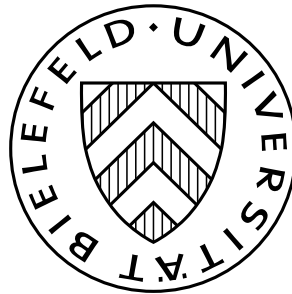


March 2009

Strong Core Equivalence Theorem in an Atomless Economy with Indivisible Commodities

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Abstract

We consider an atomless exchange economy with indivisible commodities. Every commodity can be consumed only in integer amounts. In such an economy, because of the indivisibility, the preference maximization does not imply the cost minimization. We prove that the strong core coincides with the set of cost-minimized Walras allocations which satisfy both the preference maximization and the cost minimization under the same price vector.

JEL classification: C71; D51

Keywords: Indivisible commodities; Core equivalence; Strong core; Cost-minimized Walras equilibrium

1 Introduction

We consider an atomless economy where every commodity can be consumed only in integer amounts. In such an economy, agents' consumption sets are discrete, so transfer of a small amount of any commodity is impossible; therefore, the size of the cores depends on the improvement defining them. The notion of the core we focus on is the strong core. The strong core is defined by the weak improvement which requires that at least

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[†]I acknowledge financial support from the German Research Foundation (DFG).

one agent in a coalition can be strictly better off without making other agents in the coalition worse off. This is the core that Debreu and Scarf [3] used in their limit theorem. Also, the discreteness of agents' consumption sets makes their preference relations locally satiated. Then, the preference maximization does not imply the cost minimization. A cost-minimized Walras equilibrium is a state where, under some price vector, all agents satisfy not only the preference maximization but also the cost minimization. We prove that, in our economy with indivisible commodities, the strong core coincides with the set of cost-minimized Walras allocations.

Aumann [1] proved that, in an atomless economy with perfectly divisible commodities and strongly monotone preference relations, the core defined by the strong improvement coincides with the set of Walras allocations. The strong improvement requires that all agents in a coalition can be strictly better off. We call the core defined by the strong improvement the *weak core*. Although Aumann [1] assumed neither completeness nor transitivity of preference relations, if agents' preference relations enjoy these properties as well as continuity and strong monotonicity, then the weak core is equal to the strong core. Also, from strong monotonicity of agents' preference relations, the preference maximization implies the cost minimization; therefore, Walras equilibria coincide with cost-minimized Walras equilibria. Hence, Aumann's [1] theorem implies that, in an atomless economy with perfectly divisible commodities, the strong core coincides with the set of cost-minimized Walras allocations. Our core equivalence is a counterpart to Aumann's core equivalence in an economy with indivisible commodities.

Because of the discreteness of our commodity space, the strong core and the set of cost-minimized Walras allocations can be both empty. If economy has at least one divisible commodity and if the distribution of agents' endowment vectors is dispersed, then a (cost-minimized) Walras equilibrium exists (see Mas-Colell [8] and Yamazaki [11]). Although each agent's demand correspondence may not be upper hemi-continuous at some price vector, such price vector differs among agents (because the distribution of endowment vectors is dispersed). Accordingly, by the regularizing effect of aggregation, the aggregate demand correspondence recovers upper hemi-continuity, and a (cost-minimized) Walras equilibrium exists. In our economy, however, there is no divisible commodity, and the regularizing effect does not work enough. Therefore, a cost-minimized Walras equilibrium

may not exist. As an example of an economy where the strong core and the set of cost-minimized Walras allocations are nonempty, we give the atomless version of Shapely and Scarf's [10] economy. In Shapley and Scarf's economy with finitely many agents, by the specification of agents' preference relations and endowment vectors, every agent consumes only one unit of only one commodity at any individually rational allocation. Roth and Postlewaite [9] proved that, if any agent has no indifference among consumptions of one unit of one commodity, then the strong core coincides with the set of Walras allocations (which are also cost-minimized Walras allocations by the no-indifference assumption), and these sets consist of only one allocation. In the atomless version of this economy, the strong core or, equivalently, the set of cost-minimized Walras allocations is nonempty.

Inoue [7] obtained the same core equivalence in a large finite economy; if agents' types are finite and if every type has a sufficiently large number of agents, i.e., there are many agents who have the same preference relation and the same endowment vector, then the strong core coincides with the set of cost-minimized Walras allocations. To obtain a bound of the size of economy such that the core equivalence holds for any economy whose size is larger than the bound. Inoue's [7] proof is complicated and he put a stronger assumption on agents' preference relations than ours. He assumed that any distinct two commodities are substitutable; therefore, the lexicographic preference relation is excluded. In our atomless model, however, even if some agents' preference relations are lexicographic, we can obtain the core equivalence and, by virtue of Lyapunov's convexity theorem, the proof is comprehensible.

