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Revised version of "Coalition Formation in Simple Games: The Semistrict Core"

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Stable Governments and the Semistrict Core*

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Abstract

We consider the class of proper monotonic simple games and study coalition formation when an exogenous weight vector and a solution concept are combined to guide the distribution power within winning coalitions. These distributions induce players' preferences over coalitions in a hedonic game. We formalize the notion of semistrict core stability, which is stronger than the standard core concept but weaker than the strict core notion and derive two characterization results for the semistrict core, dependent on conditions we impose on the solution concept. It turns out that a *bounded power* condition, which connects exogenous weights and the solution, is crucial. It generalizes a condition termed "absence of the paradox of smaller coalitions" that was previously used to derive core existence results.

JEL Classification: D72, C71

Keywords: coalition formation, semistrict core, simple games, winning coalitions

1 Introduction

The analysis of election results is one of the most popular applications of cooperative game theory. Thereby a game describes the parties' possibilities to form a winning coalition, respectively a government. Application of a solution concept, such as the Shapley value (or Shapley-Shubik index as it is often termed in this setup), is readily interpreted as the

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(*endogenous*) *power* that a party exerts in the parliament. This notion of power is then often used to distribute responsibilities within a government.

Two drawbacks of this approach are frequently criticized: First, this notion of power only takes into account the data of the game, i.e., it only takes into account, which coalitions may form a government. What it ignores is a party's total number of votes or its total number of seats in the parliament (cf. Snyder et al. (2005)). So, it disregards ideas of proportionality or *exogenous power distributions*. It is undoubted that a government consisting of a "small" and a "large" party does not share responsibilities (e.g., offices) equally. Second, this approach does not answer the question, which government is likely to form or, regarded from a normative point of view, should form.

In this paper we tackle these two problems. For this, we consider the class of *proper monotonic simple games*. In a simple game any possible coalition is either winning or not. Monotonicity guarantees that any supercoalition of a winning coalition is also winning and properness requires that the complementary coalition of a winning coalition is not winning. As argued above there are two sources that should play a role, when describing a power distribution among parties. We bring these endogenous and exogenous impacts together by introducing the concept of a *composite solution*. More precisely, a composite solution F takes each collection (α, S, v, φ) of an exogenous weight vector α, a coalition S, a simple game v and a cooperative solution concept φ to a distribution of power with the interpretation that $F_i(\alpha, S, v, \varphi)$ reflects player (party) i's (overall) power within coalition S, when v describes the possibilities for winning coalitions. Thereby, we will be interested in a specific composite solution, in which exogenous weights enter in a proportional fashion.

To come back to the government formation problem, we may assume that a player's incentive to take part in a (winning) coalition depends on how much power he has within this coalition according to a composite solution. In effect, we obtain preferences over coalitions. The collection of these preferences forms a *hedonic coalition formation game* (cf. Banerjee et al. (2001) and Bogomolnaia and Jackson (2002)). A solution for this (and each) hedonic game proposes a (set of) partition(s) of the set of players into coalitions. In the context of simple games this in effect means which winning coalition forms. In terms of solution the focus is on *stability* considerations, meaning that the final partition should not provide incentives for a coalition to deviate and form instead. As it can be easily seen, it is not possible for a coalition structure to be stable if it does not contain a winning coalition. Hence, the answer to the question which partitions are stable is at the same time an answer to the question which winning coalition (or government) should form with respect to stability concerns.

Depending on how restrictive conditions for coalitional deviations are formulated, we get different notions of stability. We have chosen the *semistrict core* as our stability concept for hedonic games. This stability notion is weaker than the strict core and stronger than the standard core notion. The idea of it can already be found in the work of Kirchsteiger and Puppe (1997) and, more definitive, in the works of Dimitrov and Haake (2006b) and Dimitrov (2006). The main finding in this paper is a characterization result for the semistrict core (Theorems 1 and 2). For this we require the solution concept to satisfy efficiency, symmetry, and the null player property, which are, e.g., standard requirements for *power indices*. There is an additional condition termed *bounded power* that connects exogenous weights with the solution concept. This requirement can in essence be seen as a generalization of Shenoy's (1979) *absence of the paradox of smaller coalitions*, that he used to derive an existence theorem for the core. In the specific case that external weights are equal, our semistrict core existence result can more clearly be seen as stronger and more general than the corresponding core existence result of Shenoy (1979). Here we also provide a proof for a more direct generalization (Theorem 3). If we rule out asymmetries among the players that stem from the solution concept, i.e., if we take Farrell and Scotchmer's (1988) *partnership solution* as cooperative solution concept, then again a full characterization of the semistrict core of the corresponding hedonic game can be provided (Theorem 4).

Our model is a stylized one in the sense that players' preferences are motivated by office-seeking considerations and, thus, political affinities are faded out. This line of study has a long tradition since Riker's (1962) classical monograph (see Laver and Schofield (1990) for an extensive survey). Peleg (1981) and Einy (1985) develop a theory of coalition formation in simple games with dominant players, whereas Carreras (1996) studies, among others, the formation of partnerships (cf. Kalai and Samet (1987)) in simple games. In contrast to these papers, we do not presuppose any internal structure on the winning coalition that forms; its internal structure is rather determined by the corresponding stability notion applied to the induced hedonic game. In general, our methodology is also in line with coalition formation models, where the gains from cooperation are represented by a cooperative game (see, e.g., Slikker (2001)). However, in our case a cooperative solution concept is applied to derive players' preferences over coalitions rather than to determine players' payoffs in a strategic game in which players announce their preferred coalitions. Finally, the notion of a composite solution that we introduce combines a cooperative solution and exogenous weights; it allows us to incorporate ideas of proportionality in the distribution of coalitional worth, i.e., of power within a government.

The paper is organized as follows. Section 2 includes basic notions and solution concepts from the theory of simple games and hedonic games. We define a specific composite solution and use it to induce players' preferences over coalitions in a hedonic game. Our main result is presented in Section 3, while Section 4 discusses the two special cases in which we have either equal weights or fix the partnership solution. Section 5 closes with some final remarks.

2 Preliminaries

Simple games and solutions

Let N be a finite *set of players*, which we will keep fixed throughout the paper. A (*cooperative*) *simple game with transferable utility* (a simple TU-game) is a pair (N, v), where $v : 2^N \to \{0, 1\}$ is called *characteristic function* and satisfies $v(\emptyset) = 0$. We refer to a coalition $S \subseteq N$ with $v(S) = 1$ as a *winning coalition*. In what follows we will identify a simple game (N, v) with its characteristic function v.

