partial_\varphi \widetilde{\mathcal{I}}(\tau,\varphi(\tau))) \in \mathbb{R}$$

is lower semicontinuous as well and therefore, measurable. Now we can integrate (4.25) over $(0,t)$, which combined with (4.26) finally gives the assertion. ☐

Proposition 4.17 (The energy inequality without penalty). *Let Assumption 4.1 hold. Then the limit function $\varphi \in H^1(0,T;H^1(\Omega))$ fulfills for all $t \in [0,T]$ the estimate*

$$\begin{aligned}\int_0^t \widetilde{\mathcal{R}}_\delta(\dot{\varphi}(\tau))\,d\tau + \int_0^t \widetilde{\mathcal{R}}_\delta^*\big(-\partial_\varphi \widetilde{\mathcal{I}}(\tau,\varphi(\tau))\big)\,d\tau + \widetilde{\mathcal{I}}(t,\varphi(t))& \\ \le \widetilde{\mathcal{I}}(0,\varphi(0)) + \int_0^t \partial_t \widetilde{\mathcal{I}}(\tau,\varphi(\tau))\,d\tau.&\end{aligned} \tag{4.27}$$

Proof. Let $t \in [0,T]$ be arbitrary, but fixed. Setting $s := 0$ in (3.9) yields

$$\begin{aligned}\int_0^t \mathcal{R}_\delta(\dot d_{\beta_n}(\tau))\, d\tau + \int_0^t \mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(\tau, d_{\beta_n}(\tau))\big)\, d\tau + \mathcal{I}(t, d_{\beta_n}(t))& \\ = \mathcal{I}(0, d_{\beta_n}(0)) + \int_0^t \partial_t \mathcal{I}\big(\tau, d_{\beta_n}(\tau)\big)\, d\tau \quad \forall\, n \in \mathbb{N}.&\end{aligned} \tag{4.28}$$

In view of Lemmas 4.14 and 4.16 we only need to discuss the last three terms in the above equation. To this end, we combine (3.1) and (4.16) with (4.20), (4.23), and the weakly lower semicontinuity of $\|\cdot\|_2^2$, which gives

$$\begin{aligned}\widetilde{\mathcal{I}}(t,\varphi(t)) &= -\frac{1}{2}\langle \ell(t), \mathcal{U}(t,\varphi(t))\rangle_V + \frac{\alpha}{2}\|\nabla\varphi(t)\|_2^2 \\ &\le \liminf_{n\to\infty} \Big(-\frac{1}{2}\langle \ell(t), \boldsymbol{u}_{\beta_n}(t)\rangle_V + \frac{\alpha}{2}\|\nabla\varphi_{\beta_n}(t)\|_2^2 + \underbrace{\frac{\beta_n}{2}\|\varphi_{\beta_n}(t) - d_{\beta_n}(t)\|_2^2}_{\ge 0}\Big) \\ &= \liminf_{n\to\infty} \mathcal{I}(t, d_{\beta_n}(t)),\end{aligned}$$

i.e., the desired convergence of the last term on the left hand side of (4.28). It remains to discuss the right hand side in (4.28). Thanks to Assumption 4.1 and (4.1) the initial value just vanishes, i.e.,

$$\mathcal{I}(0, d_{\beta_n}(0)) = 0 \quad \forall\, n \in \mathbb{N}, \tag{4.29}$$

and, in light of (4.16), $\ell(0) = 0$ by Assumption 4.1, and the pointwise convergence in (4.23), which gives $\nabla\varphi(0) = 0$, we obtain the same for the limit, i.e., $\widetilde{\mathcal{I}}(0,\varphi(0)) = 0$. Therefore, the formulas for the partial derivatives of $\mathcal{I}$ and $\widetilde{\mathcal{I}}$ in (3.2) and (4.17) together with the regularity of ℓ and the convergence of the displacement in (4.20) finally implies

$$\begin{aligned}\mathcal{I}(0, d_{\beta_n}(0)) &+ \int_0^t \partial_t \mathcal{I}(\tau, d_{\beta_n}(\tau))\, d\tau \\ &= \int_0^t \langle -\dot\ell(\tau), \boldsymbol{u}_{\beta_n}(\tau)\rangle\, d\tau \to \int_0^t \langle -\dot\ell(\tau), \mathcal{U}(\tau,\varphi(\tau))\rangle\, d\tau \\ &= \widetilde{\mathcal{I}}(0,\varphi(0)) + \int_0^t \partial_t \widetilde{\mathcal{I}}(\tau,\varphi(\tau))\, d\tau\end{aligned}$$

which completes the proof. ◻

In the next sections we use the energy inequality in (4.27) to show that the limit in (4.18)–(4.20) satisfies a system of equations which is equivalent to a classical viscous partial damage model containing only one single damage variable. As secondary result we will also see that the inequality (4.27) is in fact equivalent to an energy identity, see Remark 5.2 below.

5. A Single-Field Gradient Damage Model.

In this section we show that every solution of the energy inequality (4.27) satisfies an evolutionary equation and vice versa. The proof mainly follows the arguments of [13, Proposition 3.2].

