Numerical method based on extended one-step schemes for optimal control problems with time-lags

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Abstract

In this work, we develop extended one-step methods for solving optimal control problems governed by ordinary and delay differential equations. The proposed problem is reduced to either a constrained or unconstrained minimization problem according to the nature of the dynamic system and the given conditions. Pontryagin's maximum (or minimum) principle is used to characterize the optimal controls. Numerical results with simulations compared with other methods are presented to show the efficiency of the methodology.

Keywords: Extended one-step methods; Optimal Control; Ordinary differential equations, Delay differential equations; Indirect approach

1 Introduction

Control theory is application-oriented mathematics that deals with the basic principles underlying the analysis and design of (control) systems. Systems can be engineering systems (air conditioner, aircraft, CD player etcetera), economic systems, biological systems and so on. To control means that one has to influence the behavior of the system in a desirable way: for example, in the case of an air conditioner, the aim is to control the temperature of a room and maintain it at a desired level, while in the case of an aircraft, we wish to control its altitude at each point of time so that it follows a desired trajectory. Historically, optimal control is considered as an extension of the calculus of variations which is a branch of mathematics concerning problems that seek to find the path, curve, or surface for which a given function has a minimum or maximum. Optimal control problems governed by ordinary differential equations (ODEs), have a variety of applications in economics, biology and medicine [2, 3, 4, 5, 8].

On the other hand, many important economic and engineering systems do not react at once, but with a delay to changes in external influences e.g. transportation-lags and hence have been considered the optimal control problems with delays and obtaining their approximate solutions very important issues in control theory and have attracted much attention of many researchers; See [7, 9, 10, 11, 12, 13].

Most of optimal control problems, however, can not be solved analytically and consequently, reliable numerical methods are essentially required. There are two ways to solve the optimal control problems, 'direct' and 'indirect' approaches. In the direct approach, the optimal control problem is transformed into a nonlinear programming problem. While indirect approach is based on the calculus of variations or the Pontryagin maximum (or Minimum) principle, which in turn reduces to a boundary value problem.

In this paper, we adapt *extended one-step method* (EOSM) for solving the dynamic system of optimal control problem, governed by *ordinary differential equations* (ODEs) and *delay differential equations* (DDEs). The method is based on indirect approach using Forward-Backward Sweep method to find the optimal control variable. The organization of this paper is as follows: Section 2 presents the first order necessary optimality conditions for undelayed optimal control problems with bounded controls as well as with linear dependence on the control. The first order optimality conditions for delayed optimal control problems with delay on the state and the control is presentence in section 3. In section 4, the extended one step schemes, up to order five for ODEs as well as for DDEs are presented. In Section 5, we present Forward-Backward Sweep approach, which will be used to solve the undelayed and delayed control problems. Numerical results and compressions are presented in Section 6.

2 First order necessary optimality conditions for undelayed optimal control problems

The simplest Optimal Control Problem (OCP) governed by ordinary differential equations can be stated as,

$$\begin{aligned} \min_u J(y(t), u(t)) &= \int_{t_0}^{T} f(t, y(t), u(t))\, dt \\ \text{s.t.} \quad y'(t) &= g(t, y(t), u(t)), \quad y(t_0) = y_0. \end{aligned} \tag{1}$$

J is called objective functional, u is the control and y is called the state variable. The functions f and g are continuously differentiable functions in all three arguments. The control variable can be piecewise continuous, so that it can have discrete jumps. and the associated state variable Variable must be piecewise differentiable, so that it cannot have discrete jumps.

Pontryagin's Maximum(or Minimum) Principle is a powerful method for the computation of optimal controls, which has the crucial advantage that it does not require prior evaluation of the infimal cost function. We describe the method and illustrate its use in numerical examples. Pontryagin introduced the idea of adjoint functions to append the differential equation to the objective functional. Adjoint functions have a similar purpose as Lagrange multipliers in multivariate calculus, which append constraints to the function of several variables to be maximized or minimized.

Theorem 1 *(Pontryagin's Minimum Principle [2]) If $u^*(t)$ and $y^*(t)$ are optimal for problem (1), then there exists a piecewise differentiable adjoint variable $\lambda(t)$ such that*

$$H(t, y^*(t), u(t), \lambda(t)) \geq H(t, y^*(t), u^*(t), \lambda(t))$$

for all the controls u at each time, where the Hamiltonian H is

$$H = f(t, y(t), u(t)) + \lambda(t) g(t, y(t), u(t)),$$

and

$$\lambda'(t) = -\frac{\partial H(t, y^*(t), u^*(t), \lambda(t))}{\partial y} = -(f_y + \lambda(t) g_y), \tag{2}$$

with the transversality condition

$$\lambda(T) = 0.$$

and the control satisfy

$$\frac{\partial H}{\partial u} = 0 \Rightarrow f_u + \lambda(t) g_u = 0$$

on $t_0 \leq t \leq T$

Proof 1 *The proof exists in [2].*

2.1 Optimal control with bounded controls

Since many real world application problems require bounds on the controls. Consider the following optimal control problem with bounded control:

$$\begin{aligned} \min J(y(t), u(t)) = \int_{t_0}^{T} f(t, y(t), u(t))\ dt \\ \text{s.t.} \quad y'(t) =& g(t, y(t), u(t)), \\ y(t_0) =& y_0 \\ a \leq u(t) \leq b, \end{aligned} \tag{3}$$

where a and b are real constants with $a \leq b$. Pontryagin's Minimum Principle still valids for problem (3) except the minimization is over all admissible controls, that is $a \leq u(t) \leq b, \quad \forall t \in [t_0, T]$. The Hamiltonian functional is

$$H = f(t, y(t), u(t)) + \lambda(t) g(t, y(t), u(t)),$$

and necessary conditions for the state y^* and p^* are the same as in Theorem 2, namely

$$y'(t) = g(t, y(t), u(t)), \qquad y(t_0) = y_0,$$

the adjoint state equation

$$\lambda'(t) = -\frac{\partial H}{\partial y} = -(f_y + \lambda(t) g_y),$$

with the transversality condition $\lambda(T) = 0$. but with the optimal control u^* satisfies the following condition:

$$u^*(t) = \begin{cases} a & \text{if} \quad \dfrac{\partial H}{\partial u} > 0 \\ a \leq u^* \leq b & \text{if} \quad \dfrac{\partial H}{\partial u} = 0 \\ b & \text{if} \quad \dfrac{\partial H}{\partial u} < 0 \end{cases} \tag{4}$$

If we have a maxmization problem instead of a minimization problem (3), then u^* is instead chosen to maxmize H pointwise. This has the effect of reversing $>$ and $<$ in the first and third lines of (4).