Inoue [6] considered the same economic model as ours and obtained another type of core equivalence. Inoue [6] defined the core as an intermediate notion between the strong core and the weak core, and proved that the core coincides with the set of Walras allocations. It is worth noting that the usage of the separation theorem for convex sets is different between our proof and Inoue's [6] proof. To prove that a core allocation is Walrasian, Inoue [6] applied his separation theorem just once. In contrast, we use a well-known separation theorem repeatedly. First, by the separation theorem, we find a price vector under which a strong core allocation satisfies the cost minimization. As the cost minimization does not imply the preference maximization, on the budget surface of some agents, there may exist some consumption bundles which are strictly preferred

to the strong core allocation. Second, we move the price vector slightly and find a new price vector under which all agents satisfy both the preference maximization and the cost minimization. As our commodity space is discrete, even if we move the first obtained price vector slightly, any consumption bundle which is outside of the budget set under the first price vector does not enter the new budget set, and we can only put out strictly preferred consumption bundles on the old budget surface from the new budget set. When we move the first obtained price vector in an appropriate direction, we use a well-known separation theorem again.

This paper is organized as follows. In Section 2, we state the model and the main theorem. In Section 3, we prove the main theorem.

2 Model and Main Theorem

We consider an atomless exchange economy with L indivisible commodities.¹ Every commodity can be consumed in integer amounts, so the commodity space of our economy is $\mathbb{Z}^L$, where $\mathbb{Z}$ is the set of integers. Let $(A, \mathcal{A}, \nu)$ be an atomless probability space of agents. For simplicity, we assume that every agent has the same consumption set $\mathbb{Z}_+^L$.² Agent a is characterized by his preference relation $\precsim_a$ on $\mathbb{Z}_+^L$ and his endowment vector $e(a) \in \mathbb{Z}_+^L$. Every preference relation $\precsim$ is assumed to be a reflexive, transitive, and complete binary relation on $\mathbb{Z}_+^L$. For $x, y \in \mathbb{Z}_+^L$, $x \precsim y$ (or equivalently $y \succsim x$) means that consumption bundle y is at least as good as consumption bundle x with respect to the preference relation $\precsim$. We denote by $\mathcal{P}$ the space of all preference relations. By being endowed the topology of closed convergence, the space $\mathcal{P}$ is separable and complete metrizable.³ For a preference relation $\precsim \in \mathcal{P}$, we define binary relation $\succ$ as follows: $x \succ y$ if and only if not $(x \precsim y)$.

A mapping $\mathcal{E}$ from the space $(A, \mathcal{A}, \nu)$ of agents to their characteristics $\mathcal{P} \times \mathbb{Z}_+^L$, $\mathcal{E}(a) = (\precsim_a, e(a))$ for every $a \in A$, is an *economy* if $\mathcal{E}$ is $\mathcal{A}/\mathcal{B}(\mathcal{P} \times \mathbb{Z}_+^L)$ -measurable and

¹In the following model, we use the same notation and the same terminology as Inoue [6].

²This simplification can be generalized to the universal class $\mathcal{X}$ of consumptions sets such that $x \geq b$ for every $x \in X$ and every $X \in \mathcal{X}$, and $\{x \in X \mid x \leq (k, \dots, k)\} \neq \emptyset$ for every $X \in \mathcal{X}$, where b and k are a priori constants. For a more detailed discussion on the universal class, see Hildenbrand [4, pp. 84-86].

³For the justification for the adoption of this topology, see Hildenbrand [4, p. 96].

endowment mapping $e : A \rightarrow \mathbb{Z}_+^L$ is ν -integrable, where $\mathcal{B}(\mathcal{P} \times \mathbb{Z}_+^L)$ is the σ -algebra of the Borel subsets in $\mathcal{P} \times \mathbb{Z}_+^L$. A mapping $f : A \rightarrow \mathbb{Z}_+^L$ is an *allocation* for economy $\mathcal{E}$ if it is ν -integrable. An allocation $f : A \rightarrow \mathbb{Z}_+^L$ for economy $\mathcal{E}$ is *exactly feasible* if $\int_A f(a) d\nu(a) = \int_A e(a) d\nu(a)$.

Because of the indivisibility, the size of the core depends on the improvement defining it. We focus on the strong core defined by the weak improvement.

Definition 1. Let $f : A \rightarrow \mathbb{Z}_+^L$ be an allocation for economy $\mathcal{E}$. A coalition $S \in \mathcal{A}$ can *weakly improve upon* f if there exists an allocation g for $\mathcal{E}$ such that $\nu(\{a \in S \mid g(a) \succ_a f(a)\}) > 0$, $g(a) \succeq_a f(a)$ ν -a.e. $a \in S$, and $\int_S g(a) d\nu(a) = \int_S e(a) d\nu(a)$. The strong core of $\mathcal{E}$ is the set of all exactly feasible allocations that cannot be improved upon by any coalition and is denoted by $C_S(\mathcal{E})$.

The improvement defining the strong core is the same as the improvement defining Pareto-efficiency. Hence, any strong core allocation is Pareto-efficient.

The core defined by the strong improvement which requires that all agents in a coalition can be strictly better off is called the *weak core*. By definition, the strong core is a subset of the weak core. In our economy, transfer of a small amount of any commodity is impossible; the strong core can be strictly smaller than the weak core.⁴

As we will prove later, the strong core of our economy is completely characterized by cost-minimized Walras equilibria.

