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On the dynamics of capital accumulation across space *

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Abstract

We solve an optimal growth model in continuous space, continuous and bounded time. The optimizer chooses the optimal trajectories of capital and consumption across space and time by maximizing an objective function with both space and time discounting. We extract the corresponding Pontryagin conditions and prove their sufficiency. We end up with a system of two parabolic differential equations with the corresponding boundary conditions. Then, we study the roles of initial capital and technology distributions over space in various scenarios.

Keywords: Economics, Economic geography, Parabolic Differential Equations, Optimal Control

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1 Introduction

Economic geography studies the distribution over space of economic activities. In particular, it studies how and why firms and workers make their location choices, and notably the reasons underlying production agglomeration, the formation of cities and migration flows. Early contributions in economic geography were based on spatial flow equations of either goods or people (see Beckman, 1952 or more recently, Puu, 1982, Beckman and Puu, 1985, and Ten Raa, 1986). In Beckman and Puu (1985) goods production is local while goods and labor flow in neighboring locations according to a gradient process. That is, workers migrate following a preferred direction, and arrival locations provide them with a higher level of well-being.

Last decade, the emergence of the new economic geography challenged the early regional science contributions. The new economic geography models (see Krugman, 1991 and 1993, Fujita, Krugman and Venables, 1999, and Fujita and Thisse, 2002) use a refined specification of local and global market structures, and some precise assumptions on the mobility of production factors, which basically are machinery and workers. These are general equilibrium models seeking to explain production, consumption and price formation in a whole economy. Their usefulness in explaining the mechanics of economic activity agglomeration, the formation of cities, the determinants and implications of migrations, and more generally, the dynamics of the distributions of people and goods over space and time is undeniable, so undeniable that this discipline has become increasingly popular in the recent years.

The contributions quoted above have in common a discrete space structure. We believe that the alternative assumption of a continuous space structure fits better modern economies, since this structure implies that all locations have access to goods (see Beckmann, 1952). Some continuous space extensions of these models have been already studied (Krugman, 1996 and Mossay, 2003). However useful they are, there is still a major barrier to economic development. In all the papers just mentioned, economic agents consume all their revenue. This implies zero savings at any time, so there is zero investment. Therefore, these model economies are not suitable to study inter-spatial investment. Besides, they neglect economic growth since all production plants always have identical quantities of machinery and technology.

In this paper we propose a framework to study economic growth in a setting with continuous time and space. We assume that there exists an infinite number of locations where economic activity takes place, and that there is a household in each location.

We assume that there exists a policy maker who maximizes consumers well-being. A consumer well-being depends exclusively on her consumption level. There is a household at each point in space or location, who offers her labor force in exchange for a salary, which reflects the worker's marginal contribution to production. Once households receive their revenue, they decide how to spend it, or how to split it between consumption and investment. In our framework we assume that while immobile, households are allowed to invest in all firms of space. By this assumption, we are reproducing the observed feature that capital is much more mobile than labor. When the households take their investment decisions, they compare the return of their possible investment at each location (that changes with time) and choose the most profitable. Investment returns depend on the firm technology and its marginal productivity, which may vary across locations.

The policy maker maximizes consumer welfare, where welfare is measured by a concave function and the household's budget is governed by a Parabolic Differential Equation (PDE hereafter) that describes the behavior of physical capital across time and space depending on production, consumption, depreciation and the import-export balance (we offer a detailed description in section 2.1). The policy maker faces then an optimal control problem with PDE which is usually referred to as the Ramsey model with space. If the utility function is strictly concave, then this problem is ill-posed in the sense of Hadamard (1923): neither the existence nor the uniqueness of the solutions can be assured. If we want to solve the policy maker optimization problem, we would have to search for tools to overcome or deal with this problem.

The Ramsey model with continuous time and space was first proposed by Brito (2004), with a non-Benthamian utility function[1]. He studies the existence of special solutions to the dynamic system arising from optimization: travelling waves. Camacho, Zou and Boucekkine (2005) introduce spatial discounting to study a Benthamian Ramsey model. In this paper, it is assumed that households have a linear utility function. This allows the authors to distinguish three kinds of optimal solutions (two corner solutions and an interior solution) and they study convergence from a corner solution to the interior solution.

In the present paper, we keep a standard concave utility function for households and we overcome the ill-posedness by reducing the time horizon to $[0, T]$, with $T < \infty$. Under

[1] A Benthamian utility function is a weighted sum of the utilities of the consumers populating the economy.

specific assumptions, we can assure the existence and uniqueness of the solution to the optimal control problem with PDE of the policy maker. Furthermore, this solution is explicit (see Theorem 1).

In section 3 we present our computational set-up. To achieve this section, we simulate numerically the system of PDEs resulting from the Pontryagin conditions. The system includes the PDE state equation and the PDE Pontryagin condition, plus the corresponding boundary conditions. There exists some well-known methods to simulate one PDE, but the simulation of a system of PDEs is an open question. We use a relaxation method for one equation: given the initial distribution of capital, we assume a spatial distribution of consumption. We solve the state equation with a finite difference approximation. With this solution, we update the consumption trajectory using the PDE Pontryagin condition... and so forth. Thanks to this simulation method we can solve our model and obtain several interesting economic results. We start studying the long term importance of initial distribution of wealth. If locations only differ in their initial capital endowment, then the long-run distribution of capital is homogenous. In a second exercise we introduce another source of spatial heterogeneity, the geographical center would be a technological pole. In this case locations using the most advanced technologies will retain the economic leadership in the long-term.

The paper is organized as follows. Section 2 states our general spatial Ramsey model with some economic motivations. It explains the capital dynamics and derives the Pontryagin conditions associated to the optimal control problem with PDE using the recent related mathematical literature. Furthermore, it is shown that these conditions are necessary and sufficient. Section 3 presents the computational set-up and section 4 an extensive economic analysis. Section 5 concludes.