2.2 Linear dependence on the control

An optimal control problem with linear dependence on the control can be written as

$$\begin{aligned} \min_u \int_{t_0}^{T} f_1(t, y(t)) + u(t) f_2(t, y(t))\ dt \\ \text{s.t.} \quad y'(t) = g_1(t, y(t)) + u(t) g_2(t, y(t)), \\ y(t_0) = y_0 \\ a \leq u(t) \leq b. \end{aligned} \tag{5}$$

The Hamiltonian is

$$H = [f_1(t, y(t)) + \lambda(t) g_1(t, y(t))] + u(t)[f_2(t, y(t)) + \lambda(t) g_2(t, y(t))]. \tag{6}$$

The optimality condition

$$\frac{\partial H}{\partial u} = f_2(t, y) + \lambda(t) g_2(t, y)$$

has no information on the control, thus we define a switching function as

$$\psi = f_2(t, y) + \lambda(t) g_2(t, y).$$

If we are solving a minimization problem, the optimal control takes the form:

$$u^*(t) = \begin{cases} a & \text{if} \quad \psi > 0 \\ \in [a, b] & \text{if} \quad \psi = 0 \\ b & \text{if} \quad \psi < 0 \end{cases} \tag{7}$$

The control u^* is referred to as a bang-bang control if $\psi = 0$ cannot be sub-stained over the interval $[t_0, T]$ but occurs only at finitely many points. In this case, the control is either at the upper bound b or at the lower bound a.

3 First order necessary optimality conditions for delayed optimal control problems

Delayed optimal control problems variable exhibit in general a qualitatively different system dynamics compared to instantaneous optimal control problems. Consider the following retarded optimal control problem with constant delay $r \geq 0$ in the state $y(t)$ and $s \geq 0$ in the control $u(t)$

$$\text{Minimize } J(y, u) = \int_{t_0}^{T} f(t, y(t), y(t-r), u(t), u(t-s)) dt, \tag{8a}$$

subject to DDEs

$$y'(t) = g(t, y(t), y(t-r), u(t), u(t-s)), \quad t \in [t_0, T] \tag{8b}$$

$$y(t) = \phi(t), \quad t \in [t_0 - r, t_0], \tag{8c}$$

$$u(t) = \psi(t), \quad t \in [t_0 - s, t_0]. \tag{8d}$$

The Hamiltonian H for the delayed control problem is defined in analogy to the non-delayed control problem:

$$H = f(t, y, w, u, v) + \lambda \cdot g(t, y, w, u, v), \tag{9}$$

where w and v denoting the delayed state and control variables.
The first order optimality conditions for the delayed control problem obtained by applying the Pontryagin's Minimum Principle with delay which derived by Göllmann et al. [24] and consisting of:
the state differential equation

$$y'(t) = g(t, y(t), y(t-r), u(t), u(t-s)), \quad t \in [t_0, T] \tag{10}$$

the adjoint state differential equation

$$\lambda'(t) = -H_y - \chi_{[t_0, T-r]}(t) H_w(t+r) \tag{11}$$

where $\chi_{[t_0, T-r]}$ denotes the indicator function of the interval $[t_0, T-r]$ and defined by

$$\chi_{[t_0, T-r]} = \begin{cases} 1 & \text{if } t \in [t_0, T-r], \\ 0 & \text{otherwise.} \end{cases}$$

with transversality conditions

$$\lambda(T) = 0.$$

Local minimum condition for the Hamiltonian

$$H_u + \chi_{[t_0, T-s]}(t) H_v(t+s) = 0. \tag{12}$$

4 Extended one-step methods

Given the initial value problem

$$\begin{aligned} y\prime(t) &= f(t, y(t)), \quad 0 < t \le b, \\ y(0) &= y_0, \qquad\qquad t = 0. \end{aligned} \tag{13}$$

It is well known that the order of a $k-$step method cannot exceed $k+2$, therefore the A-stable *linear multistep method* (LMM) can not exceed 2 [14]. To overcome this "order barrier" imposed by A-stability, we use the so called extended one-step A-stable methods of order up to five, constructed by coupling several LMMs (see [21], to solve the dynamical system of the problem. After discretization of the problem (13), one can get

$$y_{n+1} = y_n + h[\ \alpha_0 f_n + \alpha_1 f_{n+1} + \sum_{j=2}^{m-1} \alpha_j f_{n+j}] + \kappa_n(h)\ , \tag{14}$$

with

$$y_{n+j} = \beta_{j0} y_n + \beta_{j1} y_{n+1} + h[\ \gamma_{j0} f_n + \gamma_{j1} f_{n+1} + \sum_{i=2}^{j-1} \gamma_{ji} f_{n+i}] + E_{nj}(h). \tag{15}$$

The extended one-step scheme of such problem takes the form

$$\begin{aligned} y_{n+1} &= y_n + h[\ \alpha_0 f_n + \alpha_1 f_{n+1} + \sum_{j=2}^{m-1} \alpha_j \hat{f}_{n+j}] + T_n(h), \\ \hat{y}_{n+j} &= \beta_{j0} y_n + \beta_{j1} y_{n+1} + h[\ \gamma_{j0} f_n + \gamma_{j1} f_{n+1} + \sum_{i=2}^{j-1} \gamma_{ji} \hat{f}_{n+i}] \end{aligned} \tag{16}$$

where $\alpha_j,\ \beta_{j0},\ \beta_{j1},\ \gamma_{j0},\ \gamma_{j1}$ and $\gamma_{ji},\ j = 2, 3, \ldots m-1$ are real coefficients, $f_n = f(t_n, y_n)$ and y_n is an approximation to $y(t_n)$ at a sequence of equally spced points, $t_n = nh,\ n = 0, 1, \ldots, N$. One can refer to such a methods (omitting $T_n(h)$) by Table 1.