Definition 2. A pair (p, f) of a price vector $p \in \mathbb{Q}_+^L$ and an allocation $f : A \rightarrow \mathbb{Z}_+^L$ for $\mathcal{E}$ is a *cost-minimized Walras equilibrium* for $\mathcal{E}$ if

- (i) f is exactly feasible, i.e., $\int_A f(a) d\nu(a) = \int_A e(a) d\nu(a)$,
- (ii) for ν -a.e. $a \in A$, $p \cdot f(a) \leq p \cdot e(a)$,
- (iii) for ν -a.e. $a \in A$, if $x \in \mathbb{Z}_+^L$ and $x \succ_a f(a)$, then $p \cdot x > p \cdot e(a)$, and
- (iv) for ν -a.e. $a \in A$, if $x \in \mathbb{Z}_+^L$ and $x \succeq_a f(a)$, then $p \cdot x \geq p \cdot e(a)$.

⁴Inoue [6, Examples 2.5, 2.12, and 3.4] gave an example of an economy such that the nonempty strong core is a proper subset of the weak core.

A cost-minimized Walras allocation is an allocation f for which there exists a price vector $p \in \mathbb{Q}_+^L$ such that (p, f) is a cost-minimized Walras equilibrium. The set of all cost-minimized Walras allocations for economy $\mathcal{E}$ is denoted by $W_{CM}(\mathcal{E})$.

From conditions (i) and (ii), we have $p \cdot f(a) = p \cdot e(a)$ ν -a.e. $a \in A$.⁵ Thus, condition (iii) can be rewritten as “for ν -a.e. $a \in A$, $f(a)$ maximizes $\succsim_a$ in the set $\{x \in \mathbb{Z}_+^L \mid p \cdot x \leq p \cdot e(a)\}$ ” and, therefore, this is the preference maximization condition. Also, condition (iv) can be rewritten as “for ν -a.e. $a \in A$, $f(a)$ minimizes $p \cdot x$ in the set $\{x \in \mathbb{Z}_+^L \mid x \succsim_a f(a)\}$ ” and, therefore, this is the cost minimization condition.

When a pair (p, f) of a price vector $p \in \mathbb{Q}_+^L$ and an allocation f for $\mathcal{E}$ satisfies conditions (i)-(iii), it is called a *Walras equilibrium* for $\mathcal{E}$. If agents’ preference relations are locally nonsatiated, the preference maximization implies the cost minimization. In our economy, however, any preference relation must be locally satiated, and there can exist a Walras equilibrium that is not a cost-minimized Walras equilibrium.⁶

On the other hand, if the consumption set is convex, the preference relation is continuous, and the minimum wealth condition is met, then the cost minimization implies the preference maximization (see Debreu [2, Theorem (1), Section 9, Chapter 4]). This type of argument cannot be applied to our economy, because our commodity space is discrete. Thus, in our economy, the cost minimization is not enough to guarantee the preference maximization.

We require that Walras allocations and strong core allocations are exactly feasible, that is, free disposal is not permitted. In the case where free disposal is permitted, even if agents’ preference relations are strongly monotone (in the discrete sense), there can exist a Walras allocation which is not exactly feasible.⁷ In contrast, if agents’ preference relations satisfy the following desirability condition, any strong core allocation and any

⁵This follows also from conditions (i) and (iv).

⁶In Inoue’s [6, Examples 2.5, 2.12, and 3.4] economy, the nonempty strong core is a proper subset of the set of Walras allocations. From our main theorem, the strong core coincides with the set of cost-minimized Walras allocations, so this example illustrates that there exists a Walras equilibrium that is not a cost-minimized Walras equilibrium.

⁷Inoue [6, Example 2.5] gave an example of an economy which has a Walras allocation which is not exactly feasible.

cost-minimized Walras allocation are exactly feasible.⁸

Definition 3. A preference relation $\succsim$ on $\mathbb{Z}_+^L$ is *nonsatiated in every positive direction* if for every $x \in \mathbb{Z}_+^L$ and every $\ell \in \{1, \dots, L\}$ there exists $k \in \mathbb{Z}_{++}$ such that $x + k\chi_\ell \succ x$, where χ_ℓ is the ℓ th unit vector.

Note that, even if preference relation is nonsatiated in every positive direction, two distinct commodities may not be substitutable. In particular, a lexicographic preference relation is nonsatiated in every positive direction.

We can now state our main result.

Theorem. Let $\mathcal{E} : (A, \mathcal{A}, \nu) \rightarrow \mathcal{P} \times \mathbb{Z}_+^L$ be an economy which satisfies the following conditions.

- (i) For ν -a.e. $a \in A$, preference relation $\succsim_a$ is nonsatiated in every positive direction.
- (ii) The endowment mapping $e : A \rightarrow \mathbb{Z}_+^L$ is ν -essentially bounded.

Then, the strong core coincides with the set of cost-minimized Walras allocations, i.e., $C_S(\mathcal{E}) = W_{CM}(\mathcal{E})$.

By an argument similar to the proof of the first welfare theorem, we can show that $W_{CM}(\mathcal{E}) \subseteq C_S(\mathcal{E})$ for every economy $\mathcal{E}$ even if $\mathcal{E}$ does not satisfy assumption (i) or (ii). The proof of the opposite inclusion will be given in the next section.

Assumption (ii) means that agents' endowment vectors have only finite variety. If the endowment mapping is not essentially bounded, there can exist a strong core allocation that is not a cost-minimized Walras allocation.⁹

From assumption (i), any equilibrium price vector is strictly positive. Hence, from assumption (ii), we have the following corollary.