A simple game v is *monotonic* if $v(S) = 1$ implies $v(T) = 1$ for all $T \supseteq S$, and *proper* if $v(S) = 1$ implies $v(N \backslash S) = 0$. A player $i \in N$ is a *null player* in v if $v(S) = v(S \setminus \{i\})$ for all $S \subseteq N$. Denote by $\Delta(v) \subseteq N$ the set of null players in v. Players $i, j \in N$ are *symmetric* in v, if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$. We denote by $\mathcal{W}^v = \{S \subseteq N \mid v(S) = 1\}$ the set of *winning coalitions* and by $\mathcal{MW}^v$ =$\{S \subseteq N \mid v(S) = 1 \text{ and } v(T) = 0 \text{ for all } T \subset S\}$ the set of *minimal winning coalitions* in the simple game v (cf. Shapley (1962)). For $S \subseteq N$ define the *subgame* (N, v_S) by $v_S(T) = v(S \cap T)$ for all $T \in 2^N$. Note that v_S is also a simple game with player set N (possibly with $v_S(N) = v(S) = 0$). The set of all proper monotonic simple games on the player set N will be denoted by $\mathcal{G}$. Clearly, if a game v is in the set $\mathcal{G}$, then so is any of its subgames.

For the purpose of this paper we define a *solution* (of a proper monotonic simple game) as a mapping φ: $2^N \times \mathcal{G} \to \mathbb{R}_+^N$ taking each pair $(S, v) \in 2^N \times \mathcal{G}$ to a vector in $\mathbb{R}_+^N$, i.e., it assigns a nonnegative real number $\varphi_i(S, v)$ to each player $i \in N$.[1] The set of all solutions on $\mathcal{G}$ will be denoted by $\mathcal{S}$. A solution $\varphi \in \mathcal{S}$ satisfies *efficiency* if $\sum_{i \in N} \varphi_i(S, v) = v(N)$ holds for all $S \in 2^N, v \in \mathcal{G}$, and *coalitional efficiency* if $\sum_{i \in S} \varphi_i(S, v) = v(S)$ holds for all $S \in 2^N, v \in \mathcal{G}$. A solution $\varphi \in \mathcal{S}$ is *symmetric* if $\varphi_i(S, v) = \varphi_j(S, v)$ for all $S \in 2^N, v \in \mathcal{G}$, and all $i, j \in S$ who are symmetric in v. Finally, a solution $\varphi \in \mathcal{S}$ satisfies the *null player property* if, for all $S \subseteq N$ and all $v \in \mathcal{G}$, $\varphi_i(S, v) = 0$ holds for all $i \in \Delta(v)$. Notice that coalitional efficiency implies efficiency, and efficiency and the null player property imply coalitional efficiency. Finally, we denote by $\mathcal{S}^*$ the set of efficient, symmetric solutions that satisfy the null player property.

Next, we recall two specific solutions: the *Shapley value* (cf. Shapley (1953), Shapley and Shubik (1954), and Aumann and Dréze (1974)) and the *partnership solution* (cf. Farrell and Scotchmer (1988)).

[1] Requiring nonnegativity is in accordance with the interpretation that, for all $S \in 2^N$, $\varphi_i(S, v)$ reflects player i's power in the game v.

The Shapley value $Sh : 2^N \times \mathcal{G} \longrightarrow \mathbb{R}_+^N$ is given by

$$Sh_i(S, v) := \begin{cases} \sum_{R \subseteq S} \frac{(|S|-|R|)!\,(|R|-1)!}{|S|!} \left(v(R) - v(R \setminus \{i\})\right), & \text{if } i \in S, \\ 0, & \text{otherwise} \end{cases} \qquad (i \in N).$$

The partnership solution $Pa : 2^N \times \mathcal{G} \longrightarrow \mathbb{R}_+^N$ is given by

$$Pa_i(S, v) := \begin{cases} \frac{v(S)}{|S|}, & \text{if } i \in S, \\ 0, & \text{otherwise} \end{cases} \qquad (i \in N).$$

It is easy to check that both solutions are (coalitionally) efficient and symmetric. In addition, the Shapley value also satisfies the null player property, while the partnership solution does not.

Composite solutions

In the context of simple games, solutions in $\mathcal{S}^*$, i.e., those that are efficient, symmetric and satisfy the null player property (such as the Shapley value) are often termed *power indices.* One frequent criticism to power indices is that they on the one hand measure "endogenous power", but on the other hand they cannot take "exogenous power distributions" into account. For instance, the distribution of seats in a parliament is completely ignored, when describing the corresponding majority voting game. Hence, it does not enter the solution, either.

Composite solutions, as defined below, are designed to incorporate exogenous weights as well as endogenous power. This is done by a combination of a weight vector and a solution. Thus, a composite solution does not only reflect players' opportunities to form winning coalitions, but also respects asymmetries among the players outside the game.

Formally, a *composite solution* $F : R_{++}^N \times 2^N \times \mathcal{G} \times \mathcal{S} \rightarrow \mathbb{R}_+^N$ assigns a vector of players' payoffs (powers) to each tuple (α, S, v, φ) consisting of a weight vector, a coalition, a simple game, and a solution.[2] We interpret a composite solution as follows: Suppose the game $v \in \mathcal{G}$ describes the possibilities to form winning coalitions. The vector α represents asymmetries outside the model and φ is the solution that measures players' power inherent in the game. Then (the real number) $F_i(\alpha, S, v, \varphi)$ should be viewed as player i's "overall power" within a coalition $S \subseteq N$. In the following, we concentrate on a specific composite solution Φ, which is defined by

$$\Phi_i(\alpha, S, v, \varphi) = \begin{cases} \frac{\alpha_i \varphi_i(S, v_S)}{\sum_{j \in S} \alpha_j \varphi_j(S, v_S)} \cdot v_S(S), & \text{if } i \in S, S \in \mathcal{W}^v, \\ 0, & \text{otherwise} \end{cases} \qquad (i \in N). \tag{1}$$

[2] $\alpha \in \mathbb{R}_{++}^N$ means $\alpha_i > 0$ for all $i \in N$.