Proposition 5.1. *Let Assumption 4.1 hold. Then any $\varphi \in H^1(0,T;H^1(\Omega))$, which fulfills for all $t \in [0,T]$ the energy inequality (4.27), also satisfies the following evolutionary equation*

$$-\partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t)) \in \partial\widetilde{\mathcal{R}}_\delta(\dot\varphi(t)) \quad \textit{f.a.a. } t \in (0,T). \tag{5.1}$$

The reverse assertion is true as well.

Proof. We start the proof with two auxiliary results needed for both implications stated in the Proposition. To this end let $\varphi \in H^1(0,T;H^1(\Omega))$ first be arbitrary, but fixed. Since $\widetilde{\mathcal{R}}_\delta$ is convex and proper, a classical result from convex analysis result leads to

$$\widetilde{\mathcal{R}}_\delta(\dot\varphi(t)) + \widetilde{\mathcal{R}}_\delta^*(-\partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t))) = -\langle \partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t)), \dot\varphi(t)\rangle_{H^1(\Omega)} \tag{5.2}$$

$$\Longleftrightarrow$$

$$-\partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t)) \in \partial \widetilde{\mathcal{R}}_\delta(\dot\varphi(t)) \tag{5.3}$$

Further note that (4.17), combined with (2.7), (2.2), and the boundedness assumption on g', implies

$$\|\partial_\varphi \widetilde{\mathcal{I}}(t,v)\|_{H^1(\Omega)^*} \le C\,\|v\|_{H^1(\Omega)} + c \quad \forall\,(t,v) \in [0,T]\times H^1(\Omega), \tag{5.4}$$

where $C, c > 0$ are independent of (t,v). By using the density of $C^1([0,T];H^1(\Omega))$ in $H^1(0,T;H^1(\Omega))$ **REF!!!** and the continuous Fréchet differentiability of $\widetilde{\mathcal{I}}$ by Lemma 4.12, as well as (5.4), one shows that the function $[0,T] \ni t \mapsto \widetilde{\mathcal{I}}(t,\varphi(t)) \in \mathbb{R}$ belongs to $H^1(0,T)$ with weak derivative

$$\frac{d}{dt}\,\widetilde{\mathcal{I}}(.,\varphi(.)) = \partial_t \widetilde{\mathcal{I}}(.,\varphi(.)) + \langle \partial_\varphi \widetilde{\mathcal{I}}(.,\varphi(.)), \dot\varphi(.)\rangle_{H^1(\Omega)} \in L^2(0,T). \tag{5.5}$$

Note that $\varphi \in H^1(0,T;H^1(\Omega)) \hookrightarrow C([0,T];H^1(\Omega))$ and (5.4) imply $\partial_\varphi \widetilde{\mathcal{I}}(t,v) \in L^\infty(0,T;H^1(\Omega)^*)$, which in turn renders the L^2-regularity of $\frac{d}{dt}\,\widetilde{\mathcal{I}}(.,\varphi(.))$.

Let us now assume that φ fulfills (4.27) for all $t \in [0,T]$. Due to $\widetilde{\mathcal{I}}(\cdot,\varphi(\cdot)) \in H^1(0,T)$ the energy inequality implies by setting $t = T$ that

$$\begin{aligned}&\int_0^T \widetilde{\mathcal{R}}_\delta(\dot\varphi(\tau))\,d\tau + \int_0^T \widetilde{\mathcal{R}}_\delta^*(-\partial_\varphi \widetilde{\mathcal{I}}(\tau,\varphi(\tau)))\,d\tau \\ &\quad \le -\int_0^T \Big(\frac{d}{dt}\,\widetilde{\mathcal{I}}(\tau,\varphi(\tau)) - \partial_t \widetilde{\mathcal{I}}(\tau,\varphi(\tau))\Big) d\tau = -\int_0^T \langle \partial_\varphi \widetilde{\mathcal{I}}(\tau,\varphi(\tau)), \dot\varphi(\tau)\rangle_{H^1(\Omega)}\,d\tau,\end{aligned}$$

where we used (5.5) for the last equality. Combining this with Young's inequality, i.e.,

$$\widetilde{\mathcal{R}}_\delta(\dot\varphi(t)) + \widetilde{\mathcal{R}}_\delta^*(-\partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t))) \ge -\langle \partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t)), \dot\varphi(t)\rangle_{H^1(\Omega)} \quad \text{f.a.a. } t \in (0,T),$$

leads to (5.2) and consequently (5.3), which shows the first implication.

The reverse assertion can be concluded by following the lines of the proof of Proposition 3.5. To see this, assume that $\varphi \in H^1(0,T;H^1(\Omega))$ satisfies (5.1). From the equivalence (5.3) $\Longleftrightarrow$ (5.2) and (5.5) we then obtain

$$\widetilde{\mathcal{R}}_\delta(\dot\varphi(t)) + \widetilde{\mathcal{R}}_\delta^*(-\partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t))) = -\frac{d}{dt}\widetilde{\mathcal{I}}(t,\varphi(t)) + \partial_t \widetilde{\mathcal{I}}(t,\varphi(t)) \quad \text{f.a.a. } t \in (0,T). \tag{5.6}$$

