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André Casajus

IMW · Bielefeld University
Postfach 100131
33501 Bielefeld · Germany

email: imw@wiwi.uni-bielefeld.de
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Amalgamating players, symmetry, and the Banzhaf value

André Casajus[†‡§¶]

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Abstract

We suggest new characterizations of the Banzhaf value without the symmetry axiom, which reveal that the characterizations by Lehrer (1988, International Journal of Game Theory 17, 89–99) and Nowak (1997, International Journal of Game Theory 26, 127–141) as well as most of the characterizations by Casajus (2010, Theory and Decision, forthcoming) are redundant. Further, we explore symmetry implications of Lehrer's 2-efficiency axiom.

Journal of Economic Literature Classification Number: C71.

Key Words: Banzhaf value, amalgamation, symmetry, 2-efficiency.

[†]LSI Leipziger Spieltheoretisches Institut, c/o Restaurant Zest, Bornaische Straße 54, D-04277 Leipzig, Germany; e-mail: mail@casajus.de

[‡]IMW Institute of Mathematical Economics, Bielefeld University, PF 100131, 33501 Bielefeld, Germany.

[§]Chair of Economics and Information Systems, HHL Leipzig Graduate School of Management, Jahnallee 59, D-04109 Leipzig, Germany.

[¶]Professur für Mikroökonomik, Wirtschaftswissenschaftliche Fakultät, Universität Leipzig, PF 100920, D-04009 Leipzig, Germany.

1. Introduction

As an alternative to the Shapley (1953) value, Banzhaf (1965) introduced another value for voting games, later extended to general TU games by Owen (1975). There are numerous characterizations of the Banzhaf value both on the full domain of TU games and within restricted domains. Among them, the characterizations by Lehrer (1988) stand out by the use of some appealing amalgamation properties, superadditivity, max-superadditivity[1], and 2-efficiency, where in particular the latter very nicely pinpoints the difference between the Shapley value and the Banzhaf value.

To illustrate this, let us explain superadditivity and 2-efficiency. When player j is amalgamated to player i in a TU game, he leaves the game as a genuine player, but he "sits on the shoulders" of player i, i.e., with respect to the creation of worth, player j is present in a coalition whenever player i is so. Superadditivity then requires the payoff of player i in the amalgamated game not to be smaller than the sum of the individual payoffs of players i and j in the original game, i.e., amalgamating players never hurts. 2-efficiency is more demanding; it calls for amalgamation not to make a difference, i.e., for equality to hold in the superadditivity axiom. In contrast, the Shapley value obeys N-efficiency, i.e., the sum of payoffs does not change when all players are merged into a single one.

Later on, Nowak (1997) and Casajus (2010) employ 2-efficiency to characterize the Banzhaf value, but—in contrast to Lehrer—avoid the additivity axiom by invoking marginality (Young, 1985) or differential marginality (Casajus, 2009). Besides one of the amalgamation properties and the dummy player axiom, all of the above characterizations share the symmetry axiom or differential marginality, where the latter is closely related to symmetry. While Casajus (2010, Remark 5) claims the non-redundancy of his characterization, neither Lehrer (1988) nor Nowak (1997) check for the independence of their axioms. Embarrassingly, Casajus is wrong. In most of the above mentioned characterizations, we can drop the symmetry axiom or differential marginality.

What our authors seem to have missed is that the amalgamation properties, in particular 2-efficiency, embody strong symmetry requirements. Indeed, as our first

[1]The use of "superadditivity" to denote these axioms is somewhat unfortunate. First, "additivity" is well-established as a name for another axiom for solutions of TU games. And second, "superadditivity" already refers to a common property of TU games. Nevertheless, we stick to the original names introduced by Lehrer, because it will always be clear from the context what is meant.

main result (Theorem 1), we show that the Banzhaf value already is characterized by the dummy player axiom and by 2-efficiency on the full domain of TU games. While this axiomatization also works within the domains of simple games and of superadditive simple games, this is not true within the domain of superadditive games. To hold within the latter domain, one could add either marginality or differential marginality, or the transfer axiom (Dubey and Shapley, 1979, A4). This already entails that the Nowak characterization and most of the Casajus characterizations are redundant.

Our second main result directly builds on the first one. We establish that the axioms in Lehrer's (1988, Theorem B) characterization are not independent (Theorem 6). To achieve this, we explore relations between the amalgamation properties (Lemmas 4 and 5), which may be of some interest in themselves. First, it is quite easy to show that additivity and superadditivity imply 2-efficiency on the full domain of games (Lemma 4). And second, it is little more difficult to establish that additivity, the dummy player axiom, and max-superadditivity imply superadditivity (Lemma 5). Together with Theorem 1, this already entails that one can drop symmetry from the Lehrer characterization. In order to show the latter for Lehrer (1988, Theorem A), i.e., within the class of simple games (Theorem 7), we combine ideas from the proof of the Lehrer theorem and from the proof of Theorem 1.

The plan of this note is as follows: Basic definitions and notation are given in the second section. The third and fourth section contain our results related to 2-efficiency and (max-)superadditivity, respectively. Some remarks on the relation between 2-efficiency and isomorphism invariance and on the extendability of values for two-player games by 2-efficiency conclude the paper.

2. Basic definitions and notation

A **(TU) game** is a pair (N, v) consisting of a non-empty and finite set of players N and a **coalition function** $v \in \mathbb{V}(N) := \left\{f : 2^N \to \mathbb{R} | f(\emptyset) = 0\right\}$; the domain of a coalition function frequently will be made explicit as a superscript. Subsets of N are called **coalitions**, and $v(K)$ is called the worth of coalition K. The **null game** on N is denoted $(N, \mathbf{0})$, $\mathbf{0} \in \mathbb{V}(N)$, where $\mathbf{0}(K) = 0$ for all $K \subseteq N$. For $T \in 2^N \backslash \{\emptyset\}$, the game (N, u_T), $u_T(K) = 1$ if $T \subseteq K$ and $u_T(K) = 0$ otherwise, is called a **unanimity game**; the game (N, e_T), $e_T(K) = 1$ if $T = K$ and $e_T(K) = 0$ otherwise, is called a **standard game**. For $v, w \in \mathbb{V}(N)$, $\alpha \in \mathbb{R}$, the coalition functions $v + w \in \mathbb{V}(N)$ and $\alpha \cdot v \in \mathbb{V}(N)$ are given by $(v + w)(K) = v(K) +$

$w(K)$ and $(\alpha \cdot v)(K) = \alpha \cdot v(K)$ for all $K \subseteq N$. Any $v \in \mathbb{V}(N)$ can be uniquely represented by unanimity games,

$$v = \sum_{T \subseteq N : T \neq \emptyset} \lambda_T(v) \cdot u_T, \qquad \lambda_T(v) := \sum_{S \subseteq T : S \neq \emptyset} (-1)^{|T|-|S|} \cdot v(S). \tag{1}$$

A game (N, v) is called **simple** iff $v(K) \in \{0, 1\}$ for all $K \subseteq N$; it is called **superadditive** iff $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$, $S \cap T = \emptyset$. Let $\mathbb{V}^{\text{sa}}(N)$ and $\mathbb{V}^{\text{si}}(N)$ denote the sets of superadditive and of simple coalition functions on N, respectively. For $v, w \in \mathbb{V}^{\text{si}}(N)$, we define $v \vee w, v \wedge w \in \mathbb{V}^{\text{si}}(N)$ by $(v \vee w)(K) = \max\{v(K), w(K)\}$ and $(v \wedge w)(K) = \min\{v(K), w(K)\}$ for all $K \subseteq N$. For (N, v) and $i, j \in N$, $i \neq j$, the **amalgamated games**[2] (N_{ij}, v_{ij}) and $\left(N_{ij}, v_{ij}^m\right)$ are given by $N_{ij} = N \backslash \{j\}$, $v_{ij}, v_{ij}^m \in \mathbb{V}(N \backslash \{j\})$,

$$v_{ij}(K) = \begin{cases} v(K), & i \notin K, \\ v(K \cup \{j\}), & i \in K, \end{cases} \qquad K \subseteq N_{ij} \tag{2}$$

and

$$v_{ij}^m(K) = \begin{cases} v(K), & i \notin K, \\ \max_{T \subseteq \{i,j\}, T \neq \emptyset} v(K \backslash \{i\} \cup T), & i \in K, \end{cases} \qquad K \subseteq N_{ij}. \tag{3}$$

Player $i \in N$ is called a **dummy player** in (N, v) iff $v(K \cup \{i\}) - v(K) = v(\{i\})$ for all $K \subseteq N \backslash \{i\}$; if in addition $v(\{i\}) = 0$, then i is called a **null player**; players $i, j \in N$ are called **symmetric** in (N, v) if $v(K \cup \{i\}) = v(K \cup \{j\})$ for all $K \subseteq N \backslash \{i, j\}$.

