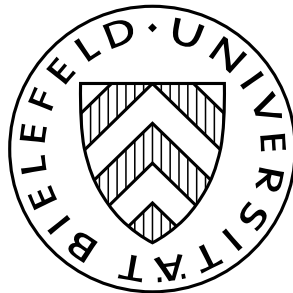# Refined best reply correspondence and dynamics

Dieter Balkenborg, Josef Hofbauer and Christoph Kuzmics

Refined best reply correspondence and dynamics

Dieter Balkenborg,[*] Josef Hofbauer,[†] and Christoph Kuzmics[‡]

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Abstract

We call a correspondence, defined on the set of mixed strategy profiles, a generalized best reply correspondence if it has (1) a product structure, is (2) upper hemi–continuous, (3) always includes a best reply to any mixed strategy profile, and is (4) convex- and closed-valued. For each generalized best reply correspondence we define a generalized best reply dynamics as a differential inclusion based on it. We call a face of the set of mixed strategy profiles a minimally asymptotically stable face (MASF) if it is asymptotically stable under some such dynamics and no subface of it is asymptotically stable under any such dynamics. The set of such correspondences (and dynamics) is endowed with the partial order of point-wise set-inclusion and, under a mild condition on the normal form of the game at hand, forms a complete lattice with meets based on point-wise intersections. The refined best reply correspondence is then defined as the smallest element of the set of all generalized best reply correspondences. We ultimately find that every Kalai and Samet's (1984) persistent retract, which coincide with Basu and Weibull's (1991) CURB sets based, however, on the refined best reply correspondence, contains a MASF. Conversely, every MASF must be a Voorneveld's (2004) prep set, again, however, based on the refined best reply correspondence.

Keywords: Evolutionary game theory, best response dynamics, CURB sets, persistent retracts, asymptotic stability, Nash equilibrium refinements, learning
JEL codes: C62, C72, C73

[*]Department of Economics, School of Business and Economics, University of Exeter, D.G.Balkenborg@exeter.ac.uk

[†]Department of Mathematics, University of Vienna, Josef.Hofbauer@univie.ac.at

[‡]Institute for Mathematical Economics, Bielefeld University, christoph.kuzmics@uni-bielefeld.de.

1 Introduction

Evolutionary support for Nash equilibrium behavior in general finite n-player games is mixed. On the one hand, strict Nash equilibria (which necessarily must be in pure strategies) are evolutionarily stable (multipopulation ESS, see e.g. (Weibull 1995, Definition 5.1)) and asymptotically stable under the multi-population replicator dynamics. In fact, strict Nash equilibria are the only asymptotically stable states under the multi-population replicator dynamics and other related imitation-based dynamics as shown by Ritzberger and Weibull (1995). Of course, many games of interest do not have a strict Nash equilibrium.

On the other hand, mixed Nash equilibria, which do exist in every finite normal form game, do not have a lot of evolutionary support in general games. To demonstrate this point Hofbauer and Swinkels (1995) and Hart and Mas-Colell (2003) consider a class of finite normal form games, in which each game has a unique but mixed Nash equilibrium, and show that any "reasonable" (deterministic continuous time) dynamic process must fail to make this unique Nash equilibrium asymptotically stable in at least one of these games.

We, thus, have to abandon the hope of obtaining Nash equilibria as the only outcomes of evolutionary processes. Yet, this is not the end of studying the outcome of evolution. One just has to accept that evolution will lead to, at least in some games, a set of states, which may also include some non-Nash equilibrium states. It may still be the case that these evolutionary stable sets are quite manageable and useful for the analysis of games in practice. Note that the switch from strategy profiles to sets of strategy profiles has also been necessary in the study of Nash equilibrium refinements (see Kohlberg and Mertens (1986)) and the study of the consequences of common knowledge of rationality in general games (see e.g. Bernheim (1984) and Pearce (1984)).

Given the evolutionary appeal of some pure strategy profiles it suggests itself to study setwise generalizations of pure strategy profiles. A useful generalization of a pure strategy profile is given by a face (of the polyhedron of mixed strategy profiles) that is simply obtained by choosing a subset of pure strategies for every player and then considering all independent mixtures over these subsets.

We are not the first to propose to study the evolutionary stability properties of such faces. Indeed Ritzberger and Weibull (1995) identify faces which are asymptotically stable under a large class of imitation-based dynamics. These faces are spanned by what Ritzberger and Weibull (1995) call cuwbr sets (these are product sets of pure strategy profiles that are closed under weakly better replies). Unless a minimal cuwbr set is a singleton, it does not contain a strict Nash equilibrium, but must contain a (possibly mixed) Nash equilibrium.

There are two (related) drawbacks of Ritzberger and Weibull's (1995) result. First, in many games even the smallest cuwbr sets are very large sets. Thus, their predictive power is limited. Second, and this is a possible reason for their limited predictive power, the dynamics that these sets are based on, while plausible in some settings, are not necessarily the most plausible in games with highly rational and highly informed players.

More rational and informed players might not adapt their strategies towards better replies so very gradually as is implicit in the class of dynamics of Ritzberger and Weibull (1995). One alternative with sharper predictions (smaller asymptotically stable sets) and more plausible adjustment behavior for highly rational and informed human beings is the best reply dynamics of Gilboa and Matsui (1991), Matsui (1992), and Hofbauer (1995) (in the spirit of fictitious play). To see the differences between Ritzberger and Weibull's (1995) imitation-based dynamics and the best reply dynamics in terms of the sharpness of their prediction consider the following 2-player game.

	H	T	C
H	4,0	0,4	$2\epsilon,\epsilon$
T	0,4	4,0	$2\epsilon,\epsilon$
C	$\epsilon,2\epsilon$	$\epsilon,2\epsilon$	ϵ,ϵ

Game 1: A game to demonstrate the difference between imitation-based and best reply dynamics.

For $\epsilon \in (0,2)$ Game 1, which is matching pennies with an additional (constant payoff) strategy, has a unique smallest cuwbr set, the set of all strategy profiles. It is easy to see that both H and T need to be in any minimal cuwbr set for both players. However, the unique minimal cuwbr set also includes pure strategy C even if ϵ is very close to 0. To see this note that when play is, for instance, sufficiently close to H,H strategy C is better than strategy H for player 2 (and, thus, better than the average strategy employed by all individuals in player population 2). Under any dynamics considered in Ritzberger and Weibull (1995) the share of C strategists in population 2 must then grow for some finite amount of time.

Yet, for all $\epsilon \in (0,2)$ strategy C is strictly dominated for both players by the mixed strategy that puts equal weight on H and T. Thus, if a human being were to play this game, and were told the current state of play and allowed to change her behavior, it seems unlikely that this human being would choose strategy C. Indeed, under the best reply dynamics strategy C will never be adopted by any revising agent. Thus, the best reply dynamics will eliminate C from any initial state. The unique minimal asymptotically stable face under the best reply dynamics is the face spanned by the unique minimal CURB set (as defined by Basu and Weibull (1991)) $\{H,T\} \times \{H,T\}$.