2 The model

We suppose that there exists one household at each location and that households do not migrate. We assume that the labor market is competitive and there is only one final good, which can be assigned to consumption or investment and plays the role of numeraire. There exists perfect mobility of capital.

2.1 Capital flows

We now turn to describe how capital flows from a location to another. Hereafter we denote by $k(x,t)$ the capital stock held by the representative household located at x at date t.

In contrast to the standard Ramsey model, the law of motion of capital does not rely entirely on the saving capacity of the economy under consideration: The net flows of capital to a given location or space interval should also be accounted for. Suppose that the technology at work in location x is simply $y(x,t) = A(x,t)f(k(x,t))$, where $A(x,t)$ stands for total factor productivity at location x and date t, and $f(.)$ is the standard neoclassical production function, which satisfies the following assumptions:

(A1) $f(\cdot)$ is non-negative, increasing and concave;

(A2) $f(\cdot)$ verifies the Inada conditions, that is,

$$f(0) = 0, \quad \lim_{k\to 0} f'(k) = +\infty, \quad \lim_{k\to+\infty} f'(k) = 0.$$

Moreover we assume that the production function is the same whatever is the location. The budget constraint of household $x \in \mathbb{R}$ is

$$\frac{\partial k(x,t)}{\partial t} = A(x,t)f\left(k(x,t)\right) - \delta k(x,t) - c(x,t) + \tau(x,t),$$

where δ is the depreciation rate[2], c (≥ 0) is the consumption and τ is the net trade balance of household x at time t. By the assumption of homogenous depreciation rate of capital, τ is also the capital account balance, without adjustment cost and no arbitrage opportunities. Since the economy is closed, we have that

$$\int_{\mathbb{R}} \left(\frac{\partial k(x,t)}{\partial t} - A(x,t)f\left(k(x,t)\right) + c(x,t) + \tau(x,t) \right) dx = 0.$$

Then for any given region $X = [a,b] \subset \mathbb{R}$:

$$\int_{X} \left(\frac{\partial k(x,t)}{\partial t} - A(x,t)f\left(k(x,t)\right) + c(x,t) + \tau(x,t) \right) dx = 0. \tag{1}$$

Recall that capital movements tend to eliminate geographical differences. We assume that there are no institutional barriers to capital flows and that adjustment is instantaneous.

[2]Depreciation rate of capital is homogenous in time t, space x, and capital level k.

[3] [4]. Without inter-regional arbitrage opportunities, capital flows from regions with lower marginal productivity of capital to the higher ones. As a consequence, for any region $X = [a, b]$,

$$\int_X \tau(x,t)dx = -\left(\frac{\partial k(b,t)}{\partial x} - \frac{\partial k(a,t)}{\partial x}\right).$$

The net trade balanced in region $[a, b]$ equals the capital flows through the boundary points a and b. We also have that

$$-\left(\frac{\partial k(b,t)}{\partial x} - \frac{\partial k(a,t)}{\partial x}\right) = -\int_X \frac{\partial^2 k}{\partial x^2}dx.$$

Substituting the above equation into equation (1), we obtain that $\forall X \subset \mathbb{R}$, and $\forall t > 0$:

$$\int_X \left(\frac{\partial k(x,t)}{\partial t} - \frac{\partial^2 k(x,t)}{\partial x^2} - (s(x,t)A(x,t)f(k(x,t)) - \delta k(x,t))\right) dx = 0.$$

By the Hahn-Banach theorem, the budget constraint can be written as:

$$\frac{\partial k(x,t)}{\partial t} - \frac{\partial^2 k(x,t)}{\partial x^2} = s(x,t)A(x,t)f\left(k(x,t)\right) - \delta k(x,t),\ \forall (x,t). \tag{2}$$

In the following, we assume that the following assumption (A3) holds:

$$(A3)\quad \int_{\mathbb{R}} \frac{e^{-\frac{(y-x)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \left[A(y,\tau)f(k(y,\tau)) - \delta k(y,\tau) - c(y,\tau)\right] dyd\tau \geq 0, \quad x \in \mathbb{R},\ t > 0.$$

Notice that investment in any location x and at any time $t(> 0)$ $(i(x,t) = A(x,t)f(k(x,t)) - \delta k(x,t) - c(x,t))$ could be positive or negative. (A3) imposes that the "average accumulated " investment in the economy is nonnegative. Indeed, (A3) computes the average accumulated investment at location x and time t assuming that it follows a normal law of distribution with mean x and variation $\sqrt{t-\tau}$.

[3] We could assume that there exists institution barriers to capital flows (see Ten Raa(1986) and Puu(1982)). If they are independent of capital k and consumption c, we obtain a linear equation with coefficients in front of the Laplacean operator. After some affine transformations, results in section 2.2 would apply to this problem. Otherwise, if the barriers are functions of k and/or c, we face *nonlinear problems*, which are not considered in this work.

[4] Nonetheless, if we consider that transportation costs exist (which are delays), then we would obtain a *differential-difference* problem. We could consider a transportation cost proportional to output (the iceberg transportation cost). In this case, results in section 2.2 would still apply. In a more general case with space velocity, we would have to deal with a non-local problem which is out of the scope of this paper.