A **value** is an operator φ that assigns a payoff vector $\varphi(N, v) \in \mathbb{R}^N$ to any game (N, v). For $K \subseteq N$, we set $\varphi_K(N, v) = \sum_{i \in K} \varphi_i(N, v)$. The **Banzhaf value** is given by

$$\text{Ba}_i(N, v) = 2^{-|N|+1} \sum_{K \subseteq N \backslash \{i\}} (v(K \cup \{i\}) - v(K)), \qquad i \in N. \tag{4}$$

Below, we list the axioms that are used later on and which are supposed to hold for all non-empty and finite player sets.

Additivity, A. For all $v, w \in \mathbb{V}(N)$, $\varphi(N, v + w) = \varphi(N, v) + \varphi(N, w)$.

Transfer, T. For all $v, w \in \mathbb{V}^{\text{si}}(N)$, $\varphi(N, v \vee w) + \varphi(N, v \wedge w) = \varphi(N, v) + \varphi(N, w)$.

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[2] The use of **SA** and **SA**^{m} by Lehrer (1988, p. 96) indicates that he must have had these definitions in mind.

Dummy player, D. For all $v \in \mathbb{V}(N)$ and all $i \in N$, who are dummy players in (N, v), $\varphi_i(N, v) = v(\{i\})$.

Symmetry, S. For all $v \in \mathbb{V}(N)$ and all $i, j \in N$, who are symmetric in (N, v), $\varphi_i(N, v) = \varphi_j(N, v)$.

Superadditivity, SA. For all $v \in \mathbb{V}(N)$ and $i, j \in N$, $i \neq j$, $\varphi_i(N_{ij}, v_{ij}) \geq \varphi_i(N, v) + \varphi_j(N, v)$.

max-Superadditivity, SA^m. For all $v \in \mathbb{V}(N)$ and $i, j \in N$, $i \neq j$, $\varphi_i\left(N_{ij}, v_{ij}^m\right) \geq \varphi_i(N, v) + \varphi_j(N, v)$.

2-Efficiency, 2E. For all $v \in \mathbb{V}(N)$ and $i, j \in N$, $i \neq j$, $\varphi_i(N_{ij}, v_{ij}) = \varphi_i(N, v) + \varphi_j(N, v)$.

Efficiency, E. For all $v \in \mathbb{V}(N)$, $\varphi_N(N, v) = v(N)$.

Marginality, M. For all $v, w \in \mathbb{V}(N)$ and all $i \in N$ such that $v(K \cup \{i\}) - v(K) = w(K \cup \{i\}) - w(K)$ for all $K \subseteq N \setminus \{i\}$, $\varphi_i(N, v) = \varphi_i(N, w)$.

Differential marginality, DM. For all $v, w \in \mathbb{V}(N)$ and $i, j \in N$, $v(K \cup \{i\}) - v(K \cup \{j\}) = w(K \cup \{i\}) - w(K \cup \{j\})$ for all $K \subseteq N \setminus \{i, j\}$ implies $\varphi_i(N, v) - \varphi_j(N, v) = \varphi_i(N, w) - \varphi_j(N, w)$.

3. 2-EFFICIENCY

Besides his main results, Lehrer (1988, Remark 3) establishes that the Banzhaf value is the unique value that satisfies 2-efficiency and that coincides with the Banzhaf value for two-player games, i.e., 2-efficiency uniquely extends the Banzhaf value from two-player games—where it coincides with the Shapley value—to the domain of all games. Since amalgamating players does not the lead out of the domain of simple games or the domain of superadditive games, respectively, this result holds within these domains and within their intersection. In the following, we suggest a new characterization of the Banzhaf value via 2-efficiency and the dummy player axiom. This characterization also works within the domains of superadditive games and of superadditive simple games, but not within the domain of simple games.

Nowak (1997) and Casajus (2010) also present characterizations of the Banzhaf value that employ 2-efficiency. Our findings shed new light on their results. Almost all of their characterizations turn out to be redundant. In the next section, we use the main idea of the proof of the following theorem to show that the characterizations of Lehrer (1988, Theorems A and B) are redundant, too.

Theorem 1. *The Banzhaf value is the unique value that satisfies* ***D*** *and* ***2E****.*

Proof. In view of Lehrer (1988, Remark 3), it suffices, w.l.o.g., to show $\varphi(\{1,2\},v) = \mathrm{Ba}(\{1,2\},v)$ for all $v \in \mathbb{V}(\{1,2\})$. Let $v = \lambda_1 \cdot u_{\{1\}}^{\{1,2\}} + \lambda_2 \cdot u_{\{2\}}^{\{1,2\}} + \lambda_{12} \cdot u_{\{1,2\}}^{\{1,2\}}$, $\lambda_1, \lambda_2, \lambda_{12} \in \mathbb{R}$ and $N = \{1,2,3,4\}$. Consider $z \in \mathbb{V}(N)$,

$$z = (\lambda_1 - \lambda_2) \cdot u_{\{4\}}^N + \lambda_2 \cdot \sum_{\ell \in N\setminus\{4\}} \left(u_{\{\ell\}}^N - u_{N\setminus\{4,\ell\}}^N\right) + (\lambda_{12} + 2 \cdot \lambda_2) \cdot u_{N\setminus\{4\}}^N. \tag{5}$$

By (2), we get $w \in \mathbb{V}(N\setminus\{3\})$,

$$w := z_{13} = z_{23} = (\lambda_1 - \lambda_2) \cdot u_{\{4\}}^{N\setminus\{3\}} + \lambda_2 \cdot \left(u_{\{1\}}^{N\setminus\{3\}} + u_{\{2\}}^{N\setminus\{3\}}\right) + \lambda_{12} \cdot u_{\{1,2\}}^{N\setminus\{3\}} \tag{6}$$

and therefore

$$\varphi_1(N\setminus\{3\},w) \overset{\mathbf{2E}}{=} \varphi_1(N,z) + \varphi_3(N,z), \tag{7}$$

$$\varphi_2(N\setminus\{3\},w) \overset{\mathbf{2E}}{=} \varphi_2(N,z) + \varphi_3(N,z). \tag{8}$$

Again by (2), we obtain $w_{12} \in \mathbb{V}(\{1,4\})$,

$$w_{12} = (2 \cdot \lambda_2 + \lambda_{12}) \cdot u_{\{1\}}^{\{1,4\}} + (\lambda_1 - \lambda_2) \cdot u_{\{4\}}^{\{1,4\}},$$

hence,

$$2 \cdot \lambda_2 + \lambda_{12} \overset{\mathbf{D}}{=} \varphi_1(\{1,4\},w_{12}) \overset{\mathbf{2E}}{=} \varphi_1(N\setminus\{3\},w) + \varphi_2(N\setminus\{3\},w) \tag{9}$$

Equations (7), (8), and (9) together entail

$$\varphi_1(N,z) + \varphi_2(N,z) + 2 \cdot \varphi_3(N,z) = 2 \cdot \lambda_2 + \lambda_{12}. \tag{10}$$

Interchanging the role of players 1, 2, and 3 in the above argument yields

$$\varphi_1(N,z) + 2 \cdot \varphi_2(N,z) + \varphi_3(N,z) = 2 \cdot \lambda_2 + \lambda_{12}, \tag{11}$$

$$2 \cdot \varphi_1(N,z) + \varphi_2(N,z) + \varphi_3(N,z) = 2 \cdot \lambda_2 + \lambda_{12}. \tag{12}$$

Now, the system of linear equations (10), (11), and (12) has the unique solution

$$\varphi_1(N,z) = \varphi_2(N,z) = \varphi_3(N,z) = \frac{\lambda_2}{2} + \frac{\lambda_{12}}{4}. \tag{13}$$