Note that any φ, which fulfills (5.1), automatically satisfies $\dot\varphi \ge 0$ in view of Definition 4.13. The latter one then also ensures the L^1-integrability of $\widetilde{\mathcal{R}}_\delta(\dot\varphi(\cdot))$. For the right hand side in (5.6) we have due to Lemma 4.12 and (5.5) that $\partial_t \widetilde{\mathcal{I}}(\cdot,\varphi(\cdot)) \in C[0,T]$

and $\frac{d}{dt}\widetilde{\mathcal{I}}(\cdot,\varphi(\cdot)) \in L^2(0,T)$, respectively. Thus, we are allowed to integrate (5.6) in time, which implies

$$\int_0^t \widetilde{\mathcal{R}}_\delta(\dot\varphi(\tau))\,d\tau + \int_0^t \widetilde{\mathcal{R}}_\delta^*(-\partial_\varphi \widetilde{\mathcal{I}}(\tau,\varphi(\tau)))\,d\tau \\ = \widetilde{\mathcal{I}}(0,\varphi(0)) - \widetilde{\mathcal{I}}(t,\varphi(t)) + \int_0^t \partial_t \widetilde{\mathcal{I}}(\tau,\varphi(\tau))\,d\tau \tag{5.7}$$

for all $t \in [0,T]$. This completes the proof. □

Remark 5.2. *An inspection of the proof of Proposition 5.1 shows that, in order to prove* (5.1)*, it suffices that the integral equation* (4.27) *holds only at* $t = T$*. Moreover, the proof shows that* (4.27) *implies* (5.1) *which in turn gives* (5.7)*. In this way we have shown that* (4.27) *is indeed an* energy identity*. Furthermore, integrating* (5.6) *over an arbitrary interval* $[s,t] \subset [0,T]$ *(instead of* $[0,t]$*) leads to an energy identity, completely analogous to* (3.9) *so that the passage to the limit* $\beta \to \infty$ *indeed preserves the structure of the energy identity. We also refer to [13, Proposition 3.2].*

We summarize our results so far in the following

Theorem 5.3 (Single-field damage model). *Let Assumption 4.1 hold and* $\{\beta_n\}_{n\in\mathbb{N}}$ *be a sequence with* $\beta_n \to \infty$ *as* $n \to \infty$*. Then there is a subsequence (denoted by the same symbol) such that*

$$\varphi_{\beta_n} \rightharpoonup \varphi \text{ in } H^1(0,T;H^1(\Omega)), \quad d_{\beta_n} \rightharpoonup \varphi \text{ in } H^1(0,T;L^2(\Omega)), \\ \boldsymbol{u}_{\beta_n} \to \mathcal{U}(\cdot,\varphi(\cdot)) \text{ in } C([0,T];V). \tag{5.8}$$

Moreover, every limit $(\varphi,\boldsymbol{u}) \in H^1(0,T;H^1(\Omega)) \times C([0,T];V)$ *with* $\boldsymbol{u} := \mathcal{U}(\cdot,\varphi(\cdot))$ *of such a sequence satisfies f.a.a.* $t \in (0,T)$ *the following PDE system:*

$$-\operatorname{div} g(\varphi(t))\varepsilon(\boldsymbol{u}(t)) = \ell(t) \quad \text{in } V^*, \tag{5.9a}$$

$$\delta\,\dot\varphi - \alpha\triangle\varphi(t) + \frac{1}{2}\,g'(\varphi(t))\mathbb{C}\,\varepsilon(\boldsymbol{u}(t)) : \varepsilon(\boldsymbol{u}(t)) \in -\partial\widetilde{\mathcal{R}}_1(\dot\varphi(t)), \quad \varphi(0) = 0, \tag{5.9b}$$

with the (non-viscous) dissipation potential $\widetilde{\mathcal{R}}_1$ *defined by*

$$\widetilde{\mathcal{R}}_1 : H^1(\Omega) \to [0,\infty], \quad \widetilde{\mathcal{R}}_1(\eta) := \begin{cases} r\int_\Omega \eta\,dx, & \text{if } \eta \ge 0 \text{ a.e. in } \Omega, \\ \infty, & \text{otherwise.} \end{cases} \tag{5.10}$$

Proof. The existence of the subsequence has already been established in Propositions 4.8 and 4.9. Furthermore, (5.9a) is just equivalent to $\boldsymbol{u} := \mathcal{U}(\cdot,\varphi(\cdot))$. It remains to verify (5.9b), which follows from (5.1). To see this, just apply (4.17) and the definition of F to the left hand side of (5.1) and use the sum rule for convex subdifferentials for the right hand side. □

The above theorem shows that (5.9) admits at least one solution. Of course, it would be desirable to have the uniqueness of the solution, too, in particular, since this guarantees the uniqueness of the limit in (5.8) and thus the (weak) convergence of the whole sequence. Unfortunately, for this purpose, we have to require the following rather restrictive assumption. We underline that this assumption is only needed to show the uniqueness, while the the rest of the analysis remains unaffected, if it is not fulfilled.