Hurkens (1995) analyzes a stochastic variant of the best reply dynamics. To be more precise, he studies a stochastic version of fictitious play, in which players play best replies to samples from their memory as in the model of Young (1993). Hurkens (1995) shows that the limiting invariant distribution of the resulting Markov chain attaches probability 1 to the set of all minimal CURB sets. Analogously one can prove that CURB sets are asymptotically stable under the best reply dynamics.[1]

Even when an imitation-based dynamics and the best reply dynamics yield the same collection of asymptotically stable sets of states, their vector fields are very different. To see this consider the following simple 2-player 2-strategy game.

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[1]This follows from Lemma 7 in this paper.

	L	R
T	1,1	1,1
B	1,1	0,0

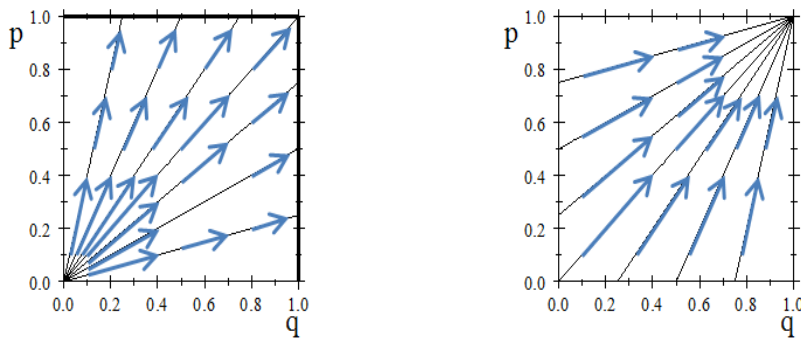Game 2: A game to demonstrate the behavioral differences in imitation-based and best reply dynamics.

Figure 1: The vector fields of the replicator and best reply dynamics for Game 2.

Note that pure strategies B and R are weakly dominated for players 1 and 2, respectively. Figure 1 sketches the vector fields of the two dynamics for this game, where p denotes the proportion of T in player population 1 and q the proportion of L in player population 2. The replicator dynamics takes play from an interior state to a possibly (weakly) dominated Nash equilibrium state on the boundary. Thus, different initial states, under the replicator dynamics, lead to different, often weakly dominated, Nash equilibrium outcomes. A consequence of this fact is that all Nash equilibrium boundary states are Lyapunov stable, yet none are asymptotically stable (not even the undominated equilibrium T,L).

The best reply dynamics, on the other hand, from any interior state converges in a straight line to the undominated Nash equilibrium T,L. However, there is something the above picture does not show. Because the best reply dynamics is a differential inclusion, there can be several trajectories emanating from the same point. This happens here precisely for all the Nash equilibria. Here, this implies that there are solutions to the best reply dynamics that move gradually along the boundary. Thus, T,L is NOT Lyapunov stable because there are trajectories starting arbitrarily nearby and leave any neighborhood. Only the whole Nash equilibrium component is asymptotically stable.

In this paper we take the view that this difficulty is due to inessential and superfluous trajectories. In fact, if one took the view that rational and informed human beings would never adopt dominated strategies, by, for instance, only playing cautious best responses (as defined by Pearce (1984)), then the resulting cautious best-reply dynamics would make T,L the unique minimal asymptotically stable face.

Hurkens (1995) considers a second stochastic variant of the best-reply dynamics, in which individuals restrict themselves to playing semi-robust best replies, as defined by Balkenborg (1992). A definition of semi-robust best replies is also given in this paper. At

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this point it suffices to say that the set of semi-robust best-replies to a particular strategy profile is a (sometimes proper) subset of the set of all best replies to this strategy profile. Hurkens (1995) shows that the limiting invariant distribution of the resulting Markov chain in this second model attaches probability 1 to the set of persistent retracts (as defined by Kalai and Samet (1984)). For the simple 2-player 2-strategy game above this implies that only the undominated pure Nash equilibrium T,L receives limiting probability 1. Generally, persistent retracts are faces that are typically smaller, never larger, than minimal CURB sets. To be more precise, every minimal CURB set contains a possibly much smaller persistent retract. Similarly every minimal cuwbr set contains a possibly much smaller minimal CURB set.

Motivated by the results of Ritzberger and Weibull (1995) and Hurkens (1995) we are, in this paper, interested in identifying and characterizing the smallest faces that are evolutionary stable under some reasonable dynamics (appropriate for highly rational and informed human beings). To make our quest more specific we restrict attention to best reply like dynamics. This is motivated by the intuitive appeal of best responding for highly rational and informed human beings as well as the fact that Ritzberger and Weibull's (1995) asymptotically stable minimal cuwbr sets are typically much larger than the minimal CURB sets or persistent retracts that Hurkens (1995) identifies as the outcome of his two variants of Young's (1993) model of best-reply learning. To perform our search in a systematic way we define and study a large class of **generalized best reply dynamics**, which is supposed to contain all reasonable best reply like dynamics.[2]

We can thus define a minimally asymptotically stable face (MASF) as a face that is asymptotically stable under some generalized best reply dynamics with the additional property that it does not contain a proper subface that is also asymptotically stable under some, possibly different, generalized best reply dynamics. Note that it may seem well possible that the dynamics that makes one MASF asymptotically stable in one game is quite different from the dynamics that makes another MASF asymptotically stable in another game. Our first main result, however, shows that this is not possible. In fact, under a mild restrictions on the class of games we can study, there is a single dynamics, the same for all games, that determines which faces are MASFs and which are not. That is, a face is a MASF if and only if it is a minimally asymptotically stable face under this particular dynamics. We are thus justified in terming this dynamics the (most) **refined best reply dynamics**.

The refined best reply dynamics is a very reasonable and intuitive dynamics (for highly rational and highly informed individuals). The following micro-story is based on and adapted from Hofbauer's (1995) story that gives rise to the best reply dynamics. For every player position there is a large population of individuals. Time is continuous and runs from 0 to infinity. Individuals always play a pure strategy. At time 0 individuals' behavior is given by some arbitrary frequency distribution of pure strategies, one distribution for each population. In every short time interval a small fraction of individuals is given the opportunity to revise their strategy. When doing so individuals know the aggregate

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[2]This class also contains some unreasonable dynamics. It will become clear in the analysis, however, that this does not pose a problem for the interpretation of our results.

distribution of play (the state of play). If there is a unique best reply a revising individual adopts it. If there are multiple best replies a revising individual considers them all, but only adopts one that is also a unique best reply in an open set of nearby states of play. One could call this a cautious myopically rational individual.[3] One similar alternative story could be that revising individuals do not know the exact state of play. Different individuals have different beliefs (which are close to the truth) about the aggregate play. If these beliefs are sufficiently diverse only a vanishing fraction of individuals will adopt a strategy that is best only on a thin set of states of play. This gives again rise to the refined best reply dynamics.