Further, we obtain

$$\varphi_1(N\setminus\{3\},w) \overset{(6),\mathbf{2E}}{=} \varphi_1(N,z) + \varphi_3(N,z) \overset{(13)}{=} \lambda_2 + \frac{\lambda_{12}}{2}, \tag{14}$$

$$\varphi_4(N\setminus\{3\},w) \overset{\mathbf{D}}{=} \lambda_1 - \lambda_2. \tag{15}$$

Since $w_{14} = v$, we thus have

$$\varphi_1(\{1,2\},v) \overset{\mathbf{2E}}{=} \varphi_1(N\setminus\{3\},w) + \varphi_4(N\setminus\{3\},w) = \lambda_1 + \frac{\lambda_{12}}{2} = \mathrm{Ba}_1(\{1,2\},v).$$

Finally, we have

$$v(\{1,2\}) \overset{\mathbf{D}}{=} \varphi_1(\{1\}, v_{12}) \overset{\mathbf{2E}}{=} \varphi_1(\{1,2\}, v) + \varphi_2(\{1,2\}, v), \tag{16}$$

which gives $\varphi_2(\{1,2\}, v) = \mathrm{Ba}_2(\{1,2\}, v)$. □

A couple of remarks on the previous result and its proof seem to be in order.

Remark 1. To some extent, the proof above illuminates the relation between 2-efficiency and symmetry. Note the idea of the proof. First, the asymmetry between players 1 and 2 in $(\{1,2\}, v)$ is shifted into the dummy player 4 in $(\{1,2,4\}, w)$. This way the symmetric "part" of v, i.e., $v - (\lambda_1 - \lambda_2) \cdot u_{\{1\}}^{\{1,2\}}$, becomes $w - (\lambda_1 - \lambda_2) \cdot u_{\{4\}}^{\{1,2,4\}}$. In order to handle the symmetric part of v in w, we add player 3 in an appropriate way and obtain (N, z), where 1, 2, and 3 are pairwise symmetric. So, the argument involving equations (6) to (13) shows how 2-efficiency breeds symmetry.

Remark 2. The superadditivity of v may not entail the superadditivity of z in (6), for example, if $\lambda_1, \lambda_2 < 0$ in (5). Therefore, the above proof does not work within the domain of superadditive games. Fortunately, one can reduce the problem to the case $\lambda_2 = 0$, where z inherits superadditivity from v.

Let v from the above proof be superadditive, i.e., $\lambda_{12} \geq 0$. Set now $N = \{1,2,3\}$ and $w \in \mathbb{V}(N)$,

$$w := \lambda_2 \cdot u_{\{2\}}^N + \lambda_1 \cdot u_{\{3\}}^N + \lambda_{12} \cdot u_{\{1,2\}}^N.$$

By (2), we obtain

$$\begin{aligned} v = w_{13} &= \lambda_1 \cdot u_{\{1\}}^{\{1,2\}} + \lambda_2 \cdot u_{\{2\}}^{\{1,2\}} + \lambda_{12} \cdot u_{\{1,2\}}^{\{1,2\}}, \\ w_{12} &= (\lambda_2 + \lambda_{12}) \cdot u_{\{1\}}^{\{1,3\}} + \lambda_1 \cdot u_{\{3\}}^{\{1,3\}}, \\ w_{23} &= (\lambda_1 + \lambda_2) \cdot u_{\{2\}}^{\{1,2\}} + \lambda_{12} \cdot u_{\{1,2\}}^{\{1,2\}}, \end{aligned}$$

hence,

$$\begin{aligned} \varphi_1(\{1,2\}, v) &\overset{\mathbf{2E}}{=} \varphi_1(N, w) + \varphi_3(N, w) \overset{\mathbf{D}}{=} \varphi_1(N, w) + \lambda_1, \\ \lambda_2 + \lambda_{12} \overset{\mathbf{D}}{=} \varphi_1(\{1,3\}, w_{12}) &\overset{\mathbf{2E}}{=} \varphi_1(N, w) + \varphi_2(N, w), \\ \varphi_2(\{1,2\}, w_{23}) &\overset{\mathbf{2E}}{=} \varphi_2(N, w) + \varphi_3(N, w) \overset{\mathbf{D}}{=} \varphi_2(N, w) + \lambda_1. \end{aligned}$$

The last three equations imply

$$\varphi_1(\{1,2\}, v) + \varphi_2(\{1,2\}, w_{23}) = 2 \cdot \lambda_1 + \lambda_2 + \lambda_{12}.$$

This way and by (16), we can determine the payoffs for v by considering a coalition function of the desired type, w_{23}. Finally, note that w, w_{12}, and w_{23} are superadditive whenever v is so.

Corollary 2. *The Banzhaf value is the unique value that satisfies* ***D*** *and* ***2E*** *within the domain of superadditive games.*

Remark 3. Our characterization does not work within the domain of simple games. Consider the value $\varphi^{(1)}$ given by

$$\varphi_i^{(1)}(N,v) = \begin{cases} \mathrm{Ba}_i(N,v), & v(N) = 1, \\ 0, & v(N) = 0, \end{cases} \qquad i \in N,\ v \in \mathbb{V}^{\mathrm{si}}(N).$$

It is easy to see that $\varphi^{(1)}$ inherits **2E** and **D** from the Banzhaf value. Just observe that for all $i,j \in N$, $i \neq j$, $v(N) = 0$ entails $v_{ij}(N) = 0$, and that for any dummy player i in (N,v) such that $v(\{i\}) = 1$, we have $v(N) = 1$. Further,

$$\varphi_1^{(1)}\left(\{1,2\}, e_{\{1\}}^{\{1,2\}}\right) = 0 \neq \frac{1}{2} = \mathrm{Ba}_1\left(\{1,2\}, e_{\{1\}}^{\{1,2\}}\right).$$

The counterexample in the previous remark involves a simple game that is not superadditive, $e_{\{1\}}^{\{1,2\}}$. Yet, if v is superadditive and simple, then $v \neq \mathbf{0}$ implies $v(N) = 1$. Hence on the domain of superadditive simple games $\varphi^{(1)}$ coincides with the Banzhaf value. Indeed, within this domain, our characterization works fine.

Corollary 3. *The Banzhaf value is the unique value that satisfies* ***D*** *and* ***2E*** *within the domain of simple superadditive games.*

Proof. In view of Lehrer (1988, Remark 3), it suffices, w.l.o.g., to show $\varphi(\{1,2\},v) = \mathrm{Ba}(\{1,2\},v)$ for $v \in \left\{\mathbf{0}^{\{1,2\}}, u_{\{1\}}^{\{1,2\}}, u_{\{2\}}^{\{1,2\}}, u_{\{1,2\}}^{\{1,2\}}\right\}$. By **D**, the claim is immediate for $\mathbf{0}^{\{1,2\}}, u_{\{1\}}^{\{1,2\}}$, and $u_{\{2\}}^{\{1,2\}}$. Careful inspection of the proof of Theorem 1 for $\lambda_1 = \lambda_2 = 0$ and $\lambda_{12} = 1$ reveals that the arguments do not leave the domain of superadditive simple games. This concludes the proof. □

Remark 4. Our characterizations actually are non-redundant. The value $\varphi^{(2)}$ given by $\varphi_i^{(2)}(N,v) = 0$ for all $i \in N$ and $v \in \mathbb{V}(N)$ meets **2E**, but not **D**. The Shapley (1953) value obeys **D**, but not **2E**.

Remark 5. Nowak (1997, Theorem) characterizes the Banzhaf value on the full domain of TU games via **2E**, **D**, **S**, and **M**, but does not check for redundancy of his characterization. Theorem 1 reveals that we can drop two axioms, **S** and **M**.

Remark 6. Casajus (2010, for short CA10) first suggests a simple proof of the Nowak characterization which employs Lehrer (1988, Remark 3). Using this proof, he shows that the Nowak characterization works within the domains of superadditive games and of simple games (CA10, Remarks 2 and 3); within the intersection of these domains one even can do without **M** (CA10, Theorem 3). Further, he shows that one can replace **S** and **M** in the Nowak characterization by **DM** (CA10, Corollary 4). Embarrassingly, we have to admit that, in view of Theorem 1, Casajus' assertion that his characterization is non-redundant (CA10, Remark 4) must be wrong. By Corollaries 2 and 3, all the Casajus characterizations on the domain of superadditive games and of superadditive simple games—both for **M** and for **DM**—also are redundant.

