On the meaning of the Nash product

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Abstract

The paper provides two alternative completions of the Pareto ordering on finite dimensional compact sets. Applied to bargaining games they lead to the Pareto efficient boundary and to the Nash solution, respectively, as sets of maximal elements. In particular, the second of these complete preorderings is represented by the Nash product. This provides an interesting "straightforward interpretation" that the Nash product according to Osborne and Rubinstein (1994, p. 303) is lacking.

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1 Introduction

The Nash bargaining solution is the most prominent solution concept for bargaining games. Introduced by John Nash (1953) it had been characterized in many different ways. It seems to be the undisputed favorite in the applied literature for labour negotiations and all kind of bargaining situations. The simplest way of defining or describing it has been provided already by Nash (1953) in addition to his axiomatic characterization, namely as the unique maximizer of the so called "Nash product".

While all other characterizations, via axioms or via support by equilibria of non-cooperative games appear to reflect different aspects of this solution and to open new ways of interpretation its most simple description via the Nash product seems to have escaped up to now a meaningful interpretation. It is true that interpreted as a social planner's utility function it represents the same social preference as the sum of the logarithms of utilities reflecting specific expected utilities. Also, it represents the social preference of a neutral benevolent planner that neglects positive affine transformations of players' utilities and a priori needs not be be either complete or transitive (cf. Trockel 1999). Yet, concerning its direct interpretation the situation is best described by the quotation of Osborne and Rubinstein (1994, p. 303):

"Although the maximization of a product of utilities is a simple mathematical operation it lacks a straightforward interpretation; we view it simply as a technical device."

It is the purpose of the present note to provide one straightforward, in fact surprising interpretation. Maximizing the Nash product is equivalent to finding maximal elements of one natural completion of the Pareto ordering. The maximal elements for the other natural completion are just the Pareto optimal points.

2 The model

We shall look at the vector ordering and complete preorderings on compact subsets of $\mathbb{R}^n$ but restrict the analysis without loss of generality to the case $n = 2$.

A complete preordering $\succsim$ on a compact set S is a complete, transitive (hence reflexive) binary relation on S. The weak vector ordering $\geq$ in contrast fails to be complete. It is, however, transitive, too.

To make things simple assume $\succsim$ on S to be continuous, hence representable by a continuous utility function $u : S \longrightarrow \mathbb{R}$. The $\succsim$-maximal elements are given by the set of maximizers of u on S, i.e. $argmax_{x \in\ S}\ u(x)$.

Let $B_{\succsim}(x)$ be the set $\{x' \in S | x' \succsim x\}$ and $W_{\succsim}(x)$ the set $\{x' \in S | x \succsim x'\}$.

For any $x', x \in S$ we obviously have:

$x' \sim x \Leftrightarrow \lambda(B_{\succsim}(x)) = \lambda(B_{\succsim}(x')) \Leftrightarrow \lambda(W_{\succsim}(x)) = \lambda(W_{\succsim}(x'))$.

λ denotes the Lebesgue measure on $\mathbb{R}^2$ the extension of the natural measure of area in $\mathbb{R}^2$ to all Lebesgue measurable sets.

The correspondences $B_{\succsim} : S \Longrightarrow S$ and $W_{\succsim} : S \Longrightarrow S$ composed with the Lebesgue measure define alternative utility functions $\lambda \circ B_{\succsim}$ and $\lambda \circ W_{\succsim}$ representing $\succsim$ as well as u.

Now consider for the vector ordering $\geq$ the analogous sets $B_{\geq}(x)$, $W_{\geq}(x)$ for arbitrary $x \in S$.

$B_{\geq}(x) = \{x' \in S | x' \geq x\}$, $W_{\geq}(x) = \{x' \in S | x \geq x'\}$.

Next, introduce the mappings $\lambda \circ B_{\geq}$ and $\lambda \circ W_{\geq}$ defined by:

$\lambda \circ B_{\geq}(x) := \lambda(B_{\geq}(x))$ and $\lambda \circ W_{\geq}(x) := \lambda(W_{\geq}(x))$.

Both are mappings from S to $\mathbb{R}$ and define therefore preference relations that are completions of $\geq$.

We have $x \geq x' \Longrightarrow \lambda(B_{\geq}(x)) \leq \lambda(B_{\geq}(x'))$ and $x \geq x' \Longrightarrow \lambda(W_{\geq}(x)) \geq \lambda(W_{\geq}(x'))$.

The two dual completions of $\geq$ are different in general.

$x \succsim_1 x' :\Longleftrightarrow \lambda(B_{\geq}(x)) \leq \lambda(B_{\geq}(x'))$

$x \succsim_2 x' :\Longleftrightarrow \lambda(W_{\geq}(x)) \geq \lambda(W_{\geq}(x'))$.

They only coincide when the binary relation one starts with is already a complete preordering. Notice, that $B_{\geq}(x)$ and $W_{\geq}(x)$ are in general proper subsets of $B_{\succsim_1}(x)$ and $W_{\succsim_2}(x)$, respectively.

3 Application to bargaining games

To keep things simple we define a normalized two-person bargaining game S as the subgraph of a concave strictly decreasing function f from $[0, 1]$ onto $[0, 1]$.

The two axes represent the players' utilities, S the feasible set of utility allocations. The vector ordering on S represents in this framework the Pareto ordering.

The efficient boundary $graphf$ of S is the set of Pareto optimal points or vector maxima. Obviously each point x in $graphf$ minimizes the value of $\lambda \circ B_{\geq}$. In fact, for $x \in graphf$ we have $\lambda(B_{\geq}(x)) = 0$

Notice that $\lambda(W_{\geq}(x))$ takes different values when x varies in $graphf$.

Now, consider the set $argmax_{x \in S}\ \lambda(W_{\geq}(x))$ of maximizers of $\lambda \circ W_{\geq}$. This set is exactly the set $\{N(S)\}$ where $N(S)$ is the Nash solution of S. Maximizing the Nash product $x_1 x_2$ for $x \in S$ means maximizing the measure of points in S Pareto dominated by x'.

Hence the two completions $\succsim_1, \succsim_2$ of the Pareto ordering $\geq$ on S have as their sets of maximizers the Pareto efficient boundary and the Nash solution, respectively.

4 Concluding remarks

We have shown that two different methods of representing complete preorderings via the measure of better sets versus worse sets may be applied as well to incomplete binary relations. Here they lead to two different functions inducing two different complete preorderings.

Applied to the non-complete Pareto ordering on a compact set S representing a bargaining situation the two completions have as their respective sets of maximizers the Pareto efficient boundary and the Nash solution of S.

This result provides a straightforward interesting interpretation of the Nash solution as a dual version of Pareto optimality. In contrast to the latter it has the advantage to single out a unique point in the efficient boundary.

The idea of defining rankings by counting the less preferred alternatives has an old tradition in social choice theory as the famous Borda Count (cf. Borda 1781) shows. In our context with a continuum of social alternatives counting is replaced by measuring. The level sets of the Nash product collect those utility allocations dominating equally large (in terms of Lebesgue measure) sets of alternatives.

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