⁸For the proof of the exact feasibility of strong core allocations, see Inoue [6, Lemma 6.1]. Although Inoue [6, Lemma 6.1] put a stronger assumption on agents' preference relations, we can weaken it to the nonsatiation in every positive direction. The exact feasibility of cost-minimized Walras allocations follows from the budget constraint (condition (ii)) and the strict positivity of cost-minimized Walras equilibrium price vector which is guaranteed by the nonsatiation in every positive direction.

⁹See Inoue [6, Example 3.3].

Corollary. *Let $\mathcal{E} : (A, \mathcal{A}, \nu) \rightarrow \mathcal{P} \times \mathbb{Z}_+^L$ be an economy which satisfies conditions (i) and (ii) of the Theorem above. Then, every strong core allocation for $\mathcal{E}$ is ν -essentially bounded.*

Because of the indivisibility, for some economy, the strong core and the set of cost-minimized Walras allocations can be both empty. In such case, the equivalence is trivial.¹⁰ The following example gives a sufficient condition for these sets to be nonempty. It is Roth and Postlewaite's [9] theorem in the atomless version of Shapley and Scarf's [10] economy.

Example. Consider an atomless economy with L indivisible commodities and L agents' types. Let $\{A_1, \dots, A_L\}$ be a measurable partition of the set A of agents. For every $\ell \in \{1, \dots, L\}$, all agents in A_ℓ has the same utility function $u_\ell : \mathbb{Z}_+^L \rightarrow \mathbb{R}$ and the same endowment vector χ_ℓ , the ℓ th unit vector. Every type has the same mass, i.e., $\nu(A_\ell) = 1/L$ for every $\ell \in \{1, \dots, L\}$. Every utility function u_ℓ satisfies that

- (a) $\min_{i \in \{1, \dots, L\}} u_\ell(\chi_i) > u_\ell(0)$, and
- (b) u_ℓ is injective on $\{\chi_1, \dots, \chi_L\}$, i.e, consuming one unit of one commodity cannot be indifferent.

The utility level at any vector in $\mathbb{Z}_+^L \setminus \{0, \chi_1, \dots, \chi_L\}$ can be specified arbitrarily. They are not important in this economy. Here, in order to apply our theorem, every u_ℓ is assumed to be nonsatiated in every positive direction. We call this economy $\mathcal{E}_0$.

In economy $\mathcal{E}_0$, from assumption (a), every individually rational exactly feasible allocation f satisfies that $f(a) \in \{\chi_1, \dots, \chi_L\}$ for ν -a.e. $a \in A$. In particular, every strong core allocation and every Walras allocation enjoy this property. In addition, as a candidate of a Walras equilibrium price vector, it suffices to consider price vectors p such that $1 < p^{(\ell)} < 2$ for every $\ell \in \{1, \dots, L\}$; any agent cannot consume more than one unit of one commodity. Hence, by assumption (b), Walras allocations are cost-minimized Walras allocations.

Let $\{a_1, \dots, a_L\}$ be agents such that $a_\ell \in A_\ell$ for every ℓ . Then, we can consider Shapley-Scarf economy $\mathcal{E}'$ with L agents $\{a_1, \dots, a_L\}$. By David Gale's top trading cycle

¹⁰For an example of the empty strong core, see Inoue [6, Example 3.2].

algorithm (see Shapley and Scarf [10]), we can find a Walras allocation $\bar{f} : \{a_1, \dots, a_L\} \rightarrow \{\chi_1, \dots, \chi_L\}$ for $\mathcal{E}'$. From assumption (b), $\bar{f}$ is the unique Walras allocation for $\mathcal{E}'$.

Define an allocation $f : A \rightarrow \mathbb{Z}_+^L$ for $\mathcal{E}_0$ by $f(a) = \bar{f}(a_\ell)$ if $a \in A_\ell$, $\ell = 1, \dots, L$. Since $\bar{f}$ is the unique Walras allocation for $\mathcal{E}'$ and since Walras allocations are cost-minimized Walras allocations, we have

$$W_{CM}(\mathcal{E}_0) = [f],$$

where $[f]$ is the set of all allocations g with $g = f$ ν -a.e. Then, by our theorem or by Roth and Postlewaite's lemma [9, Lemma 1],¹¹ we have

$$C_S(\mathcal{E}_0) = [f] = W_{CM}(\mathcal{E}_0).$$

Finally, we give a remark about the relationship with Inoue's [6] core equivalence theorem.

Remark. Inoue [6] introduced the core which is an intermediate concept between the strong core and the weak core. An exactly feasible allocation f for economy $\mathcal{E} : (A, \mathcal{A}, \nu) \rightarrow \mathcal{P} \times \mathbb{Z}_+^L$ is a *core allocation* if there exists no coalition $S \in \mathcal{A}$ and a mapping $g : S \rightarrow \mathbb{Z}_+^L$ such that $\nu(\{a \in S \mid g(a) \succ_a f(a)\}) > 0$, $g(a) = f(a)$ for ν -a.e. $a \in S \setminus \{b \in S \mid g(b) \succ_b f(b)\}$, and $\int_S g(a) d\nu(a) = \int_S e(a) d\nu(a)$. The core of $\mathcal{E}$ is the set of all core allocations and is denoted by $C(\mathcal{E})$.

