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Tomoki Inoue

IMW · Bielefeld University
Postfach 100131
33501 Bielefeld · Germany

email: imw@wiwi.uni-bielefeld.de
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Core allocations may not be Walras allocations in any large finite economy with indivisible commodities

Tomoki Inoue*†

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Abstract

We consider an exchange economy where every commodity can be consumed only in integer amounts. Inoue [Inoue, T., 2005. Do pure indivisibilities prevent core equivalence? Core equivalence theorem in an atomless economy with purely indivisible commodities only. Journal of Mathematical Economics 41, 571-601] proved that in such an economy with a continuum of agents, the core coincides with the set of Walras allocations. We show that this equivalence holds only in an atomless economy by giving two examples of the sequence of replica economies such that in any replica economy, there exists a core allocation that is not a Walras allocation.

JEL classification: C71; D51

Keywords: Indivisible commodities; Core; Walras equilibrium; Strong core; cost-minimized Walras equilibrium

1 Introduction

We consider an economy where every commodity is available only in integer amounts. In such an economy, the size of the cores depend on which notion of improvement is adopted.

*Institute of Mathematical Economics, Bielefeld University, P.O. Box 100131, 33501 Bielefeld, Germany; tomoki.inoue@gmail.com.

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The improvement defining Inoue's [2] core requires that some members in a coalition can be better off without changing the others' consumption vectors. On the other hand, the improvement defining the strong core requires that some members in a coalition can be better off without worsening the others' utilities. By definition, the core is larger (possibly strictly larger) than the strong core.

Inoue [2] proved that in an atomless economy, an economy with a continuum of agents, the core coincides with the set of Walras allocations. We show that this equivalence on the core holds only in an atomless economy by giving examples of the sequence of replica economies such that in any replica economy, there exists a core allocation that is not a Walras allocation. This is a contrasting result to the equivalence on the strong core. Inoue [4, 5] proved that, regardless of in a large finite economy or in an atomless economy, the strong core coincides with the set of cost-minimized Walras allocations. A cost-minimized Walras equilibrium is a state where, under some price vector, all agents satisfy not only the preference maximization but also the cost minimization.

In an atomless economy, finitely many agents can be negligible, but in a finite economy, any one agent cannot be negligible. The core is subject to this difference between an atomless economy and a finite economy, although the strong core is not. In our examples, in any replica economy, there exists a core allocation that is not a Walras allocation by reason that only one agent does not satisfy the preference maximization. In the limit atomless economy of the sequence of replica economies, only one agent can be negligible and, therefore, we can obtain the equivalence between the core and the set of Walras allocations.

In an economy with divisible commodities, approximate equilibrium such as pseudo-equilibrium or quasi-equilibrium is considered when we argue the convergence of the core. Anderson [1] proposed a measure of non-Walras degree of core allocations. In contrast, we focus only on whether or not a core allocation is a Walras allocation, and we do not argue the relation between the core and any approximate equilibrium. This is because in an economy where every commodity is indivisible, pseudo-equilibrium or quasi-equilibrium is not an approximate concept any longer; there exists no sufficient condition for these equilibria to be actual Walras equilibria. Strictly speaking, Anderson's [1] measure is the distance between core allocation and quasi-equilibrium and, therefore, it is not a useful measure for our model.

The paper is organized as follows. Section 2 presents the model. Section 3 gives two examples of the sequence of replica economies such that every replica economy has a core allocation that is not a Walras allocation.

2 Model

The model is essentially same as that of Inoue [2, 4, 5]. We consider an economy with L indivisible commodities, where L is a natural number with $L \geq 2$. Every commodity can be consumed only in integer amounts. Let A be the set of agents. We assume that every agent has the same consumption set $\mathbb{Z}_+^L$, the set of L-dimensional nonnegative integral vectors. Every agent a is characterized by his preference relation $\precsim_a$ and his endowment vector $e(a) \in \mathbb{Z}_+^L$. Every preference relation is assumed to be a reflexive, transitive, complete, and weakly monotone binary relation on $\mathbb{Z}_+^L$.[1] Let $\mathcal{P}$ be the set of all preference relations on $\mathbb{Z}_+^L$. An economy $\mathcal{E}$ is a mapping of the set A of agents to agents' characteristics $\mathcal{P} \times \mathbb{Z}_+^L$. Given a finite economy $\mathcal{E} : A \to \mathcal{P} \times \mathbb{Z}_+^L$, $\#A < \infty$, an allocation is a mapping of A to $\mathbb{Z}_+^L$. An allocation f is exactly feasible if $\sum_{a \in A} f(a) = \sum_{a \in A} e(a)$.

We give the definitions of the strong core, the core, a cost-minimized Walras equilibrium, and a Walras equilibrium.