2.2 The optimal control problem

The policy maker is looking for a consumption profile $c(x,t)$ to

$$\max_{c} \int_0^T \int_{\mathbb{R}} U(c(x,t),x)e^{-\rho t} dt dx + \int_{\mathbb{R}} \phi\left(k(x,T),x\right) e^{-\rho T} dx, \tag{3}$$

subject to:

$$\begin{cases} \dfrac{\partial k(x,t)}{\partial t} - \dfrac{\partial^2 k}{\partial x^2}(x,t) = A(x,t) f\left(k(x,t)\right) - \delta k(x,t) - c(x,t), \quad (x,t) \in \mathbb{R} \times [0,T], \\ k(x,0) = k_0(x) > 0, \quad x \in \mathbb{R}, \\ \lim\limits_{x \to \pm\infty} \dfrac{\partial k(x,t)}{\partial x} = 0. \end{cases} \tag{4}$$

where $c(x,t)$ is the consumption level of a representative household located at x at time t, $x \in \mathbb{R}$ and $t \geq 0$, $U(c(x,t))$ is the instantaneous utility function and $\rho > 0$ stands for the time discounting rate. Function $\phi(.)$ is taken continuously differentiable, strictly increasing with respect to its first argument, and for example rapidly decreasing with respect to its second argument to assure the convergence of the second integral term. The initial distribution of capital, $k(x,0) \in C(\mathbb{R})$, is assumed to be a known positive bounded function, that is, $0 < k(x,0) \leq K_0 < \infty$. Moreover, we assume that, if the location is far away from the origin, there is no capital flow, that is

$$\lim_{x \to \pm\infty} \frac{\partial k(x,t)}{\partial x} = 0.$$

For a given location x, the utility function is standard, ie. $\frac{\partial U}{\partial c} > 0$, $\frac{\partial^2 U}{\partial^2 c} < 0$, and checking the Inada conditions. The modelling of the objective function follows Camacho et al. (2005): suppose that $U(c,x)$ is separable, $U(c,x) = V(c)\ \psi(x)$, with $V(.)$ a strictly increasing and concave function, and $\psi(x)$ an integrable and strictly positive function such that $\int_{\mathbb{R}} \psi(x) = 1$. In such case, the presence of x *via* $\psi(x)$ in the integrand of the objective function stands for the weight assigned to location x by the central planner in a world of homogenous individual preferences. Again, this assumption is most acceptable if one has in mind that in many cases the governments' concerns and actions are not uniform in space. For example, if the government is concerned with uneven regional development, she should assign more weight to the poor regions[5] . Further assumptions on the shape of preferences with respect to x will be done along the way.

Here comes the definition of an optimal solution:

[5] $\psi(x)$ could also be interpreted as a population density function à la Clark (see Clark 1951).

Definition 1 *A trajectory* $(c(x,t), k(x,t))$*, both functions in* $C^{2,1}(\mathbb{R}\times[0,T])$*, is admissible if* $k(x,t)$ *is a solution of the equation (2) with control* $c(x,t)$ *on* $t \in [0,T]$*,* $x \in \mathbb{R}$*, and if the integral objective function (1) converges. A trajectory* $(c^\star(x,t), k^\star(x,t))$*,* $t \geq 0$*,* $(x,t) \in (\mathbb{R} \times [0,T])$ *is an optimal solution of problem (3) and (4) if it is admissible and it it is optimal in the set of admissible trajectories, ie. for any admissible trajectory* $(c(x,t), k(x,t))$*, the value of the integral (3) is not greater than its value corresponding to* $(c^\star(x,t), k^\star(x,t))$*.*

It is not very hard to see that the shape of preferences is crucial for the convergence of the integral (3) when space is **unbounded**. In particular, just like time discounting is needed to ensure the convergence of the integral objective function in the standard Ramsey model, we need a kind of *space discounting*. Hence, a natural choice of $U(c,x)$ is to take it **rapidly decreasing** with respect to the second variable. That is, $U(c,x)$, for any fixed c, defined as,

$$\{U(c,\cdot) \in C(\mathbb{R}) |\ \forall m \in \mathcal{Z}_+,\ |x^m U(c,x)| \leq M_m, \forall x \in \mathbb{R}, M > 0\}.$$

A possible choice of $U(c,x)$ checking the above mentioned characteristic is $U(c,x) = V(c)\psi(x) = V(c)\ \frac{\phi}{2}\ e^{-\phi|x|}$, where $V(.)$ is strictly increasing and concave in c, and $\phi > 0$. Whatever the interpretation is, heterogenous individual preferences or non-uniform weighting of homogenous preferences by the central planner, the specification above depicts a kind of **preference for the center of the space**. Indeed, Fujita and Thisse (2002) refer to this property by **spatially discounted accessibility**.

Pao (1992) proves the existence of solutions to this kind of equations. In order to ensure uniqueness, we need one further assumption on growth when $x \to \pm\infty$:

(A4) For any given finite T, if $(x,t) \in \mathbb{R} \times [0,T]$, there are some constants $A_0 > 0, C_0 > 0, K_0 > 0$ and $b < \frac{1}{4T}$, such that, as $x \to \pm\infty$

$$0 < A(x,t) \leq A_0 e^{b|x^2|},\ 0 < c(x,t) \leq C_0 e^{b|x^2|},\ 0 < k_0(x) \leq K_0 e^{b|x^2|}.$$

Theorem 1 *Consider state equation (4), let assumptions (A1), (A2) hold and* $A, c \in C^{2,1}(\mathbb{R} \times [0,T])$*.*

(*a*) *Suppose (A4) holds, then problem (4) has a unique solution* $k \in C^{2,1}(\mathbb{R} \times [0,T])$*, given by*

$$\begin{aligned} k(x,t) &= \int_{\mathbb{R}} \Gamma(x-y,t)k_0(y)dy \\ &+ \int_0^t \int_{\mathbb{R}} \Gamma(x-y,t-\tau)\left[A(y,\tau)f(k(y,\tau)) - \delta k(y,\tau) - c(y,\tau)\right] dyd\tau. \end{aligned} \tag{5}$$

Moreover

$$|k| \leq K e^{b'|x|^2}, \text{ as } x \to \infty,$$

where K *is a positive constant, which depends only on* $A_0,\ K_0,\ C_0,\ T,\ b' \leq \min\{b, \frac{1}{4T}\}$ *and*

$$\Gamma(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{x^2}{4t}},\ t > 0, \\ 0,\ t \leq 0. \end{cases}$$

(*b*) *If (A3) also holds, then the solution given in* (*a*) *is nonnegative. Furthermore, for any* $x \in \mathbb{R}, t \in [0,T]$*:*

$$k(x,t) \geq \inf_{x \in \mathbb{R}} k_0(x) > 0.$$

Proof: **(a)** See Camacho, Zou and Boucekkine(2005). **(b)** is a direct result of (5), because of (A3). Q.E.D.