Assumption 5.4. *To ensure uniqueness of the solution of* (5.9)*, we require that there exists some* $p > 4$ *in the two-dimensional case and* $p \ge 6$ *in the three-dimensional case* *such that the operator* $A_\varphi : \boldsymbol{W}^{1,p}_D(\Omega) \to \boldsymbol{W}^{-1,p}_D(\Omega)$ *is continuously invertible for every* $\varphi \in H^1(\Omega)$ *and the norm of its inverse is bounded uniformly w.r.t.* φ*.*

REMARK 5.5. *Assumption 5.4 is fulfilled, provided that no mixed boundary conditions are present, the domain is smooth enough, and the difference between the boundedness and ellipticity constants of the stress strain relation is sufficiently small, cf. [17, Remark 3.21] and [9, 11]. Adapted to our situation this means that the values* $\epsilon\,\gamma_{\mathbb{C}}$ *and* $\|\mathbb{C}\|_\infty$ *have to be sufficiently close to each other, which is clearly rather restrictive (beside the smoothness assumption on the domain), cf. also Remark 2.8. These assumptions on the data can be weakened, if one uses* $H^s(\Omega)$ *with* $s > N/2$ *instead* $H^1(\Omega)$ *as function space for the nonlocal damage in the penalized model* (P)*. We refer to [13, Sections 2.4 and 3.2] for details. Since the bilinear form associated with* $H^s(\Omega)$ *is harder to realize in numerical practice, we do not follow this approach.*

Before proving the unique solvability of (5.1), we need to refine the estimate (2.8).

LEMMA 5.6. *Under Assumption 5.4, we have for all* $t \in [0,T]$ *and all* $\varphi_1, \varphi_2, \psi \in H^1(\Omega)$ *the following estimate*

$$|\langle F(t,\varphi_1) - F(t,\varphi_2), \psi\rangle_{H^1(\Omega)}| \le C\,\|\varphi_1 - \varphi_2\|_{H^1(\Omega)}\|\psi\|_2, \tag{5.11}$$

with a constant $C > 0$ *independent of* t, φ_1, φ_2*, and* ψ*.*

Proof. Let $t \in [0,T]$ and $\varphi_1, \varphi_2, \psi \in H^1(\Omega)$ be arbitrary, fixed. The estimate (5.11) follows with exactly the same arguments as in [17, Lemma 3.15]. For convenience of the reader we shortly recall the arguments. We denote $\boldsymbol{u}_i := \mathcal{U}(t,\varphi_i)$ for $i = 1,2$. The definition of F in (2.7) implies

$$\begin{aligned} &|\langle F(t,\varphi_1) - F(t,\varphi_2), \psi\rangle| \\ &\quad\le \int_\Omega \big|(g'(\varphi_1) - g'(\varphi_2))\mathbb{C}\varepsilon(\boldsymbol{u}_1) : \varepsilon(\boldsymbol{u}_1)\psi\big|\,dx \\ &\qquad + \int_\Omega \big|g'(\varphi_2)[\mathbb{C}\varepsilon(\boldsymbol{u}_1) : \varepsilon(\boldsymbol{u}_1) - \mathbb{C}\varepsilon(\boldsymbol{u}_2) : \varepsilon(\boldsymbol{u}_2)]\psi\big|\,dx =: I_1 + I_2 \end{aligned} \tag{5.12}$$

Let us abbreviate $r := 2p/(p-4)$. Then Assumption 5.4 guarantees $H^1(\Omega) \hookrightarrow L^r(\Omega)$, which together with the Lipschitz continuity of g', (2.2), and Höder's inequality with $1/r + 2/p + 1/2 = 1$ implies the assertion for I_1. In case of I_2 the estimate follows from (2.2), and Lemma 2.10 with $1/\pi = 1/p + 1/r$. □

PROPOSITION 5.7. *Under Assumptions 5.4, system* (5.9) *admits a unique solution* $(\varphi, \boldsymbol{u}) \in H^1(0,T;H^1(\Omega)) \times C([0,T];V)$*.*

Proof. Let $(\varphi_i, \boldsymbol{u}_i) \in H^1(0,T;H^1(\Omega)) \times C([0,T];V)$, $i = 1,2$ be two solutions of (5.9). First note that, from the definition of F in (2.7) and $\boldsymbol{u}_i(\cdot) = \mathcal{U}(\cdot,\varphi(\cdot))$ imply that φ_i satisfies

$$\delta\dot\varphi_i(t) - \alpha\triangle\varphi_i(t) + F(t,\varphi_i(t)) \in -\partial\widetilde{\mathcal{R}}_1(\dot\varphi_i(t)), \quad i = 1,2, \tag{5.13}$$

f.a.a. $t \in (0,T)$. Therefore, $\partial\widetilde{\mathcal{R}}_1(\dot\varphi_i(t)) \neq \emptyset$, which gives $\dot\varphi_1, \dot\varphi_2 \ge 0$ f.a.a. $t \in (0,T)$. By testing (5.13) for $i = 1$ with $\dot\varphi_2 - \dot\varphi_1$ and vice versa and adding the arising inequalities, we arrive at

$$\begin{aligned} &\delta\|\dot\varphi_1(t) - \dot\varphi_2(t)\|_2^2 + \alpha(\nabla\varphi_1(t) - \nabla\varphi_2(t), \nabla\dot\varphi_1(t) - \nabla\dot\varphi_2(t))_2 \\ &\qquad\le \langle F(t,\varphi_2(t)) - F(t,\varphi_1(t)), \dot\varphi_1(t) - \dot\varphi_2(t)\rangle_{H^1(\Omega)} \quad \text{f.a.a. } t \in (0,T). \end{aligned}$$

