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An algebraic approach to general aggregation theory: Propositional-attitude aggregators as MV-homomorphisms*

Frederik Herzberg†

Abstract

This paper continues Dietrich and List's [2010] work on propositional-attitude aggregation theory, which is a generalised unification of the judgment-aggregation and probabilistic opinion-pooling literatures. We first propose an algebraic framework for an analysis of (many-valued) propositional-attitude aggregation problems. Then we shall show that systematic propositional-attitude aggregators can be viewed as homomorphisms in the category of C.C. Chang's [1958] MV-algebras. Since the 2-element Boolean algebra as well as the real unit interval can be endowed with an MV-algebra structure, we obtain as natural corollaries two famous theorems: Arrow's theorem for judgment aggregation as well as McConway's [1981] characterisation of linear opinion pools.

Key words: propositional attitude aggregation; judgment aggregation; linear opinion pooling; Arrow's impossibility theorem; many-valued logic; MV-algebra; homomorphism; Arrow's impossibility theorem; functional equation

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1 Introduction

Recently, Dietrich and List [2010] have proposed the fundamentals of a general theory of aggregation, with the aim of creating a unified theory of preference aggregation, judgment aggregation (cf. List and Puppe [2009] for a survey), probabilistic opinion pooling and more general many-valued aggregation problems. In a very general logical framework, Dietrich and List [2010] have proved that for sufficently complex aggregation problems, all independent and Paretian aggregators are already systematic (a stronger independence condition, known as Strong Setwise Property in the probabilistic opinion pooling literature following McConway [1981]). In an earlier paper, Dietrich and List [2008] had already shown that judgment aggregation can be treated as a special case of generalised probabilistic opinion pooling and that in this setting, Arrow's dictatorial impossibility theorem for judgment aggregation is a special case of a generalisation of McConway's [1981] characterisation of linear opinion pools.

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So far, however, no characterisations of general systematic many-valued propositional-attitude aggregators are known, nor has the case of infinite electorates been treated as yet. The present paper attempts to fill this gap: We generalise the main idea in Herzberg [2010] and prove that systematic many-valued propositional-attitude aggregators are homomorphisms in the category of MV-algebras (for short: MV-homomorphisms) as defined by C.C. Chang [1958a]. For the special case of a finite electorate, we obtain as natural corollaries both Arrow's dictatorial impossibility theorem for judgment aggregation and McConway's [1981] characterisation of linear opinion pools.

We should note at this point that the use of algebraic — in particular, lattice-theoretic and Boolean algebraic — methods has a long tradition in preference aggregation theory, cf. e.g. D.J. Brown [1974] and the monographs by Kim and Roush [1980] or Aleskerov [1999].[1] (A proof of Arrow's [1963] impossibility theorem using ultrafilters was published by Fishburn even as early as 1970.) Some authors have also employed filters and ultrafilters to establish impossibility theorems in judgment aggregation (e.g. Daniëls [2006], Dietrich and Mongin [2010] and Klamler and Eckert [2009]). Moreover, the relation between merging of opinions and certain functional equations — which often can be interpreted as homomorphy relations! — has long been recognised in the opinion pooling literature (cf. e.g. Aczél, Kannapan, Ng and Wagner [1983] and Aczél [1989]). Nevertheless, with the exception of the aforementioned paper (Herzberg [2010]), the published literature does not contain any systematic approaches to tackle general aggregation problems from an algebraic, lattice-theoretic perspective.

In this paper, we will first outline a formal framework for rather general many-valued aggregation problems by means of the notion of an MV-algebra (Section 2). We shall then list a number of assumptions, mainly generalisations of standard Arrovian responsiveness axioms for the aggregation functions (Section 3). Thereafter, we shall state a characterisation theorem for aggregators as MV-algebra homomorphisms and derive two well-known corollaries from judgment aggregation and probabilistic opinion pooling (Section 4); the proofs can be found in an appendix. Possible extensions of our methodology are discussed in the final Section 5.