Remark 1. The solution provided in theorem 1 is the result of "trade". At any location x and at any time t, $k(x,t)$ not only depends on its past, but it also benefits from the whole economy and its history.

Remark 2. Assumption (A3) ensures us a positive capital accumulation process. It is possible to invest (or consume) more than output at some locations. The probability of infinite investment (or consumption) is zero.

2.3 Necessary and sufficient conditions for the finite horizon optimization problem

Theorem 2 (Pontryagin Conditions). *Suppose that assumptions (A1)–(A4) hold,* $A \in C^{2,1}(\mathbb{R} \times [0,T])$ *is nonnegative and it has a finite upper-bound, that is,* $0 \leq A \leq M_A < \infty$*. Suppose that* $c \in C^{2,1}(\mathbb{R} \times [0,T])$ *is an optimal control and* $k \in C^{2,1}(\mathbb{R} \times [0,T])$

is its corresponding state. Moreover if $k(x,t) > 0$, for all $(x,t) \in (\mathbb{R} \times [0,T])$, then there exists a unique function $q(x,t) \in C^{2,1}(\mathbb{R} \times [0,T])$ satisfying the following adjoint equation (or co-state equation)

$$\begin{cases} \dfrac{\partial q}{\partial t} + \dfrac{\partial^2 q}{\partial x^2} + q(x,t)\left(Af'\left(k(x,t)\right) - \delta - \rho\right) = 0, \quad (x,t) \in \mathbb{R} \times [0,T], \\ q(x,T) = \phi_1'\left(k(x,T), x\right), \forall x \in \mathbb{R}, \\ \lim\limits_{x \to \infty} \dfrac{\partial q}{\partial x} = \lim\limits_{x \to -\infty} \dfrac{\partial q}{\partial x} = 0. \end{cases} \tag{6}$$

Moreover $c(x,t) = (U_1')^{-1}(q(x,t), x)$,

Proof: The Pontryagin condition is deduced in Camacho et al. (2005). The proof of existence of solutions to equation (6) follows the same arguments as in Theorem 1, (see also J.L. Lions, 1971, and Raymond and Zidani, 1998 and 2000). The final claim comes from the implicit function theorem, using $\frac{\partial^2 U(c,x)}{\partial c^2} \neq 0$.

Remark 3. The strict positivity of $k(x,t)$ is required for the existence result.

Remark 4. The equality $q(x,T) = \phi_1'\left(k(x,T), x\right)$ $\forall x \in \mathbb{R}$, imposes a terminal condition for the shadow price of capital.

The above conditions are not only necessary, they are also sufficient:

Theorem 3 (Necessary and Sufficient Conditions). *Provided that (A1)-(A4) hold, $q(x,t) > 0, (x,t) \in (\mathbb{R} \times [0,T])$, and that the utility function is strictly increasing with respect to its first argument, then the Pontryagin conditions, obtained in Theorem 2, are also sufficient to the original optimal control problem with finite horizon.*

Proof: See appendix (or in another context, Gozzi and Tessitore, 1998).

Furthermore, J.L.Lions (1971) and Raymond J.P. and Zidani H.(1998, 2000) showed the following existence result for finite time problem:

Theorem 4 *Suppose that (A1)-(A4) hold and $k(x,t)$ is always positive for $(x,t) \in \mathbb{R} \times [0,T]$, then the partial differential equations system (4)-(6) with the associated initial-terminal boundary conditions has at least one solution.*

3 Computational Setting

Let us analyze the dynamics of investment across space and time. Repeating (2) and (4), the evolution of physical capital and its shadow price are governed by:

$$\begin{cases} \dfrac{\partial k(x,t)}{\partial t} - \dfrac{\partial^2 k(x,t)}{\partial x^2} = A(x,t) f\left(k(x,t)\right) - \delta k(x,t) - c(x,t), \quad (x,t) \in \mathbb{R} \times [0,T], \\ \dfrac{\partial q(x,t)}{\partial t} + \dfrac{\partial^2 q(x,t)}{\partial x^2} = -q(x,t)\left(Af'\left(k(x,t)\right) - \delta - \rho\right), \quad (x,t) \in \mathbb{R} \times [0,T], \\ k(x,0) = k_0(x) > 0, \quad x \in \mathbb{R}, \\ q(x,T) = \phi_1'\left(k(x,T), x\right), \forall x \in \mathbb{R}, \\ \lim\limits_{x \to \pm\infty} \dfrac{\partial k(x,t)}{\partial x} = 0, \forall t \in [0,T], \\ \lim\limits_{x \to \pm\infty} \dfrac{\partial q(x,t)}{\partial x} = 0, \forall t \in [0,T]. \end{cases} \tag{7}$$

Since the time horizon T is finite, we can inverse time in (6) to obtain a system of parabolic partial differential equations with spatial boundary conditions where the initial level of capital and its shadow price are known.[6]

3.1 Finite Difference Approximation

We simulate system (7) using a finite difference approximation. The idea of the finite difference method is to replace the second derivative with respect to space with a central difference quotient in x, and replace the derivative with respect to time with a forward difference in time. For this purpose we set up a grid: we consider a finite but "large enough" space and time horizon $[0,J] \times [0,N]$, where boundaries do not affect results. Points in this space are $(j\Delta x, n\Delta t)$ for $j = 0,1,...,J$ and $n = 1,2,...,N$. Then, if v is a function defined on the grid, we denote by $v(j\Delta x, n\Delta t) = v_j^n$.