α_0	α_1	α_2	$\cdots$		α_{m-1}	
β_{20}	β_{21}	γ_{20}	γ_{21}			
β_{30}	β_{31}	γ_{30}	γ_{31}	γ_{32}		
$\vdots$			$\vdots$	$\vdots$	$\ddots$	
$\beta_{m-1,0}$	$\beta_{m-1,1}$	$\gamma_{m-1,0}$	$\gamma_{m-1,1}$	$\gamma_{m-1,2}$	$\cdots$	$\gamma_{m-1,m-2}$

Table 1: Coefficeints of the extended one-step methods.

Usmani and Agarwall [15] deduced an extended one-step third order A-stable scheme by requiring that $E_{n2}(h) = O(h^3)$. Later, Jacques [16] modified the method of such schemes to obtain a one parameter family of third order L-stable method by requiring that $E_{n2}(h) = O(h^3)$. Chawla *et al.* [20] obtained a two-parameter family of fourth order and A-stable methods by requiring that $E_{n2}(h)$ and $E_{n3}(h) = O(h^3)$;there exists a one-parameter sub-family of these methods which are, in addition, L-stable. Chawla *et al.* [21] extended these ideas to obtain a two-parameter family of fifth order and gave sub-families of A-stable and L-stable methods.
The general idea, for the derivation of a methods of order m , we require that $\kappa_n(h)$ and $T_n(h) = O(h^{m+1})$ while $E_{nj}(h) = O(h^{m-1})$.

$\frac{9}{24}$	$\frac{19}{24}$	$-\frac{5}{24}$	$\frac{1}{24}$	
5	-4	2	4	
28	-27	12	18	0

$\frac{9}{24}$	$\frac{19}{24}$	$-\frac{5}{24}$	$\frac{1}{24}$	
1	0	0	2	
2	-1	$-\frac{1}{2}$	4	$\frac{1}{2}$

Table 2: A-stable scheme (left) and L-stable scheme (right) of order four for the ODEs (13) [20].

$\frac{251}{720}$	$\frac{323}{360}$	$-\frac{11}{30}$	$\frac{53}{360}$	$-\frac{19}{720}$	
5	-4	2	4		
28	-27	12	18	0	
$\frac{1563}{19}$	$-\frac{1544}{19}$	$\frac{694}{19}$	$\frac{928}{19}$	$-\frac{2}{19}$	0

$\frac{251}{720}$	$\frac{323}{360}$	$-\frac{11}{30}$	$\frac{53}{360}$	$-\frac{19}{720}$	
5	-4	2	4		
28	-27	12	18	0	
$\frac{1611}{19}$	$-\frac{1592}{19}$	$\frac{712}{19}$	$\frac{966}{19}$	$-\frac{12}{19}$	$\frac{2}{19}$

Table 3: A-stable scheme (left) and L-stable scheme of order 5 for the ODEs (13) [21].

Tables 2 & 3 display the tubule of A-stable and L-stable of order four and five, respectively.

We extend the above schemes to the DDEs

$$\begin{aligned} y\prime(t) &= f(x, y(t), y(\alpha(t)), \quad a \le t \le b, \\ y(t) &= g(t), \qquad\qquad\qquad \nu \le t \le a. \end{aligned} \tag{17}$$

Here f , α and g denote given functions with $\alpha(t) \le t$ for $t \ge a$, the function α is usually called the delay or lag function and y is unknown solution for $t > a$. If the delay is a constant, it is called the constant delay, if it is a function of only time, then it is called the time dependent delay, if it is a function of time and the solution $y(t)$, then it is called the state dependent delay. The existence, uniqueness, and continuation of solutions to the above problem have been studied by Driver [1].

The extended one-step scheme for DDE (17) is given by

$$\begin{aligned} y_{n+1} = y_n + h[\ &\alpha_0 f_n + \alpha_1 f_{n+1} \\ &+ \sum_{j=2}^{m-1} \alpha_j \hat{f}_{n+j}], n = 0, 1, \dots, N-1, \end{aligned} \tag{18}$$

where $\hat{f}_{n+j} = f(t_{n+j}, \hat{y}_{n+j}, y^h(\alpha(t_{n+j})))$ and $\alpha_j,\ j = 2, 3, \dots m-1$ are real coefficients. The function y^h is computed from

$$\begin{cases} y^h(t) = g(t) \quad \text{for} \quad t \le a \\ y^h(t) = \beta_{j0} y_k + \beta_{j1} y_{k+1} + h[\ \gamma_{j0} f_k \\ \qquad + \gamma_{j1} f_{k+1} + \sum_{i=2}^{j-1} \gamma_{ji} \hat{f}_{k+i}] \ , \\ \qquad\qquad t_k < t \le t_{k+1} \quad k = 0, 1, \dots \end{cases} \tag{19}$$

where $\beta_{j0},\ \beta_{j1},\ \gamma_{j0},\ \gamma_{j1}$ and γ_{ji} are real coefficients. The function $\hat{y}_{n+j}$ are computed from (19) when $t = t_{n+j}$. In this paper, we will use $\tilde{}$ for the coefficients of $\hat{y}_{n+j}$ as in the following form

$$\hat{y}_{n+j} = \tilde{\beta}_{j0} y_n + \tilde{\beta}_{j1} y_{n+1} + h[\ \tilde{\gamma}_{j0} f_n + \tilde{\gamma}_{j1} f_{n+1} + \sum_{i=2}^{j-1} \tilde{\gamma}_{ji} \hat{f}_{n+i}] \tag{20}$$

Scheme of third order ($m = 3$)

In order to determine the coefficients α_0, α_1 and α_2, we rewrite (18) for $m = 3$ in the exact form

$$\begin{aligned} y(t_{n+1}) = y(t_n) + h\,[&\alpha_0 f(t_n, y(t_n), y(\alpha(t_n))) \\ &+ \alpha_1 f(t_{n+1}, y(t_{n+1}), y(\alpha(t_{n+1}))) \\ &+ \alpha_2 f(t_{n+2}, y(t_{n+2}), y(\alpha(t_{n+2})))] \\ &+ \kappa(t_{n+1}). \end{aligned} \tag{21}$$