Definition 1. Let $\mathcal{E} : A \to \mathcal{P} \times \mathbb{Z}_+^L$ be a finite economy. An exactly feasible allocation f is a *strong core allocation* for $\mathcal{E}$ if there exists no nonempty subset S of A and a mapping $g : S \to \mathbb{Z}_+^L$ such that

$$\begin{aligned} &g(a) \succsim_a f(a) \quad \text{for all } a \in S, \\ &g(a) \succ_a f(a) \quad \text{for some } a \in S, \text{ and} \\ &\textstyle\sum_{a \in S} g(a) = \sum_{a \in S} e(a). \end{aligned}$$

The set of all strong core allocations for $\mathcal{E}$ is called the *strong core* of $\mathcal{E}$ and is denoted by $C_S(\mathcal{E})$.

Definition 2. Let $\mathcal{E} : A \to \mathcal{P} \times \mathbb{Z}_+^L$ be a finite economy. An exactly feasible allocation f is a *core allocation* for $\mathcal{E}$ if there exists no nonempty subset S of A and a mapping

[1] A preference relation $\precsim$ on $\mathbb{Z}_+^L$ is *weakly monotone* if $x \precsim y$ for every $x, y \in \mathbb{Z}_+^L$ with $x \leq y$.

$g : S \to \mathbb{Z}_+^L$ such that

$$\begin{aligned}
&g(a) \succ_a f(a) \quad \text{for some } a \in S, \\
&g(a) = f(a) \quad \text{for all } a \in S \setminus \{b \in S \,|\, g(b) \succ_b f(b)\}, \text{ and} \\
&\textstyle\sum_{a \in S} g(a) = \sum_{a \in S} e(a).
\end{aligned}$$

The set of all core allocations for $\mathcal{E}$ is called the *core* of $\mathcal{E}$ and is denoted by $C(\mathcal{E})$.

Definition 3. Let $\mathcal{E} : A \to \mathcal{P} \times \mathbb{Z}_+^L$ be a finite economy. A pair (p, f) of a price vector $p \in \mathbb{Q}_+^L$ and an exactly feasible allocation $f : A \to \mathbb{Z}_+^L$ is called a *cost-minimized Walras equilibrium* of $\mathcal{E}$ if

(i) for all $a \in A$, $p \cdot f(a) \leq p \cdot e(a)$;

(ii) for all $a \in A$, if $x \in \mathbb{Z}_+^L$ and $x \succ_a f(a)$, then $p \cdot x > p \cdot e(a)$; and

(iii) for all $a \in A$, if $x \in \mathbb{Z}_+^L$ and $x \succsim_a f(a)$, then $p \cdot x \geq p \cdot e(a)$.

An allocation f is a *cost-minimized Walras allocation* for $\mathcal{E}$ if (p, f) is a cost-minimized Walras equilibrium for some $p \in \mathbb{Q}_+^L$. The set of all cost-minimized Walras allocations for $\mathcal{E}$ is denoted by $W_{CM}(\mathcal{E})$.

Definition 4. Let $\mathcal{E} : A \to \mathcal{P} \times \mathbb{Z}_+^L$ be a finite economy. A pair (p, f) of a price vector $p \in \mathbb{Q}_+^L$ and an exactly feasible allocation $f : A \to \mathbb{Z}_+^L$ is called a *Walras equilibrium* of $\mathcal{E}$ if (p, f) satisfies conditions (i) and (ii) in the definition of cost-minimized Walras equilibrium. The set of all Walras allocations for $\mathcal{E}$ is denoted by $W(\mathcal{E})$.

Some remarks are in order. By definition, the strong core is a subset of the core, and every cost-minimized Walras allocation is a Walras allocation, i.e., $C_S(\mathcal{E}) \subseteq C(\mathcal{E})$ and $W_{CM}(\mathcal{E}) \subseteq W(\mathcal{E})$ for every economy $\mathcal{E}$. By an argument similar to the proof of the first welfare theorem, we can show that every cost-minimized Walras allocation is a strong core allocation, and every Walras allocation is a core allocation, i.e., $W_{CM}(\mathcal{E}) \subseteq C_S(\mathcal{E})$ and $W(\mathcal{E}) \subseteq C(\mathcal{E})$ for every economy $\mathcal{E}$.

Because of the indivisibility, the core and the set of Walras allocations can be empty (see Inoue [2, Example 3.2]). Therefore, their subsets, the strong core and the set of cost-minimized Walras allocations, can be empty, too (see Inoue [4, Example 2]).

Inoue [4] proved that if agents' types are finite and if every type has a sufficiently large number of agents, then strong core allocations are cost-minimized Walras allocations, i.e., $C_S(\mathcal{E}) \subseteq W_{CM}(\mathcal{E})$ for sufficiently large economy $\mathcal{E}$.[2]

Even in an atomless economy where there exists a continuum of agents, we can define the strong core, the core, cost-minimized Walras equilibrium, and Walras equilibrium in similar manners. Inoue [2, 5] proved that in an atomless economy $\mathcal{E}_\infty$, if endowment allocation is essentially bounded and if agents' preference relations are *nonsatiated in every positive direction* [for every $x \in \mathbb{Z}_+^L$ and every $h \in \{1, \ldots, L\}$, there exists a $k \in \mathbb{Z}_{++}$ such that $x + k\chi_h \succ x$], then the core coincides with the set of Walras allocations, and the strong core coincides with the set of cost-minimized Walras allocations, i.e., $C(\mathcal{E}_\infty) = W(\mathcal{E}_\infty)$ and $C_S(\mathcal{E}_\infty) = W_{CM}(\mathcal{E}_\infty)$. Therefore, under some assumptions, regardless of in a large finite economy or in an atomless economy, the strong core coincides with the set of cost-minimized Walras allocations. In contrast, as we will show in the next section, the equivalence on the core holds only in an atomless economy and there exists an arbitrarily large finite economy whose core is strictly larger than the set of Walras allocations.