Since $\varphi^{(1)}$ fails **M**, **DM**, and **T**, the Banzhaf value is no-redundantly characterized within the domain of simple games by **2E**, **D**, and either **M**, **DM**, or **T**. For **DM**, this drops from CA10 (Remark 7). To see this for **M** or **T**, reconsider the proof of Theorem 1. Let $\lambda_1, \lambda_2, \lambda_N \in \{0,1\}$. For $\lambda_0 = \lambda_1$, the proof does not leave the domain of simple games. Remains to deal with $e_1^{\{1,2\}}$, $e_2^{\{1,2\}}$, $u_1^{\{1,2\}}$, and $u_2^{\{1,2\}}$. For $u_1^{\{1,2\}}$ and $u_2^{\{1,2\}}$, the payoffs are determined by **D**. One easily checks that 1, $v = e_{\{1\}}^{\{1,2\}}$, and $v' = e_{\{1\}}^{\{1,2\}} + e_{\{2\}}^{\{1,2\}} + e_{\{2\}}^{\{1,2\}}$ meet the hypothesis of **M**. This gives $\varphi_1(N,v) = \varphi_1(N,v')$, where $\varphi_1(N,v') = \mathrm{Ba}_1(N,v')$ was shown above. Further, (16) entails $\varphi_2(N,v) = \mathrm{Ba}_2(N,v)$. Analogously, one deals with $e_{\{2\}}^{\{1,2\}}$. Finally, we have $e_{\{1\}}^{\{1,2\}} \vee u_{\{2\}}^{\{1,2\}} = e_{\{1\}}^{\{1,2\}} + e_{\{2\}}^{\{1,2\}} + e_{\{1,2\}}^{\{1,2\}}$ and $e_{\{1\}}^{\{1,2\}} \vee u_{\{2\}}^{\{1,2\}} = \mathbf{0}^{\{1,2\}}$. Hence, by **T**, the $\varphi\left(N, e_{\{1\}}^{\{1,2\}}\right)$ is uniquely determined by the axioms. Analogously, for $e_{\{2\}}^{\{1,2\}}$.

4. Superadditivity and max-superadditivity

As one of his main results, Lehrer (1988, Theorem B) establishes that the Banzhaf value is characterized by the dummy player axiom, additivity, symmetry, and either superadditivity or max-superadditivity on the domain of all TU games. In the following, we directly employ Theorem 1 in order to show that this characterization is redundant. As in the previous section, we can drop the symmetry axiom. To prepare the announced result on the Lehrer characterization, we first explore relations between the amalgamation properties.

Lemma 4. ***A*** *and* ***SA*** *imply* ***2E****.*

Proof. We have

$$0 \overset{\mathbf{A}}{=} \varphi_i(N,\mathbf{0}) = \varphi_i(N, v + (-v)) \overset{\mathbf{A}}{=} \varphi_i(N,v) + \varphi_i(N,-v) \tag{17}$$

for all $v \in \mathbb{V}(N)$ and $i \in N$. For all $i, j \in N$, $i \neq j$, one derives

$$\begin{aligned}
-\left(\varphi_i(N,-v)+\varphi_j(N,-v)\right) &\overset{\mathbf{SA}}{\geq} -\varphi_i\left(N\backslash\{j\},(-v)_{ij}\right)\\
&\overset{(2)}{=} -\varphi_i\left(N\backslash\{j\},-(v_{ij})\right)\\
&\overset{(17)}{=} \varphi_i\left(N\backslash\{j\},v_{ij}\right)\\
&\overset{\mathbf{SA}}{\geq} \varphi_i(N,v)+\varphi_j(N,v). \qquad (18)
\end{aligned}$$

Finally, (17) and (18) entail $\varphi_i(N\backslash\{j\}, v_{ij}) = \varphi_i(N,v)+\varphi_j(N,v)$. □

Remark 7. The proof of Lemma 4 crucially rests on the fact that $(-v)_{ij} = -(v_{ij})$ for all $v \in \mathbb{V}(N)$ and N. In contrast, $(-v^m)_{ij} = -\left(v^m_{ij}\right)$ does not hold in general. Hence, the proof does not work with $\mathbf{SA}^m$ instead of $\mathbf{SA}$. Indeed, the value $\varphi^{(3)}$ given by

$$\varphi^{(3)}_i(N,v) = \left(\frac{1}{2}\right)^{-|N|+1} \cdot v(\{i\}), \qquad i \in N,\ v \in \mathbb{V}(N). \qquad (19)$$

meets $\mathbf{A}$ and $\mathbf{SA}^m$, but fails $\mathbf{SA}$. Obviously, φ meets $\mathbf{A}$. To see $\mathbf{SA}^m$, consider $|N| \geq 2$ and $v \in \mathbb{V}(N)$. Let $i, j \in N$, $i \neq j$. By (3), $v^m_{ij}(\{1\}) \geq v(\{1\})$ and $v^m_{ij}(\{1\}) \geq v(\{2\})$. Hence,

$$\begin{aligned}
\varphi^{(3)}_i\left(N\backslash\{j\},v^m_{ij}\right) &\overset{(19)}{=} \left(\frac{1}{2}\right)^{-|N\backslash\{j\}|+1} v^m_{ij}(\{i\})\\
&\overset{(3)}{\geq} \left(\frac{1}{2}\right)^{-|N|+1} \left(v(\{i\})+v(\{j\})\right)\\
&\overset{(19)}{=} \varphi^{(3)}_i(N,v)+\varphi^{(3)}_j(N,v).
\end{aligned}$$

Let $N = \{1,2\}$ and $v = -u^N_{\{1\}}$. This gives

$$\varphi^{(3)}_1(N\backslash\{2\}, v_{12}) = -1 < -\frac{1}{2}+0 = \varphi^{(3)}_1(N,v)+\varphi^{(3)}_2(N,v).$$

Thus, φ fails $\mathbf{SA}$. Yet, while $\varphi^{(3)}$ meets $\mathbf{A}$ and $\mathbf{SA}^m$, it is immediate that it does not obey $\mathbf{D}$. It turns out that the latter axiom is all we need to infer $\mathbf{SA}$ from $\mathbf{A}$ and $\mathbf{SA}^m$.

Lemma 5. ***A**, **D**, and **SA**^{m} imply **SA**.*

Proof. Let φ obey $\mathbf{A}$, $\mathbf{SA}^m$, and $\mathbf{D}$. W.l.o.g., let $1,2 \in N$ and $v \in \mathbb{V}(N)$. For $K \subseteq N\backslash\{1,2\}$ set

$$\bar{\lambda}^K_1 = v(K\cup\{2\})-v(K\cup\{1,2\}) \quad \text{and} \quad \bar{\lambda}^K_2 = v(K\cup\{1\})-v(K\cup\{1,2\}). \quad (20)$$

Let $\lambda_1 \geq \bar{\lambda}_1^K$, $\lambda_2 \geq \bar{\lambda}_2^K$, and $w = v + \lambda_1 \cdot u_{\{1\}}^N + \lambda_2 \cdot u_{\{2\}}^N$. This gives

$$\begin{aligned} w(K \cup \{1,2\}) &\geq v(K \cup \{1,2\}) + \lambda_1 + \bar{\lambda}_2^K \\ &\overset{(20)}{=} v(K \cup \{1,2\}) + \lambda_1 + v(K \cup \{1\}) - v(K \cup \{1,2\}) \\ &= w(K \cup \{1\}). \end{aligned}$$