[6] Indeed, calling $h(x,t) = q(x, T-t)$, we obtain (8):

$$\begin{cases} \dfrac{\partial h(x,t)}{\partial t} - \dfrac{\partial^2 h(x,t)}{\partial x^2} = h(x,t)\left(Af'\left(k(x,T-t)\right) - \delta - \rho\right), \quad (x,t) \in \mathbb{R} \times [0,T], \\ h(x,0) = \phi_1'\left(k(x,T), x\right), \forall x \in R, \\ \lim\limits_{x \to \pm\infty} \dfrac{\partial h(x,t)}{\partial x} = 0, \forall t \in [0,T]. \end{cases} \tag{8}$$

A general finite difference approximation for the problem $\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$ is:

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = \frac{1}{\Delta x^2}\left(\theta\left(v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}\right) + (1-\theta)\left(v_{j+1}^n - 2v_j^n + v_{j-1}^n\right)\right), \qquad (9)$$

where $0 \leq \theta \leq 1$. $\theta = 0$ gives the explicit scheme, $\theta = 1/2$ the Crank-Nicolson, and $\theta = 1$ a fully implicit backward time-difference method. As proved in Smith G.D. (1974) p. 24, the finite difference approximation method is stable and convergent for $1/2 \leq \theta \leq 1$, but for $0 \leq \theta \leq 1/2$ we must have

$$r = \delta t/(\delta x)^2 \leq \frac{1}{2(1-2\theta)}.$$

In practical terms, in an explicit method there exist a formula to obtain v_j^{n+1} for every n in terms of known values whereas with an implicit method we have to solve a system of equations to advance to the next time level. Although the explicit method may be computationally simple it has one serious drawback. The time step Δt is necessarily very small because the process is valid only for $0 < \Delta t/(\Delta x)^2 \leq 1/2$. On the other hand, the implicit method is more costly per time step, but in return we obtain a substantial benefit in the stability and convergence properties, which allows us to use a much larger time step and thus will cut the overall computing costs. Furthermore, the implicit method is unconditionally stable, meaning that it is stable without restrictions on the relative size of Δt and Δx (see Golub G.H. and Ortega J.M., 1992, p. 264).

For these reasons, we choose the implicit method to solve numerically our problem. Then, calling $(k, h) = (v, w)$ the implicit finite difference analog to (4) is:

$$\begin{cases} \frac{v_j^{n+1}-v_j^n}{\Delta t} - \frac{v_{j+1}^{n+1}-2v_j^{n+1}+v_{j-1}^{n+1}}{\Delta x^2} = A_j^n(v_j^n)^\alpha - \delta v_j^n - (U_1')^{-1}(w_j^n), \\ v_j^0 \text{ given}, \\ v_0^{n+1} = v_2^{n+1}, \quad v_J^{n+1} = v_{J-2}^{n+1} \end{cases} \qquad (10)$$

Similarly, (8):

$$\begin{cases} \frac{w_j^{n+1}-w_j^n}{\Delta t} - \frac{w_{j+1}^{n+1}-2w_j^{n+1}+w_{j-1}^{n+1}}{\Delta x^2} = w_j^n\left(A_j^{N-n}(v_j^{N-n})^{\alpha-1} - \delta - \rho\right), \\ w_j^0 = \phi_1'\left(v_j^N, j\right), \\ w_0^{n+1} = w_2^{n+1}, \quad w_J^{n+1} = w_{J-2}^{n+1} \end{cases} \qquad (11)$$

Where A_j^n is the technology matrix that describes the technological state of all points $(j\Delta x, n\Delta t)$. The last condition in each system gives the boundary condition for space. The resolution algorithm used to simulate (10) and (11) is described in the next subsection.

3.2 Algorithm

The algorithm starts with an initial guess for consumption (C_0). Since consumption is linked to the shadow price of capital by $C = (U_1')^{-1}(w)$ we can solve (10)in $[0, J] \times [0, N]$. We do it applying the algorithm for tridiagonal systems that can be found in Sewell G. (1988), p.15. Substituting the result for capital into (11) and solving it with the same method, we obtain the shadow price of capital. Using the relationship between the shadow price of capital and consumption stated just above, we actualize the value for the consumption matrix. Since (11) is the analog to the optimal necessary and sufficient condition, the optimality of the solution is ensured.

The resolution algorithm:

Step 1: *Initialization*

Choose an initial consumption matrix $C_0 = \{co_j^n\}_{j=1,\cdots,J}^{n=1,\cdots,N}$ and a stopping parameter ϵ. Compute the induced capital $V_0 = \{vo_j^n\}_{j=1,\cdots,J}^{n=1,\cdots,N}$ using (10). With this result, we compute the shadow price of capital $W_0 = \{wo_j^n\}_{j=1,\cdots,J}^{n=1,\cdots,N}$ using (11).

Step 2: *Iteration*

For $m = 1, \cdots, It$, where It is the maximum number of iterations, we define for the m^{th} iteration $C_m = \{cm_j^n\}_{j=1,\cdots,J}^{n=1,\cdots,N}$, $V_m = \{vm_j^n\}_{j=1,\cdots,J}^{n=1,\cdots,N}$ and $W_m = \{wm_j^n\}_{j=1,\cdots,J}^{n=1,\cdots,N}$ as the consumption matrix, physical capital matrix and shadow price of physical capital matrix respectively.

C_m is built from the previous iteration using the relationship linking consumption and the shadow price of physical capital:

$$C_m = (U_1')^{-1}(W_m).$$

Compute the Euclidean distance between C_m and C_{m-1}. If this distance is smaller than ϵ, then STOP and return V_m and C_m.