Suppose α, v, and φ are fixed and a winning coalition $S \in \mathcal{W}^v$ has formed. How is the worth $v_S(S) = v(S) = 1$ divided within S, i.e., how is power distributed among the players in S? First of all, any player not in S gets zero. Without external weights, player i's ($i \in S$) share of $v_S(S) = 1$ was $\varphi_i(S, v_S)/\sum_{k\in S}\varphi_k(S, v_S)$. This share is now multiplied with his external weight α_i. Normalization, i.e., the sum of shares equals 1, yields the form in (1). So the external weights enter in a proportional fashion to rearrange power within a winning coalition. Note also that Φ is homogeneous of degree zero with respect to the weights vector, which means that only agents' relative weights matter.[3]

For fixed $\alpha \in \mathbb{R}^N_{++}$ and $\varphi \in \mathcal{S}$ the mapping $\Phi(\alpha, *, \bullet, \varphi) : 2^N \times \mathcal{G} \longrightarrow \mathbb{R}^N_+$ is a solution (in the above sense). Observe further that for all $(\alpha, S, v, \varphi) \in \mathbb{R}^N_{++} \times 2^N \times \mathcal{G} \times \mathcal{S}$ the equation $\Phi(\alpha, S, v, \varphi) = \Phi(\alpha, S, v_S, \varphi)$ is valid, which means that only the subgame v_S is relevant for the overall power distribution within S.

Hedonic games and stability notions

For each player $i \in N$ we denote by $\mathcal{N}_i = \{X \subseteq N \mid i \in X\}$ the collection of all coalitions containing i. A partition Π of N is called a *coalition structure*. For each coalition structure Π and each player $i \in N$, we denote by $\Pi(i)$ the coalition in Π containing player i, i.e., $\Pi(i) \in \Pi$ and $i \in \Pi(i)$. The set of all coalition structures of N will be denoted by $\mathbf{C}^N$.

Further, we assume that each player $i \in N$ is endowed with a preference $\succeq_i$ over $\mathcal{N}_i$, i.e., a binary relation over $\mathcal{N}_i$ which is reflexive, complete, and transitive. Denote by $\succ_i$ and $\sim_i$ the strict and indifference relation associated with $\succeq_i$ and by $\succeq := (\succeq_1, \succeq_2, \ldots, \succeq_n)$ a profile of preferences $\succeq_i$ for all $i \in N$. A player's preference relation over coalitions canonically induces a preference relation over coalition structures in the following way:[4] For any two coalition structures Π and Π', player i weakly prefers Π to Π' if and only if he weakly prefers "his" coalition in Π to the one in Π', i.e., $\Pi \succeq_i \Pi'$ if and only if $\Pi(i) \succeq_i \Pi'(i)$. Hence, we assume that players' preferences over coalition structures are *purely hedonic*, i.e., they are completely characterized by their preferences over coalitions. Finally, a *hedonic game* $(N, \succeq)$ is a pair consisting of the set of players and a preference profile.

Unlike solution concepts for (simple) cooperative games do, there is no worth to distribute in hedonic games. The relevant question is rather, which coalition structure should form, taking players' preferences into account. The basic property that we require is *core stability*, which we define next in three versions.

Let $(N, \succeq)$ be a hedonic game. For any coalition $\emptyset \neq X \subseteq N$ and coalition structure Π

[3]For an axiomatic characterization of Φ we refer the interested reader to Dimitrov and Haake (2006a).

[4]With slight abuse in notation, we use the same symbol to denote preferences over coalitions and preferences over coalition structures.

of N, let $\mathcal{X}^{\Pi}(X) := \{X \cap P \mid P \in \Pi\}$. A partition Π is *strictly core stable* if there does not exist a nonempty coalition X such that $X \succeq_i \Pi(i)$ holds for all $i \in X$ and $X \succ_j \Pi(j)$ is true for some player $j \in X$. Π is *semistrictly core stable* if there does not exist a nonempty coalition X such that $X \succeq_i \Pi(i)$ for all $i \in X$ and for each $X' \in \mathcal{X}^{\Pi}(X)$ there exists a player $j \in X'$ with $X \succ_j \Pi(j)$. Π is *core stable* if there does not exist a nonempty coalition X such that $X \succ_i \Pi(i)$ holds for each $i \in X$.

Put in other words, a coalition structure Π is strictly core stable if no group of players are willing to form a coalition, so that each player is at least as well off with this new coalition and some player is better off compared to the corresponding coalitions in Π.

For semistrict core stability we again want to exclude the case that a new coalition X forms. However, the requirement for some players being better off is more subtle. For this we partition the deviating coalition X into groups that come from the same coalition in Π. Then, to make X a profitable deviation it is required that in each such group there has to be some player who is better off in the new coalition. The intuition behind this concept is that each group of deviators that comes from the same original coalition in Π is only willing to join the new coalition X, if at least one of the group members is better off. One can view this concept as strict core stability applied to original coalitions.

Clearly, the weakest notion of a coalitional deviation is incorporated in the definition of core stability - everyone in the deviating coalition should be better off. Observe that strict core stability implies semistrict core stability that, in turn, implies core stability.

In what follows, we denote by $SC(N, \succeq)$, $SSC(N, \succeq)$, and $C(N, \succeq)$ the sets of strictly core stable, semistrictly core stable, and core stable coalition structures, respectively, of a hedonic game $(N, \succeq)$. Alternatively, we call $SC(N, \succeq)$, $SSC(N, \succeq)$, and $C(N, \succeq)$ the strict core, semistrict core, and core of $(N, \succeq)$.

3 Coalition formation via composite solutions

In this section we address the following question: Given a simple game that describes the incentives for forming coalitions, which (winning) coalition should form? Clearly, a player's preferences over winning coalitions should depend on how much influence or power he has within such a coalition. In effect, preferences are based on the solution concept at hand as well as on the exogenous weight vector. In this sense, as already mentioned in the Introduction, our model is a stylized one, meaning that parties' preferences over governments are motivated by office-seeking considerations. Nonetheless, to a certain extent, dislike among parties can be put into the game by removing certain (minimal) winning coalitions, i.e., by changing the original game.

More precisely, here we consider solutions in $\mathcal{S}^*$ and take $\varphi \in \mathcal{S}^*$. Efficiency, symmetry and the null player property determine players' payoffs when they are members of minimal winning coalitions, i.e., we have

$$\varphi_i(S, v_S) = \frac{1}{|S|} \qquad \text{for all } i \in S, S \in \mathcal{MW}^v$$

and hence,

$$\Phi_i(\alpha, S, v, \varphi) = \frac{\alpha_i}{\alpha(S)} \qquad \text{for all } i \in S, S \in \mathcal{MW}^v, \tag{2}$$

where $\alpha(S) := \sum_{i \in S} \alpha_i$ is the total external weight of coalition S, $S \subseteq N$. So, within a minimal winning coalition S, all players are symmetric[5] and therefore any solution in $\mathcal{S}^*$ assigns equal power, whereas the composite solution proposes a proportional distribution according to external weights. Notice that (2) continues to hold if $\varphi \in \mathcal{S}$ is taken from the larger domain of coalitionally efficient and symmetric solutions.

