Working Papers

Institute of Mathematical Economics

425

December 2009

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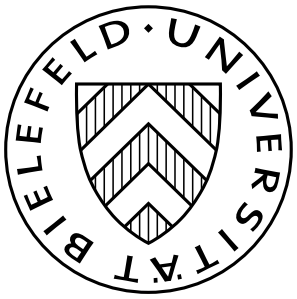

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http://www.wiwi.uni-bielefeld.de/~imw/Papers/showpaper.php?425
ISSN: 0931-6558

An Axiomatization of the Sequential Raiffa Solution

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This paper provides four axioms that uniquely characterize the Sequential Raiffa solution proposed by Raiffa (1951, 1953) for two-person bargaining games. Three of these axioms are standard and are shared by several popular bargaining solutions. They suffice to characterize these solutions on TU-bargaining games where they coincide. The fourth axiom is a weakening of Kalai's (1977) axiom of step-by-step negotiating and turns out to be sort of a dual condition to a weaker version of Nash's IIA-axiom that together with the three standard axioms suffices to characterize the Nash bargaining solution due to Nash (1950). A conclusion of this axiomatization is that in contrast to all other known bargaining solutions the Sequential Raiffa solution does not represent just another kind of fairness or equity condition in addition to the three standard axioms but rather is determined by indefinite repeated application of the three standard axioms.

Key words: Bargaining games, Raiffa solution, Nash solution, axiomatization

[1]This article is an extended version of Trockel (2001). While finishing it I became aware of a recent working paper by Anbarei and Sun (2009) dealing with robustness of intermediate agreements in bargaining. They also provide an, however quite different, set of axioms that determine the Sequential Raiffa solution.

1 INTRODUCTION

The first bargaining solution had been introduced by Nash (1950, 1953) who provided a procedural definition ("maximize the (Nash) product of players' utility levels") and an axiomatic characterization.

Almost simultaneously Raiffa (1951) offered an analysis of bargaining solutions under the name of "arbitration schemes" a shortened version of which was published in Raiffa (1953). In this work Raiffa discussed essentially four bargaining solutions [2], in today's terminology the Nash solution, the Kalai-Rosenthal solution, the Continuous Raiffa solution and the Sequential (or Discrete) Raiffa solution. The Kalai-Rosenthal solution, however, coincides in the figure on page 380 of Raiffa (1953) with the Kalai-Smorodinsky solution. This is due to identical utopia points of the analyzed bargaining problem and its individually rational sub-problem.

Of all these and other frequently used bargaining solutions only the Sequential Raiffa solution had resisted up to now an axiomatic characterization.

The various axiomatizations of bargaining solutions in the literature differ in underlying domains and do not always provide the largest possible domain. Also in my present analysis I shall not try to provide the largest possible domain for my axiomatization of the Sequential Raiffa solution. I will rather focus on providing an axiomatization on a class of bargaining games that allows a comparison with axiomatizations of other bargaining solutions.

I will provide an axiom that amounts to the requirement of repeated application of the three standard axioms for bargaining solutions that suffice to uniquely determine a solution on the class of hyperplane (hence on TU) bargaining games. Those three standard axioms are shared by several prominent bargaining solutions like the Nash, Kalai-Smorodinsky and Perles-Maschler solutions that, however, differ on general non-hyperplane bargaining problems where each of them is characterized by a different fourth axiom. It is my aim in this work to provide such a fourth axiom for the sequential Raiffa solution.

[2] I am very grateful to Hans Peters who via personal communication (Peters (2000)) had helped me to clarify some confusion concerning the Raiffa solution.

I shall derive my axiom RAS as a static restatement of Raiffa's procedural rules and will relate it to the step by step negotiation axiom of Kalai (1977). The axiom RAS and a weaker version of Nash's IIA axiom will turn out to be somehow "dual" weakenings of Kalai's axiom. An exact non-cooperative foundation of the Sequential Raiffa solution (based on an approximate one due to Myerson (1991)) is provided in Trockel (2009).

