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Representation of TU games by coalition production economies

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Abstract

We prove that every transferable utility (TU) game can be generated by a coalition production economy. Given a TU game, the set of Walrasian payoff vectors of the induced coalition production economy coincides with the core of the balanced cover of the given game. Therefore, a Walrasian equilibrium for the induced coalition production economy always exists. The induced coalition production economy has one output and the same number of inputs as agents. Every input is personalized and it can be interpreted as agent's labor. In a Walrasian equilibrium, every agent is permitted to work at several firms. In a Walrasian equilibrium without double-jobbing, in contrast, every agent has to work at exactly one firm. This restricted concept of a Walrasian equilibrium enables us to discuss which coalitions are formed in an equilibrium. If the cohesive cover or the completion of a given TU game is balanced, then the no-double-jobbing restriction does not matter, i.e., there exists no difference between Walrasian payoff vectors and Walrasian payoff vectors without double-jobbing.

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1 Introduction

A transferable utility (TU) game is a reduced form in the sense that it is derived from models with richer structure such as exchange economies or strategic coalitional games. Shapley and Shubik [18] proved that the class of totally balanced TU games coincides with the class of TU games generated by exchange economies. In an exchange economy, an allocation feasible in disjoint segmented markets is also feasible in the union of those markets. The total balancedness is a more general concept than this superadditivity. In a coalition production economy, however, different coalitions can access to different production sets and, therefore, a feasible allocation in segmented markets need not be feasible in the union of those markets any longer. A simple example is an economy with decreasing returns to scale in production. As a result, a coalition production economy can generate a TU game which is not even superadditive. On the other hand, an exchange economy is a special case of a coalition production economy. Hence, the class of TU games generated by coalition production economies must be larger than the class of totally balanced TU games, the class generated by exchange economies. Our first result says that TU games generated by coalition production economies fill up the class of all TU games. Thus, any TU game can be generated by a coalition production economy.

Given a TU game, we induce two coalition production economies generating the game. The first one is due to Sun et al. [20]. The second one is a slight modification of the first one and it is related to Billera's [2] representation of totally balanced non-transferable utility (NTU) games. Both induced economies consist of one output and the same number of inputs as agents. Every input is personalized and it can be interpreted as agents' labor. All agents have the same consumption set and the same simple utility function, where the utility is the amount of consumption of the output. The consumption set and the utility function do not depend on a given TU game. Accordingly, the difference among TU games is represented only by coalitions' production sets. Our two induced economies have the different description of production sets. In the first induced economy, agents' labor is complementary, but in the second induced economy, agents working at the same firm are required to invest their labor in the same proportion. A unique way not to waste agents' labor in the first induced economy is that agents in the same coalition work in the same proportion like the second induced economy. Thus, the difference between two economies is small; actually, they give the same set of Walrasian payoff vectors. It should be emphasized that our second induced economy describes the situation where if all agents in coalition S invest the same units λ of their labor, then they earn in total $\lambda v(S)$ units of the output, where $v(S)$ is the worth of coalition S. Recall that this story is used as an interpretation of the worth $v(S)$. Hence, our second induced economy describes this situation explicitly inside a model.

Once we obtain a coalition production economy generating a given TU game, a question of the relationship between the core and Walrasian equilibria is raised. It is known that in an exchange economy with infinitely many agents, the set of core allocations coincides with the set of Walrasian allocations (see Debreu and Scarf [7] and Aumann [1]). The current question is slightly different. We start with a TU game rather than an economy, and we compare the core of the TU game and the set of Walrasian payoff vectors of the induced coalition production economy.

Since inputs are agents' labor in our induced coalition production economies, we can define two kinds of Walrasian equilibrium. In just a Walrasian equilibrium, agents are allowed to work at several firms. In a Walrasian equilibrium without double-jobbing, in contrast, agents have to work at exactly one firm. The concept of Walrasian equilibrium without double-jobbing was considered by Sun et al. [20] to discuss which coalitions are formed in an equilibrium. A coalition is formed if all agents in the coalition work at the firm corresponding to the coalition. Our two induced economies give the same set of Walrasian payoff vectors regardless of whether the double-jobbing is allowed or not, but the coalition structure in a Walrasian equilibrium without double-jobbing can be different between two induced economies. The coalition structure in an equilibrium of our second induced economy is always a partition of the set of agents, but the coalition structure of our first induced economy due to Sun et al. [20] need not be a partition. This point was missed by Sun et al. [20].

Regarding a Walrasian equilibrium, we prove that the set of Walrasian payoff vectors of our representation coincides with the core of the balanced cover of the given TU game. Therefore, by virtue of Bondareva-Shapley theorem (see Bondareva [5] and Shapley [17]), our induced coalition production economies always have Walrasian equilibrium. The reason why the set of Walrasian payoff vectors coincides not with the core itself but with the core of the balanced cover is as follows. If the production set for the grand coalition is not efficient, then agents work at smaller coalitions' firms and they can produce more output. By working at each coalition's firm for the same ratio as the balancing weight, agents can achieve the worth of the balanced cover.

In our equivalence between Walrasian payoff vectors and the core, every core element is an input price vector in the corresponding Walrasian equilibrium. This property is not inherent in our coalition production economy. Shapley and Shubik's [19] equivalence and Qin's [13] equivalence discussed later also have the relationship between core elements and equilibrium price vectors.

Since the core of the balanced cover is equal to the core of the totally balanced cover, our equivalence has the connection with Shapley and Shubik's [19] equivalence and the TU version of Qin's [13] equivalence. Shapley and Shubik [19] proved that given a totally balanced TU game, its core coincides with Walrasian payoff vectors of their induced exchange economy. Qin [13] proved the equivalence between the inner core of a given totally balanced NTU game and Walrasian payoff vectors of Billera's [2] induced production economy. Since the inner core is equal to the core in the framework of TU games, by Qin's theorem, the core of a totally balanced TU game coincides with Walrasian payoff vectors of the TU version of Billera's representation. Together with these known results, the set of Walrasian payoff vectors of our representations, the set of Walrasian payoff vectors of

Shapley and Shubik's [18] representation of the totally balanced cover of the given TU game, and the set of Walrasian payoff vectors of Billera's [2] representation of the totally balanced cover of the given TU game are all the same.

A similar result holds for Walrasian equilibrium without double-jobbing. Sun et al. [20] induced a coalition production economy from the completion of a TU game and proved the equivalence between the core of the completion and the set of Walrasian payoff vectors without double-jobbing. The completion is the TU game in which the worth of the grand coalition is enlarged to hold the superadditivity with respect to the grand coalition. We generalize the result by Sun et al. [20]. The set of Walrasian payoff vectors without double-jobbing of the given TU game coincides with the set of Walrasian payoff vectors without double-jobbing of the completion, and this common set also coincides with the core of the completion. If the completion of a given TU game is balanced, then the no-double-jobbing restriction does not matter, i.e., the set of Walrasian payoff vectors coincides with the set of Walrasian payoff vectors without double-jobbing in our induced coalition production economies.

1.1 Outline of the paper

In Section 2, we give the description of TU games and coalition production economies. Also, we generate a TU game from a coalition production economy.

In Section 3, we give two induced coalition production economies and prove that they actually generate the original TU game (Theorem 1). The second induced economy has a connection with Billera's [2] representation of totally balanced NTU games. The connection will be clarified in Section 4.3. Theorem 2 says that every TU game can be generated by a coalition production economy.

In Section 4, we consider a Walrasian equilibrium for our induced economies. In Section 4.1, we give its precise definition (Definition 1). Theorem 3 says that our two induced economies give the same set of Walrasian payoff vectors. In Section 4.2, we review the concepts related to the core: balancedness, balanced cover, totally balanced cover, and Bondareva-Shapley theorem. In Section 4.3, we summarize Billera's [2] representation of totally balanced TU games (Theorem 4). Theorem 5 is the TU version of Qin's [13] theorem, which says that for a totally balanced TU game, its core coincides with the set of Walrasian payoff vectors of Billera's induced production economy. In Section 4.4, we summarize Shapley and Shubik's [18] representation of totally balanced TU games (Theorem 6). Theorem 7 due to Shapley and Shubik [19, Theorem 1] says that for a totally balanced TU game, its core coincides with the set of Walrasian payoff vectors of Shapley and Shubik's [18] induced exchange economy. In Section 4.5, in Theorem 8, we clarify the relationship between the core and the set of Walrasian payoff vectors of our induced coalition production economy and also clarify the relationship with Qin's equivalence theorem (Theorem 5) and Shapley and Shubik's equivalence theorem (Theorem 7).

In Section 5, we consider a Walrasian equilibrium without double-jobbing for our induced economies. Thus, we bring the indivisibility restriction of no-double-jobbing into the model analyzed in Section 4. In Section 5.1, we give the precise definition (Definition 4) of a Walrasian equilibrium without double-jobbing and discuss the coalition structure. Proposition 1 says that the coalition structure of our second induced economy is always a partition of the set of agents. On the other hand, Example 1 points out that the coalition structure of our first induced economy need not be a partition. Here, a difference between our two representations is clarified. However, our two induced economies give the same set of Walrasian payoff vectors without double-jobbing (Theorem 9). In Section 5.2, we review some concepts of a TU game: cohesive cover, completion, and superadditive cover. In Section 5.3, in Theorem 10, we clarify the relationship between the core and the set of Walrasian payoff vectors without double-jobbing of our induced economy. Theorem 11 says that if the cohesive cover or the completion of a given TU game is balanced, then the set of Walrasian payoff vectors without double-jobbing coincides with the set of Walrasian payoff vectors and, thus, the indivisibility restriction of no-double-jobbing does not matter.

In Section 6, we give some remarks concerning the extension of our results to NTU games.

2 TU games and coalition production economies

We begin with some notation. Let $N = \{1, \dots, n\}$ be a set with n elements and let $\mathbb{R}^N$ be the n-dimensional Euclidean space of vectors x with coordinates x_i indexed by $i \in N$. For a nonempty subset S of N, let $\mathbb{R}^S = \{x \in \mathbb{R}^N \mid x_i = 0 \text{ for every } i \in N \setminus S\}$, let $\mathbb{R}^S_+ = \{x \in \mathbb{R}^S \mid x_i \geq 0 \text{ for every } i \in S\}$, and let $e(S) \in \mathbb{R}^N$ be the characteristic vector of S, i.e., $e(S)_i = 1$ if $i \in S$ and 0 otherwise. We write $e(i)$ instead of $e(\{i\})$. For $x \in \mathbb{R}^N$, $x(S)$ denotes the projection of x on $\mathbb{R}^S$. The symbol 0 denotes the origin in $\mathbb{R}^N$ as well as the real number zero.

Let $\mathcal{N}$ be the family of all nonempty subsets of N, i.e., $\mathcal{N} = \{S \subseteq N \mid S \neq \emptyset\}$. Elements in $\mathcal{N}$ are called *coalitions*. A *transferable utility game* (TU game, for short) with n players is a real-valued function on $\mathcal{N}$. A typical TU game is denoted by $v : \mathcal{N} \to \mathbb{R}$.