2 Formal framework

In the following, we describe a formal model for the aggregation of many-valued propositional attitudes. The electorate will be given by some (finite or infinite) set N. In addition, a set of propositions X (*agenda*) in a sufficiently expressive language will be fixed, and the electorate as well as each individual will be supposed to display a certain *attitude* towards each proposition in the agenda (thus assigning a *truth value*). The set of possible attitudes or truth values will be denoted M (and will be assumed to possess some additional structure, viz. that of an MV-algebra). Thus, each individual expresses his or her attitudes towards the elements of the agenda through a function from X to M, called *attitude function*. Then the attitudes of all individuals can be captured by an N-sequence of attitude functions (i.e. by a map from N to M^X); such an N-sequence will be called *profile*. An *aggregator* is then simply a map from (a suitable subset of the set of) the set of profiles to the set of attitude functions.

[1] I would like to thank Professor Bernard Monjardet at this point.

2.1 Agenda syntax

Let $\mathcal{L}$ be the language of *many-valued propositional logic*. In other words, let $\mathcal{L}$ be the language whose symbols consist of countably many propositional variables, a propositional constant 0 (falsehood), a binary operation $\oplus$ (strong disjunction) and a unary operator $\neg$ (negation). The set of well-formed formulae in this language shall be denoted $\mathbf{L}$.

A number of standard abbreviations will be helpful. First, we define a new propositional constant 1 (truth) by $\neg 0$. Next, we define additional operations. The operation of weak disjunction (denoted $\vee$) will be defined via

$$\forall p, q \in \mathbf{L} \qquad p \vee q = \neg(\neg p \oplus q) \oplus q,$$

and strong conjunction (denoted $\otimes$) as well as weak conjunction (denoted $\wedge$) can then be defined through De Morgan's laws:

$$\begin{aligned} \forall p, q \in \mathbf{L} \qquad p \otimes q &= \neg(\neg p \oplus \neg q), \\ p \wedge q &= \neg(\neg p \vee \neg q). \end{aligned}$$

The implication operation (denoted $\rightarrow$) can be defined as

$$p \rightarrow q = \neg p \oplus q.$$

(In their original paper, Łukasiewicz and Tarski [1930] took $\neg$ and $\rightarrow$ as the primitive logical symbols of their language.) Łukasiewicz logic is then given by the provability relation $\vdash$, given by modus ponens (i.e. for all $p, q \in \mathbf{L}$ and $S \subseteq \mathbf{L}$, if $S \vdash p$ and $S \vdash (p \rightarrow q)$, then $S \vdash q$) and the following axiom schemes: For all $p, q \in \mathbf{L}$,

A1. For all $p, q \in \mathbf{L}$, the proposition $p \rightarrow (q \rightarrow p)$ is an axiom.

A2. For all $p, q \in \mathbf{L}$, the proposition $(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$ is an axiom.

A3. For all $p, q \in \mathbf{L}$, the proposition $((p \rightarrow q) \rightarrow q) \rightarrow ((q \rightarrow p) \rightarrow p)$ is an axiom.

A4. For all $p, q \in \mathbf{L}$, the proposition $(\neg p \rightarrow \neg q) \rightarrow (q \rightarrow p)$ is an axiom.

(Cf. Rose and Rosser [1958] and Chang [1958b].)

One can define a relation $\equiv$, called *provable equivalence*, on $\mathbf{L}$ by saying that p is *provably equivalent* to q (denoted $p \equiv q$) if and only if both $\vdash (p \rightarrow q)$ as well as $\vdash (q \rightarrow p)$ (wherein $\vdash p$ is, for all $p \in \mathbf{L}$, shorthand for $\varnothing \vdash p$). It is not difficult to verify that $\equiv$ is an equivalence relation on $\mathbf{L}$. The set of equivalence classes shall be denoted $\mathbf{L}/\equiv$. Representative-wise, one can define the constant 0, the operator $\neg$ and the operation $\oplus$ on $\mathbf{L}/\equiv$; again, it is not hard to prove that these are well-defined. Therefore, the operations $\otimes, \vee, \wedge, \rightarrow$ can be defined on $\mathbf{L}$ as well.