4 Economic Analysis

In the following subsections, we simulate the system under various scenarios and provide numerical results that help us understand the spatial problem of economic growth. Unless otherwise indicated, the parameter values and functional forms used in subsections 4.1, 4.2 and 4.3 are those in table 1.

$f(k(x,t)) = k(x,t)^\alpha$	$\alpha = 1/3$
$u(c(x,t),x) = \frac{c(x,t)^{1-\sigma}}{1-\sigma}\psi(x)$	$\sigma = 10$
$\psi(x) = e^{-\phi\|x\|}$	$\phi = 0.001$
$\delta = 0.06$	
$\rho = 0.03$	

Table 1: Functional specifications and parameter values for the numerical exercise

4.1 Uneven initial physical capital endowment

The neoclassical growth theory predicts spatial convergence in any set up with structurally homogenous space. It is simple to see why: locations initially endowed with a high level of capital will invest in poor locations, which provide a high return to investment. The latter become wealthier and their marginal productivity decreases. Notice that, rich locations have stopped investing in their own firms which work with the existent capital. Since capital depreciates, marginal productivity in rich locations increases with time. Naturally, after some time all locations reach the same level of physical endowment and they all have the same marginal productivity. Therefore, any investor is indifferent between locations and the homogenous distribution of capital is kept forever.

In our framework, we cannot prove the existence nor the uniqueness of the Steady State solution to problem (7). Nonetheless, we have obtained numerical examples where space is homogenous and the solution to (7) converges to a constant steady state. Locations have the same preferences, depreciation rate and technology level $A = 10$. The only disparity across space is introduced through the initial condition:

$$k_0(x) = \begin{cases} \bar{k}e^{(x-500)/100}, & 0 < x \leq 500, \\ \bar{k}\,(x-500)^{0.3}, & 500 < x \leq 1000, \end{cases}$$

where $\bar{k} \approx 33$. Simulation results for physical capital are shown in the left panel of figure 1. The right panel shows consumption. Space is represented on the horizontal axis and physical capital on the vertical axis. We show the level of physical capital at different times, $t = 1$, $t = 10$, $t = 50$ and $t = 150$. Points to the left of $x = 500$ have a lower initial endowment than points to the right of $x = 500$. Consequently, they initially exhibit a larger marginal productivity of capital and they are more attractive to investors. The consequences are an increase in local capital and a growth rate of about 630% at $t = 2$.

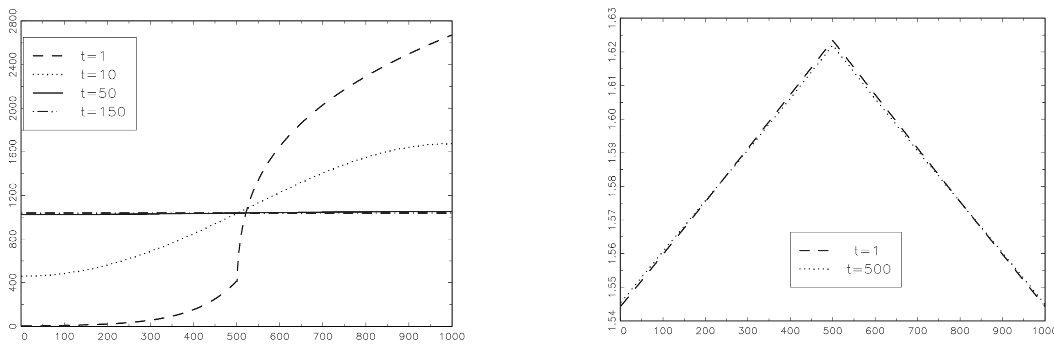


Figure 1: Heterogenous $k_0(x)$, homogenous $A(x)$. Left: Physical capital. Right: Consumption

After ten iterations, locations to the left of $x = 500$ are doing better, though the level of capital is shrinking everywhere else. In terms of investment, these locations have attracted all possible investment and they are growing fast. The initially wealthy regions have stopped home investment and as a result their stock of physical capital decreases. Notice also that since poor regions are getting wealthier, their marginal productivity decreases and so does their growth rate.

At $t = 50$ the initially richest points still keep their primacy. As poor locations are receiving investment, their marginal productivity decreases while the initially better endowed locations have increased theirs. At some point, the poor's productivity equals that of the rich locations and after this moment, these locations are equivalent in terms of foreign investment. The economy is pursuing a steady state. At $t = 150$ the level of physical capital grows at a rate smaller than 0.07%.

In what concerns consumption, the policy maker allocates more consumption to the center locations due to her spatial preferences. Interestingly, optimal consumption levels do not change with time.

We would like to compare these results with those obtained in Camacho et al.(2005), where utility is linear ($\sigma = 0$). This leads the economy to an interior solution, which is a constant steady state for capital. In the present paper consumers postpone a part of their consumption. Consuming less today and devoting an extra share of their revenue to investment, they are able to consume more in the future. This result is robust. The larger the "patient" coefficient σ, the larger the level of physical capital attaint in the steady state and the longer it takes to achieve it.

4.2 Heterogenous Technology

We can imagine now an economy with a technological center or pole, that may coincide with the geographical center without loss of generality. Technology decreases as we depart from the center. We can model this spatial technology as:

$$A(x) = 10 + 100e^{\frac{-(x-\bar{x})^2}{2\sigma_1^2}},$$

where $\sigma_1 = 100$. To study the relevance of the initial endowment, we have studied two cases. For the first one, we have chosen a bell shaped distribution:

$$k_0^1(x) = 100 + 50e^{\frac{-(x-\bar{x})^2}{2\sigma_2^2}},$$

where $\sigma_2 = 100$ and $\bar{x} = 500$ is the middle point. In the second example, initial capital endowment (k_0^2) is homogenous and equal to the average of k_0^1, that is $k_0^2 = E[k_0^1]$.