A *coalition production economy* with n agents is a collection of the commodity space $\mathbb{R}^L$, where L is the set of commodities, agents' characteristics $(X^i, u^i, \omega^i)_{i \in N}$, and coalitions' production sets $(Y^S)_{S \in \mathcal{N}}$ satisfying the following conditions. For every agent $i \in N$, consumption set $X^i \subseteq \mathbb{R}^L$ is nonempty, closed, convex, and bounded from below; utility function $u^i : X^i \to \mathbb{R}$ is continuous and concave; endowment vector ω^i is in $\mathbb{R}^L$. For every coalition $S \in \mathcal{N}$, its production set $Y^S \subseteq \mathbb{R}^L$ is nonempty, closed, convex, and satisfies $Y^S \cap \mathbb{R}^L_+ = \{0\}$ and $\left(\sum_{i \in S} X^i\right) \cap \left(\left\{\sum_{i \in S} \omega^i\right\} + Y^S\right) \neq \emptyset$. A coalition production economy is denoted by $\mathcal{E} = (\mathbb{R}^L, (X^i, u^i, \omega^i)_{i \in N}, (Y^S)_{S \in \mathcal{N}})$.

An exchange economy is a coalition production economy with $Y^S = \{0\}$ for every $S \in \mathcal{N}$. Shapley and Shubik [18] characterized the class of totally balanced TU games by exchange economies (see Section 4.4). Another special case of a coalition production economy is a production economy where every agent i has his own production set Y^i. In this case, coalition S's production set Y^S is given by $Y^S = \sum_{i \in S} Y^i$. Billera [2] characterized the class of totally balanced non-transferable utility (NTU) games by this type of production economies (see Section 4.3).

For every coalition $S \in \mathcal{N}$, $F_{\mathcal{E}}(S)$ denotes the set of feasible S-allocations for a coalition production economy $\mathcal{E}$, i.e.,

$$F_{\mathcal{E}}(S) = \left\{ (x^i)_{i \in S} \,\middle|\, x^i \in X^i \text{ for every } i \in S \text{ and } \sum_{i \in S} (x^i - \omega^i) \in Y^S \right\}.$$

Note that, for every $S \in \mathcal{N}$, $F_{\mathcal{E}}(S)$ is nonempty, compact, and convex.[1] Any coalition production economy $\mathcal{E}$ naturally generates a TU game $v_{\mathcal{E}} : \mathcal{N} \to \mathbb{R}$ by defining

$$v_{\mathcal{E}}(S) = \max \left\{ \sum_{i \in S} u^i(x^i) \,\middle|\, (x^i)_{i \in S} \in F_{\mathcal{E}}(S) \right\} \quad \text{for every } S \in \mathcal{N}.$$

3 Representation of TU games

We prove that every TU game can be generated by a coalition production economy. The next lemma enables us to restrict our attention to nonnegative TU games. It is an extension of Shapley and Shubik [18, Theorem 2] to the class of TU games generated by coalition production economies.

Lemma 1. *Let $v : \mathcal{N} \to \mathbb{R}$ be a TU game generated by a coalition production economy. Let $\lambda \in \mathbb{R}_+$ and $c \in \mathbb{R}^N$. Then, TU game $\lambda v + c$ defined by*

$$(\lambda v + c)(S) = \lambda v(S) + \sum_{i \in S} c_i \quad \textit{for every } S \in \mathcal{N}$$

is also a TU game generated by a coalition production economy.

Proof. Let $(\mathbb{R}^L, (X^i, u^i, \omega^i)_{i \in N}, (Y^S)_{S \in \mathcal{N}})$ be a coalition production economy generating v. For every agent $i \in N$, define $\bar{u}^i : X^i \to \mathbb{R}$ by $\bar{u}^i(x) = \lambda u^i(x) + c_i$. It can be easily shown that the coalition production economy $(\mathbb{R}^L, (X^i, \bar{u}^i, \omega^i)_{i \in N}, (Y^S)_{S \in \mathcal{N}})$ generates $\lambda v + c$. □

Given a TU game, we induce two coalition production economies. The first one is essentially the same as the induced coalition production economy due to Sun et al. [20].[2]

[1] For the boundedness of $F_{\mathcal{E}}(S)$, see Debreu [6, Theorem (2), p.77].

[2] In the induced coalition production economy due to Sun et al. [20], every agent gets utility from his personalized output commodity. This is not essential and, in our first induced coalition production economy, every agent gets utility from the unique output commodity. Consequently, the number of commodities is reduced from $2n$ to $n + 1$.

They constructed a coalition production economy from the completion of a TU game.[3] As we will see below, their construction can be applied to any nonnegative TU game and its generating TU game is equal to the original TU game. The second induced coalition production economy is a slight modification of the first one and it is also related to the production economy induced from an NTU game due to Billera [2] (see Section 4.3).

Let $v : \mathcal{N} \to \mathbb{R}_+$. We denote by $\mathcal{E}_1(v)$ and $\mathcal{E}_2(v)$ two coalition production economies induced by v. They have differences only in production sets and the given game v affects only production sets. For $h \in \{1, 2\}$, define $\mathcal{E}_h(v) = (\mathbb{R}^N \times \mathbb{R}, (X^i, u^i, \omega^i)_{i \in N}, (Y_h^S(v))_{S \in \mathcal{N}})$ by for every $i \in N$, $X^i = \{0\} \times \mathbb{R}_+ \subseteq \mathbb{R}^N \times \mathbb{R}$, $u^i : X^i \to \mathbb{R}$ is defined by $u^i(0, x) = x$, $\omega^i = (e(i), 0) \in \mathbb{R}^N \times \mathbb{R}$; for every $S \in \mathcal{N}$,

$$Y_1^S(v) = \left\{ (y, z) \in \mathbb{R}^N \times \mathbb{R} \;\middle|\; y \in -\mathbb{R}_+^S,\ 0 \le z \le v(S) \min_{j \in S} |y_j| \right\}$$

and

$$Y_2^S(v) = \left\{ \lambda(-e(S), z) \in \mathbb{R}^N \times \mathbb{R} \;\middle|\; \lambda \in \mathbb{R}_+,\ 0 \le z \le v(S) \right\}.$$

Note that, for every $S \in \mathcal{N}$, both $Y_1^S(v)$ and $Y_2^S(v)$ are closed convex cones with vertex $(0, 0)$ and satisfy

$$Y_h^S(v) \cap \left(\mathbb{R}_+^N \times \mathbb{R}_+ \right) = \{(0, 0)\} \quad \text{for } h = 1, 2.$$

Hence, $\mathcal{E}_1(v)$ and $\mathcal{E}_2(v)$ satisfy all the requirements for a coalition production economy.

In both coalition production economies, every agent has the same consumption set and the same utility function which do not depend on TU games. Agents are distinguished only by their endowment vectors. From the form of production sets, the first n commodities are inputs and the last commodity is an output. Since the ith input is initially owned only by agent i, input i can be interpreted as agent i's labor. For firm S to produce the output, economy $\mathcal{E}_2(v)$ requires that all agents in S invest their labor to firm S in the *same* proportion of their labor. Economy $\mathcal{E}_1(v)$ does not require the same proportion, but agents' labor is complementary. Therefore, a unique way not to waste agents' labor is that all agents in a coalition invest their labor in the same proportion like production sets of

[3] The definition of the completion of a TU game will be given in Section 5.2.

$\mathcal{E}_2(v)$. The relation of production sets between economies $\mathcal{E}_1(v)$ and $\mathcal{E}_2(v)$ is summarized in the next lemma.

Lemma 2. *Let $v : \mathcal{N} \to \mathbb{R}_+$ and let $S \in \mathcal{N}$.*

(i) $Y_2^S(v) \subseteq Y_1^S(v)$.

(ii) *If $(y, z) \in Y_1^S(v)$, then $(y, z) \leq (-\min_{j\in S} |y_j|\, e(S), z) \in Y_2^S(v)$.*

This lemma can be shown straightforwardly. The next theorem says that the TU games generated by economies $\mathcal{E}_1(v)$ and $\mathcal{E}_2(v)$ are equal to the original nonnegative game v.

Theorem 1. *Let $v : \mathcal{N} \to \mathbb{R}_+$. Then, $v = v_{\mathcal{E}_1(v)} = v_{\mathcal{E}_2(v)}$.*

Proof. Let $v : \mathcal{N} \to \mathbb{R}_+$. Since $Y_2^S(v) \subseteq Y_1^S(v)$ for every $S \in \mathcal{N}$ by Lemma 2 (i), we have $F_{\mathcal{E}_2(v)}(S) \subseteq F_{\mathcal{E}_1(v)}(S)$ and, therefore, $v_{\mathcal{E}_2(v)}(S) \leq v_{\mathcal{E}_1(v)}(S)$. Hence, it is enough to prove the inequalities $v_{\mathcal{E}_1(v)} \leq v \leq v_{\mathcal{E}_2(v)}$.[4]

We first prove that $v_{\mathcal{E}_1(v)} \leq v$. Let $S \in \mathcal{N}$. By the definition of $v_{\mathcal{E}_1(v)}$, there exists $(0, x^i)_{i\in S} \in F_{\mathcal{E}_1(v)}(S)$ such that

$$v_{\mathcal{E}_1(v)}(S) = \sum_{i\in S} u^i(0, x^i) = \sum_{i\in S} x^i.$$

Since $(0, x^i)_{i\in S} \in F_{\mathcal{E}_1(v)}(S)$, there exists $(y, z) \in Y_1^S(v)$ such that

$$(y, z) = \sum_{i\in S} \big((0, x^i) - \omega^i\big) = \left(-e(S), \sum_{i\in S} x^i\right).$$

Hence, we have

$$v_{\mathcal{E}_1(v)}(S) = \sum_{i\in S} x^i = z \leq v(S) \min_{j\in S} |y_j| = v(S).$$

We next prove that $v \leq v_{\mathcal{E}_2(v)}$. Let $S \in \mathcal{N}$. For every $i \in S$, define $(0, x^i) \in X^i$ by $x^i = v(S)/|S|$. Since

$$\sum_{i\in S} \big((0, x^i) - \omega^i\big) = \left(-e(S), \sum_{i\in S} x^i\right) = (-e(S), v(S)) \in Y_2^S(v),$$

[4] For TU games $v, w : \mathcal{N} \to \mathbb{R}$, the inequality $v \leq w$ stands for $v(S) \leq w(S)$ for every $S \in \mathcal{N}$.

we have $(0, x^i)_{i \in S} \in F_{\mathcal{E}_2(v)}(S)$. Therefore,

$$v_{\mathcal{E}_2(v)}(S) \geq \sum_{i \in S} u^i(0, x^i) = \sum_{i \in S} x^i = v(S).$$

□

Theorem 2. *Every TU game $v : \mathcal{N} \to \mathbb{R}$ can be generated by a coalition production economy.*

Proof. This follows from Lemma 1 and Theorem 1. □

4 Walrasian equilibrium

4.1 Walrasian equilibrium for our induced coalition production economies

We consider a Walrasian equilibrium for economies $\mathcal{E}_1(v)$ and $\mathcal{E}_2(v)$. In both economies, every production set is a cone. Therefore, any firm earns zero profit in a Walrasian equilibrium. Thus, we do not have to care about the problem of how to distribute the profits by firms to agents. Furthermore, in both economies $\mathcal{E}_1(v)$ and $\mathcal{E}_2(v)$, agents can transfer utilities through production; let $u \in \mathbb{R}^S$ be a utility allocation of agents in coalition S. For every $t \in \mathbb{R}^S$ with $\sum_{i \in S} t_i = 0$, since $(-e(S), \sum_{i \in S} t_i) \in Y_1^S(v) \cap Y_2^S(v)$, a utility allocation $u + t$ can be achieved through production. Therefore, we do not have to consider the presence of a commodity for transferable utility implicitly.[5]

We adopt the following equilibrium concept in our coalition production economies $\mathcal{E}_1(v)$ and $\mathcal{E}_2(v)$.