2.2 Agenda semantics

Recall that an MV-algebra M is a structure $(M, \oplus, \neg, 0)$ such that $(M, \oplus, 0)$ is a commutative monoid (i.e. $\oplus$ is a commutative and associative binary operation on M with neutral element 0) and the following identities are satisfied for all $x, y \in M$:

- $\neg\neg x = x$,
- $x \oplus 1 = 1$,
- $x \vee y = y \vee x$,

wherein 1 is shorthand for $\neg 0$ and $\vee$ is defined via

$$\forall x, y \in M \qquad x \vee y = \neg(\neg x \oplus y) \oplus y.$$

It turns out that the structure $(\mathbf{L}/\equiv, \oplus, \neg, 0)$, i.e. the set of equivalence classes of provably equivalent formulae from many-valued propositional logic with the canonical operations, is an MV-algebra, the so-called *Lindenbaum algebra* for Łukasiewicz's many-valued logic. This observation allows us to take an algebraic approach to the semantics of many-valued propositional logic, essentially due to C.C. Chang [1958a, 1959]: Let us henceforth assume that the truth values form an MV-algebra; we shall hence fix an MV-algebra M for the rest of this paper and shall refer to it as the *set of truth values*. Under these hypotheses, an *M-valuation* can be defined as an MV-algebra homomorphism from $\mathbf{L}/\equiv$ to M. If I is an M-valuation and $p \in \mathbf{L}$, we shall usually simply write $I(p)$ instead of $I([p]_\equiv)$.

Important examples of MV-algebras are the following (cf. already Chang [1958a]):

- Any Boolean algebra is an MV-algebra.
- If $M = [0,1]$, the set of all real numbers between 0 and 1, one obtains an MV-algebra with zero element 0 by setting $\neg x = 1 - x$ and $x \oplus y = \min\{x + y, 1\}$ for all $x, y \in [0,1]$. This is called the *standard MV-algebra*. It is the set of truth values for the infinite-valued logic $L_{\beth_1}$.
- With the same definitions for 0, $\neg$ and $\oplus$, the set $M = [0,1] \cap \mathbf{Q}$ (the set of all rational numbers between 0 and 1) is an MV-algebra; it is the set of truth values for Łukasiewicz's infinite-valued logic $L_{\aleph_0}$.
- Again with the same definitions for 0, $\neg$ and $\oplus$, the set $M = \{0, 1/m, \ldots, (m-1)/m, 1\}$ is an MV-algebra for every positive integer m. It is the set of truth values for Łukasiewicz's $(m+1)$-valued logic L_{m+1}.

Rose and Rosser [1958] and Chang [1959] have shown, each by a different method, the completeness of $L_{\aleph_0}$.

2.3 Attitude functions

Consider a set $X \subseteq \mathbf{L}$, henceforth called the *agenda*. *Attitude functions* are functions from X to M. An attitude function A is *rational* if and only if it can be extended to an M-valuation, i.e. there exists an M-valuation I such that $A(p) = I([p]_\equiv)$ for all $p \in X$. Therefore, any rational attitude function A is also well-defined not only on X, but on the closure of X under $\neg$ and $\oplus$. A *(rational) profile* is an N-sequence of (rational) attitude functions. An *attitude aggregator* is a map from a subset of the set of profiles to the set of attitude functions. An attitude aggregator is a *dictatorship* if and only if there exists some $i \in N$ such that $F(\underline{A}) = A_i$ for all $i \in N$.

