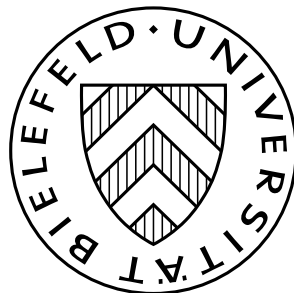


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## **Abstract**

A Cephoid is an algebraic (“Minkowski”) sum of finitely many prisms in  $\mathbb{R}^n$ . A cephoidal game is an NTU game the feasible sets of which are cephoids. We provide a version of the Shapley NTU value for such games based on the bargaining solution of Maschler–Perles.

# 1 Introduction

We wish to develop a solution concept for NTU-games based essentially on the axiom of *conditional additivity*. Our treatment will contain bargaining solutions as well as solutions for  $n$ -person cooperative NTU-games (“values”). Let us start out by defining these notions.

A **bargaining problem** is a pair  $(\underline{x}, U)$  where  $\underline{x} \in \mathbb{R}_+^n$  and  $U$  is a compact, convex, and comprehensive subset of  $\mathbb{R}^n$  containing  $\underline{x}$ .  $U$  is the **feasible set** and  $\underline{x}$  is the **status quo point**. Predominantly we assume case  $\underline{x} = 0$ , hence it suffices to consider  $U$  while  $\underline{x}$  is suppressed.

A (cooperative) **NTU-game** is a triple  $(I, \underline{P}, V)$ .  $I = \{1, \dots, n\}$  is the “set of players”,  $\underline{P} = \{S | S \subseteq I\} = \mathcal{P}(I)$  is the “system of coalitions”, and  $V := \underline{P} \rightarrow \mathcal{P}(\mathbb{R}^n)$  assigns to any coalition a closed, convex and comprehensive set of “utility vectors”  $V(S) \subseteq \mathbb{R}_S^n$  which are “feasible for the members of  $S$ ”.  $V$  obeys certain regularity conditions, see e.g. Definition 1.3, Section 1, Chapter 4 in [14] and also Section 5. As  $I$  and  $\underline{P}$  remain fixed, we frequently refer to  $V$  as to “a game”.

A bargaining problem can be seen as an NTU-game: Given  $(\underline{x}, U)$ , let  $V(S)$  be the comprehensive hull of  $\underline{x}_S$  ( $S \in \underline{P}$ ) and  $V(I) = U$ .

A **value** is a Pareto efficient, symmetric mapping  $\varphi$  from a class of games into  $\mathbb{R}^n$  that respects affine transformations of utility (a.t.u.).  $\varphi(V)$  reflects the distribution of utility considered to “solve” the game. We write  $\varphi_U$  to emphasize that we are dealing with bargaining problems. In this case we use also the term **solution**.

A **lottery** is a probability distribution over games with finite carrier. E.g., a lottery involving two bargaining problems  $U^1$  and  $U^2$  is given by a probability  $\mathbf{p} = (p_1, p_2)$ ;  $p_1 + p_2 = 1$ . The expected utility vectors are then given by  $\mathbb{E}_{\mathbf{p}}(U^\bullet) = p_1 U^1 + p_2 U^2$ ; the definition involves the algebraic sum and the multiple of a subset of  $\mathbb{R}^n$ , i.e.,  $tU := \{t\mathbf{x} | \mathbf{x} \in U\}$  and

$$U^1 + U^2 := \{\mathbf{x}^1 + \mathbf{x}^2 | \mathbf{x}^1 \in U^1, \mathbf{x}^2 \in U^2\}.$$

Generally, for any lottery  $\mathbf{p}$ , the definition of the “expected game”  $\mathbb{E}_{\mathbf{p}}V^\bullet$  runs analogously.

The combined effects of lotteries on values are reflected in the axiomatic treatment of solution concepts. SHAPLEY [16] characterizes the value (for TU-games) – among other axioms – by additivity, which (given the concept to be positively homogeneous) is equivalent to “risk neutrality”, i.e.,  $\varphi(\mathbb{E}(V^\bullet)) = \mathbb{E}\varphi(V^\bullet)$ . There is no discussion of this concept in SHAPLEY’s generalization of the value to NTU-games [17].

MASCHLER–PERLES (see [12], [7]) require their solution for bargaining problems to be **superadditive**. That is

$$\varphi(U + U') \geq \varphi(U) + \varphi(U')$$

holds true for pairs of bargaining problems. Equivalently one has

$$\varphi(\mathbb{E}_p U^\bullet) \geq \mathbb{E}_p \varphi(U^\bullet) .$$

Thus, superadditivity is interpreted to consistently favor contracting *ex ante*, thereby increasing expected utility (see [7] or [13], *p.562*, for a detailed discussion).

For a further interpretation not involving chance mechanisms see [9]. The story involves two players (corporations, governments) engaged in two “remote” bargaining problems  $U$  and  $U'$  simultaneously. Initially, these problems (to be treated on lower corporate levels) are considered to be separate affairs. Thus, there is a tendency to settle for  $\varphi(U)$  and  $\varphi(U')$  separately. Later on ranking officials in the corporations realize that combining both bargaining projects could be advantageous. The solution being superadditive, it turns out that both players/corporations profit from a *quid quo pro*.

The Maschler Perles solution works for two players only. PERLES[11] showed that for more than two players, a superadditive solution for bargaining problems does not exist. The solution presented in [9] generalizes the Maschler–Perles procedure (see also CALVO–GUTIÉRREZ [3]) and exhibits a class of games for which superadditivity prevails.

AUMANN’S[1] axiomatization of SHAPLEY’S NTU–value introduces the idea of a **conditionally additive** value. As SHAPLEY and others (see HART [5], DE CLIPPEL), he considers correspondences; the condition rephrased for functions requires that

$$\varphi(\mathbb{E}_p U^\bullet) = \mathbb{E}_p \varphi(U^\bullet) .$$

holds whenever the right hand term  $\mathbb{E}_p \varphi(U^\bullet)$  is Pareto efficient in  $\mathbb{E}_p U^\bullet$ . Equivalent is the version as follows.

**Definition 1.1.** A value  $\psi$  is **conditionally additive** if, for any two games  $V$  and  $W$  such that  $\psi(V) + \psi(W)$  is Pareto efficient in  $V(I) + W(I)$ , it follows that

$$(1.1) \quad \psi(V) + \psi(W) = \psi(V + W).$$

For two players conditional additivity is equivalent to superadditivity in order to characterize the Maschler–Perles solution. This follows easily from the construction given in [12], see also the discussion in [15].

AUMANN’S concept is based on games with smooth surfaces of each  $V(S)$  while MASCHLER and PERLES start out from a polyhedral setup. More recently, DE CLIPPEL ET AL. elaborate on the problem imposed by choosing the domain of definition for the axiomatic treatment of a value. It is obvious that in 2 dimensions conditional additivity and the IIA axiom characterizing the Nash solution are not compatible.

Within this paper we discuss a (single valued) solution concept  $\chi$  for NTU games. We provide an axiomatization of this value. The value is characterized without any version of IIA, its existence does not rely on a fixed point theorem, and it can be computed in a straightforward manner.

We focus on a particular class of polyhedral games called “cephoidal”. A cepheid is a sum of “prisms”.

To be more precise, let  $\mathbf{a} = (a_1, \dots, a_n) > 0 \in \mathbb{R}_+^n$  be a positive vector and let  $\mathbf{e}^i$  denote the  $i^{\text{th}}$  unit vector. Write  $\mathbf{a}^i := a_i \mathbf{e}^i$  ( $i \in \mathbf{I} := \{1, \dots, n\}$ ) and let  $\Pi^{\mathbf{a}} := \text{convH}(\{0, \mathbf{a}^1, \dots, \mathbf{a}^n\})$ . We call  $\Pi^{\mathbf{a}}$  a *prism*. The Pareto surface  $\partial\Pi^{\mathbf{a}}$  is the *simplex*  $\Delta^{\mathbf{a}} := \text{convH}(\{\mathbf{a}^1, \dots, \mathbf{a}^n\})$  (we wish to distinguish the two concepts). A prism represents a “primitive bargaining” problem. There is actually transferable utility in the model though depending on a “transfer rate” (the normal of  $\Delta^{\mathbf{a}}$ ). Consider an algebraic sum of prisms, i.e., let  $\mathbf{a}^\bullet = (\mathbf{a}^{(k)})_{k=1}^K = (\mathbf{a}^{(k)})_{k \in \mathbf{K}}$  be a *family* of positive vectors. Then

$$(1.2) \quad \Pi = \Pi^{\mathbf{a}^\bullet} := \sum_{k=1}^K \Pi^{\mathbf{a}^{(k)}} =: \sum_{k \in \mathbf{K}} \Pi^{(k)}$$

is called a *cephoid*. Cepheids have been introduced in [8], see also [10], [9]. Throughout this paper we assume that a cepheid is *nondegenerate*, see [8] for the details. This assumption ensures that the normals of the  $\Delta^{\mathbf{a}^{(k)}}$  (and of their boundary simplices) do not coincide. The result is a standardized structure of the Pareto surface of a cepheid. The set of (nondegenerate) cepheids in  $\mathbb{R}^n$  is denoted by  $\mathfrak{C}^n$ .

A bargaining problem is *cephoidal* if the feasible set  $U$  is a cepheid. A game is *cephoidal* if, for every  $S \in \underline{\mathbf{P}}$  the feasible set  $\mathbf{V}(S)$  is a cepheid, i.e., for every  $S \in \underline{\mathbf{P}}$ , there is a family  $(\mathbf{a}^{S,(k)})_{k \in \mathbf{K}_S}$  such that

$$(1.3) \quad \mathbf{V}(S) = \sum_{k \in \mathbf{K}_S} \Pi^{\mathbf{a}^{S,(k)}} =: \sum_{k \in \mathbf{K}_S} \Pi^{S,(k)}$$

holds true. We restrict the discussion to feasible sets in  $\mathbb{R}_+$  of cephoidal character, and affine transformation of utility (“a.t.u.”) refers to positive dilatation of the axes.

Cepheids should be considered whenever conditional additivity is an issue. In  $\mathbb{R}_+^2$ , actually *all* polyhedra are cephoidal. In  $\mathbb{R}^n$  this is not true.

Assume that players are involved in *several* (“primitive”) bargaining problems. There is either a lottery over these choosing one of them or we imagine (following the interpretation suggested in [9]) that bargaining takes place simultaneously in separate environments (countries or states). To keep both interpretations at hand, we refer to the different bargaining problems as to “states”.