By definition, the strong core is a subset of the core. For some economy, the strong core is a proper subset of the core.¹²

Inoue [6] proved that, under the same assumptions as our theorem,¹³ the core $C(\mathcal{E})$ coincides with the set $W^*(\mathcal{E})$ of exactly feasible Walras allocations. By combining this result with ours, we obtain that

$$W_{CM}(\mathcal{E}) = C_S(\mathcal{E}) \subseteq C(\mathcal{E}) = W^*(\mathcal{E})$$

¹¹From Roth and Postlewaite [9, Lemma 1], any allocation h for $\mathcal{E}'$ with $h \neq \bar{f}$ can be weakly improved upon via $\bar{f}$. Thus, any allocation h for $\mathcal{E}_0$ with $h \notin [f]$ can be weakly improved upon via f and, therefore, $C_S(\mathcal{E}_0) \subseteq [f]$. Since $W_{CM}(\mathcal{E}_0) \subseteq C_S(\mathcal{E}_0)$ holds (this inclusion holds for any economy), we have the following equalities.

¹²For an example, see Inoue [6, Examples 2.5, 2.12, and 3.4].

¹³In Inoue's [6] core equivalence theorem, the assumption on agents' preference relations can be weakened to our assumption: nonsatiated in every positive direction.

for any atomless economy $\mathcal{E}$ which satisfies the assumptions of our theorem.

In economy $\mathcal{E}_0$ from the above example, by assumption (b), we have $C_S(\mathcal{E}_0) = C(\mathcal{E}_0)$ and $W_{CM}(\mathcal{E}_0) = W^*(\mathcal{E}_0)$ and, therefore, $W_{CM}(\mathcal{E}_0) = C_S(\mathcal{E}_0) = C(\mathcal{E}_0) = W^*(\mathcal{E}_0) = [f]$.

3 Proof of Theorem

Let $\mathcal{E} : (A, \mathcal{A}, \nu) \rightarrow \mathcal{P} \times \mathbb{Z}_+^L$ be an economy which satisfies the assumptions (i) and (ii) of Theorem. We will only prove the inclusion $C_S(\mathcal{E}) \subseteq W_{CM}(\mathcal{E})$, because the opposite inclusion follows from a standard argument. For simplicity, we assume that, for every $a \in A$, $\succsim_a$ is nonsatiated in every positive direction and endowment mapping e is bounded.

Let $f \in C_S(\mathcal{E})$. For $z \in \mathbb{Z}^L$, let

$$A_z = \{a \in A \mid z + e(a) \in \mathbb{Z}_+^L \text{ and } z + e(a) \succsim_a f(a)\}$$

and let

$$B_z = \{a \in A \mid z + e(a) \in \mathbb{Z}_+^L \text{ and } z + e(a) \succ_a f(a)\}.$$

Then, $A_z, B_z \in \mathcal{A}$ for every $z \in \mathbb{Z}^L$. Define

$$A^* = A \setminus \left(\bigcup_{\substack{z \in \mathbb{Z}^L \\ \nu(A_z)=0}} A_z \cup \bigcup_{\substack{z \in \mathbb{Z}^L \\ \nu(B_z)=0}} B_z \right).$$

Then, $A^* \in \mathcal{A}$ and $\nu(A^*) = 1$. Define two correspondences $\psi : A^* \rightrightarrows \mathbb{Z}^L$ and $\varphi : A^* \rightrightarrows \mathbb{Z}^L$ as follows:

$$\psi(a) = \{z \in \mathbb{Z}^L \mid z + e(a) \in \mathbb{Z}_+^L \text{ and } z + e(a) \succsim_a f(a)\} = \{x \in \mathbb{Z}_+^L \mid x \succsim_a f(a)\} - \{e(a)\}$$

and

$$\varphi(a) = \{z \in \mathbb{Z}^L \mid z + e(a) \in \mathbb{Z}_+^L \text{ and } z + e(a) \succ_a f(a)\} = \{x \in \mathbb{Z}_+^L \mid x \succ_a f(a)\} - \{e(a)\}.$$

Note that $\varphi(a) \subseteq \psi(a)$ for every $a \in A^*$.

First, we will find a price vector p_0 under which strong core allocation f satisfies the cost minimization condition, i.e., if $x \in \mathbb{Z}_+^L$ and $x \succsim_a f(a)$, then $p_0 \cdot x \geq p_0 \cdot e(a)$. Second, we will move the price vector p_0 slightly and find a price vector $\bar{p}$ under which f satisfies

not only the cost minimization condition but also the preference maximization condition. The following claim will be used when we find price vector p_0 and when we move p_0 in an appropriate direction.

Claim 1. Let H be a linear subspace of $\mathbb{R}^L$. Assume that $\bigcup_{a \in A^*} \varphi(a) \cap H \neq \emptyset$. Then, $0 \notin \text{ri}(\text{co}(\bigcup_{a \in A^*} \psi(a) \cap H))$, where $\text{ri}(C)$ and $\text{co}(C)$ denote the relative interior and the convex hull of set C , respectively.