3 Examples

The following two examples give the sequence of replica economies where every replica economy has a core allocation that is not a Walras allocation, and only in the limit atomless economy of the sequence, the core coincides with the set of Walras allocations. In the first example, the strong core and the set of cost-minimized Walras allocations are both empty in a sufficiently large replica economy, whereas in the second example, these sets are nonempty in any replica economy.

Example 1. Let $L = 2$. Every agent has the same preference relation $\precsim_t$ and the same endowment vector e_t; there exists only one type of agents. The endowment vector e_t of

[2] Strictly speaking, to hold the inclusion $C_S(\mathcal{E}) \subseteq W_{CM}(\mathcal{E})$, Inoue [4] put a further assumption on agents' preference relations $\precsim$; there exists a $k \in \mathbb{Z}$ with $k \geq 2$ such that for every $x, y \in \mathbb{Z}_+^L$ and every $h, i \in \{1, \ldots, L\}$ with $h \neq i$, if $x^{(h)} \geq 1$, then $x - \chi_h + k\chi_i \succ x$, where χ_h is the hth unit vector. In the examples in the next section, agents' preference relations satisfy this condition and, therefore, we can apply Inoue's [4] theorem.

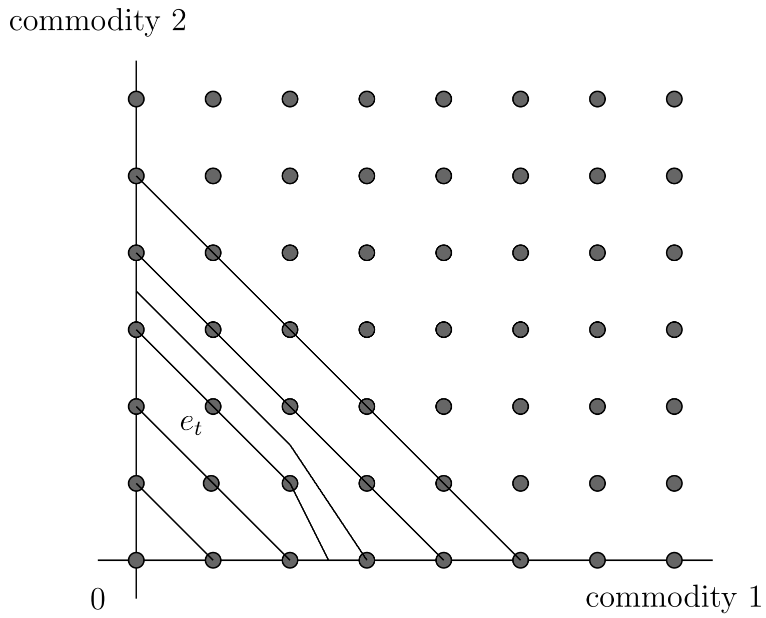

Figure 1: Endowment vector and indifference curves of agents

agents is given by $(1, 2)$. The preference relation $\precsim_t$ of agents is represented by a utility function

$$u(x, y) = \begin{cases} 3.5 & \text{if } (x, y) = (3, 0), \\ x + y & \text{otherwise.} \end{cases}$$

Although the indifference curves drawn in Figure 1 are not convex, this preference relation is *discretely convex* in the sense that for every $w \in \mathbb{Z}^2_+$, $\mathrm{co}(\{z \in \mathbb{Z}^2_+ \mid u(z) \geq u(w)\}) \cap \mathbb{Z}^2 = \{z \in \mathbb{Z}^2_+ \mid u(z) \geq u(w)\}$.[3] For every $n \in \mathbb{Z}_{++}$, economy $\mathcal{E}_n$ consists of n agents $\{(t, 1), \ldots, (t, n)\}$ who have preference relation $\precsim_t$ and endowment vector e_t. This economy is the same as the economy from Example 3 of Inoue [4].

Let e_n be the endowment allocation for $\mathcal{E}_n$, i.e., $e_n(t, i) = e_t$ for every $i \in \{1, \ldots, n\}$. For every $n \in \mathbb{Z}_{++}$, under price vector $p = (1, p^{(2)})$ with $1/2 < p^{(2)} < 1$, a pair (p, e_n) is a Walras equilibrium, but is not a cost-minimized Walras equilibrium. Since e_n is a unique Walras allocation for $\mathcal{E}_n$, we have $\emptyset = W_{CM}(\mathcal{E}_n) \subsetneq W(\mathcal{E}_n) = \{e_n\}$ for every $n \in \mathbb{Z}_{++}$. One can show that $C_S(\mathcal{E}_n) = \emptyset$ for every $n \geq 4$ and, therefore, we have the equivalence on the strong core: $C_S(\mathcal{E}_n) = \emptyset = W_{CM}(\mathcal{E}_n)$ for every $n \geq 4$.