Each state  $k$  refers to a (primitive) bargaining problem  $\Pi^{\mathbf{a}^{(k)}}$ . Now, bargaining takes place *ex ante* with respect to feasible utility assignments in each

state, so players may consider giving in by an  $\varepsilon$  with respect to state  $k$  for obtaining a  $\delta$  in state  $l$ . The final utility a player obtains is the expectation or (equivalently) the *sum* of the utilities he receives from his shares in each state. Hence, the resulting (“expected” or “global”) bargaining problem is represented by the cephoid

$$\Pi = \sum_{k \in K} \Pi^{a^{(k)}}.$$

Assignment of utility should involve Pareto efficient allocations only. Describing the shape of the *Pareto surface* amounts to indicating the maximal faces of the polyhedron  $\Pi$ . The general theory for the maximal faces of a cephoid can be found in [8].

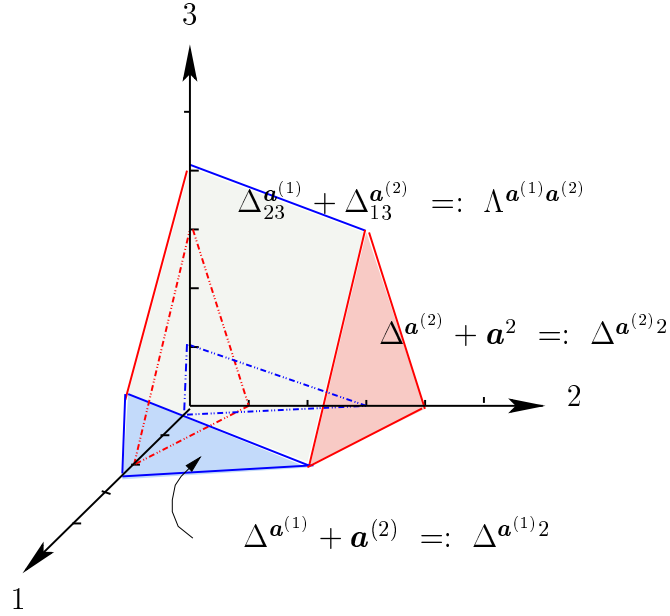


Figure 1.1: The sum of 2 prisms

Figure 1.1 describes a simple but non trivial example. We observe the sum of two prisms in  $\mathbb{R}_+^3$  (the original prisms appear in dotted lines). PERLES formulates his counterexample in terms of similar objects.

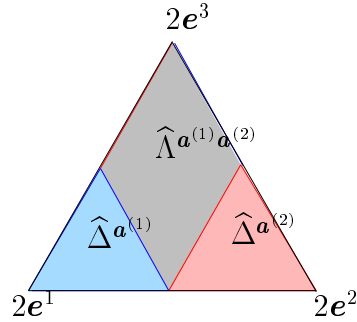


Figure 1.2: The canonical representation

The example represents a situation with three players and two states. The Pareto surface  $\partial\Pi$  of this bargaining problem shows three maximal faces,

two of them are translates of a summand by an extremal vector of the other summand. The third maximal face is the sum of two line segments each one taken from a summand.

The structure of the Pareto surface can be visualized by the “canonical representation”, see [9]. This representation is provided by a bijective mapping of the Pareto surface of  $\Pi$  onto the simplex  $2\Delta^e$  as indicated by Figure 1.2.

Figure 1.3 shows a cephaloid bargaining problem that is the sum of 4 prisms. Thus, we have three players bargaining in *four* states.

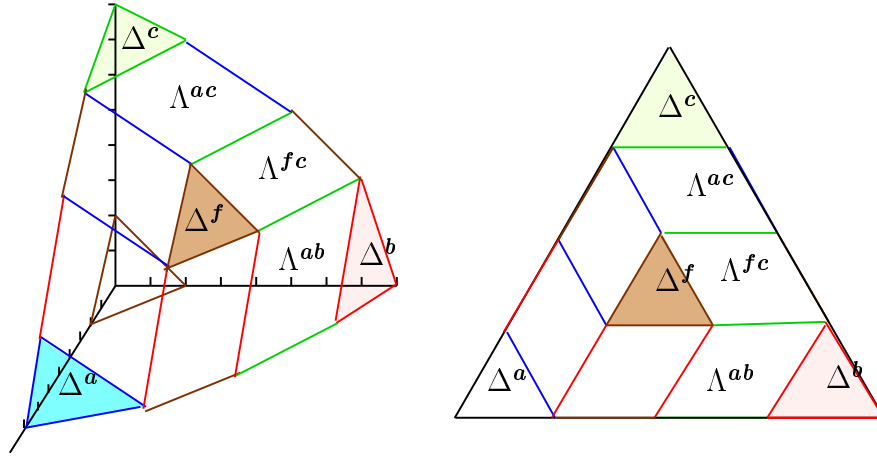


Figure 1.3: The sum of 4 prisms

For 2 dimensions (i.e., bargaining problems with 2 players) a cephaloid which is a sum of 5 prisms (hence represents 5 states) is indicated by Figure 1.4; the right hand side shows the corresponding canonical representation.

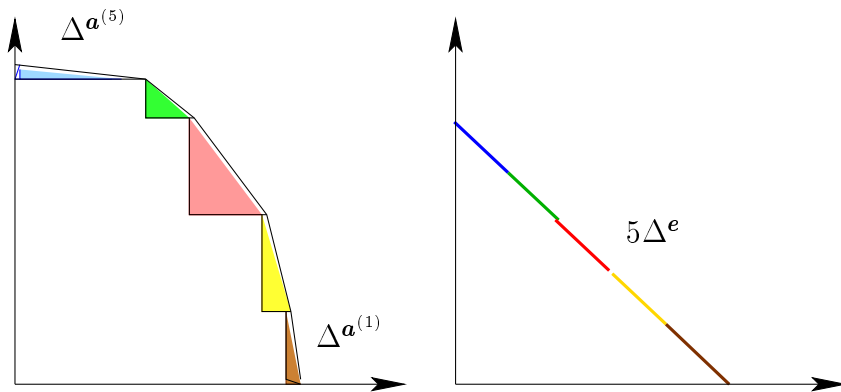


Figure 1.4: A 2 dimensional cephaloid bargaining problem

Consider the Maschler–Perles solution ([12]) in 2 dimensions. Due to the superadditivity axiom, this solution evaluates concessions of the players along maximal faces (i.e., line segments) according to the *area* of the triangles (prisms) corresponding to line segments (simplices). We are led to assign a new length measurement (“surface measure”) to a maximal face which is (in

two dimensions) the square root of the area of the corresponding prism. E.g., if the feasible set in Figure 1.4 is the cepheid

$$\Pi = \sum_{k=1}^5 \Pi^{a^{(k)}}$$

then the surface measure of (translate of) the line segment (simplex)  $\Delta^{a^{(k)}}$  is  $\tau_k := \iota_{\Delta}(\Delta^{a^{(k)}}) := \sqrt{a_1^{(k)} a_2^{(k)}}$  (see Figure 1.5).

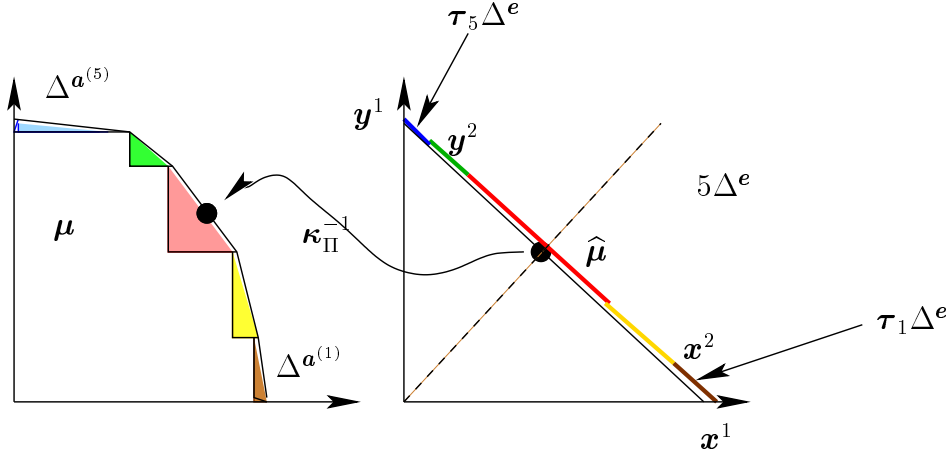


Figure 1.5: The M–P solution as the inverse image of the center point

The concession of player 1 when he moves from his bliss point  $x^1$  to  $x^2$  along  $\Delta^{a^{(1)}}$  is considered to be equal to the concession of player 2 to move from  $y^1$  to  $y^2$  along  $\Delta^{a^{(5)}}$  if and only if  $\tau_1 = \tau_5$  holds true. This results in a distribution of utility at the Maschler–Perles solution at which both players have made equal over all concessions.

In order to construct the solution, the total sum  $\tau := \sum_{k=1}^5 \tau_k$  determines the size of a new simplex  $\tau \Delta^e$ . Each line segment  $\Delta^{a^{(k)}}$  is bijectively mapped onto a copy in  $\tau \Delta^e$ , the size of this copy is the surface measure  $\tau_k$  of the line segment. This way a bijective mapping  $\kappa_{\Pi}$  of the Pareto surface  $\partial \Pi$  onto a multiple of the unit simplex (the space of adjusted commodity) appears. With respect to this representation, concessions of players along the Pareto surface are measured by Lebesgue measure. Hence the midpoint  $\hat{\mu}$  of  $\tau \Delta^e$  generates the Maschler–Perles solution  $\mu$  as the inverse image in  $\partial \Pi$ , i.e.,  $\mu = \kappa_{\Pi}^{-1}(\hat{\mu})$ .

Note that the construction is completely determined by the axiomatic of the Maschler–Perles solution. Superadditivity or conditional additivity of the solution dictates the evaluation of concessions via the area (“volume”) of the prisms involved.

Our program is to present this rationale in an axiomatic way for  $n$  players. Actually, a generalized Maschler–Perles solution has been defined in [9]. However for  $n$  players, superadditivity is, at least in full generality, not achievable



(PERLES' result [11]). The axiomatization will therefore rely on conditional additivity *and* on the surface measure.

The “surface measure” for cephoids in  $n$  dimensions is basically determined by a further consistency requirement. If two players regard line segments with equal area of the corresponding triangles as equal concessions (with respect to the construction of the solution), then for three players the relation of any two of them determines a surface measure for a three dimensional cepheid.

Consider again Figure 1.1 and Figure 1.2. Pareto efficient trade off, say in state 2, is reflected by  $\hat{\Delta}^{a^{(2)2}}$ . Here a util sacrificed by player 1 is returned to player 2 via the transfer rate (the normal of the simplex). Evaluation of concessions involves the surface measure.

For a mixed state, Pareto efficient trade off involving the *two states* 1 and 2 takes place in the diamond of Figure 1.2. The corresponding face is denoted by  $\Lambda^{a^{(1)}a^{(2)}}$  on the Pareto surface in 1.1.

