The Maschler–Perles Solution: 2 Simple Proofs for Superadditivity

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Abstract

The Maschler–Perles Solution is the unique bargaining solution which is superadditive and satisfies the usual covariance properties. We provide two proofs for supperadditivity that do not rely on the standard traveling time.

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basic

1 The MASCHLER–PERLES Solution

The MASCHLER–PERLES bargaining solution (MASCHLER–PERLES [2], [3], see also [5] for a textbook presentation) is a mapping defined on 2–dimensional bargaining problems respecting anonymity, Pareto efficiency, and affine transformations of utility. Moreover, this mapping is *superadditive* by which property it is uniquely characterized.

The solution is based on the observation that every polyhedral bargaining solution in $\mathbb{R}^2$ is an (algebraic) sum of "elementary" bargaining problems that are generated by a line segment (thus reflect constant transfer of utility).

By continuity with respect to the Hausdorff metric the solution may be transferred to bargaining problems with a smooth Pareto curve.

More precisely let $\boldsymbol{a} = (a_1, a_2) > 0 \in \mathbb{R}^n$. We introduce the unit vectors $\boldsymbol{e}^i$ as well as the vectors $\boldsymbol{a}^i := a_i \boldsymbol{e}^i \quad (i \in \boldsymbol{I})$ and associate with $\boldsymbol{a}$ the ***triangle*** $\Pi^{\boldsymbol{a}}$ which is given by

$$\Pi^{\boldsymbol{a}} := \operatorname{convH}\left(\{\boldsymbol{0}, \boldsymbol{a}^1, \boldsymbol{a}^2, \}\right). \tag{1.1}$$

The Pareto curve of this triangle is the ***line segment*** $\Delta^{\boldsymbol{a}}$ which is given by

$$\Delta^{\boldsymbol{a}} := \operatorname{convH}\left(\{\boldsymbol{a}^1, \boldsymbol{a}^2\}\right). \tag{1.2}$$

A pair $(\boldsymbol{0}, U)$ with a compact, convex, and comprehensive subset $\emptyset \neq U \subseteq \mathbb{R}^2_+$ is interpreted as a ***bargaining problem*** where $\boldsymbol{0}$ is the ***status quo point*** and U the feasible set. We can omit reference to the status quo point as all concepts are covariant with "affine transformations of utility". A bargaining problem is ***polyhedral*** if the Pareto surface consists of line segments only and this is equivalent to a feasible set given by

repr

$$\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{a^{(k)}} \tag{1.3}$$

with a family of positive vectors

$$\left(\boldsymbol{a}^{(k)}\right)_{k \in \boldsymbol{K}}, \quad \boldsymbol{K} := \{1, \ldots, K\}$$

As it is sufficient to establish matters on a dense subset with respect to the Hausdorff topology, we can restrict ourselves to generating vectors $\boldsymbol{a}^{(k)}$ which have *dyadic coordinates.*

A triangle $\Pi^{\boldsymbol{a}}$ my be represented as ("homothetic") sum of two of its copies shrinked by suitable factor. In particular, we have

$$\Pi^{\boldsymbol{a}} \;=\; \frac{1}{2}\Pi^{\boldsymbol{a}} + \frac{1}{2}\Pi^{\boldsymbol{a}} \;=\; \Pi^{\frac{1}{2}\boldsymbol{a}} + \Pi^{\frac{1}{2}\boldsymbol{a}}.$$

By this operation the ***area*** $V(\boldsymbol{a}) := a_1 a_2 = \frac{1}{2}$ area $(\Pi^{\boldsymbol{a}})$ is divided by 4, i.e.,

$$V(\frac{1}{2}\boldsymbol{a}) \;=\; \frac{1}{4}V(\boldsymbol{a}).$$

Therefore, we may assume that all triangles involved in a representation (1.3) have equal volume. The bargaining problems having this property again form a dense subset of the set of all bargaining problems. Similarly, whenever we deal with the sum of two bargaining problems, we may assume that the summands as well as the sum are dyadic w.r. to the same dyadic basis.