Now, let $\alpha \in \mathbb{R}^N_{++}$, $v \in \mathcal{G}$, and $\varphi \in \mathcal{S}$ be fixed. To simplify notation, for all $S \subseteq N$ and all $i \in N$, we write $\Phi_i(S)$ instead of $\Phi_i(\alpha, S, v, \varphi)$ to denote i's payoff according to the composite solution Φ. We are now ready to define a hedonic coalition formation game by inducing players' preferences over coalitions in the following way. For each $i \in N$ define a preference relation $\succeq_i$ over $\mathcal{N}_i$ by

$$S \succeq_i T \qquad \text{if and only if} \qquad \Phi_i(S) \geq \Phi_i(T) \qquad\qquad (S, T \in \mathcal{N}_i), \tag{3}$$

i.e., $\Phi_i(\cdot)|_{\mathcal{N}_i}$ is a representation of i's preferences. In words, player i's preferences over any two coalitions S and T that he is a member of are induced via i's power according to the composite solution Φ. Notice that paying attention to the corresponding coalitions is compatible with the very definition of a hedonic game - each player in such a game evaluates any two coalition structures based only on his preferences over the coalitions in the two partitions he belongs to (cf. Aumann and Dréze (1974) and Shenoy (1979)). In what follows, we shall use the notation $(N, \succeq)$ to denote the hedonic game induced via Φ as defined in (3).

Once preferences are clear, the question arises, which coalition structures are stable. Recall that each coalition structure that is core stable in either of the three versions necessarily has to contain exactly one winning coalition. Clearly, the best one can hope for are strictly core stable partitions. However, as the following example demonstrates, this requirement is in fact too strong to obtain existence.[6]

Example 1

Let $|N| = 3$, $\alpha \in \mathbb{R}^3_{++}$ with $\alpha_2 = \alpha_3 = a$, and let the simple game $v \in \mathcal{G}$ be given

[5] to be more precise, in the game restricted to S.

[6] We thank an anonymous referee for suggesting this example.

by its minimal winning coalitions $\mathcal{MW}^v := \{12, 13\}$[7]. Let φ be a solution in $\mathcal{S}^*$ such that $\varphi(N, v) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$, $\varphi(12, v_{12}) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$, $\varphi(13, v_{13}) = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$, and $\varphi(S', v_{S'}) = (0, 0, 0)$ for all $S' \in \{1, 2, 3, 23\}$. Now, using φ as (inherent) power index, each player forms preferences according to how much power the composite solution Φ assigns to him in a coalition. We present below players' payoffs according to Φ:

$$\Phi_1(S) = \begin{cases} \frac{\alpha_1}{\alpha_1 + a} & \text{if } S \in \{12, 13, 123\}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\Phi_2(S) = \begin{cases} \frac{a}{\alpha_1 + a} & \text{if } S = 12, \\ \frac{a}{2(\alpha_1 + a)} & \text{if } S = 123, \\ 0 & \text{otherwise.} \end{cases}$$

$$\Phi_3(S) = \begin{cases} \frac{a}{\alpha_1 + a} & \text{if } S = 13, \\ \frac{a}{2(\alpha_1 + a)} & \text{if } S = 123, \\ 0 & \text{otherwise.} \end{cases}$$

According to (3) we take these values to extract preferences over coalitions. They are given as follows:

$$12 \sim_1 13 \sim_1 123 \succ_1 1,$$
$$12 \succ_2 123 \succ_2 2 \sim_2 23,$$
$$13 \succ_3 123 \succ_3 3 \sim_3 23.$$

Collecting all preferences, we obtain a hedonic game $(N, \succeq)$ with preferences induced by the composite solution Φ. Inspecting $(N, \succeq)$, one finds that $SC(N, \succeq) = \emptyset$. □

This simple example shows that strict core stability is far too restrictive to answer the question, which coalition(s) should be formed even for $\varphi \in \mathcal{S}^*$. Consequently, we next ask for conditions that guarantee nonemptiness of the semistrict core.

In order to prove our semistrict core existence result, which is stated in Theorem 1 below, an additional condition that we call *bounded power* is needed. This condition connects players' exogenous weights with the cooperative solution concept and the simple game, hence we impose a restriction on tuples (α, v, φ). Loosely speaking, the overall power of a player within a winning coalition should be bounded from above by his relative exogenous share in minimal winning subcoalitions. Formally, the tuple $(\alpha, v, \varphi) \in \mathbb{R}^N_{++} \times \mathcal{G} \times \mathcal{S}$ satisfies *bounded power* if the following condition is met: For all $T \in \mathcal{W}^v$ and all $S \in \mathcal{MW}^v, S \subset T$ we have for all $i \in S$,

$$\Phi_i(\alpha, T, v, \varphi) = \Phi_i(T) = \frac{\alpha_i \varphi_i(T, v_T)}{\sum_{j \in T} \alpha_j \varphi_j(T, v_T)} \leq \frac{\alpha_i}{\alpha(S)}. \tag{BP}$$

[7] To simplify notation for coalitions, we write, e.g., 12 instead of $\{1, 2\}$.

So, (BP) says that a player should not have "too much" overall power in a winning coalition. Take a winning coalition T that is not minimal winning and player i in T so that there is a minimal winning coalition $S \subset T$ that contains i. Now, $\alpha_i/\alpha(S)$ is player i's relative share within S, when only exogenous weights are taken into account. And this quantity should be an upper bound for i's overall power within the original coalition T.

Let us verify condition (BP) for the game in Example 1. We only need to check the inequalities for $T = N$ and $S = 12, 13$. Straightforward calculations reveal that the tuple (α, v, φ) satisfies (BP) if and only if $\alpha_2 = \alpha_3$ is true, which is due to symmetry of players 2 and 3. A very useful implication of (BP) in the case that $\varphi \in \mathcal{S}$ is coalitionally efficient and symmetric is stated in the following lemma.