Along the boundary lines of this diamond with a simplex (i.e., a copy of a primitive situations) the utility measurement regarding concessions should be consistent – i.e., the length of the boundary segments should be determined by the length measurement in the primitive situations. As a diamond has two linear boundary segments (determined by two simplices), this induces the requirement that the area should be defined by the area in the simplices.

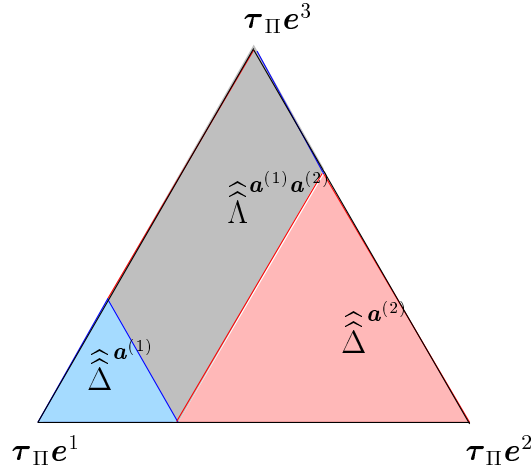


Figure 1.6: The adjusted commodity space for 3 players

The results in [9] show that one can arrange for a “measure preserving representation”. This consists of a suitable multiple  $\tau\Delta^e$  of the unit simplex plus a bijective mapping that preserves the partial ordering of maximal faces. The mapping transports the surface measure into the Lebesgue measure.

The result may look as in Figure 1.6. Accordingly, the generalized Maschler–Perles solution is the inverse image of the center point under the identifying bijection.

For a cephoidal *NTU-game* we have to imagine that each  $V(S)$  is a cepheid in  $s = |S|$  dimensions. In the context of games we wish to study NTU-values.

We shall present an axiomatic approach to the (generalized) Maschler–Perles solution as well as to a corresponding Shapley NTU–value.

## 2 The Surface Measure and the Conditionally Additive Solution

Within this section we recall the definition of the surface measure and discuss the solution concept presented in [9]. We exhibit the conditional additivity of the concept.

For  $\mathbf{a} \in \mathbb{R}_+^n$  and  $\mathbf{J} \subseteq \mathbf{I}$  we write

$$(2.1) \quad \mathbf{P}_{\mathbf{J}}^{\mathbf{a}} := \prod_{i \in \mathbf{J}} a_i .$$

For any prism  $\Pi^{\mathbf{a}}$  the *adjustment factor* is

$$(2.2) \quad \tau_{\Pi^{\mathbf{a}}} := \tau_{\mathbf{a}} := \sqrt[n]{(\mathbf{P}_{\mathbf{I}}^{\mathbf{a}})} .$$

This notion is extended to cephoids by additivity, that is, for a cephoid  $\Pi = \Pi^{\mathbf{a}^\bullet}$  we define the *adjustment factor* to be

$$(2.3) \quad \tau_{\Pi} := \sum_{k \in \mathbf{K}} \tau_{\Pi^{(k)}} .$$

Now we turn to the *surface measure* of a cephoidal face. We start out with prisms. For positive  $\mathbf{a} \in \mathbb{R}_+^n$  the *surface measure* assigned to  $\Delta^{\mathbf{a}}$  is

$$(2.4) \quad \iota_{\Delta}(\Delta^{\mathbf{a}}) := \sqrt[n]{(\mathbf{P}_{\mathbf{I}}^{\mathbf{a}})^{n-1}} = \tau_{\mathbf{a}}^{n-1} .$$

In particular, the unit simplex  $\Delta^{\mathbf{e}}$  receives surface measure 1.

Next, let  $\mathbf{F}$  be a maximal face of a cephoid  $\Pi$ . Then there is a system  $\mathcal{J} = (\mathbf{J}^{(1)}, \dots, \mathbf{J}^{(K)})$  of subsets of  $\mathbf{I}$  (the “reference system”) such that

$$(2.5) \quad \mathbf{F} = \Delta_{\mathbf{J}^{(1)}}^{(1)} + \dots + \Delta_{\mathbf{J}^{(K)}}^{(K)}$$

holds true. The numbers  $j_k := |\mathbf{J}^{(k)}|$  satisfy

$$(2.6) \quad (j_1 - 1) + \dots + (j_K - 1) = n - 1 ,$$

meaning that the dimensions of the sub-simplices involved in the construction of  $\mathbf{F}$  add up to the dimension of  $\mathbf{F}$  (see [8]). Let  $\mathbf{c}_{\mathbf{F}}$  denote the “normalizing coefficient”, i.e., quotient of the volume of  $\Delta_{\mathbf{J}^{(1)}}^{\mathbf{e}} + \dots + \Delta_{\mathbf{J}^{(K)}}^{\mathbf{e}}$  and the volume of  $\Delta^{\mathbf{e}}$  (see [9]). We can define the surface measure of  $\mathbf{F}$ .

**Definition 2.1.** *Let  $\Pi = \Pi^{\mathbf{a}^\bullet}$  be a cephoid and let  $\mathbf{F}$  be a maximal face with reference system  $\mathcal{J}$ . Then the *surface measure* of  $\mathbf{F}$  is given by*

$$(2.7) \quad \iota_{\Delta}(\mathbf{F}) = c_{\mathbf{F}} \sqrt[n]{\left[\mathbf{P}_{\mathbf{I}}^{(1)}\right]^{j_1-1} \cdot \dots \cdot \left[\mathbf{P}_{\mathbf{I}}^{(K)}\right]^{j_K-1}}$$

with  $\mathbf{P}_{\mathbf{I}}^{(k)} := \mathbf{P}_{\mathbf{I}}^{\mathbf{a}^{(k)}} \quad (k \in \mathbf{K})$ .

The *measure preserving representation* of the surface  $\partial\Pi$  consists of the multiple  $\hat{\Delta} := \tau_{\Pi}\Delta^e$  of the unit simplex and a bijective, piecewise linear mapping  $\kappa = \kappa_{\Pi}$  of  $\partial\Pi$  onto  $\hat{\Delta}$ .  $\kappa$  carries the vertices of  $\partial\Pi$  into a set of grid points on  $\hat{\Delta}$  in a way that preserves the partial ordering of faces. In addition, the Lebesgue measure of the image of a maximal face  $\mathbf{F}$  of  $\partial\Pi$  is the surface measure of  $\mathbf{F}$ . Compare Figures 1.1 and 1.6

To be more precise, let  $\Pi = \Pi^{a^\bullet}$  be a cepheid. For every  $k = 1, \dots, K$  let

$$(2.8) \quad \hat{\mathbf{a}}^{(k)} := \tau_{a^{(k)}} \mathbf{e} \quad , \quad \hat{\Delta}^{(k)} := \Delta^{(\hat{\mathbf{a}}^{(k)})}$$

such that for  $k \in \mathbf{K}$

$$(2.9) \quad \iota_{\Delta}(\hat{\Delta}^{(k)}) = \iota_{\Delta}(\Delta^{(k)})$$

is satisfied. Define

$$(2.10) \quad \hat{\Delta} := \sum_{k=1}^K \hat{\Delta}^{(k)} = \tau_{\Pi} \Delta^e .$$

We define the mapping  $\kappa_{\Pi}$  first on vertices  $\partial\Pi$  and then extend it to maximal faces by (“piecewise”) convexity. Compare the more detailed exposition in [9].

By non-degeneracy every vertex of  $\partial\Pi$  is a unique sum of vertices of the simplices  $\Delta^{(k)}$  involved. Thus, for every vertex  $\mathbf{u}$  of  $\partial\Pi$ , there is a mapping  $\mathbf{i}_{\bullet} : \mathbf{K} \rightarrow \mathbf{I}$  such that  $\mathbf{u}$  can be written via

$$(2.11) \quad \mathbf{u} = \mathbf{a}^{\mathbf{i}_{\bullet}} = \sum_{k \in \mathbf{K}} \mathbf{a}^{(k) \mathbf{i}_k} .$$

**Definition 2.2.** 1. Let  $\mathbf{u}$  be a vertex on  $\partial\Pi$  and let  $\mathbf{i}_{\bullet}$  be the corresponding mapping given by (2.11). Then

$$(2.12) \quad \hat{\mathbf{u}} := \kappa_{\Pi}(\mathbf{u}) := \sum_{k \in \mathbf{K}} \hat{\mathbf{a}}^{(k) \mathbf{i}_k} := \sum_{k \in \mathbf{K}} \tau_{a^{(k)}} \mathbf{e}^{\mathbf{i}_k} .$$

2. For a maximal face  $\mathbf{F}$  of  $\partial\Pi$ , the mapping  $\kappa_{\Pi}$  is extended by affine linearity using the vertices of  $\mathbf{F}$ . Hence we obtain the mapping

$$(2.13) \quad \kappa_{\Pi} : \partial\Pi \rightarrow \tau_{\Pi} \Delta^e = \hat{\Delta} ,$$

which constitutes a piecewise linear isomorphism. The (simplex  $\hat{\Delta}$  together with) the mapping  $\kappa_{\Pi}$  is called the *measure preserving representation* of  $\partial\Pi$ .

**Remark 2.3.** For  $n = 2$ , the surface measure coincides up to a constant with the standard travelling time along the Pareto surface as introduced by Perles–Maschler [7, 12], see also [13] for a textbook version. The travelers starting from both bliss points on the Pareto surface  $\partial\Pi$  meet at the Maschler–Perles solution after having spent equal time during the voyage. That is, in terms of the surface measure, the Perles–Maschler solution is the inverse image of the midpoint or barycenter of  $\hat{\Delta}$  under  $\kappa_{\Pi}$ . This has been used in [9] for a generalization of the Maschler–Perles solution as follows.

Now we define our solution concept which, at this stage, is a function on cephoids. A cephoid  $\Pi$  stands for the bargaining problem  $(\mathbf{0}, \Pi)$  or for the corresponding NTU-game  $\mathbf{V}$  as explained in SECTION 1. We use the notation  $\hat{\boldsymbol{\mu}} := \frac{1}{n}(1, \dots, 1) = \frac{1}{n}\mathbf{e} \in \mathbb{R}^n$ .

**Definition 2.4.** *The **conditionally additive solution** (the c.a. solution) for cephoidal bargaining problems is the mapping  $\boldsymbol{\mu} : \mathfrak{C}^n \rightarrow \mathbb{R}^n$  given by*

1.  $\boldsymbol{\mu}_{t\Delta^e} := t\hat{\boldsymbol{\mu}} = \frac{t}{n}(1, \dots, 1) \quad (t > 0)$
2.  $\boldsymbol{\mu}_\Pi := \boldsymbol{\kappa}_\Pi^{-1}(\boldsymbol{\mu}_{\tau_\Pi\Delta^e}) = \boldsymbol{\kappa}_\Pi^{-1}(\tau_\Pi\hat{\boldsymbol{\mu}})$

It should be clear that, for  $n = 2$ , this is the Maschler–Perles superadditive solution. Hence  $\boldsymbol{\mu}$  is a generalized Maschler–Perles solution. Clearly, we should justify the name and provide an axiomatization.