An important observation is the extendibility of rational attitude functions:

Remark 1. *Denote the closure of the agenda under* $\neg$ *and* $\oplus$ *by* Y*. Any rational attitude function can be uniquely extended to a function from* Y *to* M*.*

We shall always identify the extension of any rational attitude function A with its extension to Y. Thus, in the following, *all rational attitude functions are assumed to be defined on the whole of* Y*.*

2.4 Examples

Examples for this framework are the "classical" propositional judgment aggregation, where M is $2 = \{0, 1\}$ endowed with the Boolean algebra structure, or probabilistic opinion pooling, where M is the standard MV-algebra $[0, 1]$ and the agenda corresponds to the σ-algebra of events. However, this framework also encompasses aggregation problems with respect to more general many-valued logics: All that is required is that the set of truth values forms an MV-algebra; by that means, the framework proposed in this paper covers aggregation of propositions in a large class of finite- and infinite-valued logics as well. For example, voting with abstentions can easily be modeled as an aggregation problem in a three-valued logic, e.g. Łukasiewicz's L_3.

3 Aggregator responsiveness axioms

In this section, we generalise (mostly standard) terminology from aggregation theory, in order to be able to formulate our subsequent results on propositional-attitude aggregators. We shall use the abbrevation

$$\underline{A}(p) = (A_i(p))_{i \in N}$$

for all propositions $p \in X$ and all profiles $\underline{A} \in \left(M^X\right)^N$.

Definition 2. *An attitude aggregator* F *is* rational *if and only if for all rational profiles* $\underline{A}$ *in the domain of* F*,* $F(\underline{A})$ *is a rational attitude function.*

Definition 3. *An attitude aggregator* F *is* universal *if and only if its domain comprises all rational profiles.*

Independent aggregation means that the aggregate attitude towards any proposition p does not depend on the individuals' attitudes towards propositions other than p:

Definition 4. *An attitude aggregator* F *is* independent *if and only if there exists a map* $G : M^N \times X \to M$ *such that for all profiles* $\underline{A}$ *in the domain of* F *and for all* $p \in X$*,* $F(\underline{A})(p) = G\left(\underline{A}(p), p\right)$*.*

Systematic aggregation is a special case of independent aggregation, where G is constant in the second component, i.e. the aggregate attitude towards any proposition p only depends on p through the individuals' attitudes towards p:

Definition 5. *An attitude aggregator* F *is* systematic *if and only if there exists a map* $f : M^N \to M$*, called* decision criterion *of* F*, such that for all profiles* $\underline{A}$ *in the domain of* F *and for all* $p \in X$*,*

$$F(\underline{A})(p) = f\left(\underline{A}(p)\right). \tag{1}$$

Remark 6. *If the agenda contains some strictly contingent sentence p_0, then any universal systematic attitude aggregator F has a* unique *decision criterion.*

Though it appears much stronger at first sight, systematicity is under mild conditions actually equivalent to independence (cf. Dietrich and List [2010, Theorem 2]). At least as strong is the following notion (recall that rational attitude functions can be uniquely extended to the closure of the agenda under $\neg$ and $\oplus$):

Definition 7. *A systematic attitude aggregator F is* strongly systematic *if and only if Equation* (1) *holds even for all p in the closure of X under $\neg$ and $\oplus$ and all profiles $\underline{A}$ in the domain of F.*

If the agenda is closed under $\neg$ and $\oplus$, then systematicity and strong systematicity trivially coincide.

The *Pareto principle* asserts that any proposition which is rejected unanimously by all individuals, must be collectively rejected:

Definition 8. *An attitude aggregator F is* Paretian *if and only if for all profiles $\underline{A}$ in the domain of F and all $p \in X$, if $A_i(p) = 0$ for all $i \in N$ (i.e. $\underline{A}(p) = \underline{0}$), then $F(\underline{A})(p) = 0$.*

A formula $p \in \mathbf{L}$ is called *strictly contingent* if and only if there exists for all $x \in M$ some M-valuation I with $I(p) = x$. For most of the paper, we need to impose additional assumptions on the logical expressivity or complexity of the agenda.