Definition 1. A tuple $((\hat{q}, \hat{p}), (0, \hat{x}^i)_{i \in N}, (\hat{y}^S, \hat{z}^S)_{S \in \mathcal{N}})$ of a price vector $(\hat{q}, \hat{p}) \in \mathbb{R}^N \times \mathbb{R}$, agents' consumption vectors $(0, \hat{x}^i)_{i \in N}$, and coalitions' production vectors $(\hat{y}^S, \hat{z}^S)_{S \in \mathcal{N}}$ is a *Walrasian equilibrium for $\mathcal{E}_h(v)$* ($h = 1$ or 2) if

[5] This is a difference between our representation and Shapley and Shubik's [18] representation. See Section 4.4.

(1) $(\hat{q}, \hat{p}) \in \mathbb{R}^N_+ \times \mathbb{R}_+$;

(2) for every $S \in \mathcal{N}$, $(\hat{y}^S, \hat{z}^S) \in Y^S_h(v)$ and

$$(\hat{q}, \hat{p}) \cdot (\hat{y}^S, \hat{z}^S) = \max_{(y,z) \in Y^S_h(v)} (\hat{q}, \hat{p}) \cdot (y, z) = 0;$$

(3) for every $i \in N$, $(0, \hat{x}^i)$ maximizes u^i in the budget set $\{(0, x) \in X^i \mid (\hat{q}, \hat{p}) \cdot (0, x) \leq (\hat{q}, \hat{p}) \cdot \omega^i\}$; and

(4) $\sum_{i \in N}(0, \hat{x}^i) = \sum_{i \in N} \omega^i + \sum_{S \in \mathcal{N}}(\hat{y}^S, \hat{z}^S)$.

The vector $(u^i(0, \hat{x}^i))_{i \in N} = (\hat{x}^i)_{i \in N}$ is called a *Walrasian payoff vector of* $\mathcal{E}_h(v)$. The set of Walrasian payoff vectors of $\mathcal{E}_h(v)$ is denoted by $W_h(v)$.

Condition (2) is the profit maximization condition and it says that the maximum profit of any coalition is zero. Thus, in condition (3) of the utility maximization, every agent's income comes only from his endowment vector. Condition (4) represents the social feasibility of resources. Note that, by definition, agents can work at several firms in a Walrasian equilibrium. An equilibrium concept for the situation where every agent is restricted to work at exactly one firm will be given in Section 5.

Since the first n commodities are agents' personal commodities in both economies $\mathcal{E}_1(v)$ and $\mathcal{E}_2(v)$, a Walrasian equilibrium for $\mathcal{E}_1(v)$ and $\mathcal{E}_2(v)$ has the same problem as a Lindahl equilibrium for an economy with public goods; every single agent is not small relative to the size of the market for the corresponding personal commodity and, thus, he may have an incentive not to act as a price taker.[6]

The next lemma summarizes the properties of a Walrasian equilibrium for our coalition production economies $\mathcal{E}_1(v)$ and $\mathcal{E}_2(v)$.

Lemma 3. *Let* $v : \mathcal{N} \to \mathbb{R}_+$ *and let* $((\hat{q}, \hat{p}), (0, \hat{x}^i)_{i \in N}, (\hat{y}^S, \hat{z}^S)_{S \in \mathcal{N}})$ *be a Walrasian equilibrium for* $\mathcal{E}_1(v)$ *or* $\mathcal{E}_2(v)$*. Then, we have the following properties.*

[6]This problem is not inherent in our representation. The "direct market" of Shapley and Shubik [18] has also this problem (see Section 4.4).

(i) $\hat{p} > 0$.

(ii) *for every* $i \in N$, $\hat{q}_i = \hat{p}\hat{x}^i$.

(iii) *for every* $S \in \mathcal{N}$, *if* $\hat{q}(S) = 0$, *then* $v(S) = 0$.

(iv) *for every* $i \in N$, *if* $\hat{x}^i = 0$, *then* $v(\{i\}) = 0$.

Proof. Properties (i) and (ii) follow from the utility maximization condition. We prove property (iii). Suppose, to the contrary, that there exists $S \in \mathcal{N}$ such that $\hat{q}(S) = 0$ and $v(S) > 0$. Then, for every $\lambda > 0$,

$$\lambda(-e(S), v(S)) \in Y_2^S(v) \subseteq Y_1^S(v).$$

Since $(\hat{q}, \hat{p}) \cdot (\lambda(-e(S), v(S))) = \lambda\hat{p}v(S)$ can be arbitrarily large, we have a contradiction. Thus, we have obtained property (iii). Property (iv) follows from properties (ii) and (iii). □

This lemma implies the following.

Remark 1. For $h \in \{1, 2\}$ and for a Walrasian payoff vector $(\hat{x}^i)_{i\in N} \in W_h(v)$, the pair $(((\hat{x}^i)_{i\in N}, 1), (0, \hat{x}^i)_{i\in N})$ of a normalized price vector and a consumption allocation can be embedded in a Walrasian equilibrium for $\mathcal{E}_h(v)$.

Concerning the set of Walrasian payoff vectors, there exists no difference between economies $\mathcal{E}_1(v)$ and $\mathcal{E}_2(v)$.

Theorem 3. *Let* $v : \mathcal{N} \to \mathbb{R}_+$. *Then,* $W_1(v) = W_2(v)$.

This theorem enables us to denote the common set $W_1(v) = W_2(v)$ by $W(v)$. Theorem 3 will be proved with Theorem 8 in Section 4.5.

4.2 The core

A vector $(\gamma_S)_{S\in\mathcal{N}}$ such that $\gamma_S \geq 0$ for every $S \in \mathcal{N}$ and $\sum_{S\in\mathcal{N}} \gamma_S\, e(S) = e(N)$ is called a *balancing vector of weights*. A TU game $v : \mathcal{N} \to \mathbb{R}$ is said to be *balanced* if for every balancing vector $(\gamma_S)_{S\in\mathcal{N}}$ of weights, $\sum_{S\in\mathcal{N}} \gamma_S\, v(S) \leq v(N)$ holds. A TU game $v : \mathcal{N} \to \mathbb{R}$ is said to be *totally balanced* if every subgame of v is balanced, i.e., for every $S \in \mathcal{N}$ and every $(\gamma_T)_{T\in\mathcal{N}}$ with $\gamma_T \geq 0$ for every $T \in \mathcal{N}$ and $\sum_{T\in\mathcal{N}} \gamma_T\, e(T) = e(S)$, $\sum_{T\in\mathcal{N}} \gamma_T\, v(T) \leq v(S)$ holds.

Let $v : \mathcal{N} \to \mathbb{R}$ be a TU game. The *balanced cover* v^b of v is a TU game defined by $v^b(S) = v(S)$ if $S \in \mathcal{N} \setminus \{N\}$ and

$$v^b(N) = \max \left\{ \sum_{S\in\mathcal{N}} \gamma_S\, v(S) \,\middle|\, \gamma_S \geq 0 \text{ for every } S \in \mathcal{N} \text{ and } \sum_{S\in\mathcal{N}} \gamma_S\, e(S) = e(N) \right\}.$$

The *totally balanced cover* v^{tb} of v is a TU game defined by, for every $S \in \mathcal{N}$,

$$v^{tb}(S) = \max \left\{ \sum_{T\in\mathcal{N}} \gamma_T\, v(T) \,\middle|\, \gamma_T \geq 0 \text{ for every } T \in \mathcal{N} \text{ and } \sum_{T\in\mathcal{N}} \gamma_T\, e(T) = e(S) \right\}.$$

Since the set of balancing vectors of weights is a nonempty compact subset of $\mathbb{R}^{\mathcal{N}}$ (see Shapley [17, Lemma 2]), $v^b(N)$ is well-defined. Similarly, v^{tb} is well-defined. By definition, $v^b(N) = v^{tb}(N)$ and $v \leq v^b \leq v^{tb}$. Note that v^b is balanced and v^{tb} is totally balanced.

For a TU game $v : \mathcal{N} \to \mathbb{R}$, denote the *core* by $C(v)$, i.e.,

$$C(v) = \left\{ x \in \mathbb{R}^N \,\middle|\, \sum_{i\in N} x_i \leq v(N) \text{ and } \sum_{i\in S} x_i \geq v(S) \text{ for every } S \in \mathcal{N} \right\}.$$

Bondareva-Shapley theorem (Bondareva [5], Shapley [17]) says a TU game $v : \mathcal{N} \to \mathbb{R}$ is balanced if and only if $C(v) \neq \emptyset$. Thus, v is totally balanced if and only if every subgame of v has the nonempty core.

The core of the balanced cover coincides with the core of the totally balanced cover. This fact follows from Shapley and Shubik [18, Lemma 3], but the direct proof is simpler than their proof.

Lemma 4 (Shapley and Shubik)**.** *Let $v, w : \mathcal{N} \to \mathbb{R}$ be such that $v^b \leq w \leq v^{tb}$. Then, $C(v^b) = C(w) = C(v^{tb})$.*

Proof. Since $v^b(N) = v^{tb}(N)$, we have $v^b(N) = w(N) = v^{tb}(N)$. Since $v^b \leq w \leq v^{tb}$, it is clear that $C(v^{tb}) \subseteq C(w) \subseteq C(v^b)$. The inclusion $C(v^b) \subseteq C(v^{tb})$ can be shown straightforwardly. □

4.3 Billera's (1974) representation of totally balanced TU games

We summarize the TU version of Billera's [2] representation. Billera [2] characterized totally balanced NTU games by production economies where every agent has his own production set. Since the same fact as Lemma 1 holds for such production economies, we can confine to *nonnegative* totally balanced TU games.

Let $v : \mathcal{N} \to \mathbb{R}_+$ be totally balanced. Billera's [2] induced production economy $\mathcal{E}_B(v) = \big(\mathbb{R}^N \times \mathbb{R}, (X^i, u^i, \omega^i)_{i\in N}, Z(v)\big)$ has the same commodity space $\mathbb{R}^N \times \mathbb{R}$ and the same agents' characteristics $(X^i, u^i, \omega^i)_{i\in N}$ as our representations $\mathcal{E}_1(v)$ and $\mathcal{E}_2(v)$. The only difference is in the production set $Z(v)$. The set $Z(v)$ represents the production set to which every agent can access and $Z(v)$ is given by the convex cone spanned by $\bigcup_{S\in\mathcal{N}} (\{-e(S)\} \times [0, v(S)])$.[7] Since $Z(v)$ is a convex cone, every coalition's production set is also $Z(v)$. Furthermore, since every $\{-e(S)\} \times [0, v(S)]$ is convex, we have

$$\begin{aligned} Z(v) &= \left\{ \sum_{S\in\mathcal{N}} \lambda_S \left(-e(S), z^S\right) \in \mathbb{R}^N \times \mathbb{R} \;\middle|\; \lambda_S \geq 0 \text{ and } 0 \leq z^S \leq v(S) \text{ for every } S \in \mathcal{N} \right\} \\ &= \operatorname{co}\left(\bigcup_{S\in\mathcal{N}} Y_2^S(v) \right), \end{aligned}$$

where $\operatorname{co}(A)$ stands for the convex hull of set A. Therefore, by decomposing $Z(v)$, we obtain coalitions' production sets $(Y_2^S(v))_{S\in\mathcal{N}}$ in our second representation. It can be easily shown that $\mathcal{E}_B(v)$ satisfies all the requirements for a coalition production economy.