**Theorem 2.5.** *1. The adjustment factor is additive, i.e., for any two cephoids  $\Pi$  and  $\Pi'$*

$$(2.14) \quad \tau_{\Pi+\Pi'} = \tau_\Pi + \tau_{\Pi'}$$

*holds true.*

2. *The canonical representation is **conditionally additive**. I.e., for any two cephoids  $\Pi$  and  $\Pi'$  and any  $\mathbf{x} \in \partial\Pi$ ,  $\mathbf{x}' \in \partial\Pi'$  satisfying  $\mathbf{x} + \mathbf{x}' \in \partial(\Pi + \Pi')$ , it follows that*

$$(2.15) \quad \boldsymbol{\kappa}_\Pi(\mathbf{x}) + \boldsymbol{\kappa}_{\Pi'}(\mathbf{x}') = \boldsymbol{\kappa}_{\Pi+\Pi'}(\mathbf{x} + \mathbf{x}')$$

*holds true.*

**Proof:**

**1<sup>st</sup>STEP :**

The first statement is obvious from the definition, i.e., by (2.3).

**2<sup>nd</sup>STEP :**

Regarding the second statement, we start out with two extremal points  $\mathbf{u} \in \partial\Pi$  and  $\mathbf{u}' \in \partial\Pi'$ . Consider the corresponding mappings as given by (2.11), say  $\mathbf{i}_\bullet : \mathbf{K} \rightarrow \mathbf{I}$  and  $\mathbf{i}'_\bullet : \mathbf{K}' \rightarrow \mathbf{I}$ . We assume that the sum  $\mathbf{u} + \mathbf{u}'$  is Pareto efficient, hence extremal in  $\partial(\Pi + \Pi')$ . We write

$$(2.16) \quad \mathbf{u} + \mathbf{u}' = \sum_{k \in \mathbf{K}} \mathbf{a}^{(k)\mathbf{i}_k} + \sum_{k' \in \mathbf{K}'} \mathbf{a}'^{(k')\mathbf{i}'_{k'}} = \sum_{l \in \mathbf{K} \cup \mathbf{K}'} \bar{\mathbf{a}}^{l\bar{\mathbf{i}}_l}$$

with canonically defined quantities

$$(2.17) \quad \bar{\mathbf{a}}^l = \mathbf{a}^l \quad (l \in \mathbf{K}), \quad \bar{\mathbf{a}}^l = \mathbf{a}'^l \quad (l \in \mathbf{K}').$$

and

$$(2.18) \quad \bar{\iota}_l = \iota_l \quad (l \in K), \quad \bar{\iota}_l = \iota'_l \quad (l \in K').$$

Consequently

$$(2.19) \quad \kappa(\mathbf{u}) + \kappa(\mathbf{u}') = \sum_{k \in K} \hat{\mathbf{a}}^{(k)\mathbf{i}_k} + \sum_{k' \in K'} \hat{\mathbf{a}}'^{(k')\mathbf{i}'_{k'}} = \sum_{l \in K \cup K'} \hat{\mathbf{a}}^{\bar{\iota}_l}$$

and since the mapping  $\bar{\iota}_\bullet$  corresponding to  $\mathbf{u} + \mathbf{u}'$  is uniquely defined, the right hand side in (2.19) has to be  $\kappa_{\Pi+\Pi'}(\mathbf{u} + \mathbf{u}')$ .

**3<sup>rd</sup>STEP :**

Suppose now that  $\mathbf{x}$  and  $\mathbf{x}'$  sum up to a Pareto efficient point, hence admit of a joint normal. Pick extremal points of  $\partial\Pi$  and  $\partial\Pi'$  in the tangent hyperplane generated by that normal for  $\Pi$  and  $\Pi'$  respectively. Then we have convex representations, say

$$(2.20) \quad \mathbf{x} = \sum_{\rho} \alpha_{\rho} \bar{\mathbf{x}}^{\rho}, \quad \mathbf{x}' = \sum_{\sigma} \alpha'_{\sigma} \bar{\mathbf{x}}'^{\sigma}.$$

with positive coefficients adding up to 1. All extremal points admit of the same normal, hence our result from the 2<sup>nd</sup>STEP holds true for the sum of any two of them taken from the different cephoids. Also, the subfaces generated by the normal add up to a subface of the sum and all mappings behave affinely linear on these subfaces. In view of

$$(2.21) \quad \kappa_{\Pi}(\mathbf{x}) = \sum_{\rho} \alpha_{\rho} \kappa_{\Pi}(\bar{\mathbf{x}}^{\rho}) = \sum_{\rho \sigma} \alpha_{\rho} \alpha'_{\sigma} \kappa_{\Pi}(\bar{\mathbf{x}}^{\rho}),$$

we obtain

$$\begin{aligned} \kappa_{\Pi}(\mathbf{x}) + \kappa_{\Pi'}(\mathbf{x}') &= \sum_{\rho \sigma} \alpha_{\rho} \alpha'_{\sigma} (\kappa_{\Pi}(\bar{\mathbf{x}}^{\rho}) + \kappa_{\Pi'}(\bar{\mathbf{x}}'^{\sigma})) \\ &= \sum_{\rho \sigma} \alpha_{\rho} \alpha'_{\sigma} (\kappa_{\Pi+\Pi'}(\bar{\mathbf{x}}^{\rho} + \bar{\mathbf{x}}'^{\sigma})) \\ (2.22) \quad &= \kappa_{\Pi+\Pi'} \left( \sum_{\rho \sigma} \alpha_{\rho} \alpha'_{\sigma} (\bar{\mathbf{x}}^{\rho} + \bar{\mathbf{x}}'^{\sigma}) \right) \\ &= \kappa_{\Pi+\Pi'} \left( \sum_{\rho} \alpha_{\rho} \bar{\mathbf{x}}^{\rho} + \sum_{\sigma} \alpha'_{\sigma} \bar{\mathbf{x}}'^{\sigma} \right) \\ &= \kappa_{\Pi+\Pi'}(\bar{\mathbf{x}} + \bar{\mathbf{x}}'), \end{aligned}$$

**q.e.d.**

**Theorem 2.6.**  $\mu$  is conditionally additive.

**Proof:** This is an immediate consequence of Theorem 2.5 as the midpoints of multiples of the unit simplices clearly behave additively. Formally we have

$$\begin{aligned}
 \mu(\Pi) + \mu(\Pi') &= \kappa_{\Pi}^{-1} \left( \widehat{\mu}(\Delta^{\tau(\Pi)} e) \right) + \kappa_{\Pi'}^{-1} \left( \widehat{\mu}(\Delta^{\tau(\Pi')} e) \right) \\
 &= \kappa_{\Pi}^{-1} \left( \tau(\Pi) \frac{e}{n} \right) + \kappa_{\Pi'}^{-1} \left( \tau(\Pi') \frac{e}{n} \right) \\
 (2.23) \quad &= \kappa_{\Pi+\Pi'}^{-1} \left( (\tau(\Pi) + \tau(\Pi')) \frac{e}{n} \right) \\
 &= \kappa_{\Pi+\Pi'}^{-1} \left( (\tau(\Pi + \Pi')) \frac{e}{n} \right) \\
 &= \mu(\Pi + \Pi')
 \end{aligned}$$

q.e.d.

We show that  $\mu$  is a bargaining solution. Pareto efficiency is obvious. The covariance properties follow from the proper behavior of the measure preserving mapping.

**Lemma 2.7.**  $\mu$  is symmetric.

**Proof:**

**1<sup>st</sup>STEP :**

Note that a permutation  $\pi : \mathbf{I} \rightarrow \mathbf{I}$  constitutes a linear mapping on  $\mathbb{R}^n$  via  $(\pi(\mathbf{x}))_i := x_{\pi^{-1}(i)} (\mathbf{x} \in \mathbb{R}^n, i \in \mathbf{I})$ . For subsimplices this implies  $\pi(\Delta_{\mathbf{J}}^{\mathbf{a}}) = \Delta_{\pi^{-1}(\mathbf{J})}^{\mathbf{a}}$  whenever  $\mathbf{a}$  is a positive vector and  $\mathbf{J} \subseteq \mathbf{I}$ . It follows at once that a maximal face

$$(2.24) \quad \mathbf{F} = \Delta_{\mathbf{J}^{(1)}}^{(1)} + \dots + \Delta_{\mathbf{J}^{(K)}}^{(K)}$$

of a cepheid  $\Pi$  induces a maximal face

$$(2.25) \quad \pi(\mathbf{F}) := \Delta_{\pi^{-1}(\mathbf{J}^{(1)})}^{(1)} + \dots + \Delta_{\pi^{-1}(\mathbf{J}^{(K)})}^{(K)}$$

of the permuted cepheid  $\pi(\Pi)$ . Consider the surface measure of such a face as given by formula (2.7) of Section 2. We obtain for the permuted version

$$(2.26) \quad \iota_{\Delta}(\pi(\mathbf{F})) = c_{(\pi\mathcal{J})} \sqrt[n]{(v_n) [V(\pi(\Pi^{(1)}))]^{\bar{j}_1-1} \cdot \dots \cdot [V(\pi(\Pi^{(K)}))]^{\bar{j}_K-1}}.$$

Here the exponents  $\bar{j}_k$  are written for the size  $|\pi^{-1}(J^{(k)})|$  of the permuted index sets, which is for each  $k$  obviously equal to  $j_k$ . The volume of a prism does not change under a permutation, so the term under the root is actually invariant. Finally, the coefficient  $c_{(\pi\mathcal{J})}$  attached to the permuted set system equals  $c_{\mathcal{J}}$  as it depends in the size of the reference sets only (see [9]). Hence the surface measure is invariant under permutations, i.e.,

$$(2.27) \quad \iota_{\Delta}(\pi(\mathbf{F})) = \iota_{\Delta}(\mathbf{F}).$$

Since the maximal faces are being permuted, so are the extremal points of  $\partial\Pi$  and as the surface measure is invariant, we conclude that the whole p.o. structure as well as the mapping  $\kappa$  comply with the permutation.

Formally, for any cepheid  $\Pi$  and any  $\mathbf{x} \in \Pi$

$$(2.28) \quad \kappa_{\pi(\Pi)}(\pi(\mathbf{x})) = \pi(\kappa_{\Pi}(\mathbf{x}))$$

or

$$(2.29) \quad \kappa_{\pi(\Pi)} = \pi \circ \kappa_{\Pi} \circ \pi^{-1}.$$

Also,

$$(2.30) \quad \tau_{\pi(\Pi)} = \tau_{\Pi}$$

is obvious, i.e., the adjustment factor invariant under permutations.