Interestingly the refined best reply dynamics is based on Balkenborg's (1992) semi-robust best replies, and, thus, in some sense analogous to the second stochastic model of learning studied by Hurkens (1995). This then gives rise to the question whether MASFs are exactly those faces that Hurkens (1995) identifies as the long-run outcome of his learning process namely persistent retracts. Somewhat surprisingly, the answer to this question is no. There are MASFs in some games that are proper subfaces of persistent retracts as we demonstrate by example. In an effort to at least partially characterize MASFs in terms of known concepts from the literature, we ultimately show that every persistent retract contains a MASF and every MASF must be an appropriate version of a prep set (first defined by Voorneveld (2004)). The appropriate version is not the original prep set, which is based on the best reply correspondence, but such that it is based on the refined best reply correspondence, which underlies the refined best reply dynamics.

Methodologically there is some overlap of this paper with Balkenborg (1992) who, in order to analyze the properties of persistent retracts, studies the "semi-robust best reply correspondence", which differs from the refined best reply correspondences considered here by not being convex valued. Balkenborg, Jansen, and Vermeulen (2001) analyze the invariance of persistent retracts and equilibria using "sparse strategy selections". These are particulary useful when no unique minimal refined best reply correspondence exists.

The paper proceeds as follows. We first define the class of games we study in section 2. We then define the class of generalized best reply correspondences in section 3, where we also prove its lattice structure and the existence of a unique smallest element. In section 4 we study the notion of a CURB set (Basu and Weibull (1991)) and a prep set[4] (Voorneveld (2004)) for all generalized best reply correspondences and study their relationship. We there also prove that CURB sets based on the refined best reply correspondence coincide with Kalai and Samet's (1984) persistent retracts. Section 5, finally, provides the main result that persistent retracts are asymptotically stable under the refined best reply dynamics and thus contain a MASF and that every MASF must be a tight prep set based on the refined best reply correspondence. Section 6 concludes. The paper has one appendix, where we show in which sense our restriction to games with generically unique best replies is not essential.

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[3]The behavior of a revising individual is reminiscent of, yet not completely the same as playing Pearce's (1984) cautious response.

[4]A learning model which leads to the original prep sets of Voorneveld (2004) is given by Kets and Voornefeld (2008).

2 Preliminaries

Let $\Gamma = (I, S, u)$ be a finite n-player normal form game, where $I = \{1, ..., n\}$ is the set of players, $S = \times_{i \in I} S_i$ is the set of pure strategy profiles, and $u : S \to \mathbb{R}^n$ the payoff function[5]. Let $\Theta_i = \Delta(S_i)$ denote[6] the set of player i's mixed strategies, and let $\Theta = \times_{i \in I} \Theta_i$ denote the set of all mixed strategy profiles. Let $\text{int}(\Theta) = \{x \in \Theta : x_{is} > 0 \ \forall s \in S_i \ \forall i \in I\}$ denote the set of all completely mixed strategy profiles.

A strategy profile $x \in \Theta$ may also represent a population state in an evolutionary interpretation of the game in the following sense. Each player $i \in I$ is replaced by a population of agents playing in player position i and x_{is_i} denotes the proportion of players in population i who play pure strategy $s_i \in S_i$.

For $x \in \Theta$ let $\mathcal{B}_i(x) \subset S_i$ denote the set of pure-strategy best-replies to x for player i. Let $\mathcal{B}(x) = \times_{i \in I} \mathcal{B}_i(x)$. Let $\beta_i(x) = \Delta(\mathcal{B}_i(x)) \subset \Theta_i$ denote the set of mixed-strategy best-replies to x for player i. Let $\beta(x) = \times_{i \in I} \beta_i(x)$.

Two strategies $x_i, y_i \in \Theta_i$ are **own-payoff equivalent** (for player i) if $u_i(x_i, x_{-i}) = u_i(y_i, x_{-i})$ for all $x_{-i} \in \Theta_{-i} = \times_{j \neq i} \Theta_j$ (see Kalai and Samet (1984)). In contrast, Kohlberg and Mertens (1986) call two strategies $x_i, y_i \in \Theta_i$ **payoff equivalent** if $u_j(x_i, x_{-i}) = u_j(y_i, x_{-i})$ for all $x_{-i} \in \Theta_{-i}$ and **for all** players $j \in I$. We will use these concepts primarily for pure strategies.

Let $\Psi = \{x \in \Theta : \mathcal{B}(x) \text{ is a singleton}\}$. Notice that the unique best reply against a strategy combination in Ψ is necessarily a **pure** strategy. Throughout this paper we will restrict attention to games Γ for which this set Ψ is dense in Θ. Let this set of games be denoted by $\mathcal{G}^*$. A game $\Gamma \notin \mathcal{G}^*$ is given by Game 3. Player 1's best reply set is $\{A, B\}$ for any (mixed) strategy of player 2. Hence, $\beta(x)$ is never a singleton and $\Psi = \emptyset$ is not dense in Θ. This has to do with the fact that player 1 has two own-payoff equivalent pure strategies.

	C	D
A	1,1	1,0
B	1,0	1,1

Game 3: A Game in which Ψ is not dense in Θ.

Proposition 1 demonstrates that without equivalent strategies Ψ is dense in Θ. The following lemma, due to Kalai and Samet (1984), is used in the proof of Proposition 1.

Lemma 1 *Let U be a non-empty open subset of Θ. Then two strategies $x_i, y_i \in \Theta_i$ are own-payoff equivalent (for player i) if and only if $u_i(x_i, z_{-i}) = u_i(y_i, z_{-i})$ for all $z \in U$.*

Proposition 1 *Let Γ be without own-payoff equivalent pure strategies. Then Ψ is dense in Θ; i.e., $\Gamma \in \mathcal{G}^*$.*

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[5]The function u will also denote the expected utility function in the mixed extension of the game Γ.

[6]Let, generally, $\Delta(K)$ for some finite set K denote the set of all probability distributions over K.

Proof: Suppose Ψ is not dense in Θ. Then there is an open set U in Θ such that for all $y \in U$ the pure best reply set $\mathcal{B}(y)$ is not a singleton, i.e., has at least two elements. Without loss of generality, due to the finiteness of S, we can assume that there are two pure strategy-profiles $s_i, t_i \in S_i$ such that $s_i, t_i \in \mathcal{B}_i(y)$ for all $y \in U$ and some player $i \in I$. But then by Lemma 1, s_i and t_i are own-payoff equivalent for player i.   QED

Note that the converse of Proposition 1 is not true. Consider two own-payoff equivalent strategies which are strictly dominated by another strategy. If these are the only equivalent strategies in Γ then Ψ is still dense in Θ. However, the following proposition is immediate. Call $x_i \in \Theta_i$ a **robust best reply** against $x \in \Theta$ if x_i is a best reply against all strategy combinations in a neighborhood of x. Call $x_i \in \Theta_i$ a **robust strategy** if x_i is a robust best reply against **some** strategy combination $x \in \Theta$. This terminology is inspired by Okada (1983).