Definition 1.1. *We call a bargaining problem* ***standard dyadic*** *if the feasible set is a polyhedron represented as in* (1.3) *with all vectors having dyadic coordinates and generating equal volume.*

It is frequently useful to assume that the enumeration of the triangles is such that the *tangents* (i.e., the quotients $\frac{a_2^{(k)}}{a_1^{(k)}}$) are deacreasing with the index k. The Maschler–Perles solution for a standard dyadic bargaining problem is then defined inductively as follows: For $K = 1$ it is the midpoint of the line segment (the Pareto curve). For $K = 2$ (and assuming that the two triangles are not homothetic) it is the unique vertex of $\Pi \;=\; \Pi^{(1)} + \Pi^{(2)}$. For $K \geq 3$ it is defined by

MPdefs (1.4)

$$\begin{aligned} \boldsymbol{\mu}(\Pi) \;&=\; \boldsymbol{\mu}\left(\sum_{k\in\boldsymbol{K}} \Pi^{a^{(k)}}\right) \\ &:=\; \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(K)}\right) + \boldsymbol{\mu}\left(\sum_{k\in\boldsymbol{K}-\{1,K\}} \Pi^{a^{(k)}}\right). \end{aligned} \tag{1.4}$$

The last formula in fact implies uniquenes of the solution on standard dyadic bargaining problems. For, every superadditive solution $\boldsymbol{\mu}$ is necessarily additive whenever the solutions of the two summands admit of a joint normal. In our enumeration, the first polyhedron $\Pi^{(1)} + \Pi^{(K)}$ admits of a joint normal with *every* Pareto efficient point of the second polyhedron.

2 Superadditivity

super

The recursive definition of the Maschler–Perles solution shows uniqueness at once. The fact that the solution is superadditive is proved by MASCHLER–PERLES [2], [3] using the concept of the "traveling time". We wish to provide two simple proofs that do not hinge on this concept.

Theorem 2.1 (see [2], [3]). *Let* Π *be a dyadic polyhedron such that*

$$\Pi \;=\; \sum_{k\in\boldsymbol{K}} \Pi^{a^{(k)}} \tag{2.1}$$

holds true. Let

$$\Pi \;=\; \Upsilon + \Psi \tag{2.2}$$

where

$$\Upsilon \;=\; \sum_{k\in\boldsymbol{I}} \Pi^{a^{(k)}}\;, \quad \Psi \;=\; \sum_{k\in\boldsymbol{J}} \Pi^{a^{(k)}}. \tag{2.3}$$

Then

$$\boldsymbol{\mu}(\Pi) \geq \boldsymbol{\mu}(\Upsilon) + \boldsymbol{\mu}(\Psi). \tag{2.4}$$

The first proof hinges on induction, thus it is close to the definition of the solution as discussed in SECTION 1.

Proof: If Π is the sum of two polyhedra (w.l.g. not homothetic) with equal volume, then $\boldsymbol{\mu}(\Pi)$ is the unique vertex on the Pareto surface of Π while $\boldsymbol{\mu}(\Upsilon) + \boldsymbol{\mu}(\Psi)$ is a non–Pareto efficient point on the line connectiong $\mathbf{0}$ and $\boldsymbol{\mu}(\Pi)$.

In order to perform the induction step, assume that

$$\Pi \;=\; \Upsilon + \Psi, \quad \Upsilon \;=\; \sum_{k\in\boldsymbol{I}} \Pi^{a^{(k)}}\;, \quad \Psi \;=\; \sum_{k\in\boldsymbol{J}} \Pi^{a^{(k)}} \tag{2.5}$$

holds true.

1$^{\text{st}}$STEP : Assume that the indices 1 and K are jointly contained in one of the above sets, say $\{1, K\} \subseteq \boldsymbol{I}$. Then, as $\Pi^{(1)} + \Pi^{(K)}$ admits of joint normals

at $\boldsymbol{\mu}(\Pi^{(1)} + \Pi^{(K)})$ with all other polyhedra involved, we have
(2.6)

$$\begin{aligned}
\boldsymbol{\mu}(\Pi) &= \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(K)} + \sum_{k \in \boldsymbol{K}-\{1,K\}} \Pi^{a(k)}\right) \\
&= \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(K)}\right) + \boldsymbol{\mu}\left(\sum_{k \in \boldsymbol{K}-\{1,K\}} \Pi^{a(k)}\right) \\
&\qquad \text{(by Definition, see (1.4))} \\
&\geq \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(K)}\right) + \boldsymbol{\mu}\left(\sum_{k \in \boldsymbol{I}-\{1,K\}} \Pi^{a(k)}\right) + \boldsymbol{\mu}\left(\sum_{k \in \boldsymbol{J}} \Pi^{a(k)}\right) \\
&\qquad \text{(by induction hypothesis)} \\
&= \boldsymbol{\mu}\left(\sum_{k \in \boldsymbol{I}} \Pi^{a(k)}\right) + \boldsymbol{\mu}\left(\sum_{k \in \boldsymbol{J}} \Pi^{a(k)}\right) .
\end{aligned}$$