**2<sup>nd</sup>STEP** : Symmetry of the solution follows now at once; we have

$$(2.31) \quad \begin{aligned} \mu_{\pi(\Pi)} &= \kappa_{\pi(\Pi)}^{-1}(\tau_{\pi(\Pi)}\hat{\mu}) = \pi \circ \kappa_{\Pi}^{-1} \circ \pi^{-1}(\tau_{\Pi}\hat{\mu}) \\ &= \pi \circ \kappa_{\Pi}^{-1}(\tau_{\Pi}\hat{\mu}) = \pi \circ \mu_{\Pi} \end{aligned}$$

**3<sup>rd</sup>STEP** :

Now covariance with a.t.u. is verified similarly. Consider a linear mapping

$$L : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad L(\mathbf{x}) = (\alpha_1 x_1, \dots, \alpha_n x_n) \quad (\mathbf{x} \in \mathbb{R}^n)$$

for positive  $\alpha = (\alpha_1, \dots, \alpha_n)$ .

First observe that

$$(2.32) \quad \tau_{L(\Pi)} = \sum_{k \in K} \sqrt[n]{\left(\prod_{i \in I} \alpha_i a_i^{(k)}\right)} = \sqrt[n]{\prod_{i \in I} \alpha_i} \sum_{k \in K} \sqrt[n]{\left(\prod_{i \in I} a_i^{(k)}\right)} = \tau_{\alpha} \tau_{\Pi}.$$

Next define the translation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  via

$$(2.33) \quad T(\mathbf{x}) = \tau_{\alpha} \mathbf{x} \quad (\mathbf{x} \in \mathbb{R}^n)$$

such that in particular

$$(2.34) \quad T : \Delta^{\tau_{\Pi} \mathbf{e}} \rightarrow \Delta^{\tau_{L(\Pi)} \mathbf{e}}$$

holds true. Now, for some cepheid  $\Pi$ , let

$$(2.35) \quad \mathbf{u} := \sum_{k \in K} \mathbf{a}^{(k)} \mathbf{i}_k$$

be a vertex of  $\partial\Pi$  (see (2.11)). Then

$$(2.36) \quad L(\mathbf{u}) := \sum_{k \in K} L(\mathbf{a}^{(k)} \mathbf{i}_k) = \sum_{k \in K} L(\mathbf{a}^{(k)}) \mathbf{i}_k$$

is the corresponding vertex of  $L(\Pi)$ . If  $\hat{\mathbf{u}}$  is the image of  $\mathbf{u}$  under  $\kappa_\Pi$  (cf. (2.12)), then

$$\begin{aligned} \kappa_{L(\Pi)}(L(\mathbf{u})) &= \sum_{k \in \mathbf{K}} (T(\hat{\mathbf{a}}^{(k)})^{\mathbf{i}_k}) = T\left(\sum_{k \in \mathbf{K}} (\hat{\mathbf{a}}^{(k)})^{\mathbf{i}_k}\right) \\ &= T(\hat{\mathbf{u}}) = T(\kappa_\Pi(\mathbf{u})). \end{aligned} \quad (2.37)$$

Because of the linearity of  $\kappa_\bullet$  on the faces, we have

$$\kappa_{L(\Pi)}(L(\mathbf{x})) = T(\kappa_\Pi(\mathbf{x})) \quad (2.38)$$

for all  $\mathbf{x}$  in some face  $\mathbf{F}$  and then for all  $\mathbf{x} \in \partial\Pi$ . This is now reformulated to

$$\kappa_{L(\Pi)}(L(\kappa_\Pi^{-1}(\bullet))) = T(\bullet)$$

or

$$L(\kappa_\Pi^{-1}(\bullet)) = \kappa_{L(\Pi)}^{-1}(T(\bullet)). \quad (2.39)$$

**4<sup>th</sup>STEP :**

Finally, the behavior of  $\boldsymbol{\mu}$  under a.t.u. is demonstrated by

$$\begin{aligned} \boldsymbol{\mu}(L(\Pi)) &= \kappa_{L(\Pi)}^{-1}(\boldsymbol{\tau}_{L(\Pi)}\hat{\boldsymbol{\mu}}) \\ &= \kappa_{L(\Pi)}^{-1}(\boldsymbol{\tau}_\alpha\boldsymbol{\tau}_\Pi\hat{\boldsymbol{\mu}}) && \text{(by (2.32))} \\ (2.40) \quad &= \kappa_{L(\Pi)}^{-1}(T(\boldsymbol{\tau}_\Pi\hat{\boldsymbol{\mu}})) && \text{(by definition of } T) \\ &= L(\kappa_\Pi^{-1}(\boldsymbol{\tau}_\Pi\hat{\boldsymbol{\mu}})) && \text{(by (2.39))} \\ &= L(\boldsymbol{\mu}_\Pi), \end{aligned}$$

q.e.d.

### 3 Axioms for the Solution

**Definition 3.1.** An *adjustment* is a pair of mappings  $(\gamma_\bullet, \sigma)$  with the following properties:

1.  $\sigma : \mathfrak{C} \rightarrow \mathbb{R}$  (the *scaling factor*) is a positively homogeneous mapping.
2.  $\gamma_\bullet$  (the *transfer mapping*) assigns to every cepheid  $\Pi$  a bijective mapping

$$\gamma_\Pi : \partial\Pi \rightarrow \sigma_\Pi \Delta^e.$$

$\gamma_\bullet$  is positively homogeneous, i.e., satisfies

$$(3.1) \quad \gamma_{t\Pi} = t\gamma_\Pi : \partial\Pi \rightarrow \sigma_{t\Pi} \Delta^e = t\sigma_\Pi \Delta^e \quad (t > 0).$$



3. For  $\mathbf{0} < \mathbf{a} \in \mathbb{R}^n$ , the mapping

$$(3.2) \quad \gamma_{\mathbf{a}} := \gamma_{\Pi^{\mathbf{a}}} : \Delta^{\mathbf{a}} \rightarrow \sigma^{\mathbf{a}} \Delta^{\mathbf{e}}$$

is the canonical affine identification of simplices, i.e., the mapping

$$\sum_{i \in \mathbf{I}} \beta_i \mathbf{a}^i \rightarrow \sigma_{\mathbf{a}}(\beta_1, \dots, \beta_n) \quad (\beta > 0, \mathbf{e}\beta = 1).$$

For prisms we simplify the notation and write  $\gamma_{\mathbf{a}}, \sigma_{\mathbf{a}}$  instead of  $\gamma_{\Pi^{\mathbf{a}}}, \sigma_{\Pi^{\mathbf{a}}}$ .

**Remark 3.2.** A transfer mapping  $\gamma_{\bullet}$  may be conditionally additive in the sense introduced for  $\kappa_{\bullet}$  in Theorem 2.5. Note that this implies that  $\gamma_{\bullet}$  is *piecewise convex*, i.e., for any set of extremal points  $\mathbf{b}^1, \dots, \mathbf{b}^L$  of  $\partial\Pi$  and convex coefficients  $\beta = (\beta_1, \dots, \beta_L)$  ( $\beta > 0, \mathbf{e}\beta = 1$ ) with  $\sum_{l=1}^L \beta_l \mathbf{b}^l \in \partial\Pi$ , we have

$$(3.3) \quad \gamma_{\Pi} \left( \sum_{l=1}^L \beta_l \mathbf{b}^l \right) = \sum_{l=1}^L \beta_l \gamma_{\Pi}(\mathbf{b}^l)$$

**Definition 3.3.** A scaling factor  $\sigma$  is said to be *consistent*, if, there are real valued functions  $g$  and  $f$  such that for any  $i, j \in \mathbf{I}$

$$(3.4) \quad \sigma_{\mathbf{a}} = g(\sigma_{(a_i, a_j)}) f(\sigma_{(a_k)_{k \neq i, j}}) .$$

This means that the assessment of concessions by  $n$  players is consistent with the one by any two players. If  $(a_k)_{k \neq i, j}$  is fixed, then concessions between to players are evaluated according to the scaling factor for two persons. The next lemma says that  $n$  players should consistently evaluate a simplex in terms of the coordinate product. Together with positive homogeneity, this amounts to choosing  $\tau$ .

**Lemma 3.4.** Let  $\sigma$  be a consistent scaling factor. Assume that, for  $n = 2$   $\sigma$  is a function of the product. Then  $\sigma = \tau$  holds true.

**Proof:** Let  $\mathbf{0} < \mathbf{a} \in \mathbb{R}^n$ . Choose any  $i, j \in \mathbf{I}$ . Then  $\sigma_{\mathbf{a}}$  is, for fixed values of  $a_k, (k \neq i, j)$  a function of the product  $a_i a_j$ . This is true for any arbitrary choice of  $\{i, j\}$ . We show that the function  $\sigma_{\mathbf{a}}$  is exponential in the coordinate product, say

$$\sigma_{\mathbf{a}} = (a_1 \cdot \dots \cdot a_n)^r$$

Indeed, for fixed  $a_4, \dots, a_n$  write  $h^3(a_3) = f_{12}(a_3, a_4, \dots, a_n)$  etc. such that

$$\sigma_{\mathbf{a}} = g(a_1 a_2) h^3(a_3) = g(a_1 a_3) h^2(a_2) = g(a_2 a_3) h^1(a_1).$$

Then

$$\begin{aligned} \frac{\sigma_{\mathbf{a}}}{h^3(a_3) h^2(a_2) h^1(a_1)} &= \frac{g(a_1 a_2)}{h^2(a_2) h^1(a_1)} = \frac{g(a_1 a_3)}{h^3(a_3) h^1(a_1)} \\ &= \frac{g(a_3 a_2)}{h^2(a_2) h^3(a_3)} = \text{const.} \end{aligned}$$

Hence

$$\sigma_a = \text{const} \prod_{123} h^i(a_i)$$

and

$$g(a_1 a_2) = h^1(a_1) h^2(a_2)$$

Then

$$g(t) = h^1(t) h^2(1) = h^2(t) h^1(1)$$

thus

$$\frac{h^1(t)}{h^1(1)} = \frac{h^2(t)}{h^2(1)} := h(t)$$

with  $h(1) = 1$ . Consequently

$$g(t) = h(t)\alpha, \quad h(a_1 a_2)\alpha = g(a_1 a_2) = h(a_1)h(a_2)\alpha$$

meaning

$$h(a_1 a_2) = h(a_1)h(a_2).$$

Hence  $h$  is exponential and so is  $g$ . Now

$$\begin{aligned} \sigma_a &= g(a_1 a_2) f(a_3, \dots, a_n) = (a_1)^r (a_2)^r f(a_3, \dots, a_n) \\ &= (a_1)^r (a_3)^r f(a_2, a_4, \dots, a_n) = (a_2)^r (a_3)^r f(a_1, a_4, \dots, a_n), \end{aligned}$$

thus

$$\frac{\sigma_a}{(a_1)^r (a_2)^r \dots (a_n)^r} = \frac{f(a_3, \dots, a_n)}{(a_3)^r \dots (a_n)^r} = \frac{f(a_1, a_4, \dots, a_n)}{(a_1)^r (a_4)^r \dots (a_n)^r} = \dots = \text{const}$$

and

$$\sigma_a = \text{const} (a_1 \dots a_n)^r$$

Because of  $\sigma_{te} = t\sigma_e$ , hence  $r = \frac{1}{n}$ . Ignoring a constant, we come up with

$$(3.5) \quad \sigma_a = \sqrt[n]{a_1 \cdot \dots \cdot a_n} = \sqrt[n-1]{\iota_\Delta(\Delta^a)} = \tau^a.$$

**q.e.d.**

**Definition 3.5.** Let  $\eta$  be a solution, and let  $(\gamma_\bullet, \sigma)$  be an adjustment. We say that  $(\eta, \gamma_\bullet, \sigma)$  satisfies the **adjusted value axioms** if the following holds true.