Proof. Suppose, to the contrary, that $0 \in \text{ri}(\text{co}(\bigcup_{a \in A^*} \psi(a) \cap H))$.¹⁴ Let $z_0 \in \bigcup_{a \in A^*} \varphi(a) \cap H$. Since $A^* \cap B_{z_0} \neq \emptyset$, we have $\nu(B_{z_0}) > 0$. From $0 \in \text{ri}(\text{co}(\bigcup_{a \in A^*} \psi(a) \cap H))$ and $z_0 \in \bigcup_{a \in A^*} \varphi(a) \cap H$, it follows that 0 can be represented as a convex combination of z_0 and finitely many points in $\bigcup_{a \in A^*} \psi(a) \cap H$, i.e., there exist $\{z_1, \dots, z_k\} \subseteq \bigcup_{a \in A^*} \psi(a) \cap H$ and $(\alpha^{(0)}, \alpha^{(1)}, \dots, \alpha^{(k)}) \in \mathbb{R}_{++}^{k+1}$ such that $\sum_{j=0}^k \alpha^{(j)} = 1$ and $\sum_{j=0}^k \alpha^{(j)} z_j = 0$. For every $j \in \{1, \dots, k\}$, since $A^* \cap A_{z_j} \neq \emptyset$, we have $\nu(A_{z_j}) > 0$.

Define $\lambda = \min\{\nu(B_{z_0}), \nu(A_{z_1}), \dots, \nu(A_{z_k})\} > 0$. By Lyapunov's theorem, the set $\{\nu(S) \mid S \in \mathcal{A}, S \subseteq B_{z_0}\}$ is convex. Since $\nu(B_{z_0}) \geq \lambda \geq \lambda \alpha^{(0)} > 0$, there exists an $S_0 \in \mathcal{A}$ such that $S_0 \subseteq B_{z_0}$ and $\nu(S_0) = \lambda \alpha^{(0)}$.

Assume that we have chosen mutually disjoint sets $S_0, \dots, S_m \in \mathcal{A}$ such that

$$\begin{aligned} S_0 &\subseteq B_{z_0}, \\ S_j &\subseteq A_{z_j} \quad \text{for } j \in \{1, \dots, m\}, \text{ and} \\ \nu(S_j) &= \lambda \alpha^{(j)} \quad \text{for } j \in \{0, 1, \dots, m\}. \end{aligned}$$

Again, by Lyapunov's theorem, the set $\{\nu(S) \mid S \in \mathcal{A}, S \subseteq A_{z_{m+1}} \setminus \bigcup_{j=0}^m S_j\}$ is convex. Since

$$\nu\left(A_{z_{m+1}} \setminus \bigcup_{j=0}^m S_j\right) \geq \nu(A_{z_{m+1}}) - \sum_{j=0}^m \nu(S_j) \geq \lambda - \lambda \sum_{j=0}^m \alpha^{(j)} \geq \lambda \alpha^{(m+1)} > 0,$$

there exists an $S_{m+1} \in \mathcal{A}$ such that

$$S_{m+1} \subseteq A_{z_{m+1}} \setminus \bigcup_{j=0}^m S_j \quad \text{and} \quad \nu(S_{m+1}) = \lambda \alpha^{(m+1)}.$$

¹⁴The following argument is essentially the same as Aumann [1, p. 45]. See also Hildenbrand [5, pp. 843-844].

Therefore, we can obtain mutually disjoint sets $S_0, S_1, \dots, S_k \in \mathcal{A}$ such that

$$\begin{aligned} S_0 &\subseteq B_{z_0}, \\ S_j &\subseteq A_{z_j} \quad \text{for } j \in \{1, \dots, k\}, \text{ and} \\ \nu(S_j) &= \lambda \alpha^{(j)} \quad \text{for } j \in \{0, 1, \dots, k\}. \end{aligned}$$

Define $S = \bigcup_{j=0}^k S_j \in \mathcal{A}$ and define $g : S \rightarrow \mathbb{Z}_+^L$ by

$$g(a) = z_j + e(a) \quad \text{for } a \in S_j \quad (j = 0, 1, \dots, k).$$

For $a \in S_0$, since $a \in B_{z_0}$, we have $g(a) = z_0 + e(a) \succ_a f(a)$. Recall that $\nu(S_0) = \lambda \alpha^{(0)} > 0$. For $a \in S_j$, $j \in \{1, \dots, k\}$, since $a \in A_{z_j}$, we have $g(a) = z_j + e(a) \lesssim_a f(a)$. Furthermore, we have

$$\begin{aligned} \int_S g(a) d\nu(a) &= \sum_{j=0}^k z_j \nu(S_j) + \int_S e(a) d\nu(a) \\ &= \sum_{j=0}^k \lambda \alpha^{(j)} z_j + \int_S e(a) d\nu(a) \\ &= \int_S e(a) d\nu(a). \end{aligned}$$

This contradicts that f is a strong core allocation. $\square$

From assumption (i), we have $\bigcup_{a \in A^*} \varphi(a) \neq \emptyset$. Thus, by Claim 1, we have $0 \notin \text{int}(\text{co}(\bigcup_{a \in A^*} \psi(a)))$. By the separation theorem for convex sets, there exists a $p_0 \in \mathbb{R}^L \setminus \{0\}$ such that

$$p_0 \cdot z \geq 0 \quad \text{for all } z \in \text{co}\left(\bigcup_{a \in A^*} \psi(a)\right).$$

By assumption (i), $\text{co}(\bigcup_{a \in A^*} \psi(a)) = \text{co}(\bigcup_{a \in A^*} \psi(a)) + \mathbb{R}_+^L$. Hence, we have $p_0 \geq 0$.