Billera's induced production economy $\mathcal{E}_B(v)$ can be converted to an exchange economy. Define a new consumption set $\hat{X}^i$ by $\hat{X}^i = X^i - Z(v)$ and a new utility function $\hat{u}^i : \hat{X}^i \to \mathbb{R}$ by

$$\hat{u}^i(z) = \max\left\{ u^i(z+y) \;\middle|\; y \in Z(v) \cap (X^i - \{z\}) \right\}.$$

This conversion is due to Rader [15] (see also Billera and Bixby [3]). Note that $\hat{X}^i$ is nonempty, closed, and convex, but $\hat{X}^i$ is not bounded from below if $v(S) > 0$ for some $S \in \mathcal{N}$. We can show that, for every $S \in \mathcal{N}$, the set of feasible S-allocations for the

[7] Strictly speaking, Billera [2] took the convex hull rather than the convex cone. As Billera [2, p.136] remarked, in his representation, the convex hull can be replaced by the convex cone.

converted exchange economy is still nonempty and compact.[8] Also, we can show that $\hat{u}^i$ is continuous and concave.[9]

The next theorem is the TU version of Billera [2, Theorem 3.3].

Theorem 4 (Billera)**.** *Let $v : \mathcal{N} \to \mathbb{R}_+$ be totally balanced. Then, $v_{\mathcal{E}_B(v)} = v$, where $v_{\mathcal{E}_B(v)}$ is the TU game generated by economy $\mathcal{E}_B(v)$.*

It is known that the converted exchange economy also generates the given totally balanced TU game. A Walrasian equilibrium for $\mathcal{E}_B(v)$ is defined similarly to our induced coalition production economies.

Definition 2. A tuple $((\hat{q}, \hat{p}), (0, \hat{x}^i)_{i\in N}, (\hat{y}, \hat{z}))$ of a price vector $(\hat{q}, \hat{p}) \in \mathbb{R}^N \times \mathbb{R}$, agents' consumption vectors $(0, \hat{x}^i)_{i\in N}$, and production vector $(\hat{y}, \hat{z})$ is a *Walrasian equilibrium for $\mathcal{E}_B(v)$* if it satisfies conditions (1) and (3) of Definition 1 and

(2′) $(\hat{y}, \hat{z}) \in Z(v)$ and $(\hat{q}, \hat{p}) \cdot (\hat{y}, \hat{z}) = \max_{(y,z)\in Z(v)}(\hat{q}, \hat{p}) \cdot (y, z) = 0$;

(4′) $\sum_{i\in N}(0, \hat{x}^i) = \sum_{i\in N} \omega^i + (\hat{y}, \hat{z})$.

The vector $(u^i(0, \hat{x}^i))_{i\in N} = (\hat{x}^i)_{i\in N}$ is called a *Walrasian payoff vector of $\mathcal{E}_B(v)$*. The set of Walrasian payoff vectors of $\mathcal{E}_B(v)$ is denoted by $W_B(v)$.

Since $Z(v)$ is a convex cone, if $(\hat{y}^S, \hat{z}^S)$ satisfies condition (2′) for every $S \in \mathcal{N}$, then $\sum_{S\in\mathcal{N}}(\hat{y}^S, \hat{z}^S)$ also satisfies condition (2′) (see Debreu [6, Theorem (1), p.45]). Thus, for simplicity, we do not specify coalitions' optimal production vectors $(\hat{y}^S, \hat{z}^S)_{S\in\mathcal{N}}$ in the above definition.

The next theorem is the TU version of Qin [13, Theorem 1].

Theorem 5 (Qin)**.** *Let $v : \mathcal{N} \to \mathbb{R}_+$ be totally balanced. Then, $W_B(v) = C(v)$. Furthermore, for every $(\hat{x}^i)_{i\in N} \in C(v)$, $((\hat{x}^i)_{i\in N}, 1)$ can be a Walrasian equilibrium price vector for $\mathcal{E}_B(v)$.*

[8] The boundedness of the set of feasible S-allocations can be shown by an argument similar to Debreu [6, Theorem (2), p.77], although every $\hat{X}^i$ is not bounded from below.

[9] The upper semi-continuity and the concavity can be shown straightforwardly. The lower semi-continuity of $\hat{u}^i$ follows from Rockafellar [16, Theorem 10.2], because $\hat{X}^i$ is a polyhedron, i.e., $\hat{X}^i$ is the convex hull of a finite set of points and directions.

As the proof of this theorem is simple, we will give it later together with the proof of Theorem 8.

4.4 Shapley and Shubik's (1969) representation of totally balanced TU games

We summarize the representation by Shapley and Shubik [18]. In contrast to our representation and Billera's representation, given a totally balanced TU game, we can directly obtain the induced exchange economy regardless of whether it is nonnegative or not.

Let $v : \mathcal{N} \to \mathbb{R}$ be totally balanced. The *direct market* induced by v is an exchange economy $\mathcal{E}_S(v) = \big(\mathbb{R}^N, (\mathbb{R}^N_+, u_v, \omega^i)_{i\in N}\big)$ such that $\omega^i = e(i) \in \mathbb{R}^N$ for every $i \in N$ and $u_v : \mathbb{R}^N_+ \to \mathbb{R}$ is given by

$$u_v(x) = \max\left\{ \sum_{S\in\mathcal{N}} \gamma_S\, v(S) \;\middle|\; \gamma_S \geq 0 \text{ for every } S \in \mathcal{N} \text{ and } \sum_{S\in\mathcal{N}} \gamma_S\, e(S) = x \right\}$$

This common utility function among agents is continuous, concave, and homogeneous of degree one. In contrast to our representation and Billera's representation, the difference among TU games is represented by utility function. Since commodity i is initially owned only by agent i, every commodity is personalized.

Since it is implicitly assumed that there exists one commodity for transferable utility, the complete form of the utility function is

$$U_v : \mathbb{R}^N_+ \times \mathbb{R} \to \mathbb{R}, \quad U_v(x,\xi) = u_v(x) + \xi,$$

(see Shapley and Shubik [19, pp.232-233]). Hence, the total number of commodities required for Shapley and Shubik's [18] representation is the same as the one required for our representation and Billera's representation.

The definition of a Walrasian equilibrium is as follows:[10]

Definition 3. A pair $(p, (z^i)_{i\in N})$ of a price vector $p \in \mathbb{R}^N$ and an allocation $(z^i)_{i\in N}$ is a *Walrasian equilibrium for* $\mathcal{E}_S(v)$ if

[10] For the derivation of condition (i) from the standard utility maximization condition, again see Shapley and Shubik [19, pp.232-233].

(i) for every $i \in N$, z^i maximizes $z \mapsto u_v(z) - p \cdot z$ in $\mathbb{R}^N_+$;

(ii) $\sum_{i \in N} z^i = e(N) = \sum_{i \in N} \omega^i$.

A vector $(u_v(z^i) - p \cdot (z^i - \omega^i))_{i \in N}$ is called a *Walrasian payoff vector of* $\mathcal{E}_S(v)$.[11] The set of Walrasian payoff vectors of $\mathcal{E}_S(v)$ is denoted by $W_S(v)$.

It can be shown that $(z^i)_{i \in N}$ is a Walrasian allocation of $\mathcal{E}_S(v)$, i.e., $(p, (z^i)_{i \in N})$ is a Walrasian equilibrium for $\mathcal{E}_S(v)$ for some $p \in \mathbb{R}^N$ if and only if $(z^i)_{i \in N}$ maximizes $(y^i)_{i \in N} \mapsto \sum_{i \in N} u_v(y^i)$ in the set $F_{\mathcal{E}_S(v)}(N)$ of feasible N-allocations of $\mathcal{E}_S(v)$.

Shapley and Shubik [18, Theorem 5] proved that the induced exchange economy $\mathcal{E}_S(v)$ generates the original TU game v if v is totally balanced.

Theorem 6 (Shapley and Shubik)**.** *Let $v : \mathcal{N} \to \mathbb{R}$ be totally balanced. Then, $v_{\mathcal{E}_S(v)} = v$, where $v_{\mathcal{E}_S(v)}$ is the TU game generated by $\mathcal{E}_S(v)$.*

Shapley and Shubik [19, Theorem 1] proved that, for a totally balanced TU game v, Walrasian payoff vectors fill up its core. In their representation, any core element has the relation with a Walrasian equilibrium price vector.

Theorem 7 (Shapley and Shubik)**.** *Let $v : \mathcal{N} \to \mathbb{R}$ be totally balanced. Then, $W_S(v) = C(v)$. Furthermore, every $(\hat{x}^i)_{i \in N} \in C(v)$ can be a Walrasian equilibrium price vector for $\mathcal{E}_S(v)$.*

4.5 Equivalence between Walrasian payoff vectors and the core

We are now ready to state the main result of Section 4.

Theorem 8. *Let $v : \mathcal{N} \to \mathbb{R}_+$. Then,*

$$W(v) = W(v^b) = C(v^b) = C(v^{tb}) = W(v^{tb}) = W_B(v^{tb}) = W_S(v^{tb}).$$

Furthermore, by Bondareva-Shapley theorem, the above common set is nonempty.

[11] The vector $(u_v(z^i) - p \cdot (z^i - \omega^i))_{i \in N} = (U_v(z^i, -p \cdot (z^i - \omega^i)))_{i \in N}$ is an equilibrium utility allocation when every agent initially has zero amounts of the commodity for transferable utility.

Even if $v(N) < v^b(N)$, agents can produce $v^b(N)$ units of the output by working at each coalition's firm for the same ratio as the balancing vector of weights generating $v^b(N)$. Thus, although $Y_h^N(v) \subsetneq Y_h^N(v^b)$, the first equality $W(v) = W(v^b)$ holds. By applying the same argument to smaller coalitions, we obtain the equality $W(v) = W(v^{tb})$. Theorem 8 says this common set $W(v) = W(v^b) = W(v^{tb})$ coincides with the core $C(v^b)$ of the balanced cover. The other equalities $C(v^b) = C(v^{tb}) = W_B(v^{tb}) = W_S(v^{tb})$ are known results we mentioned earlier (see Lemma 4 and Theorems 5 and 7).

We prove Theorems 3, 5, and 8 simultaneously.