Then, adding $\alpha(\varphi_1(t) - \varphi_2(t), \dot{\varphi}_1(t) - \dot{\varphi}_2(t))_2$ on both sides of this estimate and applying Lemma 5.6 lead to

$$\begin{aligned}\delta\|\dot{\varphi}_1(t) - \dot{\varphi}_2(t)\|_2^2 + \alpha(\varphi_1(t) - \varphi_2(t), \dot{\varphi}_1(t) - \dot{\varphi}_2(t))_{H^1(\Omega)} \\ &\leq C\|\varphi_1(t) - \varphi_2(t)\|_{H^1(\Omega)}\|\dot{\varphi}_1(t) - \dot{\varphi}_2(t)\|_2 \\ &\leq \frac{C}{4\varepsilon}\|\varphi_1(t) - \varphi_2(t)\|_{H^1(\Omega)}^2 + C\varepsilon\|\dot{\varphi}_1(t) - \dot{\varphi}_2(t)\|_2^2 \quad \forall\, \varepsilon > 0,\end{aligned}$$

where the last estimate follows from the generalized Young inequality. By choosing $\varepsilon := \delta/(2C)$ we conclude

$$\alpha(\varphi_1(t) - \varphi_2(t), \dot{\varphi}_1(t) - \dot{\varphi}_2(t))_{H^1(\Omega)} \leq \frac{C^2}{2\delta}\|\varphi_1(t) - \varphi_2(t)\|_{H^1(\Omega)}^2 \tag{5.14}$$

f.a.a. $t \in (0, T)$. On account of [25, Lemma 3.1.43] we have for all $t \in [0, T]$

$$\int_0^t (\varphi_1(\tau) - \varphi_2(\tau), \dot{\varphi}_1(\tau) - \dot{\varphi}_2(\tau))_{H^1(\Omega)}\, d\tau = \frac{1}{2}\|\varphi_1(t) - \varphi_2(t)\|_{H^1(\Omega)}^2 - \frac{1}{2}\|\varphi_1(0) - \varphi_2(0)\|_{H^1(\Omega)}^2$$

and, due to $\varphi_1(0) = \varphi_2(0)$, we obtain after integrating (5.14) that

$$\frac{\alpha}{2}\|\varphi_1(t) - \varphi_2(t)\|_{H^1(\Omega)}^2 \leq \frac{C^2}{2\delta}\int_0^t \|\varphi_1(\tau) - \varphi_2(\tau)\|_{H^1(\Omega)}^2\, d\tau \quad \forall\, t \in [0, T],$$

which by means of Gronwall's lemma leads to

$$\|\varphi_1(t) - \varphi_2(t)\|_{H^1(\Omega)}^2 \leq 0 \quad \forall\, t \in [0, T] \tag{5.15}$$

and thus completes the proof. □

As an immediate consequence of the uniqueness result we obtain the following

COROLLARY 5.8. *If Assumption 5.4 is fulfilled, then the convergence in (5.8) is not only valid for a subsequence, but for the whole sequence* $\{(d_{\beta_n}, \varphi_{\beta_n}, \boldsymbol{u}_{\beta_n})\}$.

6. Comparison to Classical Partial Damage Models.

In this sequel we show that the one-field gradient damage model given by (5.9b) falls into the category of classical partial damage models. To be more specific, we prove that in the two-dimensional case, (5.1) is equivalent to the viscous damage model studied in [13], provided that the body is sound at the beginning of the process. In this situation, the model in [13] reads

$$-\partial_z \overline{\mathcal{I}}(t, z(t)) \in \partial\overline{\mathcal{R}}_{\bar{\delta}}(\dot{z}(t)) \quad \text{f.a.a. } t \in (0, T), \quad z(0) = 1, \tag{6.1}$$

where $\bar{\delta} > 0$ stands for the viscosity parameter. The energy functional is given by [13, (1.1)] and posseses in the two-dimensional case the exact same structure as $\widetilde{\mathcal{E}}$, see (**??**). The function spaces and the assumptions on the data are introduced in [13, Section 2.1 and 2.2], respectively, and coincide with the throughout this paper considered function spaces and standing assumptions. Although the definition of $\bar{\mathcal{E}}$ in [13, Section 2.2] does not feature some parameter $\bar{\alpha} > 0$, which denotes the degree of gradient regularization, as in (1.1), this does not cause any problems, since we can endow the space $H^1(\Omega)$ with the equivalent norm $\bar{\alpha}\|\cdot\|_{H^1(\Omega)}$. We work with homogenous Dirichlet datum $\boldsymbol{u}_D = 0$, see [13, (2.16)] and don't consider the function f, i.e. $f = 0$. Although the condition [13, (2.16)] does not allow for the function f to be the zero function, this is not problematic, since [13, (2.16)] is needed only for proving the existence of solutions for (6.1), which in our case can be concluded from Propositions 4.17 and 5.1 and the upcoming transformation of (5.9b) in (6.1). The main difference between (5.9b) and (6.1) consists in the definition of the dissipation functional, see [13, (1.3)], which implies, that unlike in our situation, the therein considered damage variable can only decrease in time. This is due to the fact that in the model analyzed in [13] the damage variable $z : [0,T]\times\Omega \to \mathbb{R}$ measures the soundness of the material, not the degree of the material rigidity loss, as in our case. That is, the larger the values of z, the sounder the body. To be more precise, one has in the designed model $z(t,x) = 0$ and $z(t,x) = 1$ when the system is fully damaged and completely sound, respectively. Moreover, in [13] it is shown under additional assumptions that for an initial datum $z_0(x) \in [0,1]$, one has $z(t,x) \in [0,1]$ throughout the whole process, for at least one of the solutions of (6.1), thus proving the viability of the mathematical model in this aspect. However one has to impose conditions on the function which measures the degree of elasticity loss, which we here call $\bar{g}$, e.g. that $\bar{g}$ is monotonically increasing, which in [13] makes perfectly sense from a practical point of view. Conditions on the (nonzero) function f are imposed as well. We refer here to [13, Proposition 4.5] for more details. The above motivates the following transformation