Claim 2. There exists a $\bar{p} \in \mathbb{R}_+^L \setminus \{0\}$ such that

- (1) $\bar{p} \cdot z > 0$ for every $z \in \bigcup_{a \in A^*} \varphi(a)$, and
- (2) $\bar{p} \cdot z \geq 0$ for every $z \in \bigcup_{a \in A^*} \psi(a)$.

Proof. Let $H_0 = \{z \in \mathbb{R}^L \mid p_0 \cdot z = 0\}$. If $\bigcup_{a \in A^*} \varphi(a) \cap H_0 = \emptyset$, then p_0 has the desired property. Assume that $\bigcup_{a \in A^*} \varphi(a) \cap H_0 \neq \emptyset$. Then, by Claim 1, we have

$$0 \notin \text{ri}\left(\text{co}\left(\bigcup_{a \in A^*} \psi(a) \cap H_0\right)\right).$$

By the separation theorem for convex sets, there exists a $p_1 \in \text{span}(\bigcup_{a \in A^*} \psi(a) \cap H_0) \setminus \{0\}$ such that $p_1 \cdot z \geq 0$ for every $z \in \bigcup_{a \in A^*} \psi(a) \cap H_0$.

Subclaim 2.1. For every $j \in \{1, \dots, L\}$, if $p_0^{(j)} = 0$, then $p_1^{(j)} \geq 0$.

Proof. Suppose, to the contrary, that, for some $j \in \{1, \dots, L\}$, $p_0^{(j)} = 0$ and $p_1^{(j)} < 0$. Since $\bigcup_{a \in A^*} \varphi(a) \cap H_0 \neq \emptyset$, there exists an agent $b \in A^*$ and a net trade vector $z_0 \in \varphi(b) \cap H_0$; therefore,

$$z_0 + e(b) \in \mathbb{Z}_+^L \quad \text{and} \quad z_0 + e(b) \succ_b f(b).$$

Since $\succsim_b$ is nonsatiated in every positive direction, for every $n \in \mathbb{Z}_{++}$, there exists an $r \geq n$ such that

$$z_0 + e(b) + r\chi_j \succ_b z_0 + e(b) \succ_b f(b).$$

Therefore, $z_0 + r\chi_j \in \varphi(b) \subseteq \bigcup_{a \in A^*} \psi(a)$. From $p_0^{(j)} = 0$ and $z_0 \in H_0$, it follows that $z_0 + r\chi_j \in H_0$ and, therefore, $z_0 + r\chi_j \in \bigcup_{a \in A^*} \psi(a) \cap H_0$. By the consequence of the separation theorem above, we have

$$0 \leq p_1 \cdot (z_0 + r\chi_j) = p_1 \cdot z_0 + rp_1^{(j)}.$$

Since we can make r arbitrarily large, this is a contradiction. $\square$

Let $E = \{z \in \bigcup_{a \in A^*} \psi(a) \mid p_0 \cdot z > 0\}$. Define the set E' of minimal elements of E as follows: $x \in E'$ if and only if $x \in E$ and there exists no $y \in E$ with $y \leq x$ and $y \neq x$. Since endowment mapping $e : A \rightarrow \mathbb{Z}_+^L$ is bounded, the set E is bounded from below. Hence, by Gordan's lemma (see, e.g., Inoue [6, Lemma 5.1]), E' is a nonempty finite set and satisfies that $E \subseteq E' + \mathbb{Z}_+^L$. Since $p_0 \geq 0$, we have

$$\inf\{p_0 \cdot z \mid z \in E\} = \min\{p_0 \cdot z \mid z \in E'\} > 0.$$

Since the mapping $p \mapsto \min\{p \cdot z \mid z \in E'\}$ is continuous, there exists an open neighborhood U_0 of p_0 such that, for every $p \in U_0$, $\min\{p \cdot z \mid z \in E'\} > 0$.

For sufficiently small $\varepsilon_1 > 0$, $p_0 + \varepsilon_1 p_1 \in U$ and, for every $j \in \{1, \dots, L\}$ with $p_0^{(j)} > 0$, $p_0^{(j)} + \varepsilon_1 p_1^{(j)} > 0$ holds. By Subclaim 2.1, we have $p_0 + \varepsilon_1 p_1 \geq 0$. By construction, if $p_0^{(j)} + \varepsilon_1 p_1^{(j)} = 0$, then $p_0^{(j)} = 0$. Therefore,

$$0 < \min\{(p_0 + \varepsilon_1 p_1) \cdot z \mid z \in E'\} = \inf\{(p_0 + \varepsilon_1 p_1) \cdot z \mid z \in E\}.$$

Hence, we have obtained that

- (a) $p_0 + \varepsilon_1 p_1 \in \mathbb{R}_+^L \setminus \{0\}$,
- (b) if $p_0^{(j)} > 0$, then $p_0^{(j)} + \varepsilon_1 p_1^{(j)} > 0$,
- (c) $(p_0 + \varepsilon_1 p_1) \cdot z > 0$ for every $z \in \bigcup_{a \in A^*} \psi(a) \setminus H_0$, and
- (d) $(p_0 + \varepsilon_1 p_1) \cdot z \geq 0$ for every $z \in \bigcup_{a \in A^*} \psi(a) \cap H_0$.