[3]Some properties of the discrete convexity are summarized in Section 4 of Inoue [3]. The discrete convexity of preference relation is related to the nonemptiness of the weak core defined by the strong improvement.

For every $n \geq 3$, define an allocation f_n for $\mathcal{E}_n$ by

$$f_n(t,i) = \begin{cases} (2,1) & \text{if } i = 1, \\ (0,3) & \text{if } i = 2, \\ (1,2) & \text{if } i \geq 3. \end{cases}$$

Note that $f_n \in C(\mathcal{E}_n)$ and $f_n \notin W(\mathcal{E}_n)$ for every $n \geq 3$. Hence, for every $n \geq 3$, $\{e_n\} = W(\mathcal{E}_n) \subsetneq C(\mathcal{E}_n)$. Note also that under price vector $p = (1, p^{(2)})$ with $1/2 < p^{(2)} < 1$, all agents but agent $(t,1)$ satisfy the preference maximization under budget constraint.

We next consider the limit economy of the sequence $(\mathcal{E}_n)_n$ of finite economies. Let $\mathcal{B}$ be the σ-algebra of the Borel subsets in $[0,1]$ and λ be the Borel measure on $[0,1]$. An atomless economy $\mathcal{E}_\infty : ([0,1], \mathcal{B}, \lambda) \to \mathcal{P} \times \mathbb{Z}^2_+$ is defined by $\mathcal{E}_\infty(a) = (\precsim_t, e_t)$ for every $a \in [0,1]$. Let $e_\infty : [0,1] \to \mathbb{Z}^2_+$ be the endowment allocation for economy $\mathcal{E}_\infty$. We require that allocations for $\mathcal{E}_\infty$ are $\mathcal{B}$-measurable. An allocation $f : [0,1] \to \mathbb{Z}^2_+$ for $\mathcal{E}_\infty$ is exactly feasible if $\int_{[0,1]} f d\lambda = \int_{[0,1]} e_\infty d\lambda$. The core and a Walras equilibrium of $\mathcal{E}_\infty$ are defined in similar manners to those of a finite economy. Note that e_∞ is a Walras allocation under the price vector $p = (1, p^{(2)})$ with $1/2 < p^{(2)} < 1$. In addition, e_∞ is a unique Walras allocation in the sense that $g = e$ λ-a.e. for every Walras allocation g for $\mathcal{E}_\infty$. Therefore, from Theorem 3.1 of Inoue [2], $\emptyset \neq W(\mathcal{E}_\infty) = C(\mathcal{E}_\infty)$ follows. This atomless economy can be regarded as the limit economy of the sequence $(\mathcal{E}_n)_n$ of finite economies in the following sense. For every $n \in \mathbb{Z}_{++}$, define $\alpha_n : [0,1] \to \{(t,1), \ldots, (t,n)\}$ by

$$\alpha_n(i) = \begin{cases} (t,1) & \text{if } i \in [0, 1/n], \\ (t,j) & \text{if } i \in](j-1)/n, j/n], \ j = 2, \ldots, n. \end{cases}$$

For every $n \in \mathbb{Z}_{++}$, let $\mathcal{B}_n$ be the algebra on $[0,1]$ generated by $\{[0,1/n],]1/n, 2/n], \ldots,](n-1)/n, 1]\}$ and λ_n be the Borel measure restricted to $\mathcal{B}_n$. For every $n \in \mathbb{Z}_{++}$, we define an economy $\tilde{\mathcal{E}}_n : ([0,1], \mathcal{B}_n, \lambda_n) \to \mathcal{P} \times \mathbb{Z}^2_+$ by $\tilde{\mathcal{E}}_n(a) = (\precsim_t, e_t)$ for every $a \in [0,1]$. We require that allocations for $\tilde{\mathcal{E}}_n$ are $\mathcal{B}_n$-measurable. Then, for every $n \in \mathbb{Z}_{++}$, economy $\tilde{\mathcal{E}}_n$ can be identified with economy $\mathcal{E}_n$. Since $\lambda \circ (\mathcal{E}_\infty)^{-1} = \lambda_n \circ (\tilde{\mathcal{E}}_n)^{-1}$ holds for every $n \in \mathbb{Z}_{++}$ and $\mathcal{B}$ is the σ-algebra generated by $\bigcup_{n=1}^\infty \mathcal{B}_n$, economy $\mathcal{E}_\infty$ can be regarded as the limit economy of the sequence $(\mathcal{E}_n)_n$ of finite economies.