Lemma 1 *Let $\alpha \in \mathbb{R}^N_{++}$, $v \in \mathcal{G}$, and $\varphi \in \mathcal{S}$ be coalitionally efficient and symmetric. The tuple (α, v, φ) satisfies (BP) if and only if $\Phi_i(S) \geq \Phi_i(T)$ holds for all $T \in \mathcal{W}^v$, $S \in \mathcal{MW}^v$ with $S \subseteq T$ and all $i \in S$.*

Proof. The assertion directly follows from (2), which shows $\frac{\alpha_i}{\alpha(S)} = \Phi_i(S)$ for all $i \in S$. □

With the help of Lemma 1, we may identify the bounded power restriction as a generalization of an assumption that Shenoy (1979) utilizes to prove an existence theorem for the core (see also Section 4.1 below). More precisely, Shenoy's condition says that a game $v \in \mathcal{G}$ *does not exhibit the paradox of smaller coalitions w.r.t.* $\varphi \in \mathcal{S}$, if for all $S, T \in \mathcal{W}^v$, $S \subseteq T$ implies $\varphi_i(S, v_S) \geq \varphi_i(T, v_T)$ for all $i \in S$. The absence of this paradox in simple games respects the fact that if players form a smaller winning coalition, then their (internal) power should not decrease since there are fewer players to share the same amount of power. Inspecting Shenoy's proof of his Theorem 7.4, one finds that the absence of the paradox is only needed for minimal winning coalitions S. Now, taking equal weights for players in Lemma 1, we in effect obtain $\varphi_i(S, v_S) = \Phi_i(S) \geq \Phi_i(T) = \varphi_i(T, v_T)$ $(T \in \mathcal{W}^v, S \in \mathcal{MW}^v, S \subseteq T, i \in S)$. Hence, when φ is coalitionally efficient and symmetric and there are no external asymmetries, (BP) reduces to (a weaker form of) Shenoy's condition.

In the remainder of this section, we provide a complete characterization of the semistrict core, when the solution is efficient, symmetric, and satisfies the null player property and the bounded power condition is in place. To structure the analysis we first prove an existence result and then complete the characterization.

It turns out that certain minimal winning coalitions are crucial for the further analysis. Let $\mathcal{A}^v$ be the set of all minimal winning coalitions with minimal total weight, i.e., $\mathcal{A}^v := \{S \in \mathcal{MW}^v \,|\, \alpha(S) \leq \alpha(T) \text{ for all } T \in \mathcal{MW}^v\}$. Denote by $\mathcal{D}^v := N \setminus \bigcup_{S \in \mathcal{A}^v} S$ the set of players that do not appear in some minimal winning coalition with minimal total weight. Now, by $\mathcal{A}^v_\Delta$ we denote the set of all winning coalitions T that consist of a minimal winning coalition in $\mathcal{A}^v$ and players in $\mathcal{D}^v$ that are moreover null players in the subgame v_T. Formally define $\mathcal{A}^v_\Delta := \{T = S \cup D \,|\, S \in \mathcal{A}^v, D \in \mathcal{D}^v, D \subseteq \Delta(v_T)\}$.

Consequently, we also define coalitions structures containing such (minimal) winning coalitions. For a set $\mathcal{B} \subseteq 2^N$ of coalitions define the set $\mathbf{P}_\mathcal{B} \subseteq \mathbf{C}^N$ of coalition structures by

$$\mathbf{P}_\mathcal{B} := \left\{\Pi \in \mathbf{C}^N \mid \Pi \cap \mathcal{B} \neq \emptyset\right\}.$$

In particular, we make use of the set $\mathbf{P}_{\mathcal{A}^v}$, the partitions of which contain a minimal winning coalition with minimal total weight. Similarly, the set $\mathbf{P}_{\mathcal{A}^v_\Delta}$ consists of all partitions with a winning coalition that contains exactly one minimal winning coalition with minimal total weight and possibly certain null players. Clearly, $\mathbf{P}_{\mathcal{A}^v} \subseteq \mathbf{P}_{\mathcal{A}^v_\Delta}$ holds.

The characterization result that we prove in this section treats the case, in which there is an efficient, symmetric solution satisfying the null player property and a vector of external weights so that the bounded power requirement is fulfilled. Here we show that the semistrict core exactly consists of those coalition structures, the winning coalition of which contains exactly one minimal winning coalition with minimal total weight and some null players from $\mathcal{D}^v$. That means, we show $SSC(N, \succeq) = \mathbf{P}_{\mathcal{A}^v_\Delta}$. We split the proof into an existence part (Theorem 1) and the characterization part (Theorem 2).

Notice that $\varphi \in \mathcal{S}^*$ implies that φ is coalitionally efficient as well and thus, we can make use of Lemma 1.

Theorem 1 (Existence) *Let $\alpha \in \mathbb{R}^N_{++}, v \in \mathcal{G}$, and $\varphi \in \mathcal{S}^*$ be given such that (BP) is satisfied.*

1. *Let $\Pi \in \mathbf{C}^N$ be a coalition structure that contains a winning coalition $W \in \Pi$ such that $W \in \mathcal{A}^v_\Delta$. Let $X \in 2^N$ be a coalition that blocks Π in either sense of stability. Then any minimal winning coalition in X is of minimal total weight, i.e., $Y \in \mathcal{MW}^v, Y \subseteq X$ imply $Y \in \mathcal{A}^v$.*
2. *We have $\mathbf{P}_{\mathcal{A}^v_\Delta} \subseteq SSC(N, \succeq)$.*

Proof. To prove the first part, write $W = W' \cup D$ with $W' \in \mathcal{A}^v$ and $D \in \mathcal{D}^v$ as in the definition of $\mathcal{A}^v_\Delta$. Using the null player property of φ and the definition of Φ we therefore have $\Phi_i(W) = \Phi_i(W') = 0$ for all $i \in D$. The players in W' are symmetric in $v_W = v_{W'}$, so that coalitional efficiency guarantees $\Phi_i(W) = \frac{\alpha_i}{\alpha(W')} = \Phi_i(W')$ for all $i \in W'$. Now, suppose X were a deviating coalition from Π. Then, irrespective of the type of stability we consider, X has to satisfy $\Phi_i(X) \geq \Phi_i(W)$ for all $i \in X \cap W$ and in particular, for all $i \in X \cap W'$ (recall that, by properness, $X \cap W' \neq \emptyset$). Then, with (BP), we obtain for any minimal winning coalition $Z \subseteq X$ and all $i \in X \cap (W' \cap Z) = W' \cap Z$,

$$\frac{\alpha_i}{\alpha(W')} = \Phi_i(W') = \Phi_i(W) \leq \Phi_i(X) \leq \Phi_i(Z) = \frac{\alpha_i}{\alpha(Z)}. \tag{4}$$

But this is only possible, if any minimal winning coalition contained in X is of minimal total weight. Moreover, any $k \in X \cap W'$ has to be a member of some minimal winning subcoalition of X, since otherwise $\Phi_k(X) = 0 < \frac{\alpha_k}{\alpha(W')} = \Phi_k(\Pi(k))$ is a contradiction to X being a deviating coalition. So, in particular (4) is satisfied with equality for all $i \in X \cap W'$.