$$z := 1 - \frac{\varphi}{\varphi_{max}} \in H^1(0,T;H^1(\Omega)). \tag{6.2}$$

In (6.2) and in the rest of the sequel φ denotes a solution of (5.1), which is assumed to satisfy $\varphi(t,x) \le \varphi_{max} \in \mathbb{R}$ for almost all $(t,x) \in (0,T)\times\Omega$. Thus, in order to be able to work with (6.2) we have to require

Assumption 6.1. *Hereafter, the limit function φ belongs to $L^\infty((0,T)\times\Omega)$.*

Remark 6.2. *As already stated in Remark 5.5, one can proceed as in [13, Section 2.4] and use the space $H^s(\Omega)$, where $s > N/2$, as function space for the nonlocal damage in the penalized model* (P). *In this situation, the limit function φ belongs to the space $H^1(0,T;H^s(\Omega))$ and due to the embedding $H^s(\Omega) \hookrightarrow C^{0,\gamma}(\bar{\Omega})$ for some $\gamma \in (0,1]$, Assumption 6.1 is then automatically fulfilled.*

Due to Assumption 6.1 we may use the notation $\varphi_{max} := \|\varphi\|_{L^\infty((0,T)\times\Omega)}$ in the rest of the section.
Our goal is to show that the variable z defined in (6.2) satisfies (6.1). Due to (6.2) we have to tranform the function g and resize α, the viscosity parameter δ and the fracture toughness r, see (6.3) and (6.6) below. Note that all these transformations are reversible, in the sense that from the viscous model analyzed in [13] one can derive (5.1). Moreover, (6.3) and (6.6) preserve the assumed properties of the transformed data. In order to distinguish between the two models, we add the symbol $\bar{}$ to the notations used for the data, functionals and operators in [13], in case that the therein used notations coincide with ours.
Recall that the coefficient function g assesses the degree of the material elasticity loss. That is why in practice this is expected to be monotonically decreasing, unlike $\bar{g}$, which should monotonically increase, see also [13, Remark 4.6.]. This is also confirmed by the transformation necessary in order to obtain the equivalence between (5.1) and

(6.1). In view of (6.2) we need to define

$$\overline{g}(x) := g(\varphi_{max}(1-x)) \quad \forall\, x \in \mathbb{R}, \tag{6.3}$$

which, as expected, leads to

$$\overline{g}(z(t)) = g(\varphi(t)) \quad \forall\, t \in [0,T]. \tag{6.4}$$

Note that $\overline{g}$ satisfies condition [13, (2.10)] due to Assumptions 2.3. With a little abuse of notation we will denote by $\overline{g}$ also the corresponding Nemytskii operator. Since in the two-dimensional case the operator $g : H^1(\Omega) \to L^\tau(\Omega)$ is continuously Fréchet differentiable for $\tau \in [1,\infty)$, the same holds for the operator $\bar{g}$ and on account of (6.3) we can write

$$\overline{g}'(z(t)) = -\varphi_{max} g'(\varphi(t)) \quad \forall\, t \in [0,T]. \tag{6.5}$$

In order to obtain the wished form of (6.1) we still need to resize the following data:

$$\bar{\alpha} := \alpha\, \varphi_{max}^2, \tag{6.6a}$$
$$\bar{\delta} := \delta\, \varphi_{max}^2, \tag{6.6b}$$
$$\kappa := r\varphi_{max}^2. \tag{6.6c}$$

Since it is interesting to see how both models behave with respect to each other under the influence of the same external load, we impose

$$\overline{\ell} := \ell. \tag{6.7}$$

We now have all the necessary tools for proving that, via the transformations (6.2), (6.3), (6.6a), (6.6b) and (6.6c), the limit model (5.1) can be converted into (6.1). We begin by enumerating some consequences of (6.2), which will turn out to be very useful in what follows. For almost all $t \in (0,T)$ we have

$$\varphi(t) = \varphi_{max}(1 - z(t)), \tag{6.8a}$$
$$\dot{\varphi}(t) = -\varphi_{max}\, \dot{z}(t), \tag{6.8b}$$
$$\nabla\varphi(t) = -\varphi_{max} \nabla z(t). \tag{6.8c}$$