For every $n \in \mathbb{Z}_{++}$, since $f_n \in C(\mathcal{E}_n)$ and $f_n \notin W(\mathcal{E}_n)$, we have $f_n \circ \alpha_n \in C(\tilde{\mathcal{E}}_n)$ and $f_n \circ \alpha_n \notin W(\tilde{\mathcal{E}}_n)$. Recall that under the price vector $p = (1, p^{(2)})$ with $1/2 < p^{(2)} < 1$, the

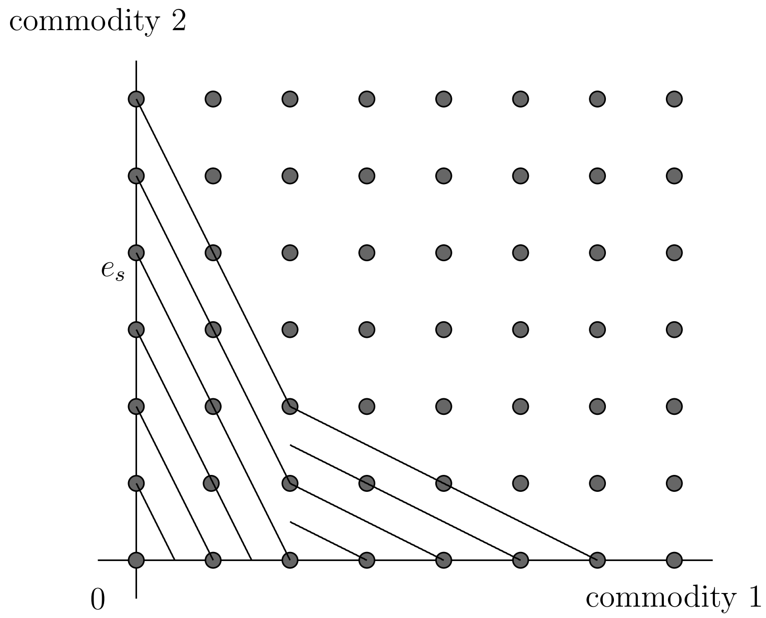

Figure 2: Endowment vector and indifference curves of agents of type s

consumption vector $f_n \circ \alpha_n(a)$ of agent $a \in [0, 1/n]$ is not the demand vector, whereas the consumption vectors $f_n \circ \alpha_n(a)$ of the other agents $a \in [0,1] \setminus [0, 1/n]$ are their demand vectors. One can show that $\lambda_n \circ (f_n \circ \alpha_n)^{-1}$ converges weakly to $\lambda \circ e_\infty^{-1}$. Therefore, $(f_n \circ \alpha_n)_{n \geq 3}$ is a sequence of core allocations that are not Walras allocations for every economy $\tilde{\mathcal{E}}_n$, but its limit allocation e_∞ is a Walras allocation for the limit economy $\mathcal{E}_\infty$ under the price vector $p = (1, p^{(2)})$ with $1/2 < p^{(2)} < 1$.

Example 2. Let $L = 2$. There exist two types $\{s, t\}$ of agents. The endowment vectors of types s and t are given by $e_s = (0, 4)$ and $e_t = (2, 0)$. The preference relations $\precsim_s$ and $\precsim_t$ of types s and t are represented by the following utility functions:

$$u_s(x, y) = \begin{cases} 2x + y & \text{if } x \leq 2, \\ (x + 2y + 6)/2 & \text{if } x \geq 3, \end{cases}$$

$$u_t(x, y) = \begin{cases} x + y & \text{if } x + y \leq 1 \text{ or } x + y \geq 4, \\ 2 & \text{if } 2 \leq x + y \leq 3. \end{cases}$$

For every $n \in \mathbb{Z}_{++}$, economy $\mathcal{E}_n$ consists of n agents of type s and n agents of type t. Let $A_n = \{(s, 1), \ldots, (s, n), (t, 1), \ldots, (t, n)\}$ be the set of agents of economy $\mathcal{E}_n$. Let e_n be the endowment allocation for $\mathcal{E}_n$.