To prove the second part, take $\Pi \in \mathbf{P}_{\mathcal{A}^v_\Delta}$ with winning coalition $W = W' \cup D$ (as above). Suppose to the contrary that X is a deviation from Π in the sense of the semistrict core. We check the conditions for players in $X \cap W$. There we know by the first part,

$$\Phi_i(X) = \Phi_i(\Pi(i)) = \Phi_i(W') \qquad (i \in X \cap W'). \tag{5}$$

Thus, there has to be $i' \in X \cap (W \setminus W') = X \cap D$ with $\Phi_{i'}(X) > \Phi_{i'}(\Pi(i')) = \Phi_{i'}(W) = 0$. Since $i' \in D$, i' is not a member of any minimal winning coalition with minimal total weight. So, by part 1, i' is not included in any minimal winning subcoalition of X, hence $i' \in \Delta(v_X)$. By the null player property of φ, $\Phi_{i'}(X) = 0$, a contradiction. □

The reader may verify that the game in Example 1 satisfies the conditions of Theorem 1. The coalition structures $\{12, 3\}$ and $\{13, 2\}$ are indeed semistrictly core stable. In both cases the winning coalition W has a minimal total weight of $\alpha(W) = \alpha_1 + a$. Note further that the core of the game from Example 1 is larger than the semistrict core as it also contains the coalition structure $\{123\}$.

Theorem 2 (Characterization Theorem) *Let $\alpha \in \mathbb{R}^N_{++}, v \in \mathcal{G}$, and $\varphi \in \mathcal{S}^*$ be given such that (BP) is satisfied. Then $SSC(N, \succeq) = \mathbf{P}_{\mathcal{A}^v_\Delta}$.*

Proof. It remains to show $SSC(N, \succeq) \subseteq \mathbf{P}_{\mathcal{A}^v_\Delta}$. For this, we take $\Pi \notin \mathbf{P}_{\mathcal{A}^v_\Delta}$ and show $\Pi \notin SSC(N, \succeq)$. Clearly, in the case that Π does not contain a winning coalition, it cannot be semistrictly core stable. Therefore, assume $\Pi \cap \mathcal{W}^v \neq \emptyset$ and denote the winning coalition in Π by W. We distinguish the following three cases:

Case 1: There are $W', W'' \in \mathcal{MW}^v, W' \neq W''$ with $W' \subseteq W, W'' \subseteq W$.

By Lemma 1,

$$\frac{\alpha_i}{\alpha(W')} = \Phi_i(W') \geq \Phi_i(W) \qquad (i \in W'), \tag{6}$$

and

$$\frac{\alpha_i}{\alpha(W'')} = \Phi_i(W'') \geq \Phi_i(W) \qquad (i \in W''). \tag{7}$$

Assuming equality in (6) and (7) yields

$$\begin{aligned}
\sum_{i \in W} \Phi_i(W) &= \sum_{i \in W'} \Phi_i(W') + \sum_{i \in W'' \setminus W'} \Phi_i(W'') + \sum_{i \in W \setminus (W' \cup W'')} \Phi_i(W) \\
&= 1 + \sum_{i \in W'' \setminus W'} \frac{\alpha_i}{\alpha(W'')} + \sum_{i \in W \setminus (W' \cup W'')} \Phi_i(W) \\
&> 1,
\end{aligned}$$

which contradicts coalitional efficiency. Therefore, strict inequality has to hold either in (6) or in (7) for some player i' in either W' or W'', which means that either W' or W'' is a deviation from Π. Hence, $\Pi \notin SSC(N, \succeq)$.

Case 2: W contains exactly one minimal winning coalition W'. Moreover, $W' \notin \mathcal{A}^v$.

Notice that each player $i \in W \setminus W'$ is a null player in v_W and that all players in W' are symmetric in v_W. Hence, we have by the null player property,

$$\Phi_i(W) = 0 \qquad (i \in W \setminus W'). \tag{8}$$

Using symmetry, coalitional efficiency, and (8) we get

$$\sum_{i \in W} \Phi_i(W) = \sum_{i \in W'} \Phi_i(W) = 1 = \sum_{i \in W'} \Phi_i(W'),$$

from which and Lemma 1

$$\Phi_i(W) = \Phi_i(W') = \frac{\alpha_i}{\alpha(W')} \qquad (i \in W') \tag{9}$$

follows. Take $T \in \mathcal{A}^v$. Since v is proper, $T \cap W' \neq \emptyset$. By (9) and $\alpha(W') > \alpha(T)$, we have on the one hand for all $i \in T \cap W'$,

$$\Phi_i(\Pi(i)) = \Phi_i(W) = \Phi_i(W') = \frac{\alpha_i}{\alpha(W')} < \frac{\alpha_i}{\alpha(T)} = \Phi_i(T).$$

On the other hand, by (8), for all $i \in N \setminus W'$, $\Phi_i(\Pi(i)) = 0$. Thus, for all $i \in T \setminus W'$,

$$\frac{\alpha_i}{\alpha(T)} = \Phi_i(T) > \Phi_i(\Pi(i)) = 0.$$

Hence, we have $\Phi_i(T) > \Phi_i(\Pi(i))$ for all $i \in T$, implying $\Pi \notin SSC(N, \succeq)$.

Case 3: W contains exactly one minimal winning coalition $W' \in \mathcal{A}^v$ and there is $j \in (W \setminus W') \setminus \mathcal{D}^v$.