Analogously, one shows $w(K \cup \{1,2\}) \geq w(K \cup \{2\})$. This implies $w_{12}^m(K) = w_{12}(K)$. Letting $\lambda_1 := \max_{K \subseteq N \setminus \{1,2\}} \bar{\lambda}_1^K$ and $\lambda_2 := \max_{K \subseteq N \setminus \{1,2\}} \bar{\lambda}_2^K$, we have (*) $w_{12}^m = w_{12}$. One obtains

$$\begin{aligned} \varphi_1(N \setminus \{2\}, v_{12}) + \lambda_1 + \lambda_2 &\overset{\mathbf{D},\mathbf{A}}{=} \varphi_1\left(N \setminus \{2\}, v_{12} + (\lambda_1 + \lambda_2) \cdot u_{\{1\}}^{N \setminus \{2\}}\right) \\ &\overset{(2)}{=} \varphi_1\left(N \setminus \{2\}, \left(v + \lambda_1 \cdot u_{\{1\}}^N + \lambda_2 \cdot u_{\{2\}}^N\right)_{12}\right) \\ &\overset{(*)}{=} \varphi_1\left(N \setminus \{2\}, \left(v + \lambda_1 \cdot u_{\{1\}}^N + \lambda_2 \cdot u_{\{2\}}^N\right)_{12}^m\right) \\ &\overset{\mathbf{SA}^m}{\geq} \varphi_1\left(N, v + \lambda_1 \cdot u_{\{1\}}^N + \lambda_2 \cdot u_{\{2\}}^N\right) \\ &\quad + \varphi_2\left(N, v + \lambda_1 \cdot u_{\{1\}}^N + \lambda_2 \cdot u_{\{2\}}^N\right) \\ &\overset{\mathbf{A},\mathbf{D}}{=} \varphi_1(N, v) + \lambda_1 + \varphi_2(N, v) + \lambda_2, \end{aligned}$$

which entails the claim. □

Since we know that the Banzhaf value meets all the axioms, the next theorem is an immediate consequence of Theorem 1 and Lemmas 4 and 5. Note that within the proof of Lehrer (1988, Theorem B) additivity is crucial even in proving uniqueness for unanimity games. In the proof of Theorem 6, additivity plays a different role. What we actually need is homogeneity for the scalar -1, i.e., $\varphi(N, -v) = -\varphi(N, v)$ for all $v \in \mathbb{V}(N)$ (Lemma 4) and additivity with respect to the addition of modular games (Lemma 5).

Theorem 6. *The Banzhaf value is the unique value that satisfies **A**, **D**, and either **SA** or **SA**^{m}.*

Remark 8. Our characterizations actually are non-redundant. The value $\varphi^{(1)}$ from (19) meets **A**, **SA**, and **SA**^{m}, but not **D**. The Shapley (1953) value obeys **A** and **D**, but not **SA** or **SA**^{m}. The value from Nowak (1997, Counterexample) satisfies **D**, **SA**, and **SA**^{m}, but not **A**.

Remark 9. Restriction of Theorem 6 to the domain of superadditive games does not work. Consider the value $\varphi^{(4)} \neq \mathrm{Ba}$ given by

$$\varphi_i^{(4)}(N,v) = v(\{i\}) = \lambda_{\{i\}}(v), \qquad i \in N,\, v \in \mathbb{V}(N).$$

Obviously, this value obeys **D** and **A**. For $v \in \mathbb{V}^{\mathrm{sa}}(N)$ and $i,j \in N$, $i \neq j$, we have

$$v_{ij}^m(\{i\}) \overset{(3)}{\geq} v_{ij}(\{i\}) \overset{(2)}{=} v(\{i,j\}) \geq v(\{i\}) + v(\{j\}).$$

Hence, $\varphi^{(4)}$ meets **SA** and $\mathbf{SA}^m$ on $\mathbb{V}^+(N)$. Since $\varphi^{(4)}$ also meets **S, M**, and **DM**, it does not help to add **S** or to replace **A** by **M** or **DM**.

Now, we turn to the domain of simple games. Lehrer (1988) also suggests a characterization of the Banzhaf value that works within this domain (Theorem A) via the transfer axiom, the dummy player axiom, symmetry, and either superadditivity or max-superadditivity. In view of Theorem 6, one might be curious whether the symmetry axiom could be dropped. As it turns out, this actually is the case. Unfortunately, however, we do not have results analogous to Lemmas 4 and 5 within the domain of simple games. Instead, we modify Lehrer's original proof.

Theorem 7. *The Banzhaf value is the unique value that satisfies* ***T****,* ***D****, and either* ***SA*** *or* $\boldsymbol{SA}^m$ *on the domain of simple games.*

Within the Lehrer (1988, Theorem A) proof, symmetry is employed only to show that a value that satisfies all the axioms in the theorem coincides with the Banzhaf value for unanimity games. Hence, the following lemma fills the gap resulting from dropping symmetry.

Lemma 8. *If φ satisfies* ***T****,* ***D****, and either* ***SA*** *or* $\boldsymbol{SA}^m$ *within the domain of simple games, then $\varphi\left(N, u_T^N\right) = \mathrm{Ba}\left(N, u_T^N\right)$ for all N and $T \subseteq N$, $T \neq \emptyset$.*

Proof. Let φ satisfy **T**, **D**, and either **SA** or $\mathbf{SA}^m$. By **D**, the claim is immediate for $i \in N \backslash T$. As in the original proof, we proceed by two inductions, first on $|N|$ and second on $|T|$.

Outer induction basis: By **D**, the claim is immediate for $|N| = 1$.

Outer induction hypothesis: Let the claim hold for all $|N| \leq n$ and $T \subseteq N$, $T \neq \emptyset$.

Outer induction step: Let $|N| = n+1$ and $T \subseteq N$, $T \neq \emptyset$.

Inner induction basis: By **D**, the claim is immediate for $|T| = 1$. Since we cannot employ **S**, we have to deal with the case $|T| = 2$ separately (see (37) below).

W.l.o.g., let $T=\{1,2\}\subseteq N$ and $3\notin N$. We have $u_{\{1\}}^{N}\wedge u_{\{2\}}^{N}=u_{T}^{N}$ and therefore

$$\varphi_i\left(N,u_{\{1\}}^{N}\vee u_{\{2\}}^{N}\right)+\varphi_i\left(N,u_{T}^{N}\right)\overset{\mathbf{T}}{=}\varphi_i\left(N,u_{\{1\}}^{N}\right)+\varphi_i\left(N,u_{\{2\}}^{N}\right)\overset{\mathbf{D}}{=}1 \tag{21}$$

for $i\in\{1,2\}$. Amalgamating 2 to 1 gives

$$\varphi_1\left(N,u_{T}^{N}\right)+\varphi_2\left(N,u_{T}^{N}\right)\overset{\mathbf{SA}\text{ or }\mathbf{SA}^m}{\leq}\varphi_1\left(N\backslash\{2\},u_{\{1\}}^{N\backslash\{2\}}\right)\overset{\mathbf{D}}{=}1 \tag{22}$$

and

$$\varphi_1\left(N,u_{\{1\}}^{N}\vee u_{\{2\}}^{N}\right)+\varphi_2\left(N,u_{\{1\}}^{N}\vee u_{\{2\}}^{N}\right)\overset{\mathbf{SA}\text{ or }\mathbf{SA}^m}{\leq}\varphi_1\left(N\backslash\{2\},u_{\{1\}}^{N\backslash\{2\}}\right)\overset{\mathbf{D}}{=}1. \tag{23}$$

Equations (21), (22), and (23) already entail

$$\varphi_1\left(N,u_{T}^{N}\right)+\varphi_2\left(N,u_{T}^{N}\right)=1. \tag{24}$$

We have $u_{\{2,3\}}^{N\cup\{3\}}\wedge u_{\{1\}}^{N\cup\{3\}}=u_{\{1,2,3\}}^{N\cup\{3\}}$ and therefore

$$\begin{aligned}&\varphi_1\left(N\cup\{3\},u_{\{2,3\}}^{N\cup\{3\}}\vee u_{\{1\}}^{N\cup\{3\}}\right)+\varphi_1\left(N\cup\{3\},u_{\{1,2,3\}}^{N\cup\{3\}}\right)\\ &\overset{\mathbf{T}}{=}\ \varphi_1\left(N\cup\{3\},u_{\{2,3\}}^{N\cup\{3\}}\right)+\varphi_1\left(N\cup\{3\},u_{\{1\}}^{N\cup\{3\}}\right)\overset{\mathbf{D}}{=}1\end{aligned} \tag{25}$$

and

$$\begin{aligned}&\varphi_2\left(N\cup\{3\},u_{\{2,3\}}^{N\cup\{3\}}\vee u_{\{1\}}^{N\cup\{3\}}\right)+\varphi_2\left(N\cup\{3\},u_{\{1,2,3\}}^{N\cup\{3\}}\right)\\ &\overset{\mathbf{T}}{=}\ \varphi_2\left(N\cup\{3\},u_{\{2,3\}}^{N\cup\{3\}}\right)+\varphi_2\left(N\cup\{3\},u_{\{1\}}^{N\cup\{3\}}\right)\\ &\overset{\mathbf{D}}{=}\ \varphi_2\left(N\cup\{3\},u_{\{2,3\}}^{N\cup\{3\}}\right).\end{aligned} \tag{26}$$