Let $H_1 = \{z \in \mathbb{R}^L \mid (p_0 + \varepsilon_1 p_1) \cdot z = 0\}$. If $\bigcup_{a \in A^*} \varphi(a) \cap H_0 \cap H_1 = \emptyset$, then $p_0 + \varepsilon_1 p_1$ has the desired property. Assume that $\bigcup_{a \in A^*} \varphi(a) \cap H_0 \cap H_1 \neq \emptyset$. Then, by the same argument as above, there exists a $p_2 \in \text{span}(\bigcup_{a \in A^*} \psi(a) \cap H_0 \cap H_1) \setminus \{0\}$ and an $\varepsilon_2 > 0$ such that

- (a') $p_0 + \varepsilon_1 p_1 + \varepsilon_2 p_2 \in \mathbb{R}_+^L \setminus \{0\}$,
- (b') if $p_0^{(j)} + \varepsilon_1 p_1^{(j)} > 0$, then $p_0^{(j)} + \varepsilon_1 p_1^{(j)} + \varepsilon_2 p_2^{(j)} > 0$,
- (c') $(p_0 + \varepsilon_1 p_1 + \varepsilon_2 p_2) \cdot z > 0$ for every $z \in \bigcup_{a \in A^*} \psi(a) \setminus (H_0 \cup H_1)$, and
- (d') $(p_0 + \varepsilon_1 p_1 + \varepsilon_2 p_2) \cdot z \geq 0$ for every $z \in \bigcup_{a \in A^*} \psi(a) \cap H_0 \cap H_1$.

Since $\dim(H_0 \cap H_1) = \dim H_0 - 1$ and since $0 \notin \bigcup_{a \in A^*} \varphi(a)$, by repeating this argument, in at most L steps, say, in m steps, we obtain that $\bigcup_{a \in A^*} \varphi(a) \cap H_0 \cap \dots \cap H_{m-1} = \emptyset$. Price vector $p_0 + \sum_{i=1}^{m-1} \varepsilon_i p_i$ has the desired property. $\square$

Let $F = \{z \in \bigcup_{a \in A^*} \psi(a) \mid \bar{p} \cdot z > 0\}$ and $V = \text{span}\{z \in \bigcup_{a \in A^*} \psi(a) \mid \bar{p} \cdot z = 0\}$. Since $f(a) - e(a) \in \psi(a)$ for every $a \in A^*$, by Claim 2, we have

$$\bar{p} \cdot (f(a) - e(a)) \geq 0 \quad \text{for every } a \in A^*.$$

Thus, by the exact feasibility of f , we have

$$\bar{p} \cdot (f(a) - e(a)) = 0 \quad \text{for } \nu\text{-a.e. } a \in A^*.$$

Therefore, $(\bar{p}, f)$ satisfies conditions (i)-(iv) of the definition of cost-minimized Walras equilibrium, but $\bar{p}$ may not be a rational vector. Finally, we find an integral price vector under which f is cost-minimized Walras equilibrium. For simplicity, we assume that $\bar{p} \cdot (f(a) - e(a)) = 0$ for every $a \in A^*$. Therefore, $f(a) - e(a) \in V$ for every $a \in A^*$. From Claim 2, it follows that $\bigcup_{a \in A^*} \varphi(a) \subseteq F$.

Since $\bar{p} \geq 0$, we have $\text{co}(F + \mathbb{Z}_+^L) \cap V = \emptyset$. Thus, $\text{co}(F) \cap (V - \mathbb{R}_+^L) = \emptyset$. Since F is bounded from below, by Inoue's [6, Theorem 5.2] separation theorem, there exists a $p^* \in V^\perp \cap \mathbb{Z}_+^L$ and an $\varepsilon > 0$ such that $p^* \cdot z \geq \varepsilon$ for every $z \in F$, where $V^\perp$ is the orthogonal complement of V .

From $p^* \in V^\perp$ and $f(a) - e(a) \in V$ for every $a \in A^*$, it follows that

$$p^* \cdot (f(a) - e(a)) = 0 \quad \text{for every } a \in A^*.$$

Let $a \in A^*$ and let $x \in \mathbb{Z}_+^L$ with $x \succ_a f(a)$. Then, $x - e(a) \in \varphi(a) \subseteq F$. Thus, $p^* \cdot (x - e(a)) \geq \varepsilon > 0$.

Let $a \in A^*$ and let $x \in \mathbb{Z}_+^L$ with $x \lesssim_a f(a)$. Then, $x - e(a) \in \psi(a) \subseteq F \cup V$. Thus, $p^* \cdot (x - e(a)) \geq 0$. Hence, (p^*, f) is a cost-minimized Walras equilibrium for economy $\mathcal{E}$. This completes the proof of Theorem.

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