Since W' is the only minimal winning subcoalition of W, we can conclude, analogous to Case 2, that $\Phi_i(W) = \Phi_i(W') = \frac{\alpha_i}{\alpha(W')}$ holds for all $i \in W'$ and $\Phi_i(W) = \Phi_i(W') = 0$ is true for all $i \in W \setminus W'$. Since $j \notin \mathcal{D}^v$ there is a minimal winning coalition with minimal total weight $T \in \mathcal{A}^v$ with $T \neq W'$ and $j \in T$. We show that T can block Π. Clearly, for all players $i \in T \cap W'$ we have $\Phi_i(T) = \frac{\alpha_i}{\alpha(T)} = \frac{\alpha_i}{\alpha(W')} = \Phi_i(W) = \Phi_i(\Pi(i))$. All players $k \in T \cap (N \setminus W')$, such as j, satisfy $\Phi_k(T) = \frac{\alpha_k}{\alpha(T)} > 0 = \Phi_k(\Pi(k))$. Hence T is a deviating coalition from Π in the semistrict sense, or put differently, $\Pi \notin SSC(N, \succeq)$.

The three cases together cover all coalition structures Π that do not belong to $\mathbf{P}_{\mathcal{A}^v_\Delta}$ and reach the conclusion that Π is not semistrictly core stable. Hence, we have shown the desired inclusion $SSC(N, \succeq) \subseteq \mathbf{P}_{\mathcal{A}^v_\Delta}$. □

From the proofs of Theorems 1 and 2 we can extract the following interesting observation.

Corollary 1 *Let $\alpha \in \mathbb{R}^N_{++}, v \in \mathcal{G}$, and $\varphi \in \mathcal{S}^*$ be given such that (BP) is satisfied. If there is only one minimal winning coalition with minimal total weight, then all three core stability notions generate the same set of stable coalition structures. More precisely, $|\mathcal{A}^v| = 1$ implies*

$$SC(N, \succeq) = SSC(N, \succeq) = C(N, \succeq) = \mathbf{P}_{\mathcal{A}^v_\Delta}.$$

Proof. (Sketch) Theorems 1 and 2 establish equality for the semistrict core. Let T denote the minimal winning coalition with minimal total weight. Part 1 of Theorem 1 shows that any deviating coalition X from $\Pi \in \mathbf{P}_{\mathcal{A}^v_\Delta}$ contains T as its unique minimal winning subcoalition so that $\Phi_i(X) = \Phi_i(T)$ for all $i \in T \cap X = T$ shows that X in fact cannot block Π in either sense. Conversely, in analogy to the three cases in the proof of Theorem 2, any $\Pi' \notin \mathbf{P}_{\mathcal{A}^v_\Delta}$ can be blocked by the coalition T. □

4 Special cases

In a composite solution asymmetries among players can either be expressed by an unequal weight vector, or by a solution concept φ that takes players' possibilities to form winning coalitions into account. In this section we analyze the two cases in which either source is ruled out: We first restrict our interest to the equal weights case, i.e., we rule out asymmetries among players that are based on external considerations and again consider hedonic games induced by the composite solution Φ as in (3). The second part of this section is devoted to the case, in which the solution concept is the partnership solution Pa as defined in Section 2. Thus, the solution ignores asymmetries stemming from endogenous considerations. Here we obtain a full characterization of the semistrict core that relies on weaker assumptions as those imposed in Theorem 2.

4.1 Equal weights and core existence

Let $v \in \mathcal{G}$ and $\varphi \in \mathcal{S}$ satisfy coalitional efficiency. Moreover, let $\alpha \in \mathbb{R}^N_{++}$ be a weight vector with equal weights, i.e., $\alpha_i = \bar{\alpha}$ for all $i \in N$. Then, for all $S \in \mathcal{W}^v$ and all $i \in S$ we have,

$$\frac{\alpha_i \varphi_i(S, v_S)}{\sum_{j \in S} \alpha_j \varphi_j(S, v_S)} \cdot v_S(S) = \frac{\bar{\alpha} \varphi_i(S, v_S)}{\bar{\alpha} \sum_{j \in S} \varphi_j(S, v_S)} \cdot v_S(S) = \varphi_i(S, v_S),$$

and therefore

$$\Phi_i(S) = \begin{cases} \varphi_i(S, v_S), & \text{if } i \in S, \\ 0, & \text{otherwise} \end{cases} \qquad (i \in N).$$

Notice that if we take equal weights then Theorem 1 can be seen as being stronger and more general than the corresponding result of Shenoy (1979). In his Theorem 7.4, Shenoy

(1979) shows that if players' preferences over coalitions are induced via the Shapley value of the corresponding subgames, and the simple game does not exhibit the paradox of smaller coalitions with respect to the Shapley value, then the core (in our terms: the core of the corresponding hedonic game) is nonempty.

As we show next, even if we do not assume equal weights, a direct generalization of Shenoy's core existence result is possible. In particular, the solution does not need to satisfy the null player property.

Theorem 3 *Let $\alpha \in \mathbb{R}^N_{++}, v \in \mathcal{G}$, and let $\varphi \in \mathcal{S}$ satisfy coalitional efficiency and symmetry. If (α, v, φ) satisfies (BP), then $C(N, \succeq) \neq \emptyset$. More precisely, $\mathbf{P}_{\mathcal{A}^v} \subseteq C(N, \succeq)$.*

Proof. Let $T \in \mathcal{A}^v$ and $\Pi \in \mathbf{C}^N$ be a partition containing T. We show that $\Pi \in C(N, \succeq)$. Suppose to the contrary that there is $X \in 2^N$ such that

$$\Phi_i(X) > \Phi_i(\Pi(i)) \qquad (i \in X). \tag{10}$$

Necessarily, $X \in \mathcal{W}^v$, so that there is $Y \in \mathcal{MW}^v$ with $Y \subseteq X$. By Lemma 1,

$$\Phi_i(Y) \geq \Phi_i(X) \text{ for all } i \in Y. \tag{11}$$

By properness and $Y \in \mathcal{W}^v$, $Y \cap T \neq \emptyset$. Then, with (2), (11), and (10), for any $i \in Y \cap T$,

$$\frac{\alpha_i}{\alpha(Y)} = \Phi_i(Y) \geq \Phi_i(X) > \Phi_i(\Pi(i)) = \Phi_i(T) = \frac{\alpha_i}{\alpha(T)},$$

which is a contradiction to $\alpha(Y) \geq \alpha(T)$. □

Notice finally that Theorem 1 shows that (BP) is in fact a sufficient condition for nonemptiness even of the semistrict core, provided that one replaces coalitional efficiency by efficiency and imposes in addition the null player property on φ. However, as shown by Dimitrov and Haake (2006b), bounded power is not a necessary condition for an induced hedonic game to have a nonempty semistrict core.