Proof. Let $v : \mathcal{N} \to \mathbb{R}_+$. It is enough to prove the following inclusions:

$$W_2(v) \subseteq W_1(v) \subseteq W_1(v^b) \subseteq C(v^b) \subseteq C(v^{tb}) \subseteq W_B(v^{tb}) \subseteq W_2(v).$$

Indeed, if we obtain these inclusions, then $W_1(v) = W_2(v)$, which means that we have proved Theorem 3. In addition, we have $W(v) = W(v^b) = C(v^b) = C(v^{tb}) = W_B(v^{tb})$. This implies that $W(v^{tb}) = C(v^{tb})$, because $(v^{tb})^b = v^{tb}$. The remaining equality $C(v^{tb}) = W_S(v^{tb})$ follows from Theorem 7.

Claim 1. $W_2(v) \subseteq W_1(v)$.

Proof. Let $(\hat{x}^i)_{i\in N} \in W_2(v)$ and let $((\hat{q},\hat{p}), (0,\hat{x}^i)_{i\in N}, (\hat{y}^S,\hat{z}^S)_{S\in\mathcal{N}})$ be a corresponding Walrasian equilibrium for $\mathcal{E}_2(v)$. By Lemma 2 (i), it is enough to prove that for every $S \in \mathcal{N}$ and every $(y,z) \in Y_1^S(v)$, $(\hat{q},\hat{p})\cdot(y,z) \leq 0$ holds. Let $S \in \mathcal{N}$ and $(y,z) \in Y_1^S(v)$. Since $(y,z) \leq (-\min_{j\in S}|y_j|\, e(S), z) \in Y_2^S(v)$ by Lemma 2 (ii), from the profit maximization condition, we have

$$0 \geq (\hat{q},\hat{p})\cdot\left(-\min_{j\in S}|y_j|\, e(S), z\right) \geq (\hat{q},\hat{p})\cdot(y,z).$$

This completes the proof of Claim 1. □

Claim 2. $W_1(v) \subseteq W_1(v^b)$.

Proof. Let $(\hat{x}^i)_{i\in N} \in W_1(v)$ and let $((\hat{q},\hat{p}), (0,\hat{x}^i)_{i\in N}, (\hat{y}^S,\hat{z}^S)_{S\in\mathcal{N}})$ be a corresponding Walrasian equilibrium for $\mathcal{E}_1(v)$. Since $Y_1^S(v) = Y_1^S(v^b)$ for every $S \in \mathcal{N}\setminus\{N\}$ and

$Y_1^N(v) \subseteq Y_1^N(v^b)$, it is enough to prove that for every $(y,z) \in Y_1^N(v^b)$, $(\hat{q},\hat{p}) \cdot (y,z) \leq 0$ holds. Let $(y,z) \in Y_1^N(v^b)$. Then, $0 \leq z \leq v^b(N) \min_{j\in N} |y_j|$. By the definition of v^b, there exists a balancing vector $(\gamma_S)_{S\in\mathcal{N}}$ of weights such that $v^b(N) = \sum_{S\in\mathcal{N}} \gamma_S \, v(S)$. Then, $0 \leq z \leq \sum_{S\in\mathcal{N}} \gamma_S \, v(S) \min_{j\in N} |y_j|$. Hence, there exists $(z^S)_{S\in\mathcal{N}}$ such that

$$0 \leq z^S \leq \gamma_S \, v(S) \min_{j\in N} |y_j| \quad \text{for every } S \in \mathcal{N} \text{ and } \quad \sum_{S\in\mathcal{N}} z^S = z.$$

Since $(-\gamma_S \min_{j\in N} |y_j| \, e(S), z^S) \in Y_2^S(v) \subseteq Y_1^S(v)$ for every $S \in \mathcal{N}$, we have

$$0 \geq (\hat{q},\hat{p}) \cdot \left(-\gamma_S \min_{j\in N} |y_j| \, e(S), z^S \right) \quad \text{for every } S \in \mathcal{N}.$$

Therefore,

$$0 \geq (\hat{q},\hat{p}) \cdot \left(-\min_{j\in N} |y_j| \sum_{S\in\mathcal{N}} \gamma_S \, e(S), \sum_{S\in\mathcal{N}} z^S \right) = (\hat{q},\hat{p}) \cdot \left(-\min_{j\in N} |y_j| \, e(N), z \right) \geq (\hat{q},\hat{p}) \cdot (y,z).$$

This completes the proof of Claim 2. □

Claim 3. $W_1(v^b) \subseteq C(v^b)$.

Proof. Let $(\hat{x}^i)_{i\in N} \in W_1(v^b)$ and let $((\hat{q},\hat{p}),(0,\hat{x}^i)_{i\in N},(\hat{y}^S,\hat{z}^S)_{S\in\mathcal{N}})$ be a corresponding Walrasian equilibrium for $\mathcal{E}_1(v^b)$. By Lemma 2, for every $S \in \mathcal{N}$,

$$(\hat{y}^S,\hat{z}^S) \leq \left(-\min_{j\in S} |\hat{y}^S| \, e(S), \hat{z}^S \right) \in Y_2^S(v^b) \subseteq Y_1^S(v^b).$$

Thus,

$$e(N) = -\sum_{S\in\mathcal{N}} \hat{y}^S \geq \sum_{S\in\mathcal{N}} \min_{j\in S} |\hat{y}_j^S| \, e(S).$$

If the above inequality is strict, $(\min_{j\in S} |\hat{y}^S|)_{S\in\mathcal{N}}$ is not a balancing vector of weights. Since $v^b \geq 0$, however, we have

$$\sum_{S\in\mathcal{N}} \min_{j\in S} |\hat{y}^S| \, v^b(S) \leq v^b(N).$$

Since $\hat{z}^S \leq \min_{j\in S} |\hat{y}_j^S| \, v^b(S)$ for every $S \in \mathcal{N}$, we have

$$\sum_{i\in N} \hat{x}^i = \sum_{S\in\mathcal{N}} \hat{z}^S \leq \sum_{S\in\mathcal{N}} \min_{j\in S} |\hat{y}_j^S| \, v^b(S) \leq v^b(N).$$

It remains to prove that for every $S \in \mathcal{N}$, $\sum_{i\in S} \hat{x}^i \geq v^b(S)$ holds. Suppose, to the contrary, that there exists $S \in \mathcal{N}$ with $\sum_{i\in S} \hat{x}^i < v^b(S)$. Since $v^b(S) = v_{\mathcal{E}_1(v^b)}(S)$ by Theorem 1, there exists $(0, x^i)_{i\in S} \in F_{\mathcal{E}_1(v^b)}(S)$ with $\sum_{i\in S} x^i = v^b(S)$. From $(0, x^i)_{i\in S} \in F_{\mathcal{E}_1(v^b)}(S)$, there exists $(y, z) \in Y_1^S(v^b)$ such that $(y, z) = \sum_{i\in S}((0, x^i) - \omega^i) = (-e(S), \sum_{i\in S} x^i)$. Since $\hat{p} > 0$ and $\sum_{i\in S} \hat{x}^i < v^b(S) = \sum_{i\in S} x^i = z$, we have

$$0 \geq (\hat{q}, \hat{p}) \cdot (y, z) > (\hat{q}, \hat{p}) \cdot \left(-e(S), \sum_{i\in S} \hat{x}^i\right) = -\sum_{i\in S} \hat{q}_i + \sum_{i\in S} \hat{p}\hat{x}^i = \sum_{i\in S} \left(\hat{p}\hat{x}^i - \hat{q}_i\right) = 0,$$

a contradiction. The last equality follows from Lemma 3 (ii). Hence, for every $S \in \mathcal{N}$, $\sum_{i\in S} \hat{x}^i \geq v^b(S)$. This completes the proof of Claim 3. □

Claim 4. $C(v^b) \subseteq C(v^{tb})$.

This follows from Lemma 4.

Claim 5. $C(v^{tb}) \subseteq W_B(v^{tb})$.

Proof. Let $(\hat{x}^i)_{i\in N} \in C(v^{tb})$. Define $\hat{q}_i = \hat{x}^i$ for every $i \in N$, $\hat{p} = 1$, and $(\hat{y}, \hat{z}) = (-e(N), \sum_{i\in N} \hat{x}^i)$. Since $(\hat{x}^i)_{i\in N} \in C(v^{tb})$, we have $\sum_{i\in N} \hat{x}^i = v^{tb}(N)$. Thus, $(\hat{y}, \hat{z}) \in Z(v^{tb})$. It is clear that conditions (1), (3), and (4′) in Definition 2 are satisfied. Also, we have

$$(\hat{q}, \hat{p}) \cdot (\hat{y}, \hat{z}) = (\hat{q}, \hat{p}) \cdot \left(-e(N), \sum_{i\in N} \hat{x}^i\right) = -\sum_{i\in N} \hat{x}^i + \sum_{i\in N} \hat{x}^i = 0.$$

It remains to prove that for every $(y, z) \in Z(v^{tb})$, $(\hat{q}, \hat{p}) \cdot (y, z) \leq 0$ holds. Let $(y, z) \in Z(v^{tb})$. Then, there exist vectors $(\lambda_S)_{S\in\mathcal{N}}$ and $(z^S)_{S\in\mathcal{N}}$ such that $\lambda_S \geq 0$ and $0 \leq z^S \leq v^{tb}(S)$ for every $S \in \mathcal{N}$, and $(y, z) = \sum_{S\in\mathcal{N}} \lambda_S(-e(S), z^S)$. For every $S \in \mathcal{N}$, we have

$$(\hat{q}, \hat{p}) \cdot (-e(S), z^S) = -\sum_{i\in S} \hat{x}^i + z^S \leq -\sum_{i\in S} \hat{x}^i + v^{tb}(S) \leq 0.$$

The last inequality follows from $(\hat{x}^i)_{i\in N} \in C(v^{tb})$. Then,

$$(\hat{q}, \hat{p}) \cdot (y, z) = \sum_{S\in\mathcal{N}} \lambda_S(\hat{q}, \hat{p}) \cdot (-e(S), z^S) \leq 0.$$

Therefore, $((\hat{q}, \hat{p}), (0, \hat{x}^i)_{i\in N}, (\hat{y}, \hat{z}))$ is a Walrasian equilibrium for $\mathcal{E}_B(v^{tb})$. Thus, $(\hat{x}^i)_{i\in N} \in W_B(v^{tb})$. This completes the proof of Claim 5 □

Claim 6. $W_B(v^{tb}) \subseteq W_2(v)$.