Proposition 2 *Let $\Gamma \in \mathcal{G}^*$. Let $s_i \in S_i$ be a robust strategy. Then player i has no distinct own-payoff equivalent strategy to s_i in S_i.*

Still, games in the class $\mathcal{G}^*$ are essentially those that do not have own-payoff equivalent strategies for any player. The semi-reduced normal form of a game is usually obtained by removing all payoff equivalent strategies. In the appendix we argue that the games in which there are own-payoff equivalent strategies which are not payoff-equivalent are exceptional. Hence the restriction to games in the class $\mathcal{G}^*$ made throughout the paper is essentially the restriction to the semi-reduced normal form[7] in the sense of Kohlberg and Mertens (1986). Since we are primarily interested in the best reply correspondence this restriction is largely without loss of generality. In fact, every trajectory of the best reply dynamics of the reduced form of a normal form game corresponds in a canonical fashion to a family of trajectories in the original game which projects onto it.

3   Generalized best reply correspondences

Definition 1 *A correspondence $\tau : \Theta \Rightarrow \Theta$ is a **generalized best reply correspondence** if*

1. *$\tau(x) = \times_{i \in I} \tau_i(x) \ \forall \ x \in \Theta$, where $\tau_i : \Theta \Rightarrow \Theta_i$ for all $i \in I$,*

2. *τ is upper hemi–continuous[8] at all $x \in \Theta$,*

3. *$\tau_i(x) \cap \beta_i(x) \neq \emptyset \ \forall \ x \in \Theta, \ \forall \ i \in I$,*

4. *$\tau(x)$ is convex and closed for all $x \in \Theta$.*

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[7]In particular, we are, for instance, not ruling out games with weakly dominated strategies.

[8]Following (Aliprantis and Border 1999, ch.17.2), or (Ritzberger 2002, Def 5.8), the correspondence τ is upper hemi–continuous at x if for every open set $V \subset \Theta$ with $\tau(x) \subset V$ there is an open subset $U \subset \Theta$ with $x \in U$ such that for all $y \in U$: $\tau(y) \subset V$.

Note that property 3 immediately implies that $\tau(x) \neq \emptyset$. Thus, a generalized best reply correspondence has the basic technical properties as the best reply correspondence β, and is minimally connected to the best reply correspondence β by the requirement that at least one best reply to some given strategy profile x, i.e. an element $\in \beta(x)$, is also available in $\tau(x)$.

A subclass of the set of generalized best reply correspondences of independent interest is one that is based on pure strategies only, in the following sense. A correspondence $\tau : \Theta \Rightarrow \Theta$ is a **generalized best reply correspondence based on pure strategies** if it is a generalized best reply correspondence and property 4 is replaced by the more stringent property 4^* that $\tau_i(x) = \Delta(T_i(x))$ for some $T_i(x) \subset S_i$ for all $x \in \Theta$ for all $i \in I$. Then property 3 is equivalent to $T_i(x) \cap \mathcal{B}_i(x) \neq \emptyset$ for all $x \in \Theta$ and for all $i \in I$.

Let $\mathcal{T} = \mathcal{T}(\Gamma)$ denote the set of all generalized best reply correspondences (of a game Γ) and let $\mathcal{T}^{PS}$ denote the subset of all generalized best reply correspondences based on pure strategies.

One natural example of a correspondence in $\mathcal{T}$ but not in $\mathcal{T}^{PS}$ is the correspondence of all mixed weakly better replies. This is given by $\tau = \times_{i \in I} \tau_i$ with $\tau_i(x) = \{y_i \in \Theta_i : u_i(y_i, x_{-i}) \geq u_i(x_i, x_{-i})\}$.

One example of a correspondences in $\mathcal{T}^{PS}$ is, of course, the best reply correspondence itself. For another let $T_i(x) = \{s_i \in S_i : u_i(s_i, x_{-i}) \geq u_i(x_i, x_{-i})\}$. I.e. $T_i(x)$ is the set of all weakly better replies to x_{-i} given x_i. The resulting correspondence is that of all mixtures of pure weakly better replies (see Ritzberger and Weibull (1995)). Another one, closely connected to the $S^\infty W$-procedure of Dekel and Fudenberg (1990), can be found by letting $T_i(x)$ be the set of all pure best replies, except weakly dominated ones.

The following example of a correspondence in $\mathcal{T}^{PS}$ is key to the subsequent analysis in this paper.

Definition 2 *For games in $\mathcal{G}^*$, for $x \in \Theta$ let*

$$\mathcal{S}_i(x) = \{s_i \in S_i : \ \exists \{x_t\}_{t=1}^\infty \in \Psi : x_t \to x \wedge \mathcal{B}_i(x_t) = \{s_i\} \ \forall t\}.$$

Then $\sigma_i(x) = \Delta(\mathcal{S}_i(x))$ and $\sigma(x) = \times_{i \in I} \sigma_i(x) \ \forall \ x \in \Theta$. We call this correspondence $\sigma : \Theta \Rightarrow \Theta$, the (most) refined best reply correspondence.

The set $\mathcal{S}_i(x)$ in the above definition is the set of pure semi-robust best replies defined in Balkenborg (1992).

Given two correspondences $\tau, \tau' \in \mathcal{T}$ let $\tau \subset \tau'$ if $\tau(x) \subset \tau'(x)$ for all $x \in \Theta$. The set $\mathcal{T}$ endowed with this notion of "smaller than" is a partial order, see e.g. (Davey and Priestley 2002, Definition 1.2). Given two correspondences $\tau, \tau' \in \mathcal{T}$ let $\tau'' = \tau \wedge \tau'$ if $\tau_i''(x) = \tau_i(x) \cap \tau_i'(x)$ and $\tau''(x) = \times_{i \in I} \tau_i''(x)$ for all $x \in \Theta$.

The first Theorem of this paper demonstrates that for games in $\mathcal{G}^*$ the set $\mathcal{T}$ of generalized best reply correspondences has a lot of structure, and especially, a smallest element. In fact the set $\mathcal{T}$ is a complete lattice, meaning that every subset of $\mathcal{T}$ has an infimum (meet) and a supremum (join), see e.g. (Davey and Priestley 2002, Definition 2.4).

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Theorem 1 *Let* $\Gamma \in \mathcal{G}^*$. *Then*

1. $\mathcal{T}$ *is a complete lattice,*

2. *with a unique smallest element, which is given by* σ, *the refined best reply correspondence,*

3. *with* $\sigma \in \mathcal{T}^{PS}$.