Further, $\left(u_{\{2,3\}}^{N\cup\{3\}}\vee u_{\{1\}}^{N\cup\{3\}}\right)_{12}=u_{\{1\}}^{(N\cup\{3\})\backslash\{2\}}$. Hence, amalgamating 2 to 1 gives

$$\begin{aligned}&\varphi_1\left(N\cup\{3\},u_{\{2,3\}}^{N\cup\{3\}}\vee u_{\{1\}}^{N\cup\{3\}}\right)+\varphi_2\left(N\cup\{3\},u_{\{2,3\}}^{N\cup\{3\}}\vee u_{\{1\}}^{N\cup\{3\}}\right)\\ &\overset{\mathbf{SA}\text{ or }\mathbf{SA}^m}{\leq}\ \varphi_1\left(N\cup\{3\}\setminus\{2\},u_{\{1\}}^{(N\cup\{3\})\backslash\{2\}}\right)\overset{\mathbf{D}}{=}1.\end{aligned} \tag{27}$$

Equations (25), (26), and (27) entail

$$\varphi_1\left(N\cup\{3\},u_{\{1,2,3\}}^{N\cup\{3\}}\right)+\varphi_2\left(N\cup\{3\},u_{\{1,2,3\}}^{N\cup\{3\}}\right)\geq\varphi_2\left(N\cup\{3\},u_{\{2,3\}}^{N\cup\{3\}}\right). \tag{28}$$

Further, amalgamating 1 to 2 gives

$$\begin{aligned}&\varphi_2\left((N\cup\{3\})\setminus\{1\},u_{\{2,3\}}^{(N\cup\{3\})\backslash\{1\}}\right)\\ &\overset{\mathbf{SA}\text{ or }\mathbf{SA}^m}{\geq}\ \varphi_1\left(N\cup\{3\},u_{\{1,2,3\}}^{N\cup\{3\}}\right)+\varphi_2\left(N\cup\{3\},u_{\{1,2,3\}}^{N\cup\{3\}}\right).\end{aligned} \tag{29}$$

Equations (28) and (29) entail

$$\begin{aligned}
&\varphi_2\left((N\cup\{3\})\setminus\{1\}, u_{\{2,3\}}^{(N\cup\{3\})\setminus\{1\}}\right)\\
&\quad\geq\ \varphi_1\left(N\cup\{3\}, u_{\{1,2,3\}}^{N\cup\{3\}}\right)+\varphi_2\left(N\cup\{3\}, u_{\{1,2,3\}}^{N\cup\{3\}}\right)\\
&\quad\geq\ \varphi_2\left(N\cup\{3\}, u_{\{2,3\}}^{N\cup\{3\}}\right).
\end{aligned}\tag{30}$$

Interchanging the role of players 2 and 3 in (30) gives

$$\begin{aligned}
&\varphi_3\left((N\cup\{3\})\setminus\{1\}, u_{\{2,3\}}^{(N\cup\{3\})\setminus\{1\}}\right)\\
&\quad\geq\ \varphi_1\left(N\cup\{3\}, u_{\{1,2,3\}}^{N\cup\{3\}}\right)+\varphi_3\left(N\cup\{3\}, u_{\{1,2,3\}}^{N\cup\{3\}}\right)\\
&\quad\geq\ \varphi_3\left(N\cup\{3\}, u_{\{2,3\}}^{N\cup\{3\}}\right).
\end{aligned}\tag{31}$$

Adding (30) and (31) and applying the appropriate version of (24) gives

$$\begin{aligned}
&2\cdot\varphi_1\left(N\cup\{3\}, u_{\{1,2,3\}}^{N\cup\{3\}}\right)\\
&\quad+\varphi_2\left(N\cup\{3\}, u_{\{1,2,3\}}^{N\cup\{3\}}\right)+\varphi_3\left(N\cup\{3\}, u_{\{1,2,3\}}^{N\cup\{3\}}\right)=1.
\end{aligned}\tag{32}$$

Interchanging the role of players 1, 2, and 3 in (32) gives

$$\begin{aligned}
&2\cdot\varphi_2\left(N\cup\{3\}, u_{\{1,2,3\}}^{N\cup\{3\}}\right)\\
&\quad+\varphi_1\left(N\cup\{3\}, u_{\{1,2,3\}}^{N\cup\{3\}}\right)+\varphi_3\left(N\cup\{3\}, u_{\{1,2,3\}}^{N\cup\{3\}}\right)=1,
\end{aligned}\tag{33}$$

and

$$\begin{aligned}
&2\cdot\varphi_3\left(N\cup\{3\}, u_{\{1,2,3\}}^{N\cup\{3\}}\right)\\
&\quad+\varphi_1\left(N\cup\{3\}, u_{\{1,2,3\}}^{N\cup\{3\}}\right)+\varphi_2\left(N\cup\{3\}, u_{\{1,2,3\}}^{N\cup\{3\}}\right)=1.
\end{aligned}\tag{34}$$

It is easy to check that the system of linear equations (32), (33), and (34) has the unique solution

$$\varphi_i\left(N\cup\{3\}, u_{\{1,2,3\}}^{N\cup\{3\}}\right)=\frac{1}{4},\qquad i\in\{1,2,3\}.\tag{35}$$

Further,

$$\frac{1}{2}=\varphi_1\left(N\cup\{3\}, u_{\{1,2,3\}}^{N\cup\{3\}}\right)+\varphi_2\left(N\cup\{3\}, u_{\{1,2,3\}}^{N\cup\{3\}}\right)\overset{\mathbf{SA}\text{ or }\mathbf{SA}^m}{\leq}\varphi_i\left(N, u_{\{1,2\}}^N\right)\tag{36}$$

for $i\in\{1,2\}$. Equations (36) and (24) imply

$$\varphi_i\left(N, u_{\{1,2\}}^N\right)=\frac{1}{2}=\mathrm{Ba}_i\left(N, u_{\{1,2\}}^N\right),\qquad i\in\{1,2\}.$$

Inner induction hypothesis: Let $\varphi_i\left(N, u_T^N\right) = \left(\frac{1}{2}\right)^{|T|-1} = \mathrm{Ba}_i\left(N, u_T^N\right),\ i \in T$ for $|T| \leq t$, $t \geq 2$.

Inner induction step: Let $|T| = t+1$. For notational parsimony, we just provide the necessary modifications of the original proof and jump into its induction step. Since we cannot make use of **S**, Lehrer (1988, Equations (4a) and (4b)) become (in our notation)

$$\begin{aligned}
\varphi_j\left(N, u_T^N\right) &= a_j, \qquad j \in T, \\
\varphi_j\left(N, u_{T\setminus\{i\}}^N \vee u_{\{i\}}^N\right) &= \begin{cases} b_j^i, & j \in T\setminus\{i\}, \\ c_i, & j = i, \end{cases} \qquad i \in T.
\end{aligned}$$

Running the argument from the proof, we obtain $a_i + a_j = \left(\frac{1}{2}\right)^t$ instead of $2a = \left(\frac{1}{2}\right)^t$. Since $|T| \geq 3$, we have a system of linear equations

$$a_i + a_j = a_i + a_k = a_j + a_k = \left(\frac{1}{2}\right)^{t-1} \tag{37}$$

for $i, j, k \in T$, $i \neq j \neq k \neq i$. Its unique solution is $a_i = a_j = a_k = \left(\frac{1}{2}\right)^t$. Thus,

$$\varphi_i\left(N, u_T^N\right) = \left(\frac{1}{2}\right)^{|T|-1} = \mathrm{Ba}_i\left(N, u_T^N\right), \qquad i \in T.$$

Done. □

Remark 10. Compare this proof with the proof of Theorem 1. Equations (32), (33), and (34) correspond to (10), (11), and (12), but one has to work harder to infer the former ones. In particular, besides **SA** or **SA**^{m}, we have to make heavy use of **D** and **T.**

Remark 11. Our characterizations are non-redundant. The value $\varphi^{(1)}$ from (19) meets **T**, **2E**, **SA**, and **SA**^{m}, but not **D**; the Shapley (1953) value obeys **T** and **D**, but neither **SA** nor **SA**^{m}; the value from Nowak (1997, Counterexample) satisfies **D**, **SA**, and **SA**^{m}, but not **T**.