2 THE SEQUENTIAL RAIFFA SOLUTION

Before laying down the model for the formal analysis of the Sequential (or Discrete) Raiffa solution I introduce and illustrate it graphically. Up to different notation I follow essentially the presentation of Luce and Raiffa (1957, pp. 136, 137).

Also Shubik (1984, pp. 196-198) refers to the Raiffa solution. However, in his section on the Kalai-Smorodinsky which he first describes by axioms he claims (p. 196) that "This scheme is essentially the same as that suggested by Raiffa (1953)..." and he continues by illustrating this solution by an example where he indeed uses the procedure of the Discrete Raiffa solution. In fact, not only are those two solutions different, they also yield different solution points if applied to Shubik's example. Shubik may have fallen victim to Raiffa's (1953) opaque presentation of four different solutions among them the Discrete Raiffa and the Kalai-Smorodinsky solutions.

Also Myerson (1991), p. 395) calls the Sequential Raiffa solution "closely related to the Kalai-Smorodinsky solution". However, this judgement appears questionable as it is neither based on a metric on bargaining solutions nor on any (even if only intuitive) comparison of the underlying axioms. The latter one would in fact require an axiomatic characterization also for the Sequential Raiffa solution!

I commented already in section 1 on the possible confusion about the Raiffa solution. Therefore a thorough analysis of this solution providing an axiomatic characterization is overdue.

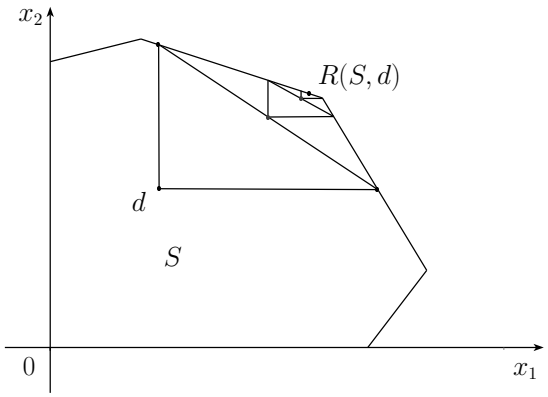


Figure 1: The Raiffa solution of (S, d)

A *two-person bargaining problem* as illustrated in Figure 1 consists of a compact, convex set $S \subset \mathbb{R}^2$ of feasible payoff vectors for the two players 1 and 2 and some point $d \in S$, the *disagreement point*, also called *threat point* or *status quo point* in the literature.

The interpretation of such a bargaining game is as follows:

If the players agree on some vector $x \in S$ they receive the payoffs x_1 and x_2, respectively. If they do not agree on any vector $x \in S$ they receive d_1 and d_2, respectively.

Notice, that in Figure 1 the individually rational part of S, i.e.
$S_d := S \cap (\{d\} + \mathbb{R}^2_+)$
is *comprehensive* in the sense that for any $x \in S_d$ also the line segment $[d, x] := co\{x, d\} \subset S_d$. This (S_d, d) is itself a bargaining problem.

The *Pareto frontier* of S_d, denoted ∂S_d, is the graph of the two functions $h_i : [d_i, \infty) \cap proj_i S_d \longrightarrow [d_{3-i}, \infty) \cap proj_{3-i} S_d : x_i \longrightarrow max\{x_{3-i} \mid x \in S_d\}$, $i = 1,\ 2$.

The Raiffa procedure determines in a first step the midpoint of the line segment $[(d_1, h_2(d_1), (h_1(d_2), d_2)]$.

In the second step the new bargaining problem (S, d^1) is considered. If $d^1 \in \partial S_d$ we are finished and d^1 is the vector of payoffs. Otherwise, as

in Figure 1, the midpoint d^2 of the line segment $[(d_1^1, h_2(d_1^1)), (h_1(d_2^1), d_2^1)]$ is chosen, leading to the new bargaining problem (S, d^2).