Further, notice that [13, (2.28)] reads

$$\partial_z \overline{\mathcal{I}}(t, z(t)) = -\bar{\alpha}\triangle z(t) + \frac{1}{2}\overline{g}'(z(t))\mathbb{C}\varepsilon\big(\overline{\mathcal{U}}(t, z(t))\big) : \varepsilon\big(\overline{\mathcal{U}}(t, z(t))\big) \quad \forall\, t \in [0,T], \tag{6.9}$$

where $\overline{\mathcal{U}}(t, z(t))$ solves the balance of momentum equation

$$-\operatorname{div}\left(\overline{g}(z(t))\mathbb{C}\varepsilon\big(\overline{\mathcal{U}}(t, z(t))\big)\right) = \overline{\ell}(t) \quad \forall\, t \in [0,T].$$

We refer here to [13, (2.13) and (2.19)]. Relying on (6.4) and (6.7), (??) gives in turn

$$\overline{\mathcal{U}}(t, z(t)) = \mathcal{U}(t, \varphi(t)) \quad \forall\, t \in [0,T]. \tag{6.10}$$

From (6.9), (6.6a), (6.8c), (6.5) and (6.10) we follow

$$\partial_z \overline{\mathcal{I}}(t, z(t)) = \varphi_{max}(\alpha\triangle\varphi(t) - \frac{1}{2}g'(\varphi(t))\mathbb{C}\varepsilon\big(\mathcal{U}(t, \varphi(t))\big) : \varepsilon\big(\mathcal{U}(t, \varphi(t))\big), \tag{6.11}$$

i.e.

$$\partial_z \overline{\mathcal{I}}(t, z(t)) = -\varphi_{max}\, \partial_\varphi \widetilde{\mathcal{I}}(t, \varphi(t)) \quad \forall\, t \in [0, T]. \tag{6.12}$$

Note that (6.12) is ensured by (**??**) and (2.7). Further, by comparing (5.10) and [13, (1.3)] and using (6.8b), we find

$$\begin{aligned} &\xi \in \partial \widetilde{\mathcal{R}}_1(\dot\varphi(t)) \\ \iff &\langle \xi, v - \dot\varphi \rangle \le \widetilde{\mathcal{R}}_1(v) - \widetilde{\mathcal{R}}_1(\dot\varphi) = r/\kappa\big(\overline{\mathcal{R}}_1(-v) - \overline{\mathcal{R}}_1(\varphi_{max}\dot z(t))\big) \\ \iff &\langle -\xi, v - \varphi_{max}\dot z(t) \rangle \le r/\kappa\big(\overline{\mathcal{R}}_1(v) - \overline{\mathcal{R}}_1(\varphi_{max}\dot z(t))\big) \ \ \forall\, v \in H^1(\Omega) \\ \iff &-\xi \in r\varphi_{max}/\kappa\, \partial \overline{\mathcal{R}}_1(\dot z(t)), \end{aligned}$$

hence in view of (6.6c)

$$\partial \widetilde{\mathcal{R}}_1(\dot\varphi(t)) = -1/\varphi_{max}\, \partial \overline{\mathcal{R}}_1(\dot z(t)) \qquad \text{f.a.a. } t \in (0, T). \tag{6.13}$$

We now make use of the formulation (5.13) of the evolutionary equation (5.1) by keeping (**??**) in mind. By means of (6.12), (6.8b), (6.6b) and (6.13) this can be rewritten as

$$1/\varphi_{max}\big(\partial_z \overline{\mathcal{I}}(t, z(t)) + \bar\delta \dot z(t)\big) \in -1/\varphi_{max} \partial \overline{\mathcal{R}}_1(\dot z(t)) \qquad \text{f.a.a. } t \in (0, T), \quad z(0) = 1.$$

By applying sum rule for convex subdifferentials we now obtain (6.1). The initial condition $z(0) = 1$ follows immediately from (5.13) and (6.2).

Remark 6.3. *Note that if one has in (6.1) the initial condition $z(0) = z_0$ a.e. in Ω, where z_0 is some constant between 0 and 1, then (5.13) and (6.1) are still equivalent by making the transformation $z := z_0\big(1 - \frac{\varphi}{\varphi_{max}}\big)$ and redefining $\bar g$ in (6.3) and the data in (6.6) accordingly.*

Remark 6.4. *The equivalency of (5.13) and (6.1) can be also proven in the three-dimensional case. One must of course proceed as in [13, Section 2.4] and work with the Sobolev-Slobodeckij space $H^{3/2}(\Omega)$ instead of $H^1(\Omega)$, as function space for the nonlocal damage in the penalized model (P). As already stated in Remarks 5.5 and 6.2, this implies that the limit function φ belongs to $H^1(0, T; H^{3/2}(\Omega))$. Note that [13, Proposition 4.5.] is then no longer applicable.*

Remark 6.5. *However, the result in this section proves the existence of solutions for (6.1) also in the situation when $f = 0$, situation which was excluded in [13], because of [13, (2.16)]. This shows that while for f as in [13, (2.16)], viscous solutions may be approximated via time-discretization and regularization, respectively, for $f = 0$, the existence of viscous solutions results via penalizing, assuming that one deals in [13] with an initial condition as in Remark 6.3.*

Since the condition [13, (2.16)] is needed in [13] only in the context of showing existence of solutions and does not affect the vanishing viscosity analysis, the equivalency of (5.13) and (6.1) allows us to conclude that, provided that Assumption 6.1 holds, the model

$$-\partial_\varphi \widetilde{\mathcal{I}}(t, \varphi(t)) \in \partial \widetilde{\mathcal{R}}_1(\dot\varphi(t)) \ \ \text{f.a.a. } t \in (0, T), \quad \varphi(0) = 0,$$

admits BV solutions, see [13, Section 5] for more details.