Proof: Let $\mathcal{T}'$ be a non-empty subset of $\mathcal{T}$. Define τ^* by $\tau^*(x) = \cap_{\tau \in \mathcal{T}'} \tau(x)$ for all $x \in \Theta$. We claim that $\tau^* \in \mathcal{T}$. Since the intersection of products, convex sets and closed sets is itself a product, convex and closed, τ^* has properties (1) and (4) of a generalized best reply correspondence. Any $\tau \in \mathcal{T}'$ is in fact compact valued. Θ is compact and Hausdorff and therefore a regular topological space (Aliprantis and Border 1999, Theorem 2.48). Hence, τ^* is upper hemi–continuous (Aliprantis and Border 1999, Theorem 17.25.3), i.e. satisfies property (2) of a generalized best reply correspondence.

It remains to show property (3) of a generalized best reply correspondence. As $\Gamma \in \mathcal{G}^*$, by definition, $\beta(x)$ is a singleton for all $x \in \Psi$, given by the pure strategy which is the unique element of $\mathcal{B}(x)$. Thus, for all $x \in \Psi$, we have that $\beta(x) \subset \tau(x)$ for any $\tau \in \mathcal{T}$. For $x \in \Psi$ let $\mathcal{S}_i(x) = \mathcal{B}_i(x)$ for all $i \in I$. As $\Gamma \in \mathcal{G}^*$ we have that Ψ is dense in Θ. For $x \notin \Psi$ let $U \subset \Theta$ be a neighborhood of x, let $\mathcal{S}_i^U(x) = \bigcup_{x' \in U \cap \Psi} \mathcal{B}_i(x')$, and let $\mathcal{S}_i(x) = \bigcap_{(U \text{ neighborhood of } x)} \mathcal{S}_i^U(x)$. Let $\sigma_i(x) = \Delta(\mathcal{S}_i(x))$ and let $\sigma(x) = \times_{i \in I} \sigma_i(x)$ $\forall$ $x \in \Theta$. Thus, σ is the refined best reply correspondence.

Then, by the properties (2) and (4) of any $\tau \in \mathcal{T}$ we must have that $\sigma(x) \subset \tau(x)$ for all $x \notin \Psi$ and thus for all $x \in \Theta$, for all $\tau \in \mathcal{T}$. Thus, $\sigma \subset \tau^*$, and, thus, τ^* satisfies property (3). It follows immediately that τ^* is the infimum of $\mathcal{T}'$ in $\mathcal{T}$. The supremum of $\mathcal{T}'$ is easily seen to be the the intersection of all upper bounds of $\mathcal{T}'$ in $\mathcal{T}$. Thus $\mathcal{T}$ is a complete lattice, proving part 1.

A complete lattice has a lowest element. Given that $\sigma \subset \tau$ for all $\tau \in \mathcal{T}$ and $\sigma \in \mathcal{T}$ this lowest element must be σ. This proves part 2. Part 3 immediately follows from the construction of σ. QED

Note that Theorem 1 also implies that the set $\mathcal{T}^{PS}$ is a complete lattice with the same smallest element, σ.

The converse of Theorem 1 is, in fact, also true, in the following sense. For any game $\Gamma \notin \mathcal{G}^*$ $\mathcal{T}$ is not a lattice and does not have a unique smallest element. To see this consider any game $\Gamma \notin \mathcal{G}^*$. This game must have at least two own-payoff equivalent pure strategies for some player which are simultaneous best replies in an open set of strategy profiles. Thus one can construct $\tau \in \mathcal{T}$ based on only one of these two pure strategies, and $\tau' \in \mathcal{T}$ based only on the other, such that $\tau(x) \cap \tau'(x) = \emptyset$ for some $x \in \Theta$ and, thus, $\tau \wedge \tau' \notin \mathcal{T}$.

Theorem 1 justifies the name **(most) refined best reply correspondence** we attached to σ, as it is the unique smallest generalized best reply correspondence and obviously satisfies $\sigma \subset \beta$.

This refined best reply correspondence σ coincides with the best reply correspondence β almost everywhere (i.e. for all $x \in \Psi$, which is dense in Θ given $\Gamma \in \mathcal{G}^*$). Furthermore

for strategy profiles $x \notin \Psi$ it is constructed in a minimal way to ensure upper hemi–continuity by requiring that $\sigma(x)$ includes all pure strategies which are best replies to some nearby $x' \in \Psi$ and no others.[9] For such x any $\sigma(x)$ must then also include all convex combinations of all pure strategies in $\tau(x)$ by property 4.

In the final paragraph of this section we provide a brief partial characterization of the refined best reply correspondence in terms of well-known objects from the theory of games. A detailed characterization of the refined best reply correspondence, its fixed points, and other objects based on it can be found in our companion paper Balkenborg, Hofbauer, and Kuzmics (2009). For 2-player games the refined best reply correspondence includes those and only those best replies that are not weakly dominated and are not equivalent to a mixture of other pure strategies. The following example (from Balkenborg, Hofbauer, and Kuzmics (2009)) demonstrates part of this claim.

	D	E	F
A	2,2	1,2	1,2
B	2,1	2,2	0,0
C	2,1	0,0	2,2

Game 4: A Game with equivalent mixed strategies.

In this game strategy A (and similarly D) is equivalent to the mixture of pure strategies B and C (E and F respectively). However, A is a best reply only on a thin set of mixed-strategy profiles. In fact, A is best against any $x \in \Theta$ in which $x_{2E} = x_{2F}$, the set of which is a thin set. Thus, while this game is in $\mathcal{G}^*$, and $y_{1A} = 0$ for all $y \in \mathcal{S}_1(x)$ for all $x \in \Theta$, i.e. A is never in the set of refined best replies. This, for instance, implies that there are strategically stable equilibria in the sense of Kohlberg and Mertens (1986) that are not fixed points of the refined best reply correspondence. For games with more than two players the set of refined best replies at a given strategy profile x can well be a proper subset of the set of best replies that are not weakly dominated and not equivalent to a mixed strategy. For a thorough discussion of this we refer the reader to our companion paper Balkenborg, Hofbauer, and Kuzmics (2009).

4 τ-CURB and τ-prep sets

A set $R \subset S$ is a **strategy selection** if $R = \times_{i \in I} R_i$ and $R_i \subset S_i$, $R_i \neq \emptyset$ for all i. For a strategy selection R let $\Theta(R) = \times_{i \in I} \Delta(R_i)$ denote set of independent strategy mixtures of the pure strategies in R. A set $\varphi \subset \Theta$ is a **face** if there is a strategy selection R such that $\varphi = \Theta(R)$. Note that $\Theta = \Theta(S)$. Note also that $\beta(x) = \Theta(\mathcal{B}(x))$ and $\sigma(x) = \Theta(\mathcal{S}(x))$. Generally $\tau(x) = \Theta(T(x))$ for some selection $T(x)$ if $\tau \in \mathcal{T}^{PS}$.