Generally, d^{n+1} is the midpoint of the line segment $[(d_1^n, h_2(d_1^n)), (h_1(d_2^n), d_2^n)]$. For any bargaining problem (S, d) the sequence of midpoints $(d^n)_{n\in\mathbb{N}}$ (that clearly depends on (S, d)) converges to some point $R(S, d)$. If ∂S is piecewise linear this convergence is generically finite.

The mapping R that associates with any bargaining problem (S, d) the point $R(S, d) \in S$ is called the *Sequential* (or Discrete) *Raiffa solution*.

The arbitration problem presented in Raiffa (1951, 1953) and in Luce and Raiffa (1957) and for which the Sequential Raiffa solution has been proposed is explicitly restricted to individually rational payoff vectors. Moreover, the whole procedure that leads to the Raiffa solution does not depend on any features of (S, d) outside its individually rational part S_d. Therefore, the Sequential Raiffa solution R automatically satisfies the axiom of Independence of Non-Individually Rational Outcomes (INIR) of Peters (1986). An important fact that is not explicitly stated in Kalai (1977) or Myerson (1977) but that has been stressed by Peters (1986) and by Peters and van Damme (1991) is, that even individually rational solutions may depend on non-individually rational parts of a bargaining problem. This is nicely documented by the difference between the individually rational Kalai-Smorodinsky and Kalai-Rosenthal solutions, where (only) the latter one does not satisfy INIR.

It is the structure of the Raiffa procedure that suggests an axiomatization of R on the domain of individually rational bargaining situations containing the disagreement point d as a feasible payoff vector.

3 THE SETTING

The subsequent analysis does not rely on details of the theory of TU and NTU coalitional games. Nevertheless, the bargaining games I am going to consider may be seen as NTU games containing TU bargaining games as special cases.

3.1 *Notation*

I will use the following notation:

$S \subset S' :\Longleftrightarrow [\forall x \in S : x \in S' \text{ and } S \neq S']$

$S \subseteq S' :\Longleftrightarrow S \subset S' \text{ or } S = S'$

$x \geq y :\Longleftrightarrow \forall i \in \{1,\ 2\} : x_i \geq y_i$

$x > y :\Longleftrightarrow x \geq y \text{ and } x \neq y$

$x >> y :\Longleftrightarrow \forall i \in \{1,\ 2\} : x_i > y_i$

$\mathbb{R}^2_+ := \{x \in \mathbb{R}^2 \mid x \geq 0\}, \mathbb{R}^2_{++} := \{x \in \mathbb{R}^2 \mid x >> 0\}$

$proj_i : \mathbb{R}^2 \longrightarrow \mathbb{R} : x = (x_1, x_2) \mapsto x_i, i = 1,\ 2$

$e := (1,\ 1), e_1 := (1,\ 0), e_2 := (0,\ 1)$

$x \cdot y = \sum_i x_i y_i$ inner product in $\mathbb{R}^2$

$x * y := (x_1 y_1, x_2 y_2)$ coordinatewise multiplication in $\mathbb{R}^2$

For $p \in \mathbb{R}^2, t \in \mathbb{R}$:

$H(p,t) := \{x \in \mathbb{R}^2 \mid p \cdot x = t\}$ hyperplane in $\mathbb{R}^2$

$H_-(p,t) := \{x \in \mathbb{R}^2 \mid p \cdot x \leq t\}$ closed lower halfspace of $H(p,t)$

For any hyperplane $H \subset \mathbb{R}^2$ with $H \cap \mathbb{R}^2_+ \neq \emptyset$ let $S^H := H_- \cap \mathbb{R}^2_+$.
For $d \in S^H$ the pair (S^H, d) is a *hyperplane bargaining game.*

coX is the convex hull of the set $X \subseteq \mathbb{R}^2$.

3.2 *Basic Concepts*

An NTU coalitional n-person bargaining game is a mapping V that associates with any coalition $T \subseteq N = \{1, ..., n\}$ of players some feasible set of payoff vectors $V(T) \subset \mathbb{R}^T := \{x : T \longrightarrow \mathbb{R} \mid x|_{N \setminus T} \equiv 0\}$.