REFERENCES

[1] E. Bonetti and G. Bonfanti. Well-posedness results for a model of damage in thermoviscoelastic materials. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25:1187–1208, 2008.

[2] E. Bonetti and G. Schimperna. Local existence to Frémond's model for damaging in elastic materials. *Contin. Mech. Thermodyn.*, 16:319–335, 2004.

[3] E. Bonetti, G. Schimperna, and A. Segatti. On a doubly nonlinear model for the evolution of damaging in viscoelastic materials. *J. Differential Equations*, 218:91–116, 2005.

[4] B.J. Dimitrijevic and K. Hackl. A method for gradient enhancement of continuum damage models. *Technische Mechanik*, 28(1):43–52, 2008.

[5] M. Efendiev and A. Mielke. On the rate–independent limit of systems with dry friction and small viscosity. *J. Convex Analysis*, 13:151–167, 2006.

[6] M. Frémond and N. Kenmochi. Damage problems for viscous locking materials. *Adv. Math. Sci. Appl.*, 16(2):697–716, 2006.

[7] M. Frémond and B. Nedjar. Damage, gradient of damage and principle of virtual power. *Int. J. Solids Struct.*, 33(8):1083–1103, 1996.

[8] E. Giusti. *Direct methods in the calculus of variations.* World Scientific Publishing Co. Inc., 2003.

[9] K. Gröger. A $W^{1,p}$-estimate for solutions to mixed boundary value problems for second order elliptic differential equations. *Math. Ann.*, 283:679–687, 1989.

[10] R. Haller-Dintelmann, C. Meyer, J. Rehberg, and A. Schiela. Hölder continuity and optimal control for nonsmooth elliptic problems. *Applied Mathematics and Optimization*, 60:397–428, 2009.

[11] R. Herzog, C. Meyer, and G. Wachsmuth. Integrability of displacement and stresses in linear and nonlinear elasticity with mixed boundary conditions. *Journal of Mathematical Analysis and Applications*, 382(2):802–813, 2011.

[12] D. Knees, A. Mielke, and C. Zanini. On the inviscid limit of a model for crack propagation. *Math. Models Methods Appl. Sci.*, 18(9):1529–1569, 2008.

[13] D. Knees, R. Rossi, and C. Zanini. A vanishing viscosity approach to a rate-independent damage model. *Mathematical Models and Methods in Applied Sciences*, 23(4):565–616, 2013.

[14] D. Knees, C. Zanini, and A. Mielke. Crack propagation in polyconvex materials. *Physica D*, 239:1470–1484, 2010.

[15] G. Lazzaroni and R. Toader. A model of crack propagation based on viscous approximation. *Math. Models Methods Appl. Sci.*, 21:1–29, 2011.

[16] J. Makowski, K. Hackl, and H. Stumpf. The fundamental role of nonlocal and local balance laws of material forces in finite elastoplasticity and damage mechanics. *International Journal of Solids and Structures*, 43:3940 – 3959, 2006.

[17] C. Meyer and L.M. Susu. Analysis of a viscous two-field gradient damage model part i: Existence and uniqueness. *?*, 25(2):?, 2016.

[18] A. Mielke. Differential, energetic, and metric formulations for rate-independent processes. In *Nonlinear PDE's and Applications*, volume 2028 of *Lect. Notes Math.*, pages 87–167. Springer, Berlin, 2011. C.I.M.E. Summer School, Cetraro, Italy 2008.

[19] A. Mielke, R. Rossi, and G. Savaré. Modeling solutions with jumps for rate-independent systems on metric spaces. *Discr. Cont. Dynam. Systems Ser. A*, 25(2):585–615, 2009.

[20] A. Mielke, R. Rossi, and G. Savaré. BV solutions and viscosity approximations of rate-independent systems. *ESAIM Control Optim. Calc. Var.*, 18:36–80, 2012.

[21] A. Mielke and S. Zelik. On the vanishing viscosity limit in parabolic systems with rate-independent dissipation terms. *Ann. Sc. Norm. Sup. Pisa Cl. Sci.*, XIII(5):67–135, 2014.

[22] B. Nedjar. Elastoplastic-damage modelling including the gradient of damage: formulation and computational aspects. *International Journal of Solids and Structures*, 38:5421 – 5451, 2001.

[23] R. Peerlings. *Enhanced damage modeling for fracture and fatigue.* Dissertation, Techische Universiteit Eindhoven, 1999.

[24] A. Simone, H. Askes, R. Peerlings, and L. Sluys. Interpolation requirements for implicit gradient-enhanced continuum damage models. *Communications in Numerical Methods in Engineering*, 19:563 – 572, 2003.

[25] G. Wachsmuth. *Optimal control of quasistatic plasticity.* Dissertation, Chemnitz University of Technology, 2011.

[26] T. Waffenschmidt, C. Polindara, A. Menzel, and S. Blanco. A gradient-enhanced large-deformation continuum damage model for fibre-reinforced materials. *Comput. Meth. Appl. Mech. Engrg.*, 268:801–842, 2014.