We expand the left and right sides of (21) in the Taylor series at the point t_{n+1}, equate the coefficients up to the third order terms $O(h^3)$ and solving the resulting system of equations, we obtain

$$\alpha_0 = \frac{5}{12}, \quad \alpha_1 = \frac{2}{3}, \quad \alpha_2 = -\frac{1}{12} \tag{22}$$

and

$$\kappa(t_{n+1}) = \frac{h^4}{24} y^{(4)}(\xi) \tag{23}$$

where $t_n < \xi < t_{n+2}$. Substituting from (22) into (18) for $m = 3$, we obtain

$$y_{n+1} = y_n + \frac{h}{12}\left[5f_n + 8f_{n+1} - \hat{f}_{n+2}\right] \tag{24}$$

where

$$y^h(t) = g(t) \quad \text{for} \quad t \le a \tag{25}$$

and $y^h(t)$ with $t > a$ is defined by

$$\begin{aligned} y^h(t) = \beta_{20} y_k + \beta_{21} y_{k+1} + h\left[\gamma_{20} f_k + \gamma_{21} f_{k+1}\right], \\ \text{for } t_k < t \le t_{k+1};\ k = 0, 1, \ldots \end{aligned} \tag{26}$$

In order to determine the coefficients β_{20}, β_{21}, γ_{20} and γ_{21}, we rewrite (26) in the exact form

$$\begin{aligned} y(t) =& \beta_{20} y(t_k) + \beta_{21} y(t_{k+1}) + h\left[\gamma_{20} f(t_k, y(t_k), y(\alpha(t_k)))\right. \\ &+ \left.\gamma_{21} f(t_{k+1}, y(t_{k+1}), y(\alpha(t_{k+1})))\right] + E(t_{k+1}). \end{aligned} \tag{27}$$

Similarly, we expand the left and right sides of (27) with Taylor series at point t_{k+1} and equate the coefficients up to the terms of second order $O(h^2)$. We obtain the resulting system of equations

$$\begin{cases} \beta_{20} + \beta_{21} = 1 \\ \beta_{20} - \gamma_{20} - \gamma_{21} = -\delta(t) \\ \beta_{20} - 2\gamma_{20} = \delta^2(t) \end{cases} \tag{28}$$

where

$$\delta(t) = \frac{1}{h}(t - t_{k+1}). \tag{29}$$

The solution of the above system (28) is

$$\begin{cases} \beta_{20} = 1 - \beta_{21} \\ \gamma_{20} = \frac{1}{2}(1 - \beta_{21} - \delta^2(t)) \\ \gamma_{21} = \frac{1}{2}(\delta^2(t) + 2\delta(t) - \beta_{21} + 1) \end{cases} \tag{30}$$

and

$$E(t_{k+1}) = \frac{h^3}{12}(2\delta^3(t) + 3\delta^2(t) + \beta_1 - 1)y^{(3)}(\eta) \tag{31}$$

where β_{21} is a free parameter and $t_k < \eta < t_{k+1}$. Substituting from (30) into (26), we obtain

$$\begin{aligned} y^h(t) =&(1 - \beta_{21})y_k + \beta_{21}y_{k+1} + \frac{h}{2}\left[(1 - \beta_{21} - \delta^2(t))f_k\right. \\ &\left.+(\delta^2(t) + 2\delta(t) - \beta_{21} + 1)f_{k+1}\right], \\ &\text{for} \quad t_k < t \leq t_{k+1};\ k = 0, 1, \dots, \end{aligned} \tag{32}$$

Finally, from (32), the approximation $\hat{y}_{n+2}$ is determined in the form

$$\hat{y}_{n+2} = (1 - \beta_{21})y_n + \beta_{21}y_{n+1} - \frac{h}{2}\left[\beta_{21}f_n + (\beta_{21} - 4)f_{n+1}\right]. \tag{33}$$

Equations (24), (32) and (33) are the basis of the third order methods (see [22]). It has proved that this method is P-stable for $\beta_{21} \in (-\infty, 2]$ (see [22])

We can estimate the parameters for schemes of order 4 and order 5 in the same manner (see [23]).

5 Numerical Methods for Optimal Control Problems

In this section, we provide the numerical algorithm for solving optimal control problems which are generally nonlinear. These problems generally do not have analytic solutions (e.g., like the linear-quadratic optimal control problem). As a result, it is necessary to employ numerical methods to solve optimal control problems. In the early years of optimal control (circa 1950s to 1980s) the forward approach for solving optimal control problems was that of indirect methods. In an indirect method, the calculus of variations is employed to obtain the first-order optimality conditions. These conditions result in a two-point (or, in the case of a complex problem, a multi-point) boundary-value problem. This boundary-value problem actually has a special structure because it arises from taking the derivative of a Hamiltonian.

The indirect methods is based on Pontryagin's Maximum Principle, in which it is necessary to explicitly get the adjoint state equation, the control equation and the transversality condition. A numerical approach using the indirect method, know as Forward-backward sweep method [2] applied here in order to solve the optimal control problem governed by ODEs. The main idea of the algorithm is described as follows:

1. Provide an initial guess for the control variable u over the interval .

2. Use the initial condition for the state variable $y_0 = y(t_0)$ and the values for u to solve for the state forward in time by using EOSM.

3. Solve the adjoint state backward in time by using EOSM, with the given state solution from the previous step and the transversality condition, $\lambda(T) = 0$.

4. Update the value of the control by entering the new values of the state and the adjoint state into the characterization of the optimal control.

5. Verify for convergence by repeating steps 1-4 until successive values of all state, adjoint, and control functions are sufficiently close.

The solution technique of the first order optimality conditions for delay optimal control problems can be considered as follows:
Let there exists a step size $h > 0$ and integers $(N, m_1, m_2) \in \mathbb{N}^3$ with $r = m_1 h$, $s = m_2 h$ and $T - t_0 = Nh$. We put $m = \max\{m_1, m_2\}$ and $\tau = \max\{r, s\}$. Then $\tau = mh$, and we consider m knots to left of t_0 and right of T. Hence, we obtain the following partition:

$$\triangle = t_{-m} = -\tau < \cdots < t_{-1} < t_0 < t_1 < \cdots < t_N = T < \cdots < t_{N+m}. \tag{34}$$

Thus, we have $t_i = t_0 + ih$, where $-m \leq i \leq N + m$. Next, we define the state and adjoint variables $y(t), \lambda(t)$ and the controls $u(t)$ in terms of nodal points y_i, λ_i, u_i. Therefore, we get the following algorithm:

Algorithm 1 Numerical algorithm for solving optimal control problem.