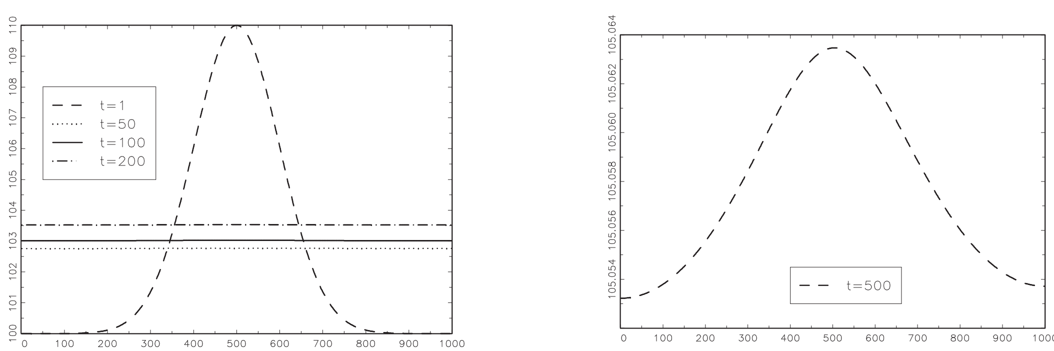


Figure 2: $k_0(x) = k_0^2(x)$, heterogenous $A(x)$. Left: Physical capital at $t = 1, 10, 100$. Right: Physical capital at $t = 500$

Disparities in k_0 should not be relevant in the long run. Therefore, we expect identical behavioral patterns in these economies. We have taken a spatial preference function that is ten times that of table 1. We wonder what the result could be if the center has everything in favor, technology and initial capital endowment. Besides the policy maker

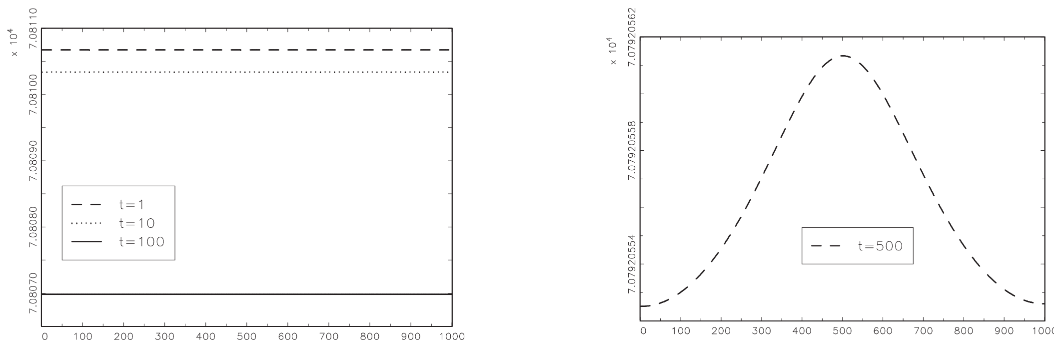


Figure 3: $k_0(x) = k_0^1(x)$, heterogenous $A(x)$. Left: Physical capital at $t = 1, 10, 100$. Right: Physical capital at $t = 500$

likes better the center. Does the central region keep its primacy forever? And if yes, have regional differences shrunk or kept over time? Simulation results are shown in figures 2 and 3, where we show the level of physical capital at different times.

The dynamics of the economy initially endowed with $k_0^1(x)$ are shown in figure 2. On the left, we can observe capital distribution at different moments ($t = 1, 10, 100$). Physical capital is an increasing function of time in every location and it keeps its initial bell shape. Differences in physical capital stocks due to the initial distribution tend to disappear, but the technological pole created around the center keeps its economic leadership (see right panel on figure 2).

Similarly, figure 3 shows physical capital times in the economy with initial capital $k_0(x) = k_0^2(x)$. As soon as $t = 2$ spatial differences in the distribution of capital arise due to the technological advantage. In effect, the central locations become wealthier. This means that their marginal productivity is kept above the other locations forever. Not only they produce more but they also consume more. And this property is true for whatever minimal spatial technological difference. Most remarkable is the decreasing behavior of capital. All locations have started with a level of physical capital above their steady state value.

4.3 The role of spatial preferences

Up to now, spatial preferences were meaningful for the convergence of the objective function. Far from this, this function shapes capital distribution and consumption. We propose two examples. We keep a constant initial distribution for capital below the Steady State solution and an heterogenous technology as in the previous subsection, given by:

$$A(x) = 10 + \frac{1}{25} e^{\frac{-(x-\bar{x})^2}{2\sigma_1^2}},$$

where $\sigma_1 = 100$. Notice that though there is a technological pole around the center, the technological gap is smaller than in the previous subsection. For simplicity, let us consider $k_0(x) = 100$. The remaining functions and parameters are as in table 1. In the first scenario $\phi = 0.001$ and for the second we chose a larger value for the parameter $\phi = 1.25$. Results are shown in figures 4 and 5. Figure 4 shows physical capital at $t = 500$ and figure 5 consumption.