Let $A \subset \Theta$. For any $\tau \in \mathcal{T}$ let $\tau(A) = \times_{i \in I} \tau_i(A)$ with $\tau_i(A) = \bigcup_{x \in A} \tau_i(x)$. The following definition is a generalized version of Basu and Weibull's (1991) CURB sets.

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[9]Strategies that are unique best replies to some x are called *inducible* in von Stengel and Zamir (2004).

For $\tau \in \mathcal{T}$ a strategy selection R is a **τ-CURB set** if $\tau(\Theta(R)) \subset \Theta(R)$. It is a **tight τ-CURB set**[10] if, in addition $\tau(\Theta(R)) \supset \Theta(R)$, and, hence, $\tau(\Theta(R)) = \Theta(R)$. It is a **minimal τ-CURB set** if it does not properly contain another τ-CURB set.

These definitions, while well-defined for all $\tau \in \mathcal{T}$ are more natural for $\tau \in \mathcal{T}^{PS}$ as they are really based on pure strategies. In fact, if $\tau \in \mathcal{T} \setminus \mathcal{T}^{PS}$ then there are typically not many tight τ-CURB sets.

Note that if $\tau = \beta$ we obtain the original definition of CURB sets of Basu and Weibull (1991). If τ is the correspondence of all mixtures of pure weakly better replies, as defined in Section 3, τ-CURB sets are Ritzberger and Weibull's (1995) **cuwbr-sets** (closed under weakly better replies). If, as mentioned in Section 3, $T_i(x)$ be the set of all pure best replies, except weakly dominated ones, and τ is the correspondence in $\mathcal{T}^{PS}$ based on these $T_i(x)$, τ-CURB sets are Basu and Weibull's (1991) **CURB*-sets**.

The lattice structure of (in, fact we only need the partial order on) $\mathcal{T}$ allows us to compare CURB sets based on different generalized best reply correspondences.

Lemma 2 *Let $\Gamma \in \mathcal{G}^*$. Let $\tau, \tau' \in \mathcal{T}$ with $\tau \subset \tau'$. Then any τ'-CURB set is also a τ-CURB set. Furthermore, any τ-CURB set for some $\tau \in \mathcal{T}$ is also a σ-CURB set.*

Proof: For the first part, let R be a τ'-CURB set. Thus, by definition, $\tau'(\Theta(R)) \subset \Theta(R)$. But as $\tau \subset \tau'$ we have that $\tau(\Theta(R)) \subset \tau'(\Theta(R)) \subset \Theta(R)$. The second part follows from the first part of this lemma and the second part of Theorem 1.   QED

Lemma 2 implies that the smallest of all τ-CURB sets for any $\tau \in \mathcal{T}$ are the minimal σ-CURB sets. It turns out, we can characterize minimal σ-CURB sets.

The following definitions are due to Kalai and Samet (1984). A set $\varphi \subset \Theta$ is a **retract** if $\varphi = \times_{i \in I} \varphi_i$, where $\varphi_i \subset \Theta_i$ is nonempty, compact, and convex. A set $\varphi \subset \Theta$ **absorbs** another set $\varphi' \subset \Theta$ if for all $x \in \varphi'$ we have that $\beta(x) \cap \varphi \neq \emptyset$. A retract φ is an **absorbing retract** if it absorbs a neighborhood of itself. It is a **persistent retract** if it does not properly contain another absorbing retract. Kalai and Samet (1984) show that, for games without equivalent strategies, and, hence, for games in $\mathcal{G}^*$, persistent retracts have to be faces.

Lemma 3 *Let $\Gamma \in \mathcal{G}^*$. A strategy selection $R \subset S$ is a σ-CURB set if and only if $\Theta(R)$ is an absorbing retract.*

Proof: "$\Leftarrow$": Let the strategy selection $R \subset S$ be such that $\Theta(R)$ is an absorbing retract, i.e., it absorbs a neighborhood of itself. Let U be such a neighborhood of $\Theta(R)$. We then have that for every $y \in U$ there is an $r \in R$ such that $r \in \mathcal{B}(y)$. For all $r \in R$ let $U^r = \{y \in U : r \in \mathcal{B}(y)\}$. We obviously have $\bigcup_{r \in R} U^r = U$. Suppose R is not a σ-CURB set. Then there is a player $i \in I$ and a pure strategy $s_i \in S_i \setminus R_i$ such that $s_i \in \mathcal{S}_i(x)$ for

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[10]Note that, for any $\tau \in \mathcal{T}$, the set of tight τ-CURB sets, together with the empty set, also forms a (finite and thus complete) lattice. This follows from the fact that the set of all pure strategy selections is a lattice if we include the empty set. Then τ as a function from subsets of the set of pure strategy sets to itself is order-preserving, i.e. if $R \subset R'$ then $\tau(R) \subset \tau(R')$. Thus, by Tarski's fixed point theorem the set of all fixed points of τ also forms a lattice. These are tight τ-CURB sets (and the emptyset).

some $x \in \Theta(R)$. By the definition of $\mathcal{S}_i$ we must then have that $s_i \in \beta(y)$ for all $y \in O$ for some open set O the closure of which contains x. But then, by the finiteness of R, there is a strategy profile $r \in R$ such that U^r and O have an intersection which contains an open set. On this set s_i and r_i are now both best replies. But then, by Lemma 1, s_i and r_i are equivalent for player i, which, by Proposition 2, contradicts our assumption. "$\Rightarrow$": Suppose $R \subset S$ is a σ-CURB set. Suppose that $\Theta(R)$ is not an absorbing retract. Then for every neighborhood U of $\Theta(R)$ there is a $y_U \in U$ such that $\beta(y_U) \cap \Theta(R) = \emptyset$. In particular for every such y_U there is a player $i \in I$ and a pure strategy $s_i \in S_i \setminus R_i$ such that $s_i \in \mathcal{B}_i(y_U)$. By the finiteness of the number of players and pure strategies and by the compactness of Θ, this means that there is a convergent subsequence of $y_U \in \text{int}(\Theta)$ such that $y_U \to x$ for some $x \in \Theta(R)$ and there is an $i \in I$ and an $s_i \in S_i \setminus R_i$ such that $s_i \in \mathcal{B}_i(y_U)$ for all such y_U. Now one of two things must be true. Either s_i is a best reply in an open set with closure intersecting $\Theta(R)$, in which case $s_i \in R_i$ given the definition of σ and a σ-CURB set, which gives rise to a contradiction. Or there is no open set with closure intersecting $\Theta(R)$ such that s_i is best on the whole open set, in which case there must be a strategy $r_i \in R_i$ which is such that $r_i \in \beta(y_U)$ at least for a subsequence of all such y_U (converging to x), which again gives rise to a contradiction. $\hspace{2em}$ QED

Lemma 3 immediately implies the following theorem.