If all coalitions except the grand one and the singletons become irrelevant one has an n-person bargaining game. The simplest way to express this irrelevance is by assuming $V(T) = \{0\}$ for all $T \subset N$ with $|T| > 1$. Thus, 2-person NTU games are always bargaining games. In my present framework of $N = \{1,\ 2\}$ I rename the set $V(N)$ by S and denote the optimal payoffs for $i \in \{1,\ 2\}$ in $V(\{i\})$ by d_i.

Superadditivity reduces here to *essentiality* requesting that
$\exists\ x \in S : x >> d$, such that negotiation becomes meaningful.

If the boundary ∂S_d of the closed convex set S_d is part of a line in $\mathbb{R}^2$ we call (S_d, d) a *hyperplane bargaining game.*

If $e = (1,\ 1)$ is a normal vector to ∂S_d the hyperplane game (S_d, d) is called a *TU bargaining game.* Because of utility transfers at a rate 1/1 our TU bargaining game can be identified with the TU game in standard notation defined by $v(\{i\}) = d_i, i = 1,\ 2$ and $v(N) := max_{x \in S_d}\ e \cdot x$

DEFINITION 1:

A two person *bargaining situation* (or *game*) is a pair (S, d), where $d \in S \subset \mathbb{R}^2_+$ and S is compact, convex and $x \notin \partial S$.

Let $\mathcal{B}$ be the set of all two person bargaining games and $\mathcal{B}^H$ the set of hyperplane games in $\mathcal{B}$. Let $\mathcal{B}$ and $\mathcal{B}^H_0$ be the subsets of $\mathcal{B}$ and $\mathcal{B}^H$, respectively, that have disagreement point $d := 0 \in \mathbb{R}^2$.

DEFINITION 2:

For any set $\mathcal{C} \subseteq \mathcal{B}$ a *solution on* $\mathcal{C}$ is a mapping
$F : \mathcal{C} \longrightarrow \mathbb{R}^2 : (S, d) \mapsto f(S, d) \in S$.

First, I shall formulate three axioms that are shared by axiomatic characterizations of several popular bargaining solutions. These three axioms define a unique bargaining solution on the domain $\mathcal{B}^H$ of two person bargaining games. This *standard solution* always picks the middle point of the line segment ∂S_d.

Let $\mathcal{L}$ be any set of bargaining solutions on $\mathcal{C} \subseteq \mathcal{B}$.

Pareto Optimality $(PO) : \forall (S, d) \in \mathcal{C} : (\{F(S, d)\} + \mathbb{R}^2_+) \cap S = \{F(S, d)\}$

Covariance w.r.t. positive affine transformations (COV):

$\forall g := g^{a,b} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : x \mapsto a * x + b \ (a \in \mathbb{R}^2_{++}, b \in \mathbb{R}) :$
$F(g(S), g(d)) = g(F(S, d))$

Symmetry $(SYM) : \forall (S, d) \in \mathcal{C} :$

$[\forall (x_1, x_2) \in S : (x_2, x_1) \in S] \Longrightarrow F_1(S, d) = F_2(S, d)$

Any solution that satisfies these three axioms is called a *standard solution on* $\mathcal{C}$.

On $\mathcal{C} = \mathcal{B}^H$ there exists a unique standard solution that on the set of TU bargaining games coincides with the Shapley value.

On $\mathcal{C} = \mathcal{B}$ or $\mathcal{C} = \mathcal{B}_0$ there exist various different standard solutions, among them the Nash, Kalai-Smorodinsky, Perles-Maschler and the two Raiffa solutions.

For my purpose it is crucial to see that the Sequential Raiffa solution is a standard solution. But this is obvious from the Raiffa procedure by which it is defined.

I shall use the letter L for denoting arbitrary solutions on $\mathcal{C} \subseteq \mathcal{B}$ that are standard solutions.

4 AXIOMATIZATION

I start by formulating four further axioms. Two of them I shall apply later on,the other two are for comparison and interpretational purposes.

Step-by-Step Negotiaton (SSN):

Let F be a solution on $\mathcal{B}_0$.