Remark 12. One easily checks that the proof of Lemma 8 does not leave the domain of superadditive games. Further, the arguments of Dubey and Shapley (1979, p. 105) employed by Lehrer (1988, p. 98) also stay within this domain. Hence, Theorem 7 works within the domain of superadditive simple games.

5. Concluding remarks

In a sense, one main point of this paper is that amalgamation properties have strong symmetry implications. Indeed, 2-efficiency entails even isomorphism invariance (below), which implies symmetry. Hence, the redundancy of the characterization of Lehrer (1988, Theorem B) also drops from Lemmas 4 and 5, and Theorem 9 below.

Isomorphism invariance, II. For any bijection $\pi : N \to N'$ and $v \in \mathbb{V}(N)$, we have $\varphi_{\pi(i)}(N', v \circ \pi^{-1}) = \varphi_i(N, v)$ for all $i \in N$, where $v \circ \pi^{-1} \in \mathbb{V}(N')$ is given by

$$\left(v \circ \pi^{-1}\right)(K') = v\left(\pi^{-1}(K')\right), \qquad K' \subseteq N'. \tag{38}$$

Theorem 9. ***2E*** *implies* ***II***.

Proof. Let φ obey **2E** and $v \in \mathbb{V}(N)$. For $i \in N$ and $k \notin N$, let $v^{ki} \in \mathbb{V}((N \setminus \{i\}) \cup \{k\})$ be given by

$$v^{ki}(K) = \begin{cases} v(K), & k \notin K, \\ (K \setminus \{k\}) \cup \{i\}, & k \in K, \end{cases} \qquad K \subseteq (N \setminus \{i\}) \cup \{k\} \tag{39}$$

and let $v^{(k)} \in \mathbb{V}(N \cup \{k\})$ be given by

$$v^{(k)}(K) = v(K \setminus \{k\}), \qquad K \subseteq N \cup \{k\}. \tag{40}$$

Equations (2), (39), and (40) imply $v^{(k)}_{ik} = v$ and $v^{(k)}_{ki} = v^{ki}$ for $i \in N$. Hence,

$$\varphi_i(N, v) \overset{\mathbf{2E}}{=} \varphi_i\left(N \cup \{k\}, v^{(k)}\right) + \varphi_k\left(N \cup \{k\}, v^{(k)}\right) \overset{\mathbf{2E}}{=} \varphi_k\left((N \setminus \{i\}) \cup \{k\}, v^{ki}\right). \tag{41}$$

By (2) and (39), we have $v_{ji} = v^{ki}_{jk}$ for $j \in N \setminus \{i\}$. Therefore,

$$\begin{aligned} \varphi_i(N, v) + \varphi_j(N, v) &\overset{\mathbf{2E}}{=} \varphi_j(N \setminus \{i\}, v_{ji}) = \varphi_j\left(N \setminus \{i\}, v^{ki}_{jk}\right) \\ &\overset{\mathbf{2E}}{=} \varphi_j\left((N \setminus \{i\}) \cup \{k\}, v^{ki}\right) + \varphi_k\left((N \setminus \{i\}) \cup \{k\}, v^{ki}\right). \end{aligned}$$

In view of (41), this entails

$$\varphi_j(N, v) = \varphi_j\left((N \setminus \{i\}) \cup \{k\}, v^{ki}\right). \tag{42}$$

Let now $\pi : N \to N'$ be bijective. Since π factors through some player set N'' such that $N \cap N'' = N' \cap N'' = \emptyset$, w.l.o.g., we assume that $N \cap N' = \emptyset$ and $N = \{1, \ldots, n\}$, $n \in \mathbb{N}$. By (39) and (38), we have

$$v \circ \pi^{-1} = \left(\left(v^{\pi(1)1}\right)^{\pi(2)2} \cdots\right)^{\pi(n)n}.$$

Successive application of (41) and (42) finally gives

$$\varphi_i(N,v) = \varphi_{\pi(i)}\left(N', v \circ \pi^{-1}\right), \qquad i \in N. \qquad \square$$

By the proof of Lehrer (1988, Remark 3), we know that any value ψ for two-player games has at most one **2E**-extension, i.e., some value φ that obeys **2E** and that coincides with ψ for two-player games. Quite naturally now the question comes to mind which values for two-player games actually allow for a **2E**-extension. While Theorem 9 already entails that such values must obey **II**, the following example reveals that **II** alone does not suffice. As it turns out, we do not need too much more.

Let φ be given by $\varphi_i(N,v) = \lambda_N(v)$, $i \in N$, $v \in \mathbb{V}(N)$ for $|N| = 2$. It is quite immediate that φ satisfies **II**. While φ can be (uniquely) extended by **2E** to three-player games, this extension is impossible downwards to one-player games or upwards up to games with four or more players. To see the former, one easily checks that φ given by

$$\varphi_i(N,v) = \lambda_{N\setminus\{i\}}(v) + \frac{\lambda_N(v)}{2}, \qquad i \in N, \ |N| = 3, \ v \in \mathbb{V}(N), \tag{43}$$

does the job.

Consider $N = \{1,2\}$, $N' = \{1,3\}$, $v = u_{\{1\}}^{\{1,2\}}$, and $v' = u_{\{1,3\}}^{\{1,3\}}$, and observe $v_{12} = v'_{13} = u_{\{1\}}^{\{1\}}$. If φ were extendable by **2E** to one-player games, we had

$$0 = \varphi_N(N,v) \overset{\mathbf{2E}}{=} \varphi_1(\{1\}, v_{12}) = \varphi_1\left(\{1\}, u_{\{1\}}^{\{1\}}\right) = \varphi_1(\{1\}, v'_{13}) \overset{\mathbf{2E}}{=} \varphi_{N'}(N', v') = 2.$$

Contradiction.

Let now $N = \{1,2,3,4\}$ and $v = u_{\{1,2,3\}}^N + u_{\{2,4\}}^N$. Suppose φ were extendable by **2E** upwards up to four-player games. We then had

$$\varphi_i(N,v) + \varphi_j(N,v) \overset{\mathbf{2E}}{=} \varphi_i(N, v_{ij}), \qquad (i,j) \in N \times N, \ i \neq j. \tag{44}$$

Solving the system of equations (44), for $(i,j) = (1,2), (1,3), (2,3)$ for $\varphi_1(N,v)$ and applying (2) and (43) gives

$$\varphi_1(N,v) = \frac{\varphi_1(N,v_{12}) + \varphi_1(N,v_{13}) - \varphi_2(N,v_{23})}{2} = \frac{1}{2};$$

and for $(i,j) = (1,2), (1,4), (2,4)$, we obtain

$$\varphi_1(N,v) = \frac{\varphi_1(N,v_{12}) + \varphi_1(N,v_{14}) - \varphi_2(N,v_{24})}{2} = 0.$$

Contradiction.

Note what goes wrong with our example. It fails the following property which is easy to be checked to be necessary and sufficient for extendability to one-player games. Moreover, this is all we need in addition to **II** in order to make a value for two-player games to be **2E**-extendable.

1-Extendability, 1X. For $|N| = |N'| = 2$, $v \in \mathbb{V}(N)$, $v' \in \mathbb{V}(N')$, and $v(N) = w(N')$, we have $\varphi_N(N, v) = \varphi_{N'}(N', v')$.

Theorem 10. *If ψ satisfies **1X** and **II** for two-player games, then there is a unique value φ that satisfies **2E** and that coincides with ψ for two-player games.*

Proof. Let ψ obey **1X** and **II** and let $\psi = \varphi$ for two-player games. Remains to show the existence of such φ. For $v \in \mathbb{V}(\{i\})$, set $\varphi_i(\{i\}, v) = \psi_N(N, w)$, where $|N| = 2$, and $w \in \mathbb{V}(N)$ such that $w(N) = v(\{i\})$. By **1X**, φ is well-defined for one-player games; and by construction, φ meets **2E** for two-player games.