Definition 9. *The agenda X is called* complex *if and only if there exists a strictly contingent propositions p_0 in X as well as strictly contingent propositions p_1, p_2, p_3, q_1, q_2 in the closure of X under $\neg$ and $\oplus$ such that for all M-valuations I, one has $I(p_1) \oplus I(p_2) = I(p_3)$ and $\neg I(q_1) = I(q_2)$. If p_1, p_2, p_3, q_1, q_2 are even in X, then X is said to be* rich.

Any conceivable combination of truth values can be obtained through M-valuations of elements of rich agendas, hence their name. This will be the key to the proof of our main result, via the notion of *strongly systematisable* aggregators:

Definition 10. *In this paper, a systematic attitude aggregator F for a complex agenda X is called* strongly systematisable *if and only if F is either strongly systematic or the agenda X is rich.*

4 Results

Note that M^N is — as the direct product of card(N) identical copies of M — again an MV-algebra; the strong disjunction $\oplus_N$ and negation $\neg_N$ are defined componentwise, the zero element 0_N is just the N-sequence $\underline{0}$ of 0's.

Theorem 11. *If F is a rational, universal, Paretian and strongly systematisable attitude aggregator, then the decision criterion of F is an MV-homomorphism.*

Conversely, if f is an MV-homomorphism and F is defined by Equation (1) *for all rational profiles $\underline{A}$ and all $p \in X$, then F is a rational, universal, Paretian and systematic attitude aggregator.*

(The uniqueness of the decision criterion had already been noted in Remark 6.)

If $M = \{0, 1\}$ with the usual Boolean structure, then M^N is again a Boolean algebra and isomorphic to the power-set Boolean algebra of N. This allows us to deduce, as an easy corollary to Theorem 11, the recent result in Herzberg [2010]. Ultimately, this leads to Arrow's impossibility theorem for judgment aggregation (recall the previous remark about the equivalence of systematicity and independence under mild conditions).

Corollary 12. *Suppose F is a rational, universal, Paretian and strongly systematisable attitude aggregator. If the algebra of truth values is Boolean, then the decision criterion of F is a Boolean homomorphism.*

If the algebra of truth values is just the Boolean algebra $\{0, 1\}$ and the electorate N is finite, then F is a dictatorship.

If M is the standard MV-algebra, then Theorem 11 yields McConway's [1981] characterisation of linear opinion pools ("weighted averaging") as a second corollary:

Corollary 13. *Let F be a rational, universal, Paretian and strongly systematisable aggregator, and let the algebra of truth values be the standard MV-algebra $[0, 1]$. If the electorate N is finite, then the decision criterion of F is a linear map from $[0, 1]^N$ to $[0, 1]$.*

5 Discussion

We have seen that one can neatly formulate an aggregation theory for general many-valued propositional attitudes based on the theory of MV-algebras. Aggregators satisfying common responsiveness axioms (agenda complexity resp. richness, collective rationality, universality, systematicity resp. strong systematicity, Pareto principle) then simply correspond to MV-homomorphisms from M^N to M (M being the MV-algebra of truth values). For special cases of M, one can use classical classification results for such homomorphisms to obtain a classification of Paretian systematic aggregators, e.g. if M is the Boolean algebra $2 = \{0, 1\}$ (which leads to the judgment-aggregation analogue of Arrow's [1963] impossibility theorem, cf. Dietrich and List [2007]) or $M = [0, 1]$ (which entails McConway's [1981] characterisation of linear opinion pooling). More general aggregator classifications might be derived from MV-algebra classifications (cf. Chang [1959], Mundici [1986], Cignoli and Mundici [1997]).

This algebraic approach to aggregation theory could be taken further by taking Heyting algebras or BL-algebras as sets of truth values. By that means, aggregation of intuitionistic resp. fuzzy propositional attitudes could be studied in full generality.

A powerful alternative to algebraic aggregation theory is the model-theoretic approach pioneered by Lauwers and Van Liedekerke [1995], as it allows to study aggregation problems for predicate logic in a natural manner as well. It remains to be seen whether even many-valued aggregation problems can be studied by model-theoretic methods; such an approach could pave the way for a systematic analysis of aggregation problems in many-valued predicate logic. The algebraic approach to many-valued model theory proposed by Zlatoš [1981] might be a first starting point for such an endeavour.