$\forall (S',0),(S,0) \in \mathcal{B}_0, S \subseteq S'$ with $((S' - F(S,0)) \cap \mathbb{R}^2_+, 0) \in \mathcal{B}_0$:

$F(S',0) = F(S,0) + F((S' - F(S,0)) \cap \mathbb{R}^2_+, 0)$

This axiom has been introduced by Kalai (1977) for his axiomatic characterization of proportional solutions. It plays a crucial role as a reference axiom in my present analysis.

For similar reasons I formulate now Nash's (1951) famous IIA axiom and a weakening of it that I call SHA.

Independence of Irrelevant Alternatives(IIA):

Let F be a solution on $\mathcal{B}$.

$\forall (S',d),(S,d) \in \mathcal{B}, S \subseteq S'$ with $F(S',d) \in S : F(S',d) = F(S,d)$

Supporting Hyperplane Approximation(SHA):

Let F be a solution on $\mathcal{B}$.

$\forall (S,d) \in \mathcal{B}, (S^H,d) \in \mathcal{B}^H, S \subseteq S^H$ with $F(S^H,d) \in S : F(S^H,d) = F(S,d)$

Notice that S^H is a hyperplane game where H is a hyperplane supporting S in the point $F(S^H,d)$.

It is closely related to the λ-transfer principle as introduced in Shapley (1969) by which the Nash solution may be alternatively derived as Shapley's NTU value on bargaining games. I will not elaborate on this relation here. But the axiom SHA may also be seen as a reference to Shapley.

It is obvious that SHA is a weakening of IIA as it is a restriction of IIA to

those pairs $((S,d),(S',d))$ as defined in IIA for which $S' \in \mathcal{B}^H$.

Now I turn from outside approximation of S to the "dual" approximation from inside. Accordingly, I consider for a given game $(S,d) \in \mathcal{B}$ the unique associated maximal hyperplane game (S^H,d) with $S^H \subseteq S$. The analogon to axiom SHA in this new situation will be axiom RAS that may simultaneously serve as a mnemonic for "Raiffa Sequential".

Repeated Application of the Standard Axioms (RAS):

Let L be a standard solution on $\mathcal{B}$.

$\forall (S,d) \in \mathcal{B} : L(S,d) = L(S^H,d) + L(S - L(S^H,d),0)$.

Notice, that due to COV, that is satisfied for standard solutions, one gets

$L(S - L(S^H,d),0) = L(S_{L(S^H,d)},\ L(S^H,d)) - L(S^H,d)$

Hence under COV, so for any standard solution L, the axiom RAS is equivalent to the following axiom:

Invariance under Change of Disagreement Points by the Raiffa Procedure (ICDR):

$\forall (S,d) \in \mathcal{B} : L(S,d) = L(S_{L(S^H,d)},\ L(S^H,d)$.

I can now state my first main result:

THEOREM 1:

There exists a unique standard solution R on $\mathcal{B}$ that satisfies RAS. This is the Sequential Raiffa solution.

PROOF:

First I verify that R as defined by Raiffa's procedure satisfies ICDR, hence, as R is a standard solution, also RAS.

For any $(S,d) \in \mathcal{B}$ the point $R(S,d)$ is defined as the unique limit of a sequence of points in S specified by the Raiffa procedure.

As $(S_{L(S^H,d)}, L(S^H,d))$ as a starting game for this procedure coincides with the second game reached by this procedure, when (S,d) is the starting game, both sequences finally agree and converge to the same limit, namely $\{R(S,d)\}$. Hence

$R(S,d) = R(S_{L(S^H,d)}), L(S^H,d))$.

Next, I show that there is no other standard solution L on $\mathcal{B}$ that satisfies RAS.

So, let L be any standard solution on $\mathcal{B}$ that satisfies RAS. Denote the games in the sequence defined by the Raiffa procedure and starting with (S,d) by:

$(S^{\circ}_{d^{\circ}}, d^{\circ}) := (S_d, d), (S^1_{d^1}, d^1) := (S_{L(S^H,d)}, L(S^H,d)$ etc.