We proceed by induction on $|N|$.

Induction basis: As shown above and by assumption, φ meets **2E** and **II** for two-player games.

Induction hypothesis: Let now φ be defined for games with at most $n \geq 2$ players and let φ satisfy **2E** and **II**.

Induction step: We show that φ can be extended to $n+1$-player games such that **2E** and **II** are met. Let $|N| = n+1$. Fix $i \in N$ and pick some $j, k \in N\setminus\{i\}$, $j \neq k$. We define $\varphi_i(N, v)$ to be the unique solution of the system of equations

$$\varphi_\alpha(N, v) + \varphi_\beta(N, v) = \varphi_\alpha(N, v_{\alpha\beta}), \qquad (\alpha, \beta) \in N \times N,\ \alpha \neq \beta \tag{45}$$

for $(\alpha, \beta) = (i, j), (i, k), (j, k)$, which turns out to be

$$\varphi_i(N, v) = \frac{\varphi_i(N\setminus\{j\}, v_{ij}) + \varphi_i(N\setminus\{k\}, v_{ik}) - \varphi_j(N\setminus\{k\}, v_{jk})}{2}. \tag{46}$$

In order to show that $\varphi_i(N, v)$ is well-defined, we have to show (i) that $\varphi_\alpha(N, v_{\alpha\beta}) = \varphi_\beta(N, v_{\beta\alpha})$ in (45) and that (ii) $\varphi_i(N, v)$ does not depend on the choice of j and k for $|N| > 3$. The former, i.e., (45), drops from (2) and the induction hypothesis, in particular **II**. With respect to (ii), it suffices to establish that $\varphi_i(N, v)$ remains unchanged when we replace k by $\ell \in N\setminus\{i, j, k\}$ in (46). So, it remains to show

$$\begin{aligned} A &:= \varphi_i(N\setminus\{k\}, v_{ik}) + \varphi_j(N\setminus\{\ell\}, v_{j\ell}) \\ &\quad = \varphi_i(N\setminus\{\ell\}, v_{i\ell}) + \varphi_j(N\setminus\{k\}, v_{jk}) =: B. \end{aligned} \tag{47}$$

By the induction hypothesis, in particular **2E**, the analogon of (46) holds for games with n players. Hence, we have

$$\varphi_i\left(N\backslash\left\{k\right\},v_{ik}\right)=\frac{\varphi_i\left(N\backslash\left\{j,k\right\},\left(v_{ik}\right)_{ij}\right)+\varphi_i\left(N\backslash\left\{k,\ell\right\},\left(v_{ik}\right)_{i\ell}\right)-\varphi_j\left(N\backslash\left\{k,\ell\right\},\left(v_{ik}\right)_{j\ell}\right)}{2},\tag{48}$$

$$\varphi_j\left(N\backslash\left\{\ell\right\},v_{j\ell}\right)=\frac{\varphi_i\left(N\backslash\left\{j,\ell\right\},\left(v_{j\ell}\right)_{ij}\right)-\varphi_i\left(N\backslash\left\{k,\ell\right\},\left(v_{j\ell}\right)_{ik}\right)+\varphi_j\left(N\backslash\left\{k,\ell\right\},\left(v_{j\ell}\right)_{jk}\right)}{2},\tag{49}$$

$$\varphi_i\left(N\backslash\left\{\ell\right\},v_{i\ell}\right)=\frac{\varphi_i\left(N\backslash\left\{j,\ell\right\},\left(v_{i\ell}\right)_{ij}\right)+\varphi_i\left(N\backslash\left\{k,\ell\right\},\left(v_{i\ell}\right)_{ik}\right)-\varphi_j\left(N\backslash\left\{k,\ell\right\},\left(v_{i\ell}\right)_{jk}\right)}{2},\tag{50}$$

$$\varphi_j\left(N\backslash\left\{k\right\},v_{jk}\right)=\frac{\varphi_i\left(N\backslash\left\{j,k\right\},\left(v_{jk}\right)_{ij}\right)-\varphi_i\left(N\backslash\left\{k,\ell\right\},\left(v_{jk}\right)_{i\ell}\right)+\varphi_j\left(N\backslash\left\{k,\ell\right\},\left(v_{jk}\right)_{j\ell}\right)}{2}.\tag{51}$$

By (2), we have $\left(v_{ik}\right)_{ij}=\left(v_{jk}\right)_{ij}$, $\left(v_{j\ell}\right)_{ij}=\left(v_{i\ell}\right)_{ij}$, $\left(v_{ik}\right)_{i\ell}=\left(v_{i\ell}\right)_{ik}$, $\left(v_{j\ell}\right)_{jk}=\left(v_{jk}\right)_{j\ell}$, $\left(v_{ik}\right)_{j\ell}=\left(v_{j\ell}\right)_{ik}$, and $\left(v_{i\ell}\right)_{jk}=\left(v_{jk}\right)_{i\ell}$. Hence by (47)–(51),

$$\begin{aligned}A-B &= \frac{\varphi_i\left(N\backslash\left\{k,\ell\right\},\left(v_{i\ell}\right)_{jk}\right)+\varphi_j\left(N\backslash\left\{k,\ell\right\},\left(v_{i\ell}\right)_{jk}\right)}{2}\\ &\quad -\frac{\varphi_i\left(N\backslash\left\{k,\ell\right\},\left(v_{ik}\right)_{j\ell}\right)+\varphi_j\left(N\backslash\left\{k,\ell\right\},\left(v_{ik}\right)_{j\ell}\right)}{2}.\end{aligned}$$

The induction hypothesis, in particular **2E**, now entails

$$A-B=\frac{\varphi_i\left(N\backslash\left\{j,k,\ell\right\},\left(\left(v_{i\ell}\right)_{jk}\right)_{ij}\right)-\varphi_i\left(N\backslash\left\{j,k,\ell\right\},\left(\left(v_{ik}\right)_{j\ell}\right)_{ij}\right)}{2}.$$

By (2), we have $\left(\left(v_{i\ell}\right)_{jk}\right)_{ij}=\left(\left(v_{ik}\right)_{j\ell}\right)_{ij}$, hence, $A=B$. Note that it is this step where we need **2E**-extendability to one-player games in order to show upwards **2E**-extendability to four-player games. By construction, φ meets **2E** for $n+1$-player games.

Remains to show that φ obeys **II** for $n+1$-player games. Consider $|N| = |N'| = n+1$ and some bijection $\pi : N \to N'$. For $i, j \in N$, $i \neq j$, we have

$$\begin{aligned}
&\varphi_{\pi(i)}\left(N', v\circ\pi^{-1}\right) + \varphi_{\pi(j)}\left(N', v\circ\pi^{-1}\right)\\
\overset{\mathbf{2E}}{=}\;& \varphi_{\pi(i)}\left(N'\backslash\left\{\pi\left(j\right)\right\}, \left(v\circ\pi^{-1}\right)_{\pi(i)\pi(j)}\right)\\
\overset{(2)}{=}\;& \varphi_{\pi|_{N\backslash\{j\}}(i)}\left(\pi|_{N\backslash\{j\}}\left(N\backslash\left\{j\right\}\right), v_{ij}\circ\pi^{-1}|_{\pi(N\backslash\{j\})}\right)\\
\overset{\mathbf{II}}{=}\;& \varphi_i\left(N\backslash\left\{j\right\}, v_{ij}\right)\\
\overset{\mathbf{2E}}{=}\;& \varphi_i\left(N, v\right) + \varphi_j\left(N, v\right).
\end{aligned} \tag{52}$$

Since $|N| > 2$, the system of linear equations (52), $i, j \in N$, $i \neq j$ has the unique solution, $\varphi_{\pi(i)}\left(\pi\left(N\right), v\circ\pi^{-1}\right) = \varphi_i\left(N, v\right), i \in N$. Thus, φ obeys **II**. □

Remark 13. From the proof of Theorem 10 it is clear that **II** suffices to guarantee **2E**-extendability to three-player games.

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