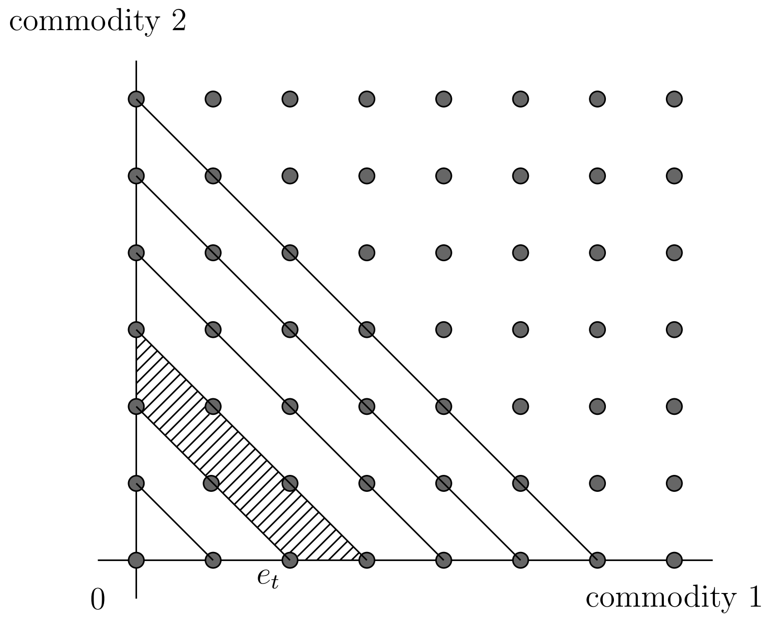

Figure 3: Endowment vector and indifference curves of agents of type t

For every $n \geq 2$, define an allocation f_n for $\mathcal{E}_n$ by

$$f_n(s,i) = \begin{cases} (1,3) & \text{if } i = 1, \\ (2,1) & \text{if } i \geq 2, \end{cases}$$

$$f_n(t,i) = \begin{cases} (1,1) & \text{if } i = 1, \\ (0,3) & \text{if } i \geq 2. \end{cases}$$

Then, for every $n \geq 2$, $f_n \notin W(\mathcal{E}_n)$. Note that under price vector $p = (3,2)$, all agents but agent $(s,1)$ satisfy the preference maximization under budget constraint. We prove that f_n is a core allocation.

Claim 1. For every $n \geq 2$, $f_n \in C(\mathcal{E}_n)$.

Proof. Suppose, to the contrary, that $f_n \notin C(\mathcal{E}_n)$ for some $n \geq 2$. Then, there exists a coalition $\emptyset \neq S \subseteq A_n$ and a mapping $g : S \to \mathbb{Z}^2_+$ such that

$$g(r^*, i^*) \succ_{r^*} f_n(r^*, i^*) \quad \text{for some } (r^*, i^*) \in S,$$
$$g(r,i) = f_n(r,i) \quad \text{for all } (r,i) \in S \setminus \{(r',i') \in S \,|\, g(r',i') \succ_{r'} f_n(r',i')\}, \text{ and}$$
$$\textstyle\sum_{(r,i)\in S} g(r,i) = \sum_{(r,i)\in S} e_n(r,i).$$

Let $p = (3, 2)$. Then, we have

$$\begin{aligned}
&p \cdot (f_n(s,1) - e_n(s,1)) = 1, \\
&p \cdot (f_n(s,i) - e_n(s,i)) = 0 \quad \text{for every } i \geq 2, \\
&p \cdot (f_n(t,1) - e_n(t,1)) = -1, \quad \text{and} \\
&p \cdot (f_n(t,i) - e_n(t,i)) = 0 \quad \text{for every } i \geq 2.
\end{aligned}$$

Let $B = \{(r,i) \in S \mid g(r,i) \succ_r f_n(r,i)\}$. Since $g(r,i) = f_n(r,i)$ for every $(r,i) \in S \setminus B$, we have

$$\sum_{(r,i) \in S \setminus B} p \cdot (g(r,i) - e_n(r,i)) \geq -1.$$

From Figures 2 and 3 of agents' indifference curves, it follows that for every $(r,i) \in B$,

$$p \cdot (g(r,i) - e_n(r,i)) \geq 2.$$

Since $B \neq \emptyset$, we have

$$\sum_{(r,i) \in S} p \cdot (g(r,i) - e_n(r,i)) \geq 1,$$

which contradicts the exact feasibility of g within S. □

Thus, for every $n \geq 2$, $W(\mathcal{E}_n) \subsetneq C(\mathcal{E}_n)$.

For every $n \in \mathbb{Z}_{++}$, define allocations g_n and h_n for $\mathcal{E}_n$ by

$$g_n(r,i) = \begin{cases} (2,0) & \text{if } r = s, \\ (0,4) & \text{if } r = t, \end{cases}$$

$$h_n(r,i) = \begin{cases} (2,2) & \text{if } r = s, \\ (0,2) & \text{if } r = t. \end{cases}$$

It can be shown without difficulty that for every $n \in \mathbb{Z}_{++}$, $W_{CM}(\mathcal{E}_n) = \{g_n, h_n\}$. Thus, by Inoue's [4] theorem, $W_{CM}(\mathcal{E}_n) = \{g_n, h_n\} = C_S(\mathcal{E}_n)$ for n large enough. As we will show in the following, this equality holds for every $n \in \mathbb{Z}_{++}$.

Claim 2. For every $n \in \mathbb{Z}_{++}$, $C_S(\mathcal{E}_n) = \{g_n, h_n\}$.

Proof. Let $n \in \mathbb{Z}_{++}$ and let $f \in C_S(\mathcal{E}_n)$. We first assume that there exists an agent (t, i_0) of type t such that $2 \leq f^{(1)}(t, i_0) + f^{(2)}(t, i_0) \leq 3$. We will show that $f = g_n$.