Proof. Let $(\hat{x}^i)_{i\in N} \in W_B(v^{tb})$ and let $((\hat{q},\hat{p}),(0,\hat{x}^i)_{i\in N},(\hat{y},\hat{z}))$ be a corresponding Walrasian equilibrium for $\mathcal{E}_B(v^{tb})$. Since

$$Z(v^{tb}) \ni (\hat{y},\hat{z}) = \sum_{i\in N}\big((0,\hat{x}^i)-\omega^i\big) = \left(-e(N), \sum_{i\in N}\hat{x}^i\right),$$

there exist vectors $(\lambda_S)_{S\in\mathcal{N}}$ and $(z^S)_{S\in\mathcal{N}}$ such that $\lambda_S \geq 0$ and $0 \leq z^S \leq v^{tb}(S)$ for every $S \in \mathcal{N}$, and $(-e(N), \sum_{i\in N}\hat{x}^i) = \sum_{S\in\mathcal{N}}\lambda_S(-e(S), z^S)$. By the definition of v^{tb}, for every $S \in \mathcal{N}$, there exists $(\gamma_{S,T})_{T\in\mathcal{N}}$ such that $\gamma_{S,T} \geq 0$ for every $T \in \mathcal{N}$, $\sum_{T\in\mathcal{N}}\gamma_{S,T}\, e(T) = e(S)$, and $v^{tb}(S) = \sum_{T\in\mathcal{N}}\gamma_{S,T}\, v(T)$. For every $S \in \mathcal{N}$, since $0 \leq z^S \leq v^{tb}(S) = \sum_{T\in\mathcal{N}}\gamma_{S,T}\, v(T)$, there exists $(z^{S,T})_{T\in\mathcal{N}}$ such that $0 \leq z^{S,T} \leq v(T)$ for every $T \in \mathcal{N}$ and $z^S = \sum_{T\in\mathcal{N}}\gamma_{S,T}\, z^{S,T}$. Then, for every $S \in \mathcal{N}$,

$$(-e(S), z^S) = \left(-\sum_{T\in\mathcal{N}}\gamma_{S,T}\, e(T), \sum_{T\in\mathcal{N}}\gamma_{S,T}\, z^{S,T}\right) = \sum_{T\in\mathcal{N}}\gamma_{S,T}\,(-e(T), z^{S,T}).$$

Note that for every $S, T \in \mathcal{N}$, $(-e(T), z^{S,T}) \in Y_2^T(v)$. Then,

$$\begin{aligned}(\hat{y},\hat{z}) &= \sum_{S\in\mathcal{N}}\lambda_S(-e(S), z^S) = \sum_{S\in\mathcal{N}}\lambda_S\sum_{T\in\mathcal{N}}\gamma_{S,T}(-e(T), z^{S,T}) \\ &= \sum_{T\in\mathcal{N}}\left(\sum_{S\in\mathcal{N}}\lambda_S\,\gamma_{S,T}(-e(T), z^{S,T})\right).\end{aligned}$$

For every $T \in \mathcal{N}$, define

$$(\hat{y}^T, \hat{z}^T) = \sum_{S\in\mathcal{N}}\lambda_S\,\gamma_{S,T}(-e(T), z^{S,T}).$$

Then, $(\hat{y},\hat{z}) = \sum_{T\in\mathcal{N}}(\hat{y}^T,\hat{z}^T)$ and $(\hat{y}^T,\hat{z}^T) \in Y_2^T(v)$, because $Y_2^T(v)$ is a convex cone. Since $Y_2^T(v) \subseteq Z(v) \subseteq Z(v^{tb})$, we have $(\hat{q},\hat{p})\cdot(\hat{y}^T,\hat{z}^T) \leq 0$ for every $T \in \mathcal{N}$. Since

$$0 = (\hat{q},\hat{p})\cdot(\hat{y},\hat{z}) = \sum_{T\in\mathcal{N}}(\hat{q},\hat{p})\cdot(\hat{y}^T,\hat{z}^T),$$

we have $(\hat{q},\hat{p})\cdot(\hat{y}^T,\hat{z}^T) = 0$ for every $T \in \mathcal{N}$.

Let $T \in \mathcal{N}$ and $(y,z) \in Y_2^T(v)$. Again, since $Y_2^T(v) \subseteq Z(v) \subseteq Z(v^{tb})$, we have $(\hat{q},\hat{p})\cdot(y,z) \leq 0$.

Hence, $((\hat{q},\hat{p}),(0,\hat{x}^i)_{i\in N},(\hat{y}^T,\hat{z}^T)_{T\in\mathcal{N}})$ is a Walrasian equilibrium for $\mathcal{E}_2(v)$. Thus, $(\hat{x}^i)_{i\in N}\in W_2(v)$. This completes the proof of Claim 6. □

From Claims 1-6, we have proved Theorems 3, 5, and 8. □

From the proof above, we have the following.

Remark 2. A payoff vector $(\hat{x}^i)_{i\in N}\in C(v^b)=C(v^{tb})$ can be a Walrasian equilibrium input price vector for economies $\mathcal{E}_1(v)$, $\mathcal{E}_2(v)$, and $\mathcal{E}_B(v)$.

5 Walrasian equilibrium without double-jobbing

In the previous section, agents are allowed to work at several firms. Here, we bring the indivisibility restriction of no-double-jobbing into the model.

5.1 Walrasian equilibrium without double-jobbing and the coalition structure

We first give the precise definition of a Walrasian equilibrium without double-jobbing for our induced coalition production economies.

Definition 4. A tuple $((\hat{q},\hat{p}),(0,\hat{x}^i)_{i\in N},(\hat{y}^S,\hat{z}^S)_{S\in\mathcal{N}})$ is a *Walrasian equilibrium without double-jobbing for* $\mathcal{E}_h(v)$ ($h=1$ or 2) if it is a Walrasian equilibrium for $\mathcal{E}_h(v)$ and it also satisfies

(5) for every $i\in N$, there exists a unique $S\in\mathcal{N}$ such that $i\in S$, $\hat{y}^S_i=-1$, and $\hat{y}^T_i=0$ for every $T\neq S$.

The vector $(u^i(0,\hat{x}^i))_{i\in N}=(\hat{x}^i)_{i\in N}$ is called a *Walrasian payoff vector without double-jobbing of* $\mathcal{E}_h(v)$. The set of Walrasian payoff vectors without double-jobbing of $\mathcal{E}_h(v)$ is denoted by $W^*_h(v)$.

In this definition, condition (5) can be replaced by

(5′) for every $S \in \mathcal{N}$, $\hat{y}^S \in \{-1, 0\}^N$.

It is clear that condition (5) implies condition (5′). Conversely, condition (5′) together with conditions (2) and (4) in Definition 1 implies condition (5). Note that, by definition, $W_h^*(v) \subseteq W_h(v)$ for $h = 1, 2$.

This notion of Walrasian equilibrium without double-jobbing was considered by Sun et al. [20] in order to discuss the endogenous coalition formation of a core element. If a TU game $v : \mathcal{N} \to \mathbb{R}_+$ is balanced, by Theorem 8, its core coincides with the set of Walrasian payoff vectors. In a Walrasian equilibrium, however, agents may invest their labor *fractionally* in any coalition's production plan. Thus, we cannot discuss which coalitions are formed in an equilibrium. The no-double-jobbing condition (5) requires that any agent can contribute only one firm. If all agents in coalition S work at firm S, we can interpreted that coalition S is formed endogenously in an equilibrium. Consequently, we can discuss which coalitions are formed in a core element by considering the corresponding Walrasian equilibrium without double-jobbing.

Let $((\hat{q}, \hat{p}), (0, \hat{x}^i)_{i \in N}, (\hat{y}^S, \hat{z}^S)_{S \in \mathcal{N}})$ be a Walrasian equilibrium without double-jobbing for $\mathcal{E}_h(v)$. The *coalition structure in the equilibrium* is the set of coalitions actually formed in the equilibrium. Thus, $\{S \in \mathcal{N} \mid \hat{y}_i^S = -1 \text{ for every } i \in S\}$ is the coalition structure. By no-double-jobbing condition (5), any two distinct coalitions in the coalition structure are disjoint.

There exists a difference in the coalition structure between our induced economies $\mathcal{E}_1(v)$ and $\mathcal{E}_2(v)$. The coalition structure of $\mathcal{E}_2(v)$ is always a partition of N, but the coalition structure of $\mathcal{E}_1(v)$ need not be a partition.

Proposition 1. *The coalition structure in a Walrasian equilibrium without double-jobbing for $\mathcal{E}_2(v)$ is a partition of N.*

Proof. Let $((\hat{q}, \hat{p}), (0, \hat{x}^i)_{i \in N}, (\hat{y}^S, \hat{z}^S)_{S \in \mathcal{N}})$ be a Walrasian equilibrium without double-jobbing for $\mathcal{E}_2(v)$ and let $\mathcal{B} = \{S \in \mathcal{N} \mid \hat{y}_i^S = -1 \text{ for every } i \in S\}$ be the associated coalition structure. Since any elements in $\mathcal{B}$ are disjoint by no-double-jobbing condition, it is enough to prove that $N \subseteq \bigcup_{S \in \mathcal{B}} S$. Let $i \in N$. Then, by no-double-jobbing condition, there exists a unique $S \in \mathcal{N}$ such that $i \in S$, $\hat{y}_i^S = -1$, and $\hat{y}_i^T = 0$ for every $T \neq S$. By the definition of $Y_2^S(v)$, we have $\hat{y}_j^S = -1$ for every $j \in S$ and, therefore, $S \in \mathcal{B}$. Hence, $\mathcal{B}$ is a partition of N. □

The next proposition says that in a Walrasian equilibrium without double-jobbing for $\mathcal{E}_1(v)$, if coalition S contains an agent who works at firm S and whose Walrasian payoff is strictly positive, then the coalition S is actually formed.

Proposition 2. *Let $((\hat{q}, \hat{p}), (0, \hat{x}^i)_{i\in N}, (\hat{y}^S, \hat{z}^S)_{S\in\mathcal{N}})$ be a Walrasian equilibrium without double-jobbing for $\mathcal{E}_1(v)$. If $\hat{x}^i > 0$ and $\hat{y}_i^S = -1$ for some $i \in S$, then coalition S is formed, i.e., $\hat{y}_j^S = -1$ for every $j \in S$. Therefore,*

$$N \setminus \{i \in N \mid \hat{x}^i = 0\} \subseteq \bigcup_{S\in\mathcal{B}} S,$$

where $\mathcal{B} = \{S \in \mathcal{N} \mid \hat{y}_i^S = -1 \text{ for every } i \in S\}$ is the coalition structure in the equilibrium.

Proof. Suppose, to the contrary, that there exists a coalition $S \in \mathcal{N}$ and agents $i, j \in S$ such that $\hat{x}^i > 0$, $\hat{y}_i^S = -1$, and $\hat{y}_j^S = 0$. From $(\hat{y}^S, \hat{z}^S) \in Y_1^S(v)$, it follows that $0 \leq \hat{z}^S \leq v(S) \min_{k\in S} |\hat{y}_k^S| = 0$. Then, from the profit maximization condition, we have

$$0 = (\hat{q}, \hat{p}) \cdot (\hat{y}^S, \hat{z}^S) = \hat{q} \cdot \hat{y}^S + \hat{p}\hat{z}^S = \hat{q} \cdot \hat{y}^S.$$

Since $\hat{q} \geq 0$ and $\hat{y}^S \leq 0$, we have $\hat{q}_k\hat{y}_k^S = 0$ for every $k \in S$. Thus, by Lemma 3 (ii) and (i), we have

$$0 = \hat{q}_i\hat{y}_i^S = -\hat{q}_i = -\hat{p}\hat{x}^i < 0,$$

a contradiction. □

This proposition implies that if a Walrasian payoff vector without double-jobbing of $\mathcal{E}_1(v)$ is strictly positive, then the corresponding coalition structure is a partition of N. If agent i's Walrasian payoff without double-jobbing is zero, however, it is indifferent for agent i between working at firm $\{i\}$ and working at an *inactive* firm S with $\hat{y}_j^S = 0$ for some $j \in S \setminus \{i\}$. When agent i works at firm $\{i\}$, the maximum output from the production is zero, because $v(\{i\}) = 0$ by Lemma 3 (iv). When agent i works at firm S with $\hat{y}_j^S = 0$ for some $j \in S \setminus \{i\}$, the maximum output is also zero, because $v(S) \min_{k \in S} |\hat{y}_k^S| = 0$. In the latter case, the associated coalition structure is not a partition.[12] The following example illustrates this point.