Step 1:
for $i = -m, ..., 0,$ **do**
  $y_i = \phi(t_i)$ and $u_i = \psi(t_i)$,
end for
for $i = N, ..., N + m,$ **do**
  $\lambda_i = 0$,
end for
Step 2:
for $i = 0, ..., N - 1,$ **do**
  Solve the state forward in time by using EOSM.
end for
Step 3:
for $i = 0, ..., N - 1,$ **do**
  Solve he adjoint state backward in time by using EOSM, with the given state solution from the previous step and the transversality condition, $\lambda(T) = 0$.
end for
Step 4:
Update the value of the control by entering the new values of the state and the adjoint state into the characterization of the optimal control.
Step 5:
Verify for convergence by repeating steps 1-4 until successive values of all state, adjoint, and control functions are sufficiently close.

6 Numerical Examples

In this section, we present various examples of unconstrained and constrained optimal control problems governed by ordinary and delay differential equations to show the efficiency of the extended one step method. All the results obtained by applying the third order extended one step method with $\beta_{21} = 0$

Example 1 *[14] Consider the Feldaum problem of minimizing*

$$\min_u \; J(y,u) = \frac{1}{2}\int_0^1 (y+u^2)dt \tag{35}$$

subject to

$$y'(t) = -y(t) + u(t), \quad y(0) = 1 \tag{36}$$

and analytical solution:

$$y(t) = -\frac{1}{2} + \frac{1}{4}e^{t-1} + e^{-t}(\frac{3}{2} - \frac{1}{4}e^{-1}) \tag{37}$$

$$u(t) = -0.5(1 - e^{t-1}) \tag{38}$$

In Table 4, we compare the results of the third order extended one step method by the analytical solution, results obtained in [18] and RK methods of order 3. The numerical simulations of the optimal state and optimal control are given in Figure 1.

t	exact		EOSM		[18]		R-K	
	$u(t)$	$y(t)$	$u(t)$	$y(t)$	$u(t)$	$y(t)$	u(t)	y(t)
0	-0.3161	1.0000	-0.3151	1.0000	-0.38582	1.0000	-0.3952	1.0000
0.2	-0.2753	0.7651	-0.2744	0.7666	-0.27687	0.75939	-0.3569	0.7842
0.4	-0.2256	0.5810	-0.2248	0.5837	-0.19023	0.57994	-0.3044	0.6091
0.6	-0.1648	0.4403	-0.1642	0.4438	-0.11889	0.44720	-0.2326	0.4710
0.8	-0.0906	0.3374	-0.0902	0.3416	-0.05714	0.35047	-0.1344	0.3677
1	0	0.2680	0	0.2728	0	0.28196	0	0.2984

Table 4: values of the controls and states for example 1.

Example 2 *([2]) Consider the following optimal control problem*

$$\min_u \; J(y,u) = \int_0^1 y_2(t) + u(t)^2 dt \tag{39}$$

subject to

$$y_1'(t) = y_2(t), \quad y_1(0) = 0, y_1(1) = 1. \tag{40}$$

$$y_2'(t) = u(t), \quad y_2(0) = 0.$$

In this example, we have 2 state variables y_1, y_2 and control u. For each state equation, there is one associated adjoint equation.If each state variable has two conditions (as an initial and a final time condition), then the adjoint variable associated with that state trajectory will have no transversality condition. The optimal states y_1 and y_2 are shown in Figure 2.

Example 3 *([2]) Consider the following optimal control with bounded control*

$$\max_{u1,u_2} \int_0^1 y(t) - \frac{1}{8}u_1(t)^2 - \frac{1}{2}u_2(t)^2 dt \tag{41}$$

subject to

$$y_1'(t) = u_1(t) + u_2(t), \quad y(0) = 0, \tag{42}$$

$$1 \le u_1(t) \le 2.$$

The optimal controls u_1 and u_2 are given in Figure 3.

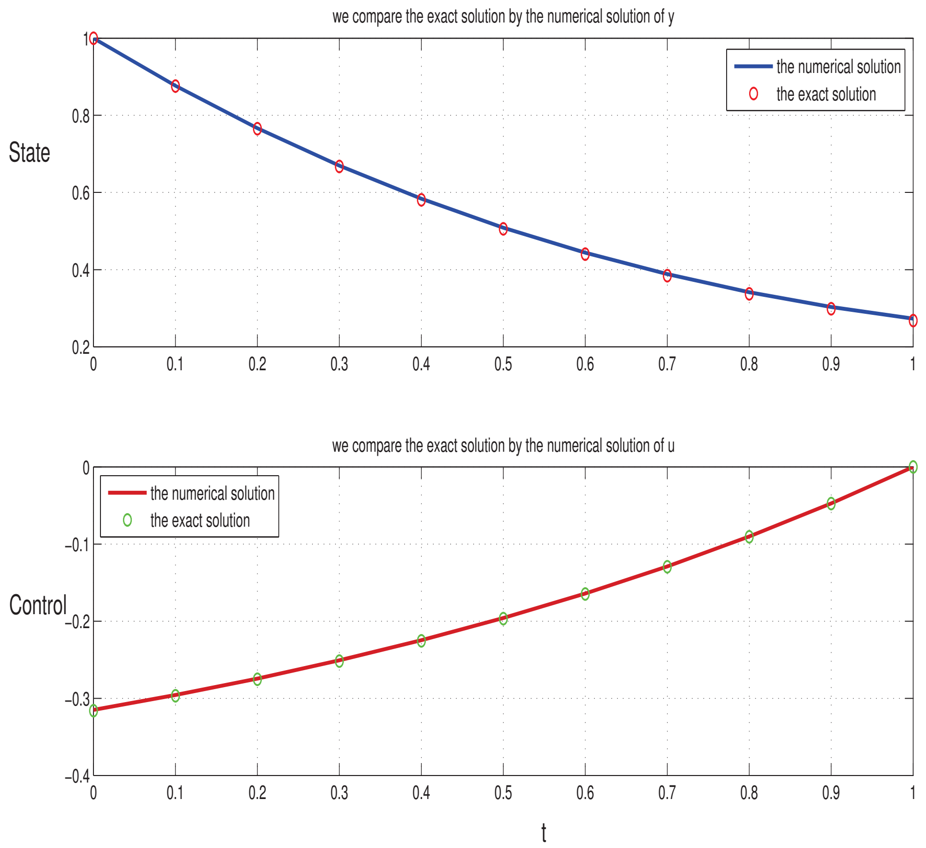

Figure 1: Optimal control and state for example 1.