A Proofs

Proof of Remark 1. Let A be a rational attitude function. Consider any two M-valuations I, I' such that $A(p) = I(p) = I'(p)$ for all $p \in X$ or, more precisely, $A(p) = I([p]_\equiv) = I'([p]_\equiv)$ for all $p \in X$. In other words, I and I' are homomorphisms from $\mathbf{L}/\equiv$ to M which agree on the set $[X]_\equiv$ of $\equiv$-equivalence classes of elements of X. Thus, they must agree on the closure of $[X]_\equiv$ under the operations $\neg$ and $\oplus$ in $\mathbf{L}/\equiv$. Call this closure $[Y]_\equiv$. Since the operations $\neg$ and $\oplus$ in $\mathbf{L}/\equiv$ are defined representative-wise, $[Y]_\equiv$ equals the set of equivalence classes of elements of Y (the closure of X under the operations $\neg$ and $\oplus$ in $\mathbf{L}$). It follows that $I([p]_\equiv) = I'([p]_\equiv)$ for all $p \in Y$. □

Proof of Remark 6. Let p_0 be as in the statement of Remark 6, let f and f' be decision criteria of F, and let $\underline{x} = (x_i)_{i \in N} \in M^N$. Then there exists for each $i \in N$ some M-valuation I_i such that $I_i(p_0) = x_i$. Now each I_i induces a rational attitude function A_i defined by $A_i(p) = I_i(p)$ for all $p \in X$, so that in particular $A_i(p_0) = x_i$ for all $i \in M$. As F is universal, the profile $\underline{A} = (A_i)_{i \in N}$ is in the domain of F. Hence

$$f(\underline{x}) = f\left(\underline{A}(p_0)\right) = F(\underline{A})(p_0) = f'\left(\underline{A}(p_0)\right) = f'(\underline{x}).$$

□

Proof of Theorem 11. Let F be a rational, universal, Paretian and strongly systematisable attitude aggregator, and let f be the decision criterion of F.

Consider any two elements of M^N, $\underline{x} = (x_i)_{i \in N}$ and $\underline{y} = (y_i)_{i \in N}$. Let p_1, p_2, p_3, q_1, q_2 be as in the definition of agenda complexity. Then on the one hand, since p_1, p_2, p_3, q_1, q_2 are strictly contingent by assumption, there exists for each $i \in N$ some M-valuations I_i, I_i', I_i'' such that

- $I_i(p_1) = x_i$ and $I_i(p_2) = y_i$,
- $I_i'(q_1) = x_i$,
- $I_i''(p_1) = 0$.

On the other hand, since p_1, p_2, p_3, q_1, q_2 were assumed to be as in the definition of agenda complexity, it follows for each $i \in N$,

- not only $I_i(p_1) = x_i$ and $I_i(p_2) = y_i$, but also $I_i(p_3) = x_i \oplus y_i$ and $I(p_3) = I(p_1) \oplus I(p_2)$ for every M-valuation I,
- not only $I_i'(q_1) = x_i$, but also $I_i'(q_2) = \neg x_i$ and $I'(q_2) = \neg I'(q_1)$ for every M-valuation I',
- $I_i''(p_1) = 0$.

In other words, there exists an N-sequence $\underline{I} = (I_i)_{i \in N}$ of M-valuations such that

- $\underline{I}(p_1) = \underline{x}$, $\underline{I}(p_2) = \underline{y}$, $\underline{I}(p_3) = \underline{x} \oplus_N \underline{y}$ and $I(p_3) = I(p_1) \oplus I(p_2)$ for every M-valuation I,
- $\underline{I}'(q_1) = \underline{x}$, $\underline{I}'(q_2) = \neg_N \underline{x}$ and $I'(q_2) = \neg I'(q_1)$ for every M-valuation I',

- $\underline{I}''(p_1) = \underline{0} = 0_N$.