Example 1. Let $N = \{1, 2\}$ and let v be such that $v(\{1\}) = 1$, $v(\{2\}) = 0$, and $v(\{1, 2\}) = 1$. Let $\hat{q} = (1, 0)$, $\hat{p} = 1$, $\hat{x}^1 = 1$, and $\hat{x}^2 = 0$.[13] Then, there exist three possibilities of production vectors with which $((\hat{q}, \hat{p}), (0, \hat{x}^i)_{i \in N})$ is a Walrasian equilibrium without double-jobbing for $\mathcal{E}_1(v)$.

	case 1	case 2	case 3
$(\hat{y}_1^{\{1\}}, \hat{y}_2^{\{1\}}, \hat{z}^{\{1\}})$	(-1,0,1)	(0,0,0)	(-1,0,1)
$(\hat{y}_1^{\{2\}}, \hat{y}_2^{\{2\}}, \hat{z}^{\{2\}})$	(0,-1,0)	(0,0,0)	(0,0,0)
$(\hat{y}_1^{\{1,2\}}, \hat{y}_2^{\{1,2\}}, \hat{z}^{\{1,2\}})$	(0,0,0)	(-1,-1,1)	(0,-1,0)

In cases 1 and 2, the associated coalition structure is a partition of $\{1, 2\}$. In case 3, however, the coalition structure is $\{\{1\}\}$, which is not a partition of $\{1, 2\}$.

Notice that the coalition structure in a Walrasian equilibrium without double-jobbing for $\mathcal{E}_1(v)$ need not be a partition as in case 3, but there exists a coalition structure of the form of a partition as in cases 1 and 2. This fact holds in general.[14]

The next theorem says that the sets $W_1^*(v)$ and $W_2^*(v)$ of Walrasian payoff vectors without double-jobbing for $\mathcal{E}_1(v)$ and $\mathcal{E}_2(v)$ are the same.

Theorem 9. *Let $v : \mathcal{N} \to \mathbb{R}_+$. Then, $W_1^*(v) = W_2^*(v)$.*

This theorem enables us to denote the common set $W_1^*(v) = W_2^*(v)$ by $W^*(v)$. Theorem 9 will be proved with Theorem 10 in Section 5.3.

[12] This point was missed by Sun et al. [20].

[13] Note that $(\hat{x}^1, \hat{x}^2) = (1, 0)$ is a core element of v. By an argument similar to the proof of Claim 4 of Theorem 10, the pair $(((\hat{x}^1, \hat{x}^2), 1), ((0, \hat{x}^1), (0, \hat{x}^2)))$ of a price vector and a consumption allocation can be embedded in a Walrasian equilibrium without double-jobbing. Recall Remark 2.

[14] By an argument similar to the proof of Theorem 10, we can prove that if $((\hat{q}, \hat{p}), (0, \hat{x}^i)_{i \in N})$ is the pair of a price vector and a consumption allocation of a Walrasian equilibrium without double-jobbing for $\mathcal{E}_1(v)$, then there exist coalitions' production vectors $(\hat{y}^S, \hat{z}^S)_{S \in \mathcal{N}}$ such that $((\hat{q}, \hat{p}), (0, \hat{x}^i)_{i \in N}, (\hat{y}^S, \hat{z}^S)_{S \in \mathcal{N}})$ is a Walrasian equilibrium without double-jobbing for $\mathcal{E}_2(v)$. Then, by virtue of Proposition 1, the resulting coalition structure is a partition of N.

5.2 Cohesive cover, completion, and superadditive cover of TU games

Let $v : \mathcal{N} \to \mathbb{R}$. The *cohesive cover* or the *completion* v^c of v is a TU game defined by $v^c(S) = v(S)$ if $S \in \mathcal{N} \setminus \{N\}$ and

$$v^c(N) = \max \left\{ \sum_{j=1}^{k} v(S_j) \,\middle|\, \{S_1, \ldots, S_k\} \text{ is a partition of } N \right\}.$$

The term "cohesive" is due to Osborne and Rubinstein [12] and the term "completion" is due to Sun et al. [20].

In the standard definition of the core, the feasibility with respect to the grand coalition N is required (see Section 4.2). If for a TU game $v : \mathcal{N} \to \mathbb{R}$ and $x \in \mathbb{R}^N$, $v(N) < \sum_{i \in N} x_i \le \sum_{j=1}^{k} v(S_j)$ holds for some partition $\{S_1, \ldots, S_k\}$ of N, payoff vector x is not feasible with respect to the grand coalition but it can be achieved, provided that agents form the coalitions $S_1, \ldots, S_k$. In the core of the completion v^c, payoff vectors achievable by disjoint coalitions of N are regarded as *possible* payoff vectors. Accordingly, the feasibility with respect to the grand coalition N is distinguished from the possibility of payoff vectors.[15]

The *superadditive cover* v^{sa} of v is a TU game defined by, for every $S \in \mathcal{N}$,

$$v^{sa}(S) = \max \left\{ \sum_{j=1}^{k} v(T_j) \,\middle|\, \{T_1, \ldots, T_k\} \text{ is a partition of } S \right\}.$$

By definition, $v^c(N) = v^{sa}(N)$ and $v \le v^c \le v^{sa}$. The core of the completion coincides with the core of the superadditive cover. This fact is due to Guesnerie and Oddou [9, Lemma 1].

Lemma 5 (Guesnerie and Oddou). *Let $v, w : \mathcal{N} \to \mathbb{R}$ be such that $v^c \le w \le v^{sa}$. Then, $C(v^c) = C(w) = C(v^{sa})$.*

This can be shown straightforwardly.

[15] This kind of distinction was made in the early period of the research of the core by Gillies [8]. Boehm [4], Guesnerie and Oddou [9], and Sun et al. [20] emphasize this distinction.

5.3 Equivalence between Walrasian payoff vectors without double-jobbing and the core

We are now ready to state the main result of Section 5. The next theorem is an extension of Sun et al. [20, Theorem 3.2] that proved the equality $W_1^*(v^c) = C(v^c)$ for every $v : \mathcal{N} \to \mathbb{R}_+$.[16]

Theorem 10. *Let $v : \mathcal{N} \to \mathbb{R}_+$. Then, $W^*(v) = W^*(v^c) = C(v^c) = C(v^{sa}) = W^*(v^{sa})$.*

Concerning the equalities $W^*(v) = W^*(v^c) = W^*(v^{sa})$, we can give the intuition similar to the equalities $W(v) = W(v^b) = W(v^{tb})$ in Theorem 8. In the case where $v(N) < v^c(N)$, agents can earn $v^c(N)$ units of the output by working at disjoint smaller coalitions' firms. Thus, although $Y_h^N(v) \subsetneq Y_h^N(v^c)$, the equality $W^*(v) = W^*(v^c)$ holds. By applying the same argument to smaller coalitions, we have the equality $W^*(v) = W^*(v^{sa})$.

We simultaneously prove Theorems 9 and 10. The proof is similar to the proof of Theorems 3 and 8. Its idea originates from the proof of Theorem 3.2 of Sun et al. [20], but we need a modification because the coalition structure in an equilibrium for $\mathcal{E}_1(v)$ need not be a partition.

Proof. Let $v : \mathcal{N} \to \mathbb{R}_+$. It is enough to prove the following inclusions:

$$W_2^*(v) \subseteq W_1^*(v) \subseteq W_1^*(v^c) \subseteq C(v^c) \subseteq W_2^*(v).$$

Indeed, if we obtain these inclusions, we have $W^*(v) = W^*(v^c) = C(v^c)$ for every $v : \mathcal{N} \to \mathbb{R}_+$. Since $(v^{sa})^c = v^{sa}$, these equalities imply $W^*(v^{sa}) = C(v^{sa})$. The remaining equality $C(v^c) = C(v^{sa})$ follows from Lemma 5.

Claim 1. $W_2^*(v) \subseteq W_1^*(v)$.

Proof. Let $(\hat{x}^i)_{i\in N} \in W_2^*(v)$ and let $((\hat{q}, \hat{p}), (0, \hat{x}^i)_{i\in N}, (\hat{y}^S, \hat{z}^S)_{S\in\mathcal{N}})$ be a corresponding Walrasian equilibrium without double-jobbing for $\mathcal{E}_2(v)$. Since $Y_2^S(v) \subseteq Y_1^S(v)$ for every

[16] From the proof of Theorem 3.2 of Sun et al. [20], as far as the equality $W_1^*(v^c) = C(v^c)$ is concerned, their assumption of the balancedness of v^c is dispensable.

$S \in \mathcal{N}$ by Lemma 2 (i), it suffices to prove that for every $S \in \mathcal{N}$ and every $(y,z) \in Y_1^S(v)$, $(\hat{q},\hat{p}) \cdot (y,z) \leq 0$. Let $S \in \mathcal{N}$ and $(y,z) \in Y_1^S(v)$. Then, by Lemma 2 (ii),

$$(y,z) \leq \left(-\min_{j \in S} |y_j|\, e(S), z\right) \in Y_2^S(v)$$

and, therefore, we have

$$(\hat{q},\hat{p}) \cdot (y,z) \leq (\hat{q},\hat{p}) \cdot \left(-\min_{j \in S} |y_j|\, e(S), z\right) \leq 0.$$

This completes the proof of Claim 1. □

Claim 2. $W_1^*(v) \subseteq W_1^*(v^c)$.

Proof. Let $(\hat{x}^i)_{i \in N} \in W_1^*(v)$ and let $((\hat{q},\hat{p}),(0,\hat{x}^i)_{i \in N},(\hat{y}^S,\hat{z}^S)_{S \in \mathcal{N}})$ be a corresponding Walrasian equilibrium without double-jobbing for $\mathcal{E}_1(v)$. Since $v(N) \leq v^c(N)$, from $(\hat{y}^N,\hat{z}^N) \in Y_1^N(v)$, it follows that $(\hat{y}^N,\hat{z}^N) \in Y_1^N(v^c)$. It remains to prove that for every $(y,z) \in Y_1^N(v^c)$, $(\hat{q},\hat{p}) \cdot (y,z) \leq 0$ holds. Let $(y,z) \in Y_1^N(v^c)$. By the definition of v^c, there exists a partition π of N with $v^c(N) = \sum_{S \in \pi} v(S)$. Since $(y,z) \leq (-\min_{j \in N} |y_j|\, e(N), z) \in Y_2^N(v^c)$ by Lemma 2 (ii), we have

$$0 \leq z \leq v^c(N) \min_{j \in N} |y_j| = \sum_{S \in \pi} \min_{j \in N} |y_j|\, v(S).$$

Thus, there exists $(z^S)_{S \in \pi}$ such that

$$0 \leq z^S \leq \min_{j \in N} |y_j|\, v(S) \quad \text{for every } S \in \pi \text{ and} \quad \sum_{S \in \pi} z^S = z.$$

Since $\left(-\min_{j \in N} |y_j|\, e(S), z^S\right) \in Y_2^S(v) \subseteq Y_1^S(v)$, we have

$$(\hat{q},\hat{p}) \cdot \left(-\min_{j \in N} |y_j|\, e(S), z^S\right) \leq 0 \quad \text{for every } S \in \pi.$$

Hence,

$$(\hat{q},\hat{p}) \cdot (y,z) \leq (\hat{q},\hat{p}) \cdot \left(-\min_{j \in N} |y_j|\, e(N), z\right) = \sum_{S \in \pi} (\hat{q},\hat{p}) \cdot \left(-\min_{j \in N} |y_j|\, e(S), z^S\right) \leq 0.$$

This completes the proof of Claim 2. □

Claim 3. $W_1^*(v^c) \subseteq C(v^c)$.