4.2 Partnerships

According to Farrell and Scotchmer (1988), a partnership is a coalition that divides its output equally, i.e., in our context that divides its power equally. In the following, we use $\varphi = Pa$ as cooperative solution and show next that the core and the semistrict core of $(N, \succeq)$ coincide with the set of all partitions containing a minimal winning coalition with minimal total weight.

Consequently, with $\varphi = Pa$ we have for all $S \in \mathcal{W}^v$ and all $i \in S$,

$$\frac{\alpha_i Pa_i(S,v)}{\sum_{j\in S}\alpha_j Pa_j(S,v)} = \frac{\alpha_i}{\alpha(S)},$$

and hence

$$\Phi_i(S) = \begin{cases} \frac{\alpha_i}{\alpha(S)} \cdot v_S(S), & \text{if } i \in S, \\ 0, & \text{otherwise} \end{cases} \qquad (i \in N).$$

In other words, any winning coalition S distributes its power in this case proportionally to the exogenously given weights. Parties' possibilities to form winning coalitions are ignored.

Remark 1 *With $\varphi = Pa$, bounded power reduces to $\alpha(S) \leq \alpha(T)$ for all $T \in \mathcal{W}^v, S \in \mathcal{MW}^v, S \subset T$. This condition is trivially satisfied, since $\alpha_i > 0$ for all $i \in N$.*

Recall that the partnership solution satisfies efficiency and symmetry but not the null player property. Thus, we cannot directly apply Theorem 1 to deduce nonemptiness of the semistrict core. Nevertheless, the next theorem shows that in this context the semistrict core and the core coincide with the set of coalition structures containing a minimal winning coalition with minimal total weight.

Theorem 4 *Let $\alpha \in \mathbb{R}^N_{++}, v \in \mathcal{G}$, and $\varphi = Pa$. Then, $SSC(N,\succeq) = C(N,\succeq) = \mathbf{P}_{\mathcal{A}^v}$.*

Proof. Notice first that Pa satisfies coalitional efficiency and symmetry. Moreover, by Remark 1, (BP) is satisfied as well. Hence, we can apply Theorem 3 and get $\mathbf{P}_{\mathcal{A}^v} \subseteq C(N,\succeq)$.

To show that $C(N,\succeq) \subseteq \mathbf{P}_{\mathcal{A}^v}$ holds, we take a coalition structure $\Pi \in \left(\mathbf{C}^N \setminus \mathbf{P}_{\mathcal{A}^v}\right)$ and show $\Pi \notin C(N,\succeq)$. If there is no winning coalition in Π this is certainly true. Otherwise, let W be the winning coalition contained in Π and let $X \in \mathcal{A}^v$. Recall that $X \cap W \neq \emptyset$ holds by properness of v. Then,

$$\Phi_i(X) = \frac{\alpha_i}{\alpha(X)} > 0 = \Phi_i(\Pi(i)) \qquad (i \in X \setminus W),$$

and

$$\Phi_i(X) = \frac{\alpha_i}{\alpha(X)} > \frac{\alpha_i}{\alpha(W)} = \frac{\alpha_i}{\alpha(\Pi(i))} = \Phi_i(\Pi(i)) \qquad (i \in X \cap W),$$

where the last inequality follows from $X \in \mathcal{A}^v$ and $W \notin \mathcal{A}^v$. Hence, $\Phi_i(X) > \Phi_i(\Pi(i))$ for each $i \in X$, and so X is a deviation (in the sense of the core) from Π, showing $\Pi \notin C(N,\succeq)$.

It remains then to show $C(N,\succeq) \subseteq SSC(N,\succeq)$ (the reverse inclusion is fulfilled by definition). Let $\Pi \notin SSC(N,\succeq)$ and W be the winning coalition in Π.[8] Then there is a

[8]Again, if Π does not a winning coalition it is not in the core.

deviating coalition Y (in the sense of the semistrict core) and a player $i' \in Y \cap W$ such that

$$\frac{\alpha_{i'}}{\alpha(Y)} = \Phi_{i'}(Y) > \Phi_{i'}(\Pi(i') = \Phi_{i'}(W) = \frac{\alpha_{i'}}{\alpha(W)},$$

from which $\alpha(Y) < \alpha(W)$ and hence $W \notin \mathcal{A}^v$ follow. Therefore, $\Pi \notin \mathbf{P}_{\mathcal{A}^v}$, which means in view of the first part of the proof, $\Pi \notin C(N, \succeq)$ and we have the desired inclusion. $\square$

In order to obtain a complete characterization of the core, Shenoy (1979) considers the class of symmetric monotonic simple games. A simple game is *symmetric* if the worth of a coalition, i.e., whether it is winning or not, only depends on its size. Notice then that if $\varphi \in \mathcal{S}$ is coalitionally efficient and symmetric, we have $\varphi_i(S, v_S) = \frac{1}{|S|}$ for each player i who is a member of a winning coalition S (S needs not be minimal winning). Thus, any such solution coincides with the partnership solution on this class and therefore Theorem 4 tells us, how semistrict core and core look like.

Corollary 2 *Let $\alpha \in \mathbb{R}^N_{++}$, and $v \in \mathcal{G}$ be a symmetric game. If $\varphi \in \mathcal{S}$ is coalitionally efficient and symmetric, then $SSC(N, \succeq) = C(N, \succeq) = \mathbf{P}_{\mathcal{A}^v}$.*

We should not fail to mention that even if we restrict ourselves to equal weights, then Corollary 2 generalizes Proposition 7.6 of Shenoy (1979).

5 Conclusion

In this paper we studied conditions that guarantee semistrict core stability in hedonic games, provided that players' preferences are derived from an underlying simple game. Using a a multiplicative composite solution we were able to generalize previous results in Shenoy (1979) by enlarging the domain of solution concepts applied to a simple game and by using a stronger stability notion. The use of the specific composite solution allowed us to incorporate the influence of both exogenous and endogenous factors on players' preferences over coalitions. The main insight from our analysis is that those coalition structures containing a minimal winning coalition with minimal total weight are semistrictly core stable. And, to get all partitions in the semistrict core, one may add certain null players to the winning coalition. Moreover, in some interesting special cases, the semistrict core consists only of such partitions. Hence, our results with respect to the above analyzed special cases can be seen as a formal proof of Riker's (1962) 'size principle' in a more general setting. Notice finally that nonemptiness of the semistrict core for the case of the partnership solution was already indicated by Kirchsteiger and Puppe (1997). However, to the best of our knowledge, our analysis is the first rigorous account using the semistrict core concept that takes into account both a large domain of solutions on simple games and exogenously given weight vectors.

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