2nd STEP : Suppose now, that we have $1 \in \boldsymbol{I}$ and $K \in \boldsymbol{J}$. Let L denote the largest index in $\boldsymbol{I}$, i.e., the one wich induces the largest slope (in absolute value) of a line segment involved in Υ. Then we obtain

$$\begin{aligned}
\boldsymbol{\mu}(\Pi) &= \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(L)} + \sum_{k \in \boldsymbol{I}-\{1,L\}} \Pi^{a(k)} + \sum_{k \in \boldsymbol{J}} \Pi^{a(k)}\right) \\
&\geq \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(L)} + \sum_{k \in \boldsymbol{J}} \Pi^{a(k)}\right) + \boldsymbol{\mu}\left(\sum_{k \in \boldsymbol{I}-\{1,L\}} \Pi^{a(k)}\right) \\
&\qquad \text{(by the } 1^{st} STEP \text{ as} \{1,K\} \subseteq \boldsymbol{J} + \{1,L\}) \\
&\geq \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(L)}\right) + \boldsymbol{\mu}\left(\sum_{k \in \boldsymbol{J}} \Pi^{a(k)}\right) + \boldsymbol{\mu}\left(\sum_{k \in \boldsymbol{I}-\{1,L\}} \Pi^{a(k)}\right) \qquad (2.7) \\
&\qquad \text{(by induction hypothesis)} \\
&= \boldsymbol{\mu}\left(\sum_{k \in \boldsymbol{J}} \Pi^{a(k)}\right) + \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(L)} \sum_{k \in \boldsymbol{I}-\{1,L\}} \Pi^{a(k)}\right) \\
&\qquad \text{(by Definition applied to } \Upsilon \text{, see (1.4))} \\
&= \boldsymbol{\mu}\{\Upsilon\} + \boldsymbol{\mu}\{\Psi\},
\end{aligned}$$

q.e.d.

The second proof refers to the construction of the solution.

Proof: The enumeration is such that the tangent slope decreases with the index k. Since the *products* $a_1^{(k)}a_2^{(k)}$ are all equal, it follows that the enumeration satisfies

$$\begin{array}{c} a_1^{(1)} \geq a_1^{(2)} \ \ldots \quad \geq a_1^{(K)} \\ a_2^{(1)} \leq a_2^{(2)} \ \ldots \quad \leq a_2^{(K)} \end{array} \quad , \tag{2.8}$$

W.l.o.g we may assume that K is even (otherwise split every polyhedron homothetically in two). Then we know that

largest

$$\boldsymbol{\mu}(\Pi) = \left(\sum_{k=1}^{\frac{K}{2}} a_1^{(k)} \ , \ \sum_{k=\frac{K}{2}+1}^{K} a_2^{(k)} \right) , \tag{2.9}$$

that is, $\boldsymbol{\mu}(\Pi)$ *collects the* $\frac{K}{2}$ *largest vectors* with respect to each coordinate.

Now with respect to Υ we may as well assume that $|\boldsymbol{I}|$ is even. Thus, there is a decomposition $\boldsymbol{I} = \boldsymbol{I}_1 + \boldsymbol{I}_2$ with $|\boldsymbol{I}_1| = |\boldsymbol{I}_2|$ such that

$$\boldsymbol{\mu}(\Upsilon) = \left(\sum_{k \in \boldsymbol{I}_1} a_1^{(k)} \ , \ \sum_{k \in \boldsymbol{I}_2} a_2^{(k)} \right) . \tag{2.10}$$

The same holds true concerning Ψ with respect to a decomposition $\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2$. Clearly, $|\boldsymbol{I}_1 + \boldsymbol{J}_1| = |\boldsymbol{I}_2 + \boldsymbol{J}_2| = \frac{K}{2}$ and hence

$$\boldsymbol{\mu}_1(\Upsilon) + \boldsymbol{\mu}_1(\Psi) = \sum_{k \in \boldsymbol{I}_1} a_1^{(k)} + \sum_{k \in \boldsymbol{J}_1} a_1^{(k)} = \sum_{k \in \boldsymbol{I}_1 + \boldsymbol{J}_1} a_1^{(k)} \leq \sum_{k=1}^{\frac{K}{2}} a_1^{(k)} \tag{2.11}$$

as the last sum collects the largest $\frac{K}{2}$ coordinates (see (2.9)),

q.e.d.

References

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