Because of ICDR we have

$$L(S,d) = L(S^n_{d^n}, d^n) \quad \forall\ n \in \mathbb{N}.$$

In particular for the Raiffa solution that is standard we have

$$R(S,d) = R(S^n_{d^n}, d^n) \quad \forall\ n \in \mathbb{N} \text{ and, moreover,}$$

$$lim_{n \longrightarrow \infty}\ R(S^n_{d^n}, d^n) = R(S_d, d) \equiv R(S,d).$$

As the sequence $(S^n_{d^n})_{n \in \mathbb{N}}$ converges in the Hausdorff metric to $\{R(S,d)\}$ also the sequence $L(S^n_{d^n}, d^n)$ must converge to $R(S,d)$. But as it is a constant sequence we get:

$$L(S,d) = lim_{n \longrightarrow \infty}\ L(S^n_{d^n}, d^n) = R(S,d)$$

Q.E.D.

5 APPRAISAL OF THE AXIOM RAS

The goal of the present section is it to demonstrate that the axioms RAS and SHA are somehow dual weakenings of Kalai's SSN axiom. For this sake it is convenient to consider explicitly also singleton bargaining sets. For these the points of disagreement and of solution coincide.

For any given game $(S,d) \in \mathcal{B}$ there exist minimal hyperplane games $(\bar{S}^H, d)$ with $S \subseteq \bar{S}^H$ and maximal hyperplane games $(\underline{S}^H, d)$ with $\underline{S}^H \subseteq S$.

$(\bar{S}^H, d)$ is *minimal* in this sense if for any other hyperplane game $(\tilde{S}^H, d)$ the relation $\bar{S}^H \supset \tilde{S}^H \supseteq S$ is impossible.

Likewise $(\underline{S}^H, d)$ is *maximal* if for any other hyperplane game $(\tilde{S}^H, d)$ the relation $\underline{S}^H \subset \tilde{S}^H \subseteq S$ is impossible.

Any $(S,d) \in \mathcal{B}$ has a unique $(\underline{S}^H, d)$ but generally many $(\bar{S}^H, d)$. Denote the set of those minimal games by $\mathcal{H}^S$.

I will apply now the axiom SSN with its general $S, S', S \subseteq S'$ to the following situations where L is supposed to be a standard solution:

case 1: $S' := \bar{S}^H, \bar{S}^H \in \mathcal{H}^S, L(\bar{S}^H, 0) \in S$

case 2: $S := \underline{S'}^H$

In case 1 the axiom SSN reduces to:

$$\forall (S,0),\ (\bar{S}^H, 0) \in \mathcal{B}_0 \text{ with } L(\bar{S}^H, 0) \in S : L(\bar{S}^H, 0) = L(S,0)$$

This is exactly the restatement of SHA for this case. Indeed, as $L(\bar{\mathbf{S}}^H, 0) \in \partial S$ and $L(S,0) \in \partial S$ the condition $((\bar{S}^H - L(S,0)) \cap \mathbb{R}^2_+, 0) \in \mathcal{B}_0$ in SSN is automatically satisfied.

The statement in SSN, namely:

$$L(\bar{S}^H, 0) = L(S,0) + L((\bar{S}^H - L(S,0)) \cap \mathbb{R}^2_+, 0)$$

can equivalently written as

$$L(\bar{S}^H, 0) - L(S, 0) = L((\bar{S}^H - L(S, 0)) \cap \mathbb{R}^2_+, 0).$$

As the right hand term is non-negative, the equality requires this to be the case also for the left hand term.

But for $L(\bar{\mathbf{S}}^H, 0) \neq L(S, 0)$ this term has one positive and one negative coordinate. Hence we must have $L(\bar{S}^H, 0) = L(S, 0)$. This effect is illustrated in Figure 2.