Next note that by restricting each I_i, I_i' and I_i'' to the set of equivalence classes of elements of X (recall that $I(p)$ is shorthand for $I([p]_\equiv)$ for any M-valuation I and any $p \in \mathbf{L}$), one obtains rational attitude functions A_i, A_i' and A_i''. All A_i, A_i' and A_i'' are rational attitude functions and thus can be uniquely extended to Y by Remark 1. Hence, we have constructed rational profiles $\underline{A} = (A_i)_{i\in N}$, $\underline{A}' = (A_i')_{i\in N}$ and $\underline{A}'' = (A_i'')_{i\in N}$ such that

- $\underline{A}(p_1) = \underline{x}$, $\underline{A}(p_2) = \underline{y}$, $\underline{A}(p_3) = \underline{x} \oplus_N \underline{y}$, and $I(p_3) = I(p_1) \oplus I(p_2)$ for every M-valuation I,
- $\underline{A}'(q_1) = \underline{x}$, $\underline{A}'(q_2) = \neg_N \underline{x}$, and $I'(q_2) = \neg I'(q_1)$ for every M-valuation I',
- $\underline{A}''(p_1) = 0_N$.

Note that since F is universal, the profiles $\underline{A}, \underline{A}', \underline{A}''$ must be in the domain of F. Since F is rational, $F(\underline{A})$, $F(\underline{A}')$ and $F(\underline{A}'')$ are rational attitude functions and thus can be uniquely extended to Y by Remark 1. Moreover, there exist M-valuations I and I' such that $F(\underline{A})(p) = I(p)$ as well as $F(\underline{A}')(p) = I'(p)$ for all $p \in X$ and hence, by the homomorphy of I, also for all $p \in Y$. From here, it follows that

- $F(\underline{A})(p_3) = F(\underline{A})(p_1) \oplus F(\underline{A})(p_2)$,
- $F(\underline{A}')(q_2) = \neg F(\underline{A}')(q_1)$.

Let us next exploit the choice of p_1, p_2, p_3, q_1, q_2 as in the definition of agenda complexity and the strong systematicity of F or the richness of X. This yields for any M-valuation I which extends $F(\underline{A})$,

$$\begin{aligned} f(\underline{x} \oplus_N \underline{y}) &= f\left(\underline{A}(p_3)\right) = I(p_3) \\ &= I(p_1) \oplus I(p_2) = f\left(\underline{A}(p_1)\right) \oplus f\left(\underline{A}(p_2)\right) = f(\underline{x}) \oplus f(\underline{y}). \end{aligned}$$

Similarly (this time applying the formulae in the definition of agenda complexity to an M-valuation I which extends $F(\underline{A}')$),

$$\begin{aligned} f(\neg_N \underline{x}) &= f\left(\underline{A}'(q_2)\right) = I(q_2) \\ &= \neg I(q_1) = \neg f\left(\underline{A}'(q_1)\right) = \neg f(\underline{x}). \end{aligned}$$

Thus, f preserves the operators $\neg$ and $\oplus$ and maps the zero element 0_N of M^N to $0 \in M$; hence, f is an MV-homomorphism.

Conversely, let f be an MV-homomorphism. Clearly, the F defined by Equation (1) for all rational profiles $\underline{A}$ and all $p \in X$ is both systematic and universal. Moreover, since f is a homomorphism, any composition of f with an N-sequence of MV homomorphisms from $\mathbf{L}$ to M will again be a homomorphism from $\mathbf{L}$ to M. In other words, the composition of f with an N-sequence of valuations is again a valuation. This shows that the composition of f with a rational profile is a rational attitude function. Hence, the F defined by Equation (1) is rational. Since $f(0_N) = f(\underline{0}) = 0$, it is clear that F is Paretian. □

Proof of Corollary 12. If M is even a Boolean algebra, then so is M^N. By Theorem 11, the decision criterion f is an MV-homomorphism. Since any