Example 4 *([17]) This is an example for the Bang-Bang Controls*

$$\max_{u} \ J(y,u) = \int_0^2 2y(t) - 3u(t)dt \tag{43}$$

subject to

$$y_1'(t) = y(t) + u(t), \quad y(0) = 5 \tag{44}$$
$$0 \leq u(t) \leq 2$$

If we view this as a simple population model with exponential growth, our aim to increase the population as much as possible and keeping the cost of the control down. The optimal control and state are shown in figure 4.

Example 5 *(see[24])*
Consider the following optimal control problem governed by delay differential equation with the delay $r = 1$ in the state and $s = 2$ in the control

$$\texttt{Minimize}\ J(y,u) = \int_0^3 (y^2 + u^2)dt \tag{45}$$

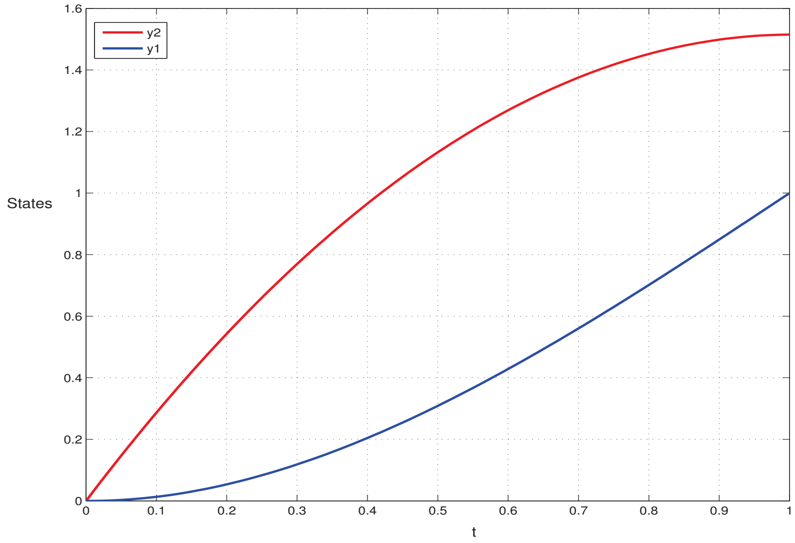

Figure 2: Optimal states for example 2.

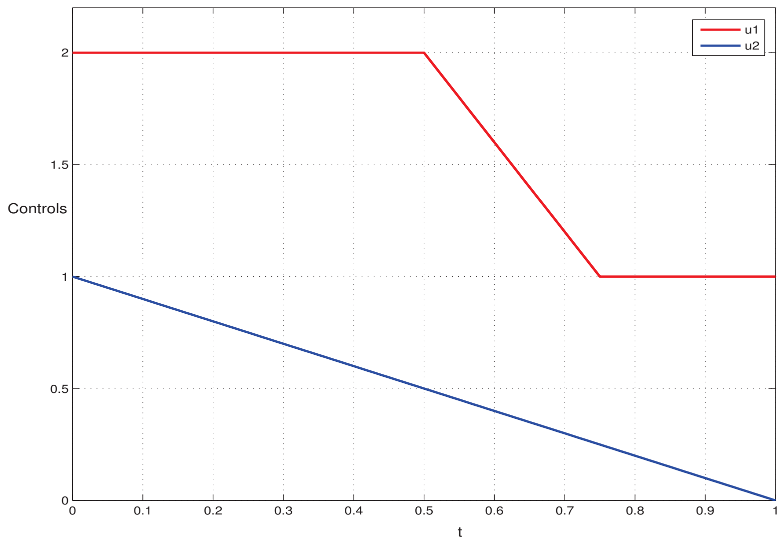

Figure 3: Optimal controls for Example 3.

subject to

$$y'(t) = y(t-1)u(t-2), \qquad t \in [0,3] \tag{46}$$

$$y(t) = 1, \qquad -1 \le t \le 0 \tag{47}$$

$$u(t) = 0, \qquad -2 \le t \le 0 \tag{48}$$

The optimality system is:

$$y'(t) = y(t-1)u(t-2), \qquad t \in [0,3] \tag{49}$$

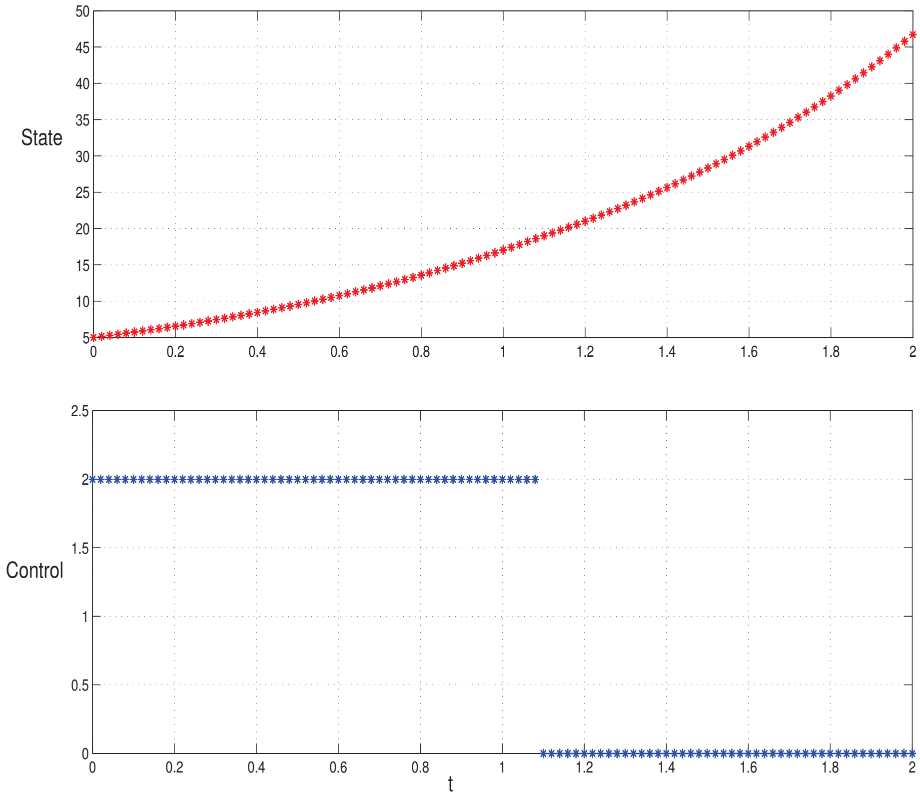

Figure 4: Optimal Control and state for example 4.

with initial state profile

$$y(t) = 1, \qquad -1 \le t \le 0 \tag{50}$$

The co-state equations

$$\lambda'(t) = \begin{cases} -2y(t) - \lambda(t+1)u(t-1), & t \in [0,2] \\ -2y(t), & t \in [2,3] \end{cases} \tag{51}$$

with the transversality condition

$$\lambda(3) = 0$$

and the control

$$u(t) = \begin{cases} -\frac{1}{2}\lambda(t+2), & t \in [0,1] \\ 0, & t \in [1,3] \end{cases} \tag{52}$$

The optimal state and optimal obtained by applying the extended one step method comapred by the analytical solution are given in Figure 5

Example 6 *(see[25])*

$$\texttt{Minimize } J(y,u) = \frac{1}{2}\int_0^5 (10y_1^2 + y_2^2 + u^2)dt \tag{53}$$

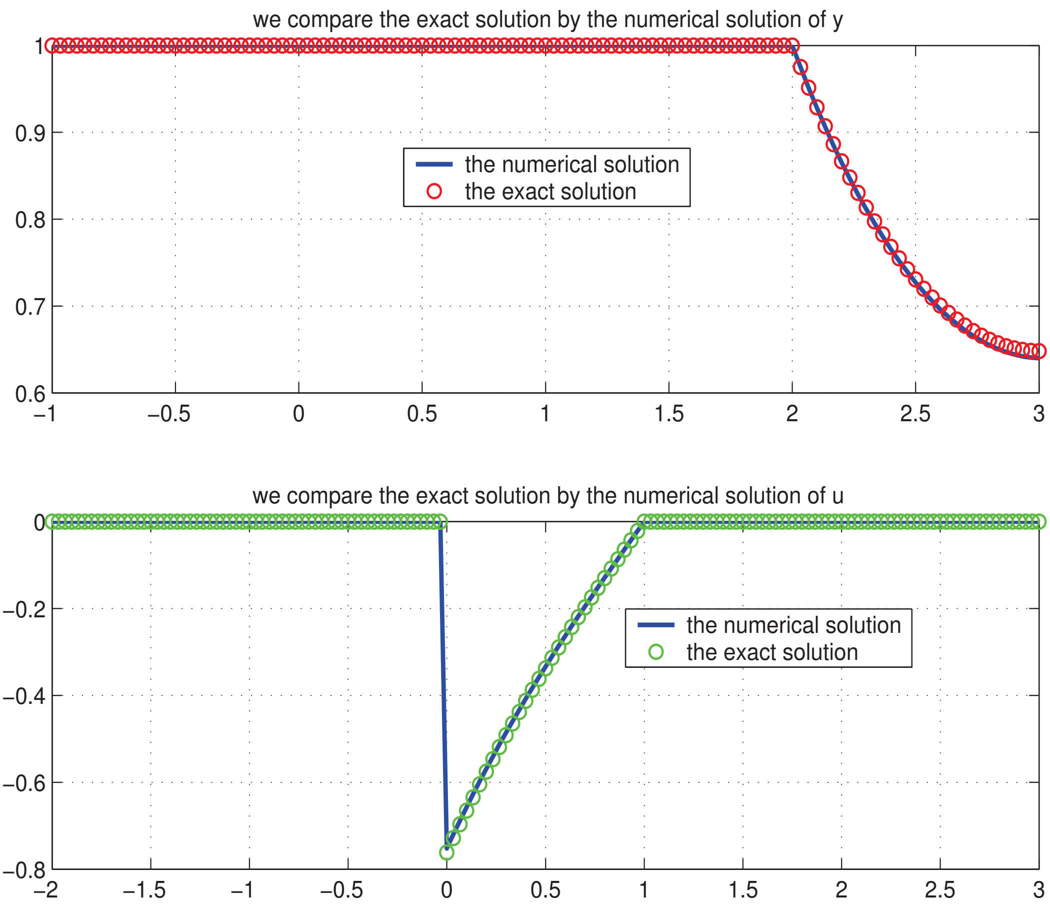

Figure 5: Comparison of the numerical solution of the optimal control and state by the exact soltion

subject to

$$y_1'(t) = y_2(t) \tag{54}$$

$$y_2'(t) = -10y_1(t) - 5y_2(t) - 2y_1(t-\tau) - y_2(t-\tau) + u(t) \tag{55}$$

with initial state profile

$$y_1(t) = y_2(t) = 1, \qquad -\tau \leq t \leq 0 \tag{56}$$

The optimality system is:

$$y_1'(t) = y_2(t) \tag{57}$$

$$y_2'(t) = -10y_1(t) - 5y_2(t) - 2y_1(t-\tau) - y_2(t-\tau) + u(t) \tag{58}$$

with initial state profile

$$y_1(t) = y_2(t) = 1, \qquad -\tau \leq t \leq 0 \tag{59}$$

The co-state equations

$$\lambda_1'(t) = \begin{cases} -10y_1(t) + 10\lambda_2(t) + 2\lambda_2(t+\tau), & t \in [0, 5-\tau] \\ -10y_1(t) + 10\lambda_2(t), & t \in [5-\tau, 5] \end{cases} \tag{60}$$