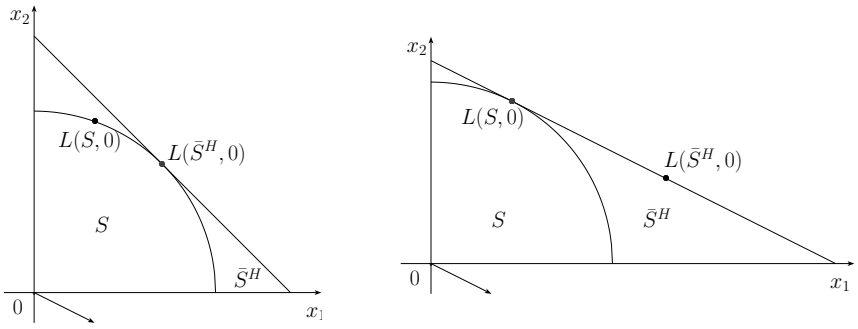


Figure 2: $L(\bar{S}^H, 0) - L(S, 0) \not\geq 0$

Next I consider *case* 2.

This time the smaller S in axiom SSN equals $\underline{S}^H \subset S'$ where $(\underline{S}^H, 0) \in \mathcal{B}^H_0$. Therefore, the condition $((S' - L(\underline{S}^H, 0)) \cap \mathbb{R}^2_+, 0) \in \mathcal{B}_0$ is automatically satisfied.

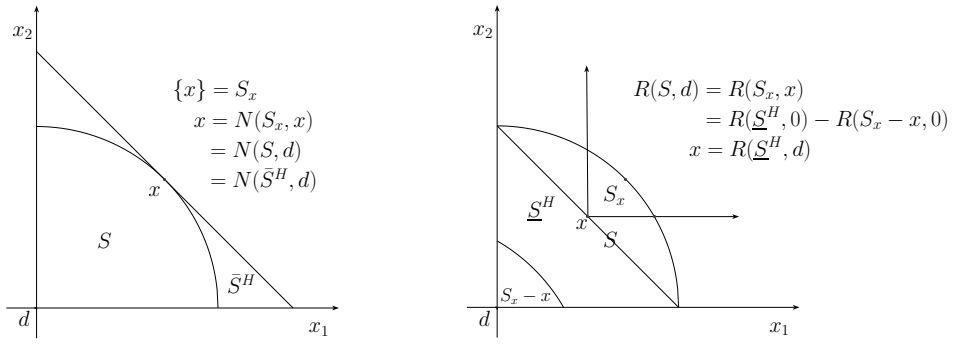
Thus, for $d = 0$ axiom SSN reduces in *case* 2 exactly to RAS.

The duality of inner and outer hyperplane approximation as represented by RAS and SHA is visualized by Figure 3 that is based on the following equivalent reformulations of axioms ICDR (hence RAS) and SHA.

RAS*: Let $L(\underline{S}^H, d) \in S$. $\quad \forall (S, d) \in \mathcal{B}: \; L(S) = L(S_{L(\underline{S}^H, d)}, L(\underline{S}^H, d))$

SHA*: Let $L(\bar{S}^H, d) \in S$. $\quad \forall (S, d) \in \mathcal{B}: \; L(S) = L(S_{L(\bar{S}^H, d)}, L(\bar{S}^H, d))$

Note, that $L(\underline{S}^H, d) \in S$ is obviously always satisfied.



a. Nash:
$N(S, d) = N\left(S_{N(\bar{S}^H, d)}, N(\bar{S}^H, d)\right)$

b. Raiffa:
$R(S, d) = R\left(S_{R(\underline{S}^H, d)}, R(\underline{S}^H, d)\right)$

Figure 3:

The structural similarity between RAS* and SHA* that is illustrated in Figure 3 may at this point appear as a mere curiosity. It is, however, of considerable importance as it will turn out in the next section that SHA* (or SHA) substituted for the stronger axiom IIA still suffices to characterize the Nash solution uniquely among the standard solutions on $\mathcal{B}$.

6 A DETOUR TO THE NASH SOLUTION

The startling "duality" of the axioms SHA and RAS immediately leads to the question of duality between the bargaining solutions N and R.