Let $p = (1, 1)$. Let (s, j_0) be an agent of type s such that $p \cdot (f(s, j_0) - e_n(s, j_0)) \leq 0$. We will prove that $f(s, j_0) = (2, 2)$. Suppose, to the contrary, that $f(s, j_0) \neq (2, 2)$. Let $S = \{(s, j_0), (t, i_0)\}$. Define $k : S \to \mathbb{Z}^2_+$ by

$$k(s, j_0) = (2, 2) \quad \text{and} \quad k(t, i_0) = (0, 2).$$

Since $2 \leq f^{(1)}(t, i_0) + f^{(2)}(t, i_0) \leq 3$, we have $k(t, i_0) \sim_t f(t, i_0)$. Since $p \cdot (f(s, j_0) - e_n(s, j_0)) \leq 0$ and $f(s, j_0) \neq (2, 2)$, we have

$$k(s, j_0) = (2, 2) \succ_s f(s, j_0).$$

Also, we have $k(s, j_0) + k(t, i_0) = e_n(s, j_0) + e_n(t, i_0)$. This contradicts that $f \in C_S(\mathcal{E}_n)$. Thus, $f(s, j_0) = (2, 2)$.

Therefore, $p \cdot (f(s, j) - e_n(s, j)) \geq 0$ for every $j \in \{1, \ldots, n\}$. By the individual rationality of f, $p \cdot (f(t, i) - e_n(t, i)) \geq 0$ for every $i \in \{1, \ldots, n\}$. Thus, by the exact feasibility of f, we have

$$p \cdot (f(s, j) - e_n(s, j)) = 0 \quad \text{for every } j \in \{1, \ldots, n\},$$

and, therefore, from the argument in the previous paragraph, we have $f(s, j) = (2, 2)$ for every $j \in \{1, \ldots, n\}$. Again, by the exact feasibility of f, we have $f(t, i) = (0, 2)$ for every $i \in \{1, \ldots, n\}$. Thus, $f = g_n$.

We next assume that $f^{(1)}(t, i) + f^{(2)}(t, i) \geq 4$ for every $i \in \{1, \ldots, n\}$. We will show that $f = h_n$. Let $q = (2, 1)$. Since $f^{(1)}(t, i) + f^{(2)}(t, i) \geq 4$ for every $i \in \{1, \ldots, n\}$, we have $q \cdot (f(t, i) - e_n(t, i)) \geq 0$ for every $i \in \{1, \ldots, n\}$. By the individual rationality of f, $q \cdot (f(s, j) - e_n(s, j)) \geq 0$ for every $j \in \{1, \ldots, n\}$. By the exact feasibility of f, we have

$$q \cdot (f(t, i) - e_n(t, i)) = 0 \quad \text{for every } i \in \{1, \ldots, n\}.$$

Therefore, $f(t, i) = (0, 4)$ for every $i \in \{1, \ldots, n\}$. Again, by the exact feasibility of f, we have $f(s, j) = (2, 0)$ for every $j \in \{1, \ldots, n\}$. Therefore, $f = h_n$. Hence, for every $n \in \mathbb{Z}_{++}$, $C_S(\mathcal{E}_n) = \{g_n, h_n\}$. $\square$

For every $n \in \mathbb{Z}_{++}$, endowment allocation e_n is a Walras allocation under price vector $p = (1, p^{(2)})$ with $1/2 < p^{(2)} < 2/3$, but e_n is not a cost-minimized Walras allocation. By summing up, for every $n \geq 2$, $\emptyset \neq C_S(\mathcal{E}_n) = W_{CM}(\mathcal{E}_n) \subsetneq W(\mathcal{E}_n) \subsetneq C(\mathcal{E}_n)$.

We consider the limit economy of the sequence $(\mathcal{E}_n)_n$ of finite economies. Let $\mathcal{B}$ be the σ-algebra of the Borel subsets in $[0,1]$ and λ be the Borel measure on $[0,1]$. An atomless economy $\mathcal{E}_\infty : ([0,1], \mathcal{B}, \lambda) \to \mathcal{P} \times \mathbb{Z}^2_+$ is defined by

$$\mathcal{E}_\infty(a) = \begin{cases} (\precsim_s, e_s) & \text{if } a \in [1, 1/2], \\ (\precsim_t, e_t) & \text{if } a \in]1/2, 1]. \end{cases}$$

From Theorem 3.1 of Inoue [2], $W(\mathcal{E}_\infty) = C(\mathcal{E}_\infty)$ follows, and from Theorem of Inoue [5], $W_{CM}(\mathcal{E}_\infty) = C_S(\mathcal{E}_\infty)$ follows. Let e_∞ be the endowment allocation for economy $\mathcal{E}_\infty$. Define allocations g_∞ and h_∞ by

$$g_\infty(a) = \begin{cases} (2,0) & \text{if } a \in [0, 1/2], \\ (0,4) & \text{if } a \in]1/2, 1], \end{cases}$$

$$h_\infty(a) = \begin{cases} (2,2) & \text{if } a \in [0, 1/2], \\ (0,2) & \text{if } a \in]1/2, 1]. \end{cases}$$

Then, we have $g_\infty, h_\infty \in W_{CM}(\mathcal{E}_\infty)$, $e_\infty \in W(\mathcal{E}_\infty)$, and $e_\infty \notin W_{CM}(\mathcal{E}_\infty)$. Therefore, $\emptyset \neq C_S(\mathcal{E}_\infty) = W_{CM}(\mathcal{E}_\infty) \subsetneq W(\mathcal{E}_\infty) = C(\mathcal{E}_\infty)$.