Theorem 2 *Let $\Gamma \in \mathcal{G}^*$. A strategy selection $R \subset S$ is a minimal σ-CURB set if and only if $\Theta(R)$ is a persistent retract.*

Proof: follows from Lemma 3. $\hspace{20em}$ QED

Theorem 2, together with Lemma 2, implies that the smallest τ-CURB sets for any $\tau \in \mathcal{T}$ are Kalai and Samet's (1984) persistent retracts.

The largest tight β-CURB is the set of rationalizable strategies (Bernheim (1984) and Pearce (1984)). We can similarly define, for any $\tau \in \mathcal{T}$, the set of **τ-rationalizable strategies** as the largest tight τ-CURB set.

Alternatively we can define τ-rationalizable, more in the original spirit of Bernheim (1984) and Pearce (1984), in the following way. We shall do this only for $\tau \in \mathcal{T}^{PS}$. If $\tau \in \mathcal{T}^{PS}$ then there are correspondences T_i for each player such that $\tau_i(x) = \Delta(T_i(x))$ for all $x \in \Theta$ and for all $i \in I$. For $A \subset \Theta$ let $T_i(A) = \bigcup_{x \in A} T_i(x)$. Let $\tau_i(A) = \Delta(T_i(A))$. Let $\tau(A) = \times_{i \in I} \tau_i(A)$. For $k \geq 2$ let $\tau^k(A) = \tau(\tau^{k-1}(A))$. For $A = \Theta$, $\tau^k(A)$ is a decreasing sequence, and we denote $\tau^\infty(\Theta) = \bigcap_{k=1}^\infty \tau^k(\Theta)$. A pure strategy profile $s \in S$ is **τ-rationalizable** if it is an element of the strategy selection $R \subset S$ which satisfies $\Theta(R) = \tau^\infty(\Theta)$.

We thus have notions of rationalizability for any generalized best reply correspondence (based on pure strategies).

Lemma 2 immediately implies the following result.

Corollary 1 *Let $\Gamma \in \mathcal{G}^*$. Let $\tau, \tau' \in \mathcal{T}$ such that $\tau \subset \tau'$. Then every τ-rationalizable strategy is also τ'-rationalizable. In particular, every σ-rationalizable strategy is also τ-rationalizable for any $\tau \in \mathcal{T}$.*

This Corollary thus states that the smallest set of τ-rationalizable strategies is obtained when $\tau = \sigma$.

It turns out that not only such τ-CURB sets play a role in our analysis of generalized best reply dynamics in the next section, but also τ-versions (especially the σ-version) of Voorneveld's (2004) prep sets. We shall define τ-prep sets only for $\tau \in \mathcal{T}^{PS}$. Let $\tau \in \mathcal{T}^{PS}$. A strategy selection R is a τ-**prep set** if for all $x \in \Theta(R)$ and for all $i \in I$ $\tau_i(x) \cap \Delta(R_i) \neq \emptyset$. A τ-prep set is minimal if it does not properly contain any other τ-prep set. Thus, any pure fixed point of τ is a minimal τ-prep set, just as every pure Nash equilibrium is a minimal (β-)prep set. We shall call a τ-prep set **tight** if for every $s_i \in R_i$ $s_i \in \tau_i(x)$ for some $x \in \Theta(R)$. Minimal τ-prep sets are necessarily tight.

Analogously to Lemma 2 we can compare τ-prep sets for different τ's in $\mathcal{T}^{PS}$. Only the comparison is reversed as the next Lemma states.

Lemma 4 *Let $\Gamma \in \mathcal{G}^*$. Let $\tau, \tau' \in \mathcal{T}^{PS}$ with $\tau \subset \tau'$. Then any τ-prep set is also a τ'-prep set. Furthermore, any σ-prep set is also a τ-prep set for every $\tau \in \mathcal{T}^{PS}$.*

Proof: For the first part, let R be a τ-prep set. Thus, by definition, for all $x \in \Theta(R)$ and for all $i \in I$ $\tau_i(x) \cap \Theta(R)_i \neq \emptyset$. But then, as $\tau \subset \tau'$, we also have for all $x \in \Theta(R)$ and for all $i \in I$ $\tau'_i(x) \cap \Theta(R)_i \neq \emptyset$. The second part follows from the first part of this lemma and the second part of Theorem 1. QED

Thus, Lemmas 2 and 4 imply that the smaller $\tau \in \mathcal{T}$ (in the lattice) the more τ-CURB sets and the fewer τ-prep sets there are.

5 Generalized best reply dynamics

Gilboa and Matsui (1991), Matsui (1992) and Hofbauer (1995) introduced and studied the continuous time best reply dynamics (1), which is, modulo a time change, equivalent to Brown (1951)'s continuous time version of fictitious play. This best reply dynamics is given by the differential inclusion

$$\dot{x} \in \beta(x) - x. \tag{1}$$

Similarly we can define generalized τ-best reply dynamics, for some $\tau \in \mathcal{T}$, given by the differential inclusion

$$\dot{x} \in \tau(x) - x. \tag{2}$$

The best reply dynamics (1) is obviously a special case of (2) for $\tau = \beta$. A solution to (2) is an absolutely continuous function $\xi(t, x_0)$ through initial state $x_0 \in \Theta$, defined for at least all $t \geq 0$, that satisfies (2) for almost all t. [11]

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[11]Gilboa and Matsui (1991) and Matsui (1992) require additionally the right differentiability of solutions. Hofbauer (1995) argued that all solutions in the sense of differential inclusions should be admitted. This is natural for applications to discrete approximations (fictitious play, see Hofbauer and Sorin (2006)) or stochastic approximations, see Benaim, Hofbauer, and Sorin (2005). Note that any absolutely continuous solution is automatically Lipschitz, since the right hand side of (1) is bounded. Hofbauer (1995)

Since the right hand side of (2) is upper-hemi continuous with compact and convex values, existence of at least one Lipschitz-continuous solution $\xi(t, x_0)$ through each initial state x_0 is guaranteed for any $\tau \in \mathcal{T}$, see (Aubin and Cellina 1984, Chapter 2, Section 1, Theorem 3, p.98). In general, several solutions can exist through a given initial state.

The objects we are after in this paper can now be defined as follows.