1.  $\eta$  is conditionally additive.
2.  $\gamma_\bullet$  is conditionally additive.
3.  $\sigma$  is additive and consistent.
4. The solution concept respects the adjustment. That is,

$$(3.6) \quad \gamma_\Pi(\eta(\Pi)) = \eta(\gamma_\Pi(\Pi)) = \eta(\sigma_\Pi \Delta^e).$$

**Theorem 3.6.** *If  $(\eta, \gamma_\bullet, \sigma)$  satisfies the adjusted value axioms, then*

1.  $\eta$  is the generalized superadditive solution  $\mu$ ,

*Up to some positive common constant,*

2.  $\gamma_\bullet$  is the measure preserving mapping  $\kappa_\bullet$ , and
3.  $\sigma$  is the assessment function  $\tau$ .

**Proof:**

For arbitrary  $n$ , every bargaining solution yields the center-point whenever the bargaining problem is a simplex. Therefore, with respect to the last axiom, equation (3.6) can be rewritten

$$(3.7) \quad \gamma_\Pi(\eta(\Pi)) = \hat{\mu}(\gamma_\Pi(\Pi)) = \hat{\mu}(\sigma_\Pi \Delta^e) = \sigma_\Pi\left(\frac{1}{n}, \dots, \frac{1}{n}\right).$$

**1<sup>st</sup>STEP :**

For  $n = 2$  all polyhedral bargaining problems are cephoids. There is one and only one solution which is conditionally additive on polyhedral bargaining problems, this is the Maschler–Perles solution  $\mu$ , see [7],[12] [15]. Hence we have  $\eta = \mu$ .

**2<sup>nd</sup>STEP :**

We prove that, for  $n = 2$ ,  $\sigma$  and  $\tau$  coincide up to a constant. Let  $\mathbb{R}^2 \ni \mathbf{a}, \mathbf{b} > \mathbf{0}$  be positive vectors and let  $\Delta^a$  and  $\Delta^b$  be the corresponding simplices (line segments) in  $\mathbb{R}^2$  (we assume non-degeneracy). Assume that  $a_1 a_2 \geq b_1 b_2$ . Also, choose  $\alpha \leq 1$  such that  $\alpha a_1 a_2 = b_1 b_2$ . Furthermore, let  $\Pi := \Pi^a + \Pi^b$  and  $\Pi^\alpha = \Pi^{\alpha a} + \Pi^b$ . Then  $\Pi^\alpha$  is symmetric up to an affine transformation, so  $\eta(\Pi^\alpha) = \mu(\Pi^\alpha)$  is the unique vertex. Hence,  $\gamma_{\Pi^\alpha}$  maps the two line

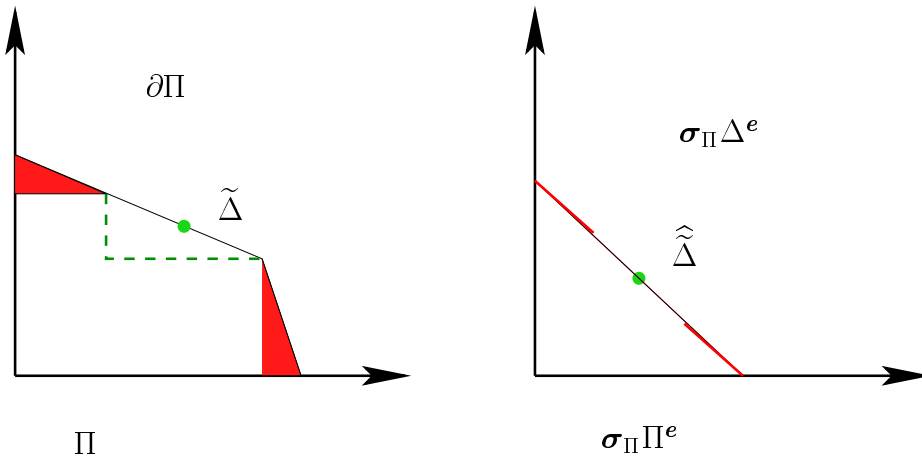


Figure 3.1: A sum of two prisms and the  $\gamma$ -image

segments of  $\partial\Pi^\alpha$  bijectively linear onto the two line segments of  $2\sigma_{\Pi^\alpha}\Delta^e$  that are generated by the midpoint  $(\sigma_{\Pi^\alpha}, \sigma_{\Pi^\alpha})$ . We conclude that  $\gamma_{\Pi^\alpha} = \kappa_{\Pi^\alpha}$  and  $\gamma_{(1-\alpha)\Pi^\alpha} = \kappa_{(1-\alpha)\Pi^\alpha}$  holds true. Now any  $\mathbf{x}$  on the “left side” of  $\partial\Pi^\alpha$  and any  $\mathbf{x}'$  on  $(1-\alpha)\Delta^a$  add up to a Pareto efficient sum  $\mathbf{x} + \mathbf{x}'$ . By conditional additivity we have

$$\gamma_\Pi(\mathbf{x} + \mathbf{x}') = \gamma_{\Pi^\alpha}(\mathbf{x}) + \gamma_{(1-\alpha)\Pi^\alpha}(\mathbf{x}') = \kappa_{\Pi^\alpha}(\mathbf{x}) + \kappa_{(1-\alpha)\Pi^\alpha}(\mathbf{x}') = \kappa_\Pi(\mathbf{x} + \mathbf{x}').$$

That is,  $\gamma_\Pi$  and  $\kappa_\Pi$  coincide on the “left side” of  $\partial\Pi$ . In particular  $\mu$  behaves additively, i.e.,  $\mu_\Pi = \mu_{\Pi^\alpha} + \mu_{(1-\alpha)\Pi^\alpha}$ . That is, the midpoint of  $\tilde{\Delta}$  (cf. Figure 3.1) is mapped onto the midpoint of  $\sigma_\Pi$ . Hence, all line segments on  $\partial\Pi$  are mapped onto the corresponding line segments on  $\sigma_\Pi\Delta^e$  in the same ratio of length as is the case with the mapping  $\kappa_\Pi$ . We conclude that

$$(3.8) \quad \gamma_\Pi = \frac{\sigma_\Pi}{\tau_\Pi} \kappa_\Pi =: r_\Pi \kappa_\Pi$$

holds true indeed.

We claim that the ratio  $r_\Pi$  does not depend on  $\Pi$ . Indeed, change  $\mathbf{b}$  to  $\mathbf{b}'$  in the above argument such that the product  $\alpha a_1 a_2 = b_1 b_2 = b'_1 b'_2$  is the same. Then the length of the line segments involving  $\Delta^a$  and  $\alpha\Delta^a$  does not change. As the ratios are again the ones indicated above, the total length of the image  $\sigma_\Pi\Delta^e$  does not change.

The above procedure is naturally extended to a sum of  $K$  prisms in  $\mathbb{R}^2$  (see [7],[12] or [13]). Hence, for some positive  $r$ , we have  $\gamma_\bullet = r\kappa_\bullet$ ,  $\sigma_\bullet = r\tau_\bullet$ . We assume that the constant is 1, hence

$$(3.9) \quad \sigma_{\Pi^\alpha} = \tau_{\Pi^\alpha} = \iota_\Delta(\partial\Pi^\alpha) = \sqrt{a_1 a_2}.$$

which is well known from the Maschler–Perles solution [7],[12].

**3<sup>rd</sup>STEP** : Now we turn to bargaining problems in  $\mathbb{R}^n$ . First of all we determine the nature of  $\sigma_\Pi$ . By the previous step,  $\sigma$  equals  $\tau$  on two dimensional simplices. By Lemma 3.4 it follows that  $\sigma$  equals  $\tau$  on all  $n$ -dimensional prisms. As both functions are additive, they coincide necessarily on all cephoids.

**4<sup>th</sup>STEP** : Let  $\Pi$  be a cephoid and let  $\mathbf{u}$  be a vertex of  $\partial\Pi$ . By non-degeneracy  $\mathbf{u}$  is a unique sum of vertices of the simplices  $\Delta^{(k)}$  say

$$(3.10) \quad \mathbf{u} = \mathbf{a}^{\mathbf{i}_\bullet} = \sum_{k \in K} \mathbf{a}^{(k)\mathbf{i}_k}.$$

with suitable  $\mathbf{i}_\bullet : K \rightarrow I$  (see (2.11)). By *item 3* we know that

$$\gamma_{\mathbf{a}^{(k)}}(\mathbf{a}^{(k)\mathbf{i}_k}) = \sigma_{\mathbf{a}^{(k)}} \mathbf{e}^{\mathbf{i}_k} \quad (k \in K).$$

As  $\sigma$  and  $\tau$  coincide on prisms, we use conditional additivity in order to

conclude that

$$\begin{aligned}
 \gamma_{\Pi}(\mathbf{u}) &= \gamma_{\Pi} \left( \sum_{k \in K} \mathbf{a}^{(k)} \mathbf{i}_k \right) = \sum_{k \in K} \gamma_{\mathbf{a}^{(k)}} (\mathbf{a}^{(k)} \mathbf{i}_k) \\
 &= \sum_{k \in K} \sigma_{\mathbf{a}^{(k)}} \mathbf{e}^{\mathbf{i}_k} = \sum_{k \in K} \tau^{\mathbf{a}^{(k)}} \mathbf{e}^{\mathbf{i}_k} \\
 &= \sum_{k \in K} \kappa_{\mathbf{a}^{(k)}} (\mathbf{a}^{(k)} \mathbf{i}_k) = \kappa_{\Pi} \left( \sum_{k \in K} \mathbf{a}^{(k)} \mathbf{i}_k \right) \\
 &= \kappa_{\Pi}(\mathbf{u}).
 \end{aligned}
 \tag{3.11}$$

Thus,  $\gamma_{\Pi}$  and  $\kappa_{\Pi}$  coincide on the extremal points of  $\partial\Pi$ . By piecewise convexity, they coincide necessarily on all of  $\partial\Pi$ .

q.e.d.