MV-homomorphism between two Boolean algebras is a Boolean homomorphism (because the Boolean operations $\vee$ and $\wedge$ as well as the constant 1 can be defined through $\neg$ and $\oplus$: $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$ for all x, y and $1 = \neg 0$), it follows that f is actually a Boolean homomorphism. Boolean algebra teaches that the shell of f, i.e. $f^{-1}\{1\} = f^{-1}\{\neg 0\}$, is a filter in 2^N (which is isomorphic to the power-set Boolean algebra of N), and if $M = \{0, 1\}$, then the shell of f is even an ultrafilter on N. Now if N is finite, this means — as all ultrafilters on finite sets are principal — that there exists some $i_0 \in N$ such that $f^{-1}\{1\} = \{C \subseteq N : i_0 \in C\}$. This, however, implies that F is a dictatorship, the dictator being i_0. □

Proof of Corollary 13. Without loss of generality, we may assume $N = \{1, \ldots, n\}$ for some positive integer n. By Theorem 11, the decision criterion $f : [0, 1]^N \to [0, 1]$ is an MV-homomorphism. This implies, if M is the standard MV-algebra $[0, 1]$, that

$$f(x_1, \ldots, x_n) \oplus f(y_1, \ldots, y_n) = f(x_1 \oplus y_1, \ldots, x_n \oplus y_n)$$

for all $x_1, y_1, \ldots, x_n, y_n \in [0, 1]$. Hence (by the definition of $\oplus$ in the Łukasiewicz algebra, i.e. $x \oplus y = \min\{x + y, 1\}$ for all $x, y \in [0, 1]$ and the componentwise definition of $\oplus$ in the direct power $[0, 1]^N$) one has for all $x_1, y_1, \ldots, x_n, y_n \in [0, 1]$ with $x_i + y_i \leq 1$ for all $i \in N$,

$$f(x_1, \ldots, x_n) + f(y_1, \ldots, y_n) = f(x_1 + y_1, \ldots, x_n + y_n). \qquad (2)$$

One can now emulate McConway's [1981] original argument: An iterated application of the preceding equation yields for all $z_1, \ldots, z_n \in [0, 1]$,

$$\begin{aligned} f(z_1, \ldots, z_n) &= f(z_1, 0, \ldots, 0) + f(0, z_2, \ldots, z_n) \\ &= f(z_1, 0, \ldots, 0) + f(0, z_2, 0, \ldots, 0) + f(0, 0, z_3, \ldots, z_n) \\ &= \sum_{i=1}^{n} f\left(\underbrace{0, \ldots, 0}_{i-1}, z_i, \underbrace{0, \ldots, 0}_{n-i}\right). \end{aligned}$$

Hence, definining f_i by

$$f_i(z) = f\left(\underbrace{0, \ldots, 0}_{i-1}, z, \underbrace{0, \ldots, 0}_{n-i}\right)$$

for every $z \in [0, 1]$ and each $i \in N$, we obtain

$$f(z_1, \ldots, z_n) = \sum_{i=1}^{n} f_i(z_i)$$

for all $z_1, \ldots, z_n \in [0, 1]$. Moreover, Equation (2) also implies $f_i(x + y) = f_i(x) + f_i(y)$ for all $x, y \in [0, 1]$ with $x + y \leq 1$ and each $i \in N$. Therefore, every f_i satisfies Cauchy's functional equation. Also, the range of every f_i is by definition contained in the range of f and thus in $[0, 1]$, whence $f_i(x)$ is nonnegative for all $x \in [0, 1]$ and every $i \in N$. Therefore, there exists for every $i \in N$ some α_i such that $f_i(x) = \alpha_i x$ for all $x \in [0, 1]$ (cf. Aczél [1961, 1966, Section 2.1.1, Theorem 1]), and this α_i must be nonnegative. Thus, $f(z_1, \ldots, z_n) = \sum_{i=1}^{n} \alpha_i z_i$ for all $z_1, \ldots, z_n \in [0, 1]$. □

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