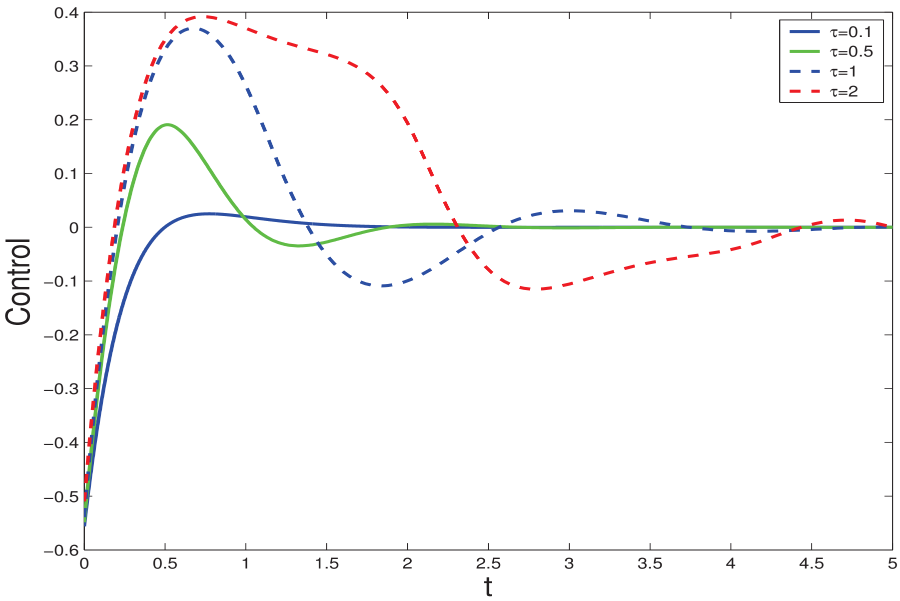

Figure 6: The curve of optimal control for example 6 under different values of delay.

$$\lambda_2'(t) = \begin{cases} -y_2(t) - \lambda_1(t) + 5\lambda_2(t) + 2\lambda_2(t+\tau), & t \in [0, 5-\tau] \\ -y_2(t) - \lambda_1(t) + 5\lambda_2(t), & t \in [5-\tau, 5] \end{cases} \tag{61}$$

with the transversality condition

$$\lambda_i(t_f)^* = 0, \qquad i = 1, 2$$

and the control equation

$$u(t) = -\lambda_2(t), \qquad t \in [0, 5] \tag{62}$$

This optimal control problem is solved for different values of τ , namely 0.1,0.5, 1 and 2 using the extended one-step method; See Figure 7-6

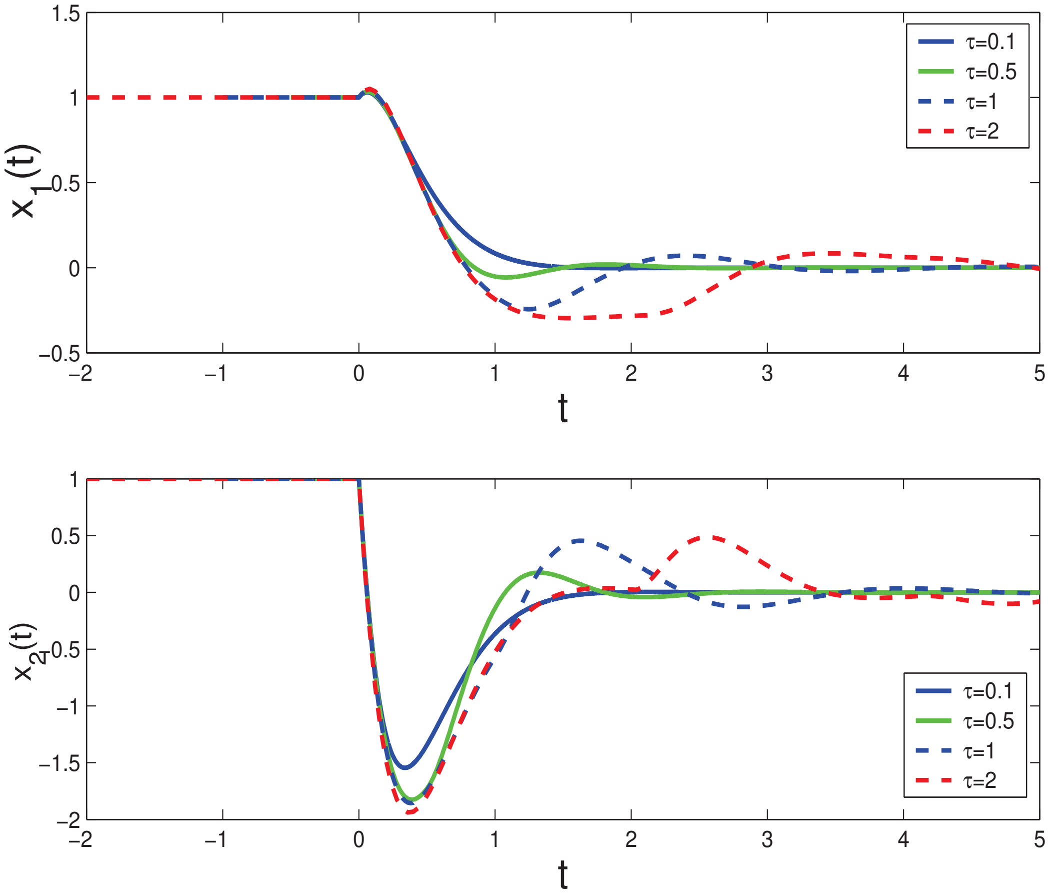

Figure 7: The curve of states for example 6 with different values of delay.

7 Conclusion

In this paper, we provided a new numerical method, based on extended one-step schemes, for solving optimal control problems governed by ODEs and DDEs. All the results have been obtained by applying the third order extended one step method with $\beta_{21} = 0$. We presented different problem reformulations and compared the performance w.r.t exact solutions. The suggested method is suitable and efficient for optimal control systems governed by both ordinary and delay differential equations.

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