## 4 The TU-game

Within this section we introduce the TU game derived from an NTU-game. To this end, let  $\mathbf{V}$  be a cephoidal NTU-game. Our approach suggests that players evaluate concessions and gains in accordance with the coordinate product. It seems plausible that a “side payment game” derived from an NTU situation has to be calibrated accordingly.

The foremost candidate is suggested by the surface measure and the adjustment factor. Accordingly, coalition  $S$  considers its “worth” implied by  $\mathbf{V}$  to be given as follows.

**Definition 4.1.** *Let  $\mathbf{V}$  be a cephoidal NTU-game. The **TU-game induced by  $\mathbf{V}$**  is*

$$(4.1) \quad \widehat{\mathbf{v}} = \widehat{\mathbf{v}}^{\mathbf{V}} : \underline{\mathbf{P}} \rightarrow \mathbb{R}$$

$$(4.2) \quad \widehat{\mathbf{v}}(S) = \widehat{\mathbf{v}}^{\mathbf{V}}(S) = \tau_{\mathbf{V}(S)} \quad (S \in \underline{\mathbf{P}}).$$

**Example 4.2.** A *hyperplane NTU-game* illustrates the relevance of our version. Let  $\mathbf{v} : \underline{\mathbf{P}} \rightarrow \mathbb{R}_+$  be a nonnegative TU-game and let  $\mathbf{a} \in \mathbb{R}_+^n$  be a positive vector. Define  $\mathbf{V} = \mathbf{V}_a^{\mathbf{v}}$  by

$$\mathbf{V}(S) = \mathbf{V}_a^{\mathbf{v}}(S) = \mathbf{v}(S) \Delta_S^{\mathbf{a}} \quad (S \in \underline{\mathbf{P}}).$$

We have (using  $s := |S|$ )

$$\begin{aligned}
 \tau_{\mathbf{V}(S)} &= \sqrt[s]{\left( \prod_{i \in S} \mathbf{v}(S) a_i \right)} = \mathbf{v}(S) \sqrt[s]{\prod_{i \in S} a_i} \\
 &= \mathbf{v}(S) \tau_{\Delta_S^{\mathbf{a}}} = \mathbf{v}(S) \tau_{\mathbf{a}}(S)
 \end{aligned}$$

It follows at once that for  $S \in \underline{\mathbf{P}}$

$$\hat{\mathbf{v}}(S) = \mathbf{v}(S)\boldsymbol{\tau}_{\mathbf{a}} .$$

The worth of coalition  $S \in \underline{\mathbf{P}}$  is adjusted or rescaled by means of the adjustment factor. Clearly,  $\hat{\mathbf{v}}$  and  $\mathbf{v}$  coincide whenever  $\mathbf{a}$  is the unit vector  $\mathbf{e}$  and hence  $\mathbf{V}_{\mathbf{e}}^{\mathbf{v}}$  is the *embedding* of the side payment game  $\mathbf{v}$  into the NTU framework.

**Lemma 4.3.** *The TU game mapping  $\hat{\mathbf{v}}^{\bullet}$  is additive. That is, for  $\mathbf{V}, \mathbf{W} \in \mathfrak{V}^n$*

$$(4.3) \quad \hat{\mathbf{v}}^{\mathbf{V}} + \hat{\mathbf{v}}^{\mathbf{W}} = \hat{\mathbf{v}}^{\mathbf{V}+\mathbf{W}} .$$

**Proof:** The proof follows immediately from the additivity of the adjustment factor, i.e., from Theorem 2.5, **q.e.d.**

Next we wish to assess the behavior of side payment games under affine transformations. Let

$$L : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad L(\mathbf{x}) := (\alpha_1 x_1, \dots, \alpha_n x_n) \quad (\mathbf{x} \in \mathbb{R}^n)$$

specify such a transformation, then (2.32) implies obviously

$$(4.4) \quad \boldsymbol{\tau}_{L(\mathbf{V}(S))} = \boldsymbol{\tau}_{\alpha_S} \boldsymbol{\tau}_{\mathbf{V}(S)} .$$

Accordingly, we have to define the action of a.t.u. regarding the admittance of side payments. The appropriate version of the transformed game is given as follows.

**Definition 4.4.** *Let  $\mathbf{V}$  be an NTU-game and let  $L$  be an a.t.u.. Define the transformed game  $\hat{L}\mathbf{V}$  by*

$$(4.5) \quad (\hat{L}\mathbf{V})(S) := \frac{\boldsymbol{\tau}_{\alpha}}{\boldsymbol{\tau}_{\alpha_S}} L(\mathbf{V}(S)) \quad (S \in \underline{\mathbf{P}}).$$

A transformation of utility refers to the grand coalition. The Shapley value assumes that agreement eventually takes place within the grand coalition. When a rescaling of the axes is applied, coalitions considering  $\hat{\mathbf{v}}$  should take the different measurements into account when rescaling is applied.

**Lemma 4.5.** *Let  $L$  be an a.t.u.. Then the TU-game mapping  $\hat{\mathbf{v}}^{\bullet}$  satisfies*

$$(4.6) \quad \hat{\mathbf{v}}^{\hat{L}\mathbf{V}}(S) = \boldsymbol{\tau}_{\alpha} \hat{\mathbf{v}}^{\mathbf{V}}(S)$$

for all  $\mathbf{V} \in \mathfrak{V}^n$  and  $S \in \underline{\mathbf{P}}$ . Thus, an affine transformation is reflected in the corresponding TU-game by a rescaling via the factor

$$(4.7) \quad \boldsymbol{\tau}_{\alpha} = \sqrt[n]{\left( \prod_{i \in I} \alpha_i \right)} .$$

The **Proof** is obvious.

## 5 The Conditionally Additive Value

Within this section we describe an NTU Shapley value based on Shapley's seminal paper [16] but also referring to the Maschler–Perles solution ([7]). Thus, it is appropriate to say that we define a Maschler–Perles–Shapley value.

Let  $\mathbf{V}$  be a cephoidal NTU-game. Recall the TU game  $\hat{\mathbf{v}} = \hat{\mathbf{v}}^{\mathbf{V}}$  given by formula (4.2) of Section 4. The (TU –) Shapley value of  $\hat{\mathbf{v}}$  is denoted by  $\Phi(\hat{\mathbf{v}})$  and satisfies

$$(5.1) \quad (\Phi(\hat{\mathbf{v}}))(\mathbf{I}) = \sum_{i \in \mathbf{I}} \Phi_i(\hat{\mathbf{v}}) = \hat{\mathbf{v}}(\mathbf{I}) = \tau_{\mathbf{V}(\mathbf{I})}$$

that is

$$(5.2) \quad \Phi(\hat{\mathbf{v}}) \in \Delta^{\tau_{\mathbf{V}(\mathbf{I})}} ;$$

thus,  $\Phi(\hat{\mathbf{v}})$  is located in the range of  $\kappa_{(\mathbf{V}(\mathbf{I}))}$ . This justifies the following definition.

**Definition 5.1.** *The **conditionally additive value** (the **c.a. value** or the **MPS value**) is the mapping  $\chi$  defined on cephoidal games by*

$$(5.3) \quad \chi(\mathbf{V}) := \kappa_{(\mathbf{V}(\mathbf{I}))}^{-1}(\Phi(\hat{\mathbf{v}}^{\mathbf{V}})) .$$

**Theorem 5.2.** *The MPS value*

1. *is Pareto efficient,*
2. *is symmetric,*
3. *respects a.t.u.*
4. *is conditionally superadditive.*

**Proof: 1<sup>st</sup>STEP :**

In order to deal with functions depending on (cephoidal) NTU-games, we extend  $\kappa$  in a canonical way; we introduce

$$(5.4) \quad \kappa^{\mathbf{V}} := \kappa_{\mathbf{V}(\mathbf{I})} : \partial \mathbf{V}(\mathbf{I}) \rightarrow \tau_{\mathbf{V}(\mathbf{I})} \Delta^e = \hat{\mathbf{v}}^{\mathbf{V}}(\mathbf{I}) \Delta^e .$$

**2<sup>nd</sup>STEP :**

Pareto efficiency is obvious from the definition. We prove symmetry. Recall that, for a TU game  $\mathbf{v}$  we have  $\pi \mathbf{v} = \mathbf{v} \circ \pi^{-1}$  while for NTU games the appropriate definition is  $\pi \mathbf{V} := \pi \circ \mathbf{V} \circ \pi^{-1}$ .

Now, in view of (2.30) we have for  $S \in \underline{\mathbf{P}}$

$$(5.5) \quad \begin{aligned} \hat{\mathbf{v}}^{\pi \mathbf{V}}(S) &= \tau_{(\pi \mathbf{V})(S)} = \tau_{\pi(\mathbf{V}(\pi^{-1}(S)))} \\ &= \tau_{\mathbf{V}(\pi^{-1}(S))} = \tau_{\pi^{-1}(\pi(\mathbf{V}(\pi^{-1}(S))))} \\ &= \tau_{(\pi^{-1} \mathbf{V})(S)} \end{aligned}$$

that is,

$$(5.6) \quad \widehat{\mathbf{v}}^{\pi \mathbf{V}} = \pi \widehat{\mathbf{v}}^{\mathbf{V}}$$

proving symmetry of the function  $\widehat{\mathbf{v}}^\bullet$ . Analogously, we prove the symmetry of  $\kappa^\bullet$ . In view of (2.29) we have

$$(5.7) \quad \begin{aligned} \kappa^{\pi \mathbf{V}} &= \kappa_{(\pi \mathbf{V})(\mathbf{I})} &= \kappa_{(\pi(\mathbf{V}(\pi^{-1}(\mathbf{I})))} \\ &= \pi \circ \kappa_{\mathbf{V}(\mathbf{I})} \circ \pi^{-1} &= \pi \circ \kappa^{\mathbf{V}} \circ \pi^{-1}. \end{aligned}$$

Combining this we obtain

$$(5.8) \quad \begin{aligned} \chi(\pi \mathbf{V}) &= (\kappa^{\pi \mathbf{V}})^{-1} \left( \Phi(\widehat{\mathbf{v}}^{\pi \mathbf{V}}) \right) \\ &= \pi \circ \kappa^{\mathbf{V}} \circ \pi^{-1} \left( \Phi(\pi \widehat{\mathbf{v}}^{\mathbf{V}}) \right) \\ &= \pi \circ \kappa^{\mathbf{V}} \circ \pi^{-1} \pi \left( \Phi(\widehat{\mathbf{v}}^{\mathbf{V}}) \right) \\ &= \pi \circ \kappa^{\mathbf{V}} \circ \left( \Phi(\widehat{\mathbf{v}}^{\mathbf{V}}) \right) \\ &= \pi(\chi(\mathbf{V})), \end{aligned}$$