Axioms RAS and IIA uniquely determine the solutions R and N, respectively, among the standard bargaining solutions. If SHA which is a strict weakening of IIA still characterizes N uniquely among the standard solutions then R and N are somehow dual solutions.

THEOREM 2:

There exist a unique standard solution on $\mathcal{B}$ that satisfies SHA. This is the Nash solution N.

PROOF:

As N is known to satisfy IIA it satisfies in particular SHA on $\mathcal{B}$.

Now consider any standard solution L on $\mathcal{B}$ with $L \neq N$.

As both, L and N, are standard solutions they coincide on $\mathcal{B}^H$. Therefore, there exists at least one game $(S,d) \in \mathcal{B} \setminus \mathcal{B}^H$ with $L(S,d) \neq N(S,d)$. W.l.o.g. we choose $d = 0$.

Consider a hyperplane game $(\bar{S}^H, 0)$ with $L(\bar{S}^H, 0) \in S$.

As L is standard we have $L(\bar{S}^H, 0) = N(\bar{S}^H, 0) \in S$.

By SHA, that holds for N, we conclude that $L(\bar{S}^H, 0) = N(\bar{S}^H, 0) = N(S, 0)$ and thus that $L(S, 0) \neq L(\bar{S}^H, 0)$. Hence L violates SHA.

Q.E.D.

This result says that the difference between the Nash solution N and the Sequential Raiffa solution R amounts to the difference between the two "dual" axioms SHA and RAS as both, R and N, are standard solutions.

Due to a lack of a duality theory for NTU games in the literature, however, it remains an open problem whether this is really a duality in the usual mathematical sense.

7 SEQUENTIAL RAIFFA VERSUS NASH: CONCLUDING REMARKS

The Nash and the Sequential Raiffa solution are both determined by the three standard axioms and some transfer principle that describes how to proceed from a standard solution on $\mathcal{B}^H$ to get a unique solution on $\mathcal{B}$.

For the Nash solution this transfer is defined by the axiom SHA that is

closely related to Shapley's λ-transfer. Among the hyperplane bargaining games supporting (from outside) a given game $(S, d) \in \mathcal{B}$ there is exactly one whose solution is attainable in (S, d). This results from a Brower type fixed point argument used by Shapley (1969) together with convexity of bargaining games. The Nash solution is the unique no trade equal distribution Walrasian equilibrium of the bargaining game interpreted as an Arrow-Debreu economy (c.f. Trockel (1996)). For any other supporting hyperplane game the Nash solution would not be attainable in (S, d). Rather transfers (i.e. trade) would be necessary to realize it. A hint to a close relation between the Nash bargaining solution and the equal division Walrasian equilibrium via a shared consistency axiom called *Multilateral Stability* had already been made by Thomson and Lensberg (1989, p. 101).

The transfer principle for the Sequential Raiffa solution is quite different. Here it is not a single hyperplane game that provides the solution but a whole sequence of them. And these do not support (S, d). Rather together they approach the Sequential Raiffa solution via the sequence of their solutions. The repeated generation of smaller and smaller remaining bargaining games converging, like the sequence of their disagreement points, to $R(S,\ d)$ results from repeated mutual concessions ("trades") at certain transfer rates till nothing remains to be exchanged to mutual advantage. This is reminiscent of consecutive non-market clearing exchange activities followed by new exchange at different prices until some kind of non-Walrasian equilibrium is reached (cf. Benassy (1989)).

Also $R(S, d)$ may be written as a unique fixed point. But this results from the Banach Fixed point theorem for contractions. In fact, repeated application of some contracting map T is generating the sequence $(S_{d^n},\ d^n)$ of bargaining games from $(S_{d^\circ}, d^\circ) := (S_d, d)$.

While $N(S, d)$ corresponds to a one shot Walrasian equilibrium that may derived from some tatonnement process, $R(S, d)$ is the result of some non-tatonnement process.

I believe that working out a rigorous economic approach representing the Raiffa procedure will also help to clarify the exact meaning of the "duality" between N and R.

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