Let $\mathcal{B}_n$ be the algebra on $[0,1]$ generated by $\{[0, 1/(2n)],]1/(2n), 2/(2n)], \ldots,](2n-1)/(2n), 1]\}$ and λ_n be the Borel measure restricted to $\mathcal{B}_n$. Also, for every $n \in \mathbb{Z}_{++}$, define $\alpha_n : [0,1] \to A_n$ by

$$\alpha_n(a) = \begin{cases} (s,1) & \text{if } a \in [0, 1/(2n)], \\ \quad\vdots & \\ (s,n) & \text{if } a \in](n-1)/(2n), 1/2], \\ (t,1) & \text{if } a \in]1/2, (n+1)/(2n)], \\ \quad\vdots & \\ (t,n) & \text{if } a \in](2n-1)/(2n), 1]. \end{cases}$$

For every $n \in \mathbb{Z}_{++}$, define an economy $\tilde{\mathcal{E}}_n : ([0,1], \mathcal{B}_n, \lambda_n) \to \mathcal{P} \times \mathbb{Z}^2_+$ by $\tilde{\mathcal{E}}_n(a) = \mathcal{E}_n \circ \alpha_n(a)$ for every $a \in [0,1]$. Since we require that allocations for $\tilde{\mathcal{E}}_n$ must be $\mathcal{B}_n$-measurable, economy $\tilde{\mathcal{E}}_n$ can be identified with economy $\mathcal{E}_n$. Since $\lambda \circ (\mathcal{E}_\infty)^{-1} = \lambda_n \circ (\tilde{\mathcal{E}}_n)^{-1}$ holds for every $n \in \mathbb{Z}_{++}$ and $\mathcal{B}$ is the σ-algebra generated by $\bigcup_{n=1}^{\infty} \mathcal{B}_n$, economy $\mathcal{E}_\infty$ can be regarded as the limit economy of the sequence $(\tilde{\mathcal{E}}_n)_n$. By an argument above, for every $n \geq 2$, $f_n \circ \alpha_n \notin W(\tilde{\mathcal{E}}_n)$, $f_n \circ \alpha_n \in C(\tilde{\mathcal{E}}_n)$, $C_S(\tilde{\mathcal{E}}_n) = \{g_n \circ \alpha_n, h_n \circ \alpha_n\} = W_{CM}(\tilde{\mathcal{E}}_n)$, $e_n \circ \alpha_n \in W(\tilde{\mathcal{E}}_n)$, and $e_n \circ \alpha_n \notin W_{CM}(\tilde{\mathcal{E}}_n)$. Thus, for every $n \geq 2$, $\emptyset \neq C_S(\tilde{\mathcal{E}}_n) = W_{CM}(\tilde{\mathcal{E}}_n) \subsetneq W(\tilde{\mathcal{E}}_n) \subsetneq C(\tilde{\mathcal{E}}_n)$. Hence, the core coincides with the set of Walras allocations only in the limit atomless economy, whereas the strong core coincides with the set of cost-minimized Walras allocations regardless of in a finite economy or in the limit atomless economy.

Define an allocation f_∞ for $\mathcal{E}_\infty$ by

$$f_\infty(a) = \begin{cases} (2,1) & \text{if } a \in [0, 1/2], \\ (0,3) & \text{if } a \in]1/2, 1]. \end{cases}$$

Note that $\lambda_n \circ (f_n \circ \alpha_n)^{-1}$ converges weakly to $\lambda \circ f_\infty^{-1}$ and f_∞ is a Walras allocation for $\mathcal{E}_\infty$ under price vector $(3, 2)$. Thus, $(f_n \circ \alpha_n)_{n \geq 2}$ is a sequence of allocations, each of which is not a Walras allocation for $\tilde{\mathcal{E}}_n$, but its limit allocation f_∞ is a Walras allocation for the limit atomless economy $\mathcal{E}_\infty$.

References

[1] Anderson, R.M., 1978. An elementary core equivalence theorem. Econometrica 46, 1483-1487.

[2] Inoue, T., 2005. Do pure indivisibilities prevent core equivalence? Core equivalence theorem in an atomless economy with purely indivisible commodities only. Journal of Mathematical Economics 41, 571-601.

[3] Inoue, T., 2008. Indivisible commodities and the nonemptiness of the weak core. Journal of Mathematical Economics 44, 96-111.

[4] Inoue, T., 2009. Indivisible commodities and an equivalence theorem on the strong core. IMW Working Paper 417, Institute of Mathematical Economics, Bielefeld University.

[5] Inoue, T., 2009. Strong core equivalence theorem in an atomless economy with indivisible commodities. IMW Working Paper 418, Institute of Mathematical Economics, Bielefeld University.