**3<sup>rd</sup>STEP :**

Next, covariance with a.t.u is verified. To this end, let  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) > \mathbf{0}$  and let

$$\begin{aligned} L : \quad \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ L(\mathbf{x}) &= (\alpha_1 x_1, \dots, \alpha_n x_n) \quad (\mathbf{x} \in \mathbb{R}^n) \end{aligned}$$

be the corresponding positive linear mapping. Note that we have to apply the mapping  $\widehat{L}$  (see (4.5)) when transforming an NTU-game. Also, recall the translation  $T$  defined in (2.33) that satisfies

$$(5.9) \quad \tau_{L(\Pi)} = \tau_{\boldsymbol{\alpha}} \tau_{\Pi} = T(\tau_{\Pi})$$

(see (2.32)). Also, Lemma 4.5 reads

$$(5.10) \quad \widehat{\mathbf{v}}^{\widehat{L}\mathbf{V}} = \tau_{\boldsymbol{\alpha}} \widehat{\mathbf{v}}^{\mathbf{V}}.$$

The relation of the mappings  $\kappa_{\Pi}$  and  $\kappa_{L(\Pi)}$  is explained by (2.38). Therefore the applications of  $\chi$  and  $L$  commute as

$$(5.11) \quad \begin{aligned} \chi(\widehat{L}\mathbf{V}) &= \kappa_{(\widehat{L}\mathbf{V})(\mathbf{I})}^{-1} \left( \Phi(\widehat{\mathbf{v}}^{\widehat{L}\mathbf{V}}) \right) && \text{(by definition of } \chi) \\ &= \kappa_{(\widehat{L}\mathbf{V})(\mathbf{I})}^{-1} \left( \Phi(\tau_{\boldsymbol{\alpha}} \widehat{\mathbf{v}}^{\mathbf{V}}) \right) && \text{(by (4.6) of Lemma 4.5)} \\ &= \kappa_{L(\mathbf{V}(\mathbf{I}))}^{-1} \left( \Phi(\tau_{\boldsymbol{\alpha}} \widehat{\mathbf{v}}^{\mathbf{V}}) \right) && \text{as } L(\mathbf{V}(\mathbf{I})) = (\widehat{L}\mathbf{V})(\mathbf{I}) \\ &= \kappa_{L(\mathbf{V}(\mathbf{I}))}^{-1} \left( \tau_{\boldsymbol{\alpha}} \Phi(\widehat{\mathbf{v}}^{\mathbf{V}}) \right) && \text{(as } \Phi \text{ is linear)} \\ &= \kappa_{L(\mathbf{V}(\mathbf{I}))}^{-1} \left( T(\Phi(\widehat{\mathbf{v}}^{\mathbf{V}})) \right) && \text{(by definition of } T) \\ &= L \left( \kappa_{\mathbf{V}(\mathbf{I})}^{-1} \left( \Phi(\widehat{\mathbf{v}}^{\mathbf{V}}) \right) \right) && \text{(by (2.39))} \\ &= L(\chi(\mathbf{V})). \end{aligned}$$



**4<sup>th</sup>STEP** : Finally, the proof for conditional additivity runs quite analogously to the one of Lemma 2.6. If, for two games  $\mathbf{V}$  and  $\mathbf{W}$  the values  $\chi(\mathbf{V})$  and  $\chi(\mathbf{W})$  yield a Pareto efficient sum, then they are located within faces that admit of a joint normal and  $\kappa$  behaves additively. Consequently

$$\begin{aligned}
 (5.12) \quad & \chi(\mathbf{V}) + \chi(\mathbf{W}) \\
 &= \kappa_{(\mathbf{V}(I))}^{-1}(\Phi(\hat{\mathbf{v}}^{\mathbf{V}})) + \kappa_{(\mathbf{W}(I))}^{-1}(\Phi(\hat{\mathbf{v}}^{\mathbf{W}})) \text{ by definition,} \\
 &= \kappa_{(\mathbf{V}(I)) + (\mathbf{W}(I))}^{-1} \left( \Phi(\hat{\mathbf{v}}^{\mathbf{V}}) + \Phi(\hat{\mathbf{v}}^{\mathbf{W}}) \right) \text{ by (2.15) Theorem 2.5,} \\
 &= \kappa_{(\mathbf{V}(I)) + (\mathbf{W}(I))}^{-1} \left( \Phi(\hat{\mathbf{v}}^{\mathbf{V}} + \hat{\mathbf{v}}^{\mathbf{W}}) \right) , \text{ as the Shapley value is additive,} \\
 &= \kappa_{(\mathbf{V}(I)) + (\mathbf{W}(I))}^{-1} \left( \Phi(\hat{\mathbf{v}}^{\mathbf{V} + \mathbf{W}}) \right) \text{ by Lemma (4.3),} \\
 &= \chi(\mathbf{V} + \mathbf{W}),
 \end{aligned}$$

q.e.d.

## 6 Axioms for the Conditionally Additive Value

Within this section, a *value* is a mapping  $\psi : \mathfrak{V}^n \rightarrow \mathbb{R}^n$  which is Pareto efficient, symmetric, and a.t.u. covariant.

An adjustment is a pair  $(\gamma_{\bullet}, \sigma)$  consisting of a scaling factor and a transfer mapping as given by Definition 3.1. Clearly adjustments induce mappings on games, we have

$$(6.1) \quad \gamma^{\mathbf{V}} := \gamma_{\mathbf{V}(I)} , \quad \sigma^{\mathbf{V}} := \sigma_{\mathbf{V}(I)}$$

and

$$(6.2) \quad \mathring{\mathbf{v}}(S) = \mathring{\mathbf{v}}^{\mathbf{V}}(S) = \mathring{\mathbf{v}}^{\sigma, \mathbf{V}}(S) := \sigma_{\mathbf{V}(S)} \quad (S \in \underline{\mathbf{P}}) .$$

Finally, a value is required to obey the *null player axiom*, that is, any null player of  $\mathring{\mathbf{v}}^{\sigma, \mathbf{V}}$  receives  $\psi_i(\mathbf{V}) = 0$ .

**Definition 6.1.** We say that  $(\psi, \gamma_{\bullet}, \sigma)$  satisfies the *adjusted value axioms* if the following holds true.

1.  $\psi$  is conditionally additive.
2.  $\gamma_{\bullet}$  is conditionally additive.
3.  $\sigma$  is additive and consistent.
4. The solution concept respects the adjustment. That is,

$$(6.3) \quad \gamma^{\mathbf{V}}(\psi(\mathbf{V})) = \psi(\mathring{\mathbf{v}}^{\sigma, \mathbf{V}}(\bullet) \Delta_{\bullet}^e) \quad (\mathbf{V} \in \mathfrak{V}^n).$$

**Theorem 6.2.** *If  $(\psi, \gamma_\bullet, \sigma)$  satisfies the adjusted value axioms, then  $\psi$  is the MPS value  $\chi$ , and (up to some positive common constant)*

1.  $\gamma_\bullet$  is the measure preserving mapping  $\kappa_\bullet$ , and
2.  $\sigma$  is the adjustment factor  $\tau$ .

**Proof:**

**1<sup>st</sup>STEP :**

For bargaining problems, we know that the c.a. value  $\chi$  equals the generalized Maschler–Perles solution  $\mu$ . Now, the axiomatic of Definition 6.1 is the one presented in Definition 3.5 when restricted to bargaining problems. It follows from Theorem 3.6 that  $(\gamma, \sigma) = (\kappa, \tau)$  holds true. Thus, statements 1 and 2 are immediately verified..

**2<sup>nd</sup>STEP :**

It follows that the derived side payment game is

$$(6.4) \quad \overset{\circ}{v}^V = \overset{\circ}{v}^{\sigma, V} = \widehat{v}^{\tau, V} = \widehat{v}^V \quad (V \in \mathfrak{V}).$$

We claim that  $\psi$  has to coincide with the Shapley value (more precisely: with  $\chi$ ) on hyperplane games.

Indeed, consider the function  $\vartheta$  on TU–games defined by

$$\vartheta(v) := \psi(v(\bullet)\Delta_\bullet^e) = \psi(V_e^v).$$

As  $\psi$  is conditionally additive it follows that  $\vartheta$  is additive. Also, by (6.4) and Example 4.2 we have  $\overset{\circ}{v} = \widehat{v} = v$ , hence null players of  $\overset{\circ}{v}$  and  $v$  coincide. Consequently,  $\vartheta$  satisfies the axioms of the Shapley value.

**3<sup>rd</sup>STEP :**

In particular, the fourth axiom (formula (6.3) in Definition 6.1) can be replaced by

$$(6.5) \quad \gamma^V(\psi(V)) = \Phi(\overset{\circ}{v}^{\sigma, V}) = \Phi(\widehat{v}^{\tau, V}) \quad (V \in \mathfrak{V}^n).$$

As  $\gamma^V = \kappa^V$ , this implies

$$(6.6) \quad \psi(V) = (\kappa^V)^{-1} \left( \Phi(\widehat{v}^{\tau, V}) \right) = \chi(V) \quad (V \in \mathfrak{V}^n).$$

**q.e.d.**

**Remark 6.3.** *Our solution or value respectively is point valued and does not require a fixed point theorem. With this respect it differs from other NTU–Shapley values given previously. See e.g. AUMANN [1], who axiomatizes the Shapley’s transfer value and HART [5], who axiomatizes Harsanyi’s [4] NTU–value. See also DE*

CLIPPEL-PETERS-ZANK [2] who discuss the dependence on regularity conditions and add axiomatizations for several values including the Consistent NTU-Shapley value of MASCHLER-OWEN [6]. Our solution concept is “constructive”. The maximal faces of a cepheid can be determined by a recursive procedure (see [8]), thus the scaling factor and the surface measure are attainable by computational methods. We do not know as yet whether these problems are “NP-hard”.

**Remark 6.4.** We believe that, with a suitable topology on Pareto surfaces, cepheids are dense within a large class of smooth polyhedra. In two dimensions this is well known assuming that no line segments parallel to an axis appears in the Pareto surface. In  $n$  dimensions a more restrictive condition may be necessary. Quite likely, the surface measure can be extended to certain smooth Pareto surfaces, a plausible candidate would be obtained by integrating the “volume element” over the Pareto surface, vaguely

$$\iota_{\Delta}(\partial U) := \int_{\partial U} \sqrt{(dx_1 \cdot \dots \cdot dx_n)^{n-1}}.$$

Yet, the continuity properties of the c.a. value are to be studied carefully. This task clearly exceeds the scope of our present framework.

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