Proof. Let $(\hat{x}^i)_{i\in N} \in W_1^*(v^c)$ and let $((\hat{q}, \hat{p}), (0, \hat{x}^i)_{i\in N}, (\hat{y}^S, \hat{z}^S)_{S\in\mathcal{N}})$ be a corresponding Walrasian equilibrium without double-jobbing for $\mathcal{E}_1(v^c)$. Let $\mathcal{B} = \{S \in \mathcal{N} \,|\, \hat{y}_i^S = -1 \text{ for every } i \in S\}$ be the coalition structure and let $S_0 = N \setminus \bigcup_{S\in\mathcal{B}} S$. Then, by no-double-jobbing condition, $\mathcal{B} \cup \{S_0\}$ is a partition of N and, by Proposition 2, $\hat{x}^i = 0$ for every $i \in S_0$. Note that for every $S \in \mathcal{N} \setminus \mathcal{B}$, there exists $i \in S$ with $\hat{y}_i^S = 0$. Thus, for every $S \in \mathcal{N} \setminus \mathcal{B}$, $\hat{z}^S = 0$ holds. In particular, $\hat{z}^{S_0} = 0 \leq v^c(S_0)$. Since $(\hat{y}^S, \hat{z}^S) = (-e(S), \hat{z}^S) \in Y_1^S(v^c)$ for every $S \in \mathcal{B}$, we have $\hat{z}^S \leq v^c(S)$ for every $S \in \mathcal{B}$. Then,

$$\sum_{i\in N} \hat{x}^i = \sum_{S\in\mathcal{N}} \hat{z}^S = \sum_{S\in\mathcal{B}\cup\{S_0\}} \hat{z}^S \leq \sum_{S\in\mathcal{B}\cup\{S_0\}} v^c(S) \leq v^c(N).$$

Let $S \in \mathcal{N}$. Since $(-e(S), v(S)) \in Y_1^S(v^c)$, from the profit maximization condition and Lemma 3 (ii), we have

$$0 \geq (\hat{q}, \hat{p}) \cdot (-e(S), v(S)) = -\sum_{i\in S} \hat{q}_i + \hat{p}v(S) = \hat{p}\left(-\sum_{i\in S} \hat{x}^i + v(S)\right).$$

Since $\hat{p} > 0$ by Lemma 3 (i), we have $\sum_{i\in S} \hat{x}^i \geq v(S)$ for every $S \in \mathcal{N}$. Hence,

$$\sum_{i\in S} \hat{x}^i \geq v(S) = v^c(S) \quad \text{for every } S \in N \setminus \{N\}.$$

By the definition of v^c, there exists a partition π of N with $v^c(N) = \sum_{S\in\pi} v(S)$. Since $\sum_{i\in S} \hat{x}^i \geq v(S)$ for every $S \in \pi$, we have

$$\sum_{i\in N} \hat{x}^i = \sum_{S\in\pi}\sum_{i\in S} \hat{x}^i \geq \sum_{S\in\pi} v(S) = v^c(N).$$

Thus, $(\hat{x}^i)_{i\in N} \in C(v^c)$ and, therefore, we have proved Claim 3. □

Claim 4. $C(v^c) \subseteq W_2^*(v)$.

Proof. Let $(\hat{x}^i)_{i\in N} \in C(v^c)$. By the definition of v^c, there exists a partition π of N with $v^c(N) = \sum_{S\in\pi} v(S)$. Define $\hat{p} = 1$, $\hat{q}_i = \hat{x}^i$ for every $i \in N$, $(\hat{y}^S, \hat{z}^S) = (-e(S), v(S))$ if $S \in \pi$, and $(\hat{y}^S, \hat{z}^S) = (0, 0)$ if $S \in \mathcal{N} \setminus \pi$. We prove that $((\hat{q}, \hat{p}), (0, \hat{x}^i)_{i\in N}, (\hat{y}^S, \hat{z}^S)_{S\in\mathcal{N}})$

is a Walrasian equilibrium without double-jobbing for $\mathcal{E}_2(v)$. The nonnegativity of price vector, the utility maximization, and the no-double-jobbing are all satisfied. Since

$$\begin{aligned}\sum_{S\in\mathcal{N}}(\hat{y}^S,\hat{z}^S) &= \sum_{S\in\pi}(\hat{y}^S,\hat{z}^S) = \sum_{S\in\pi}(-e(S),v(S)) = \left(-e(N),\sum_{S\in\pi}v(S)\right)\\ &= \sum_{i\in N}(0,\hat{x}^i) - \sum_{i\in N}\omega^i,\end{aligned}$$

we have the feasibility. The last equality follows from $\sum_{i\in N}\hat{x}^i = v^c(N) = \sum_{S\in\pi}v(S)$, because $(\hat{x}^i)_{i\in N}\in C(v^c)$.

It is clear that for every $S\in\mathcal{N}$, $(\hat{y}^S,\hat{z}^S)\in Y_2^S(v)$. Let $S\in\pi$. Then,

$$\begin{aligned}(\hat{q},\hat{p})\cdot(\hat{y}^S,\hat{z}^S) &= (\hat{q},\hat{p})\cdot(-e(S),v(S)) = -\sum_{i\in S}\hat{q}_i + v(S) = -\sum_{i\in S}\hat{x}^i + v(S)\\ &\leq -\sum_{i\in S}\hat{x}^i + v^c(S) \leq 0.\end{aligned}$$

Since

$$\begin{aligned}\sum_{S\in\pi}(\hat{q},\hat{p})\cdot(\hat{y}^S,\hat{z}^S) &= (\hat{q},\hat{p})\cdot(-e(N),v^c(N)) = -\sum_{i\in N}\hat{q}_i + v^c(N)\\ &= -\sum_{i\in N}\hat{x}^i + v^c(N) = 0,\end{aligned}$$

we have

$$(\hat{q},\hat{p})\cdot(\hat{y}^S,\hat{z}^S) = 0 \quad \text{for every } S\in\pi.$$

It is clear that for every $S\in\mathcal{N}\setminus\pi$, $(\hat{q},\hat{p})\cdot(\hat{y}^S,\hat{z}^S)=0$ holds.

It remains to prove that for every $S\in\mathcal{N}$ and every $(y,z)\in Y_2^S(v)$, $(\hat{q},\hat{p})\cdot(y,z)\leq 0$. Let $S\in\mathcal{N}$ and $(y,z)\in Y_2^S(v)$. Then, there exists $\lambda\geq 0$ with $y=-\lambda e(S)$. Then, from $(y,z)=(-\lambda e(S),z)\in Y_2^S(v)$, it follows that $z\leq\lambda v(S)$. Hence,

$$\begin{aligned}(\hat{q},\hat{p})\cdot(y,z) &\leq (\hat{q},\hat{p})\cdot(-\lambda e(S),\lambda v(S)) = \lambda\left(-\sum_{i\in S}\hat{q}_i + v(S)\right) \leq \lambda\left(-\sum_{i\in S}\hat{x}^i + v^c(S)\right)\\ &\leq 0.\end{aligned}$$

The last inequality follows from $(\hat{x}^i)_{i\in N}\in C(v^c)$. Thus, $((\hat{q},\hat{p}),(0,\hat{x}^i)_{i\in N},(\hat{y}^S,\hat{z}^S)_{S\in\mathcal{N}})$ is a Walrasian equilibrium without double-jobbing for $\mathcal{E}_2(v)$. This completes the proof of Claim 4. □

From Claims 1-4, we have proved Theorems 9 and 10. □

If the completion of a TU game is balanced, there exists no difference between Walrasian payoff vector and Walrasian payoff vector without double-jobbing.

Theorem 11. *Let* $v : \mathcal{N} \to \mathbb{R}_+$.

(i) *If* v^c *is balanced, then* $v^c = v^b$ *and*

$$\begin{aligned} & W^*(v) = W^*(v^c) = W^*(v^{sa}) = W^*(v^{tb}) \\ = \; & C(v^c) = C(v^{sa}) = C(v^{tb}) \\ = \; & W(v) = W(v^c) = W(v^{sa}) = W(v^{tb}) = W_B(v^{tb}) = W_S(v^{tb}) \neq \emptyset. \end{aligned}$$

(ii) *If* $W^*(v) \neq \emptyset$*, then* v^c *is balanced.*

Proof. We first prove (i). Let $v : \mathcal{N} \to \mathbb{R}_+$ be such that v^c is balanced. Then, $v^c(N) = v^b(N)$ and, therefore, $v^c = v^b$. By Theorem 8, we have

$$W(v) = W(v^c) = C(v^c) = C(v^{tb}) = W(v^{tb}) = W_B(v^{tb}) = W_S(v^{tb}) \neq \emptyset.$$

Since $(v^{sa})^b = v^{sa}$, we have $W(v^{sa}) = C(v^{sa})$. By Theorem 10, we have

$$W^*(v) = W^*(v^c) = C(v^c) = C(v^{sa}) = W^*(v^{sa}).$$

Since $(v^{tb})^c = v^{tb}$, we have $W^*(v^{tb}) = C(v^{tb})$. Thus, we have proved (i).

Statement (ii) immediately follows from Theorem 10 and Bondareva-Shapley theorem. □

6 Concluding remarks

First, we proved that every TU game can be generated by a coalition production economy (Theorem 2). Second, we clarify the relationship between the core of a given TU game and the set of Walrasian payoff vectors of our induced coalition production economies

(Theorem 8). Third, we consider a Walrasian equilibrium without double-jobbing and extend the theorem by Sun et al. [20] (Theorems 10 and 11).

Our second induced economy $\mathcal{E}_2(v)$ has the connection with Billera's [2] induced production economy. Billera [2] gave a representation theorem on NTU games. By the same idea as the present paper, we can prove that every compactly generated NTU game can be generated by a coalition production economy without transferable utility (see Inoue [10]). Furthermore, we can prove that for every compactly generated NTU game, the inner core of its balanced cover coincides with the set of Walrasian payoff vectors of the NTU representation (see Inoue [10]), which is also equal to the set of Walrasian payoff vectors of Billera's [2] NTU representation of the totally balanced cover (see Qin [13]). This common set is, however, possibly empty, because the balancedness is not sufficient for the nonemptiness of the inner core. Qin [14] and Inoue [11] give sufficient conditions for the nonemptiness of the inner core.

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