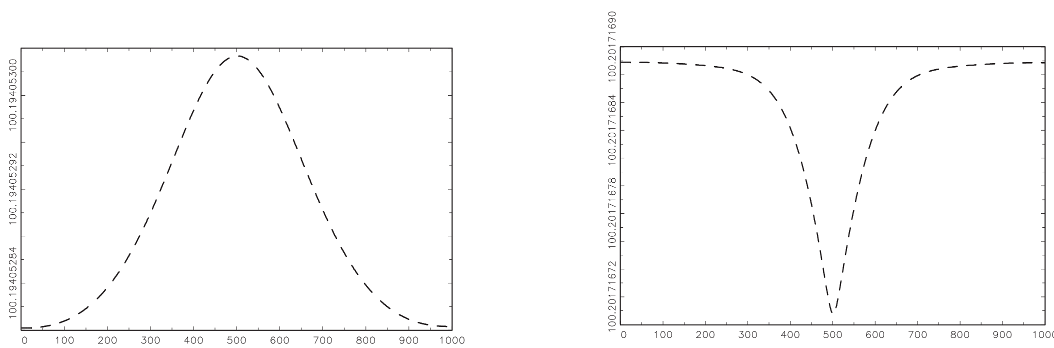
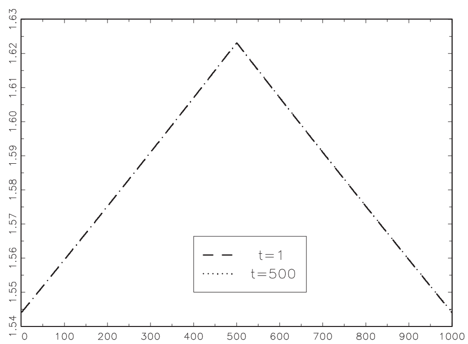


Figure 4: Physical capital at $t = 500$. Left: $\phi = 0.001$. Right:$\phi = 1.25$

In both cases, physical capital is increasing with time. If preferences for the center are strong, then the policy maker privileges their consumption. This produces a funny effect. Consumers on the spatial borders consume less than the center. They invest more than the center, accumulate more capital and grow faster. As a result, physical capital displays an inverted bell shape. The high level of consumption in the center area impedes capital accumulation despite of the permanent technological advantage.

On the other hand, when preferences for the center are lower central locations consume less and invest more with respect to the previous example. Then, they are able to keep the spatial primacy in terms of the level of physical capital. Figure 5 points out that consumption is more egalitarian when preferences are less steep. Not only the center



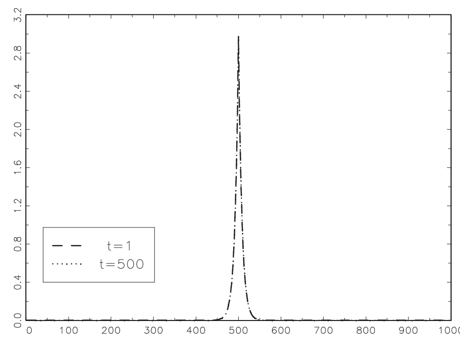


Figure 5: Initial and final consumption. Left: $\phi = 0.001$. Right:$\phi = 1.25$

consumes less, but also the difference between consumption in the center and the borders is smaller[7].

5 Conclusion

We have formulated a prototype of spatial Ramsey model with continuous space. In particular, we have reconsidered the Benthamian-Ramsey model of Camacho et al.(2005) endowing consumers with a strictly concave utility function. We have studied the induced dynamic problem and shown the optimal control of the resulting parabolic partial differential equations. The simulation of systems of PDE is not treated in the literature to our knowledge. We have provided a simple numerical set-up that allows to do so in the economicly appealing case of the new economic geography models.

[7]If we had interpreted ψ as a population density function, then the numerical results point out at the existence of critical population density functions that impede economic growth.

6 Appendix

6.1 Proof of Theorem 3

We write the value function in a different way and separate the state variable k and the optimal control variable c. Then we have

$$\begin{aligned} V &= \int_0^T \int_{\mathbb{R}} [U(c(x,t),x)e^{-\rho t} - \lambda(x,t)c(x,t)]dxdt \\ &- \int_0^T \int_{\mathbb{R}} \lambda(x,t) \left(\frac{\partial k(x,t)}{\partial t} - \frac{\partial^2 k(x,t)}{\partial x^2} - A(x,t) f\left(k(x,t)\right) - \delta k(x,t) \right) dxdt \\ &= \int_0^T \int_{\mathbb{R}} F(c,\lambda) dtdx - \int_0^T \int_{\mathbb{R}} G(k,\lambda) dxdt, \end{aligned}$$

where $q(c,t)$ is the solution of the costate equation mentioned in last proposition and

$$F(c,\lambda) = U(c(x,t),x)e^{-\rho t} - \lambda(x,t)c(x,t),$$

and

$$G(k,\lambda) = \lambda(x,t) \left(\frac{\partial k(x,t)}{\partial t} - \frac{\partial^2 k(x,t)}{\partial x^2} - A(x,t) f\left(k(x,t)\right) - \delta k(x,t) \right).$$

Hence the control only appears in function F. It follows that

$$V \leq \int_0^T \int_{\mathbb{R}} \max_c F(c,\lambda) dx\, dt - \int_0^T \int_{\mathbb{R}} G(k,\lambda) dxdt.$$

So if for any $(x,t) \in (\mathbb{R} \times [0,T))$, we can find $c^\star(x,t) \in C^{2,1}(\mathbb{R} \times [0,T))$, such that,

$$F(c^\star,\lambda) = \max_c F(c,\lambda),$$

then the above $c^\star$ is optimal. In fact we have that

$$\frac{\partial F(c,\lambda)}{\partial c} = U_1'(c,x)e^{-\rho t} - \lambda(x,t), \text{ and } \frac{\partial^2 F(c,\lambda)}{\partial c^2} = U_1''(c,x)e^{-\rho t}.$$

By assumption, $U_1''(c,x) < 0$ and $U_1'(c,x) > 0$, so function $F(c,\lambda)$ is strictly concave. Hence there is unique maximum point, such that,

$$\frac{\partial F(c,\lambda)}{\partial c} = 0,$$

or equivalently, we have

$$U_1'(c,x)e^{-\rho t} = \lambda(x,t),$$

if and only if $\lambda(x,t) > 0, (x,t) \in (\mathbb{R} \times [0,T])$. Define $q(x,t) = e^{\rho t}\lambda(x,t)$, we finish the proof. Q.E.D

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