Definition 3 *Let $\Gamma \in \mathcal{G}^*$. A face $\Theta(R)$ (spanned by a strategy selection R) is a **minimal asymptotically stable face (MASF)** if there is a $\tau \in \mathcal{T}$ such that $\Theta(R)$ is asymptotically stable[12] under $\dot{x} \in \tau(x) - x$ and for all proper subfaces $\Theta(R') \subset \Theta(R)$, with R' a strategy selection, and for all $\tau' \in \mathcal{T}$ $\Theta(R')$ is not asymptotically stable under $\dot{x} \in \tau'(x) - x$.*

Since $\sigma(x) \subset \tau(x)$ for all x, for all $\tau \in \mathcal{T}$, every solution of the σ-best reply dynamics

$$\dot{x} \in \sigma(x) - x. \tag{3}$$

is also a solution of the τ-best reply dynamics (2). This means that if there is any substantial difference in the dynamics it is due to the multiplicity of trajectories. In fact, this is worth stating as a Lemma.

Lemma 5 *Let $\Gamma \in \mathcal{G}^*$. Let $\tau, \tau' \in \mathcal{T}$ such that $\tau \subset \tau'$. Let $x^0 \in \Theta$ be an arbitrary initial state. Then every solution to $\dot{x} \in \tau(x) - x$ through x^0 is also a solution to $\dot{x} \in \tau'(x) - x$ through x^0.*

Proof: This follows immediately from the partial order on $\mathcal{T}$. QED

This Lemma immediately implies another Lemma.

Lemma 6 *Let $\Gamma \in \mathcal{G}^*$. Let $\tau, \tau' \in \mathcal{T}$ such that $\tau \subset \tau'$. Let $A \subset \Theta$. If A is asymptotically stable under $\dot{x} \in \tau'(x) - x$, then A is also asymptotically stable under $\dot{x} \in \tau(x) - x$. Furthermore, if A is asymptotically stable under $\dot{x} \in \tau(x) - x$ for some $\tau \in \mathcal{T}$, then A is also asymptotically stable under $\dot{x} \in \sigma(x) - x$.*

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also provides an explicit construction of all piecewise linear solutions (for 2 person games) and provides conditions when these constitute all solutions. See also Hofbauer and Sigmund (1998), Cressman (2003) and Sandholm (2010).

[12] We call A asymptotically stable if it is Lyapunov stable and attractive. A is Lyapunov stable if for every neighborhood U of A there exists a neighborhood V of A such that all solutions $\xi(t, x_0)$ with $x_0 \in V$ satisfy $\xi(t, x_0) \in U$ for all $t \geq 0$. A is attractive if there is a neighborhood U of A such that for every solution $\xi(t, x_0)$ with $x_0 \in U$ its ω–limit set is contained in A: $\bigcap_{T \geq 0} \overline{\{\xi(t, x_0) : t \geq T\}} \subset A$.
Note that in contrast to Def IX on p.339 in Benaim, Hofbauer, and Sorin (2005), we drop here requirement (i) invariance. A is invariant if for every $x_0 \in A$ there is a complete solution $\xi(t, x_0) \in A$ (i.e., defined for all positive and negative times $t \in \mathbb{R}$). Consider as example the matching pennies game with the best reply dynamics. Then the only invariant set is the unique NE which is the minimal asymptotically stable set of this game. In particular there are no invariant faces under the best reply dynamics. The only CURB set is the whole strategy space. But this is not invariant, only forward invariant. Therefore it is not reasonable to require invariance in the definition of MASF.

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Proof: The first part follows directly from Lemma 5. The second part follows from the first part and the second part of Theorem 1. QED

Lemma 6, thus, implies that minimal asymptotically stable faces (MASF) are those and only those faces which are the smallest faces that are asymptotically stable under the (most) refined best reply dynamics $\dot{x} \in \sigma(x) - x$. In the remainder of this section we provide partial characterizations of such faces. We first show that, for any $\tau \in \mathcal{T}^{PS}$ the τ-best reply dynamics converges to the set of τ-rationalizable strategies[13]. Furthermore, every τ-CURB set is asymptotically stable under this dynamics. In particular Basu and Weibull's (1991) CURB sets are asymptotically stable under the best reply dynamics and Kalai and Samet's (1984) persistent retracts are asymptotically stable under the refined best reply dynamics. These results are similar to the results of Hurkens (1995), who for a stochastic learning model a la Young (1993) showed that recurrent sets coincide with CURB sets or persistent retracts depending on the details of the model. These results are also similar to that of Ritzberger and Weibull (1995) who show that any strategy selection which is closed under weakly better replies is asymptotically stable under any deterministic payoff-positive dynamics.

We then, however, give an example of a game in which a proper sub-face of a persistent retract is asymptotically stable under the refined best reply dynamics. I.e. a MASF can be smaller than a persistent retract. We show that a necessary condition for a face to be asymptotically stable under the refined best reply dynamics (to be a MASF) is that it constitutes a tight σ-prep set (and must thus be a τ-prep set for all $\tau \in \mathcal{T}$).

Theorem 3 *Let $\Gamma \in \mathcal{G}^*$. Let $\tau \in \mathcal{T}^{PS}$. Let R be the strategy selection of S which spans the set of τ-rationalizable strategies, i.e., $\Theta(R) = \tau^{\infty}(\Theta)$. Let $s_i \in S_i \setminus R_i$. Then $x_{is_i}(t) \to 0$ for any solution $x(\cdot)$ to $\dot{x} \in \tau(x) - x$ for any initial state $x(0) \in \Theta$.*

Proof: The proof is by induction on k, the iteration in the deletion process, i.e., the k in $\tau^{\infty}(\Theta) = \bigcap_{k=1}^{\infty} \tau^k(\Theta)$. Let R^k denote the strategy selection of S which spans $\tau^k(\Theta)$, i.e., $\Theta(R^k) = \tau^k(\Theta)$. For $k = 1$ consider an arbitrary strategy $s_i \in S_i \setminus R_i^1$. By definition then $s_i \notin \tau_i(x)$ for any $x \in \Theta$. Hence its growth rate according to $\dot{x} \in \tau(x) - x$ is $\dot{x}_{is_i} = 0 - x_{is_i}$, and therefore

$$x_{is_i}(t) = e^{-t} x_{is_i} \tag{4}$$

for all $t \geq 0$, i.e., $x_{is_i}(t)$ shrinks exponentially to zero. This proves the statement of the theorem for $s_i \in S_i \setminus R_i^1$. Now assume the statement of the theorem is true for $s_i \in S_i \setminus R_i^{k-1}$, i.e., for any such s_i we have that $x_{is_i}(t) \to 0$ for any solution $x(\cdot)$ to $\dot{x} \in \tau(x) - x$ for any initial state $x(0) \in \Theta$. Then for any such s_i and for any $x(0) \in \Theta$ there is a finite T such that $x_{is_i}(t) < \epsilon$ for all $t \geq T$. Now by the definition of τ, $s_i \in S_i \setminus R_i^k$ implies that $s_i \notin \tau_i(x(t))$ provided ϵ is small enough (or t large enough). But then for all $t \geq T$ we again have that $\dot{x}_{is_i} = 0 - x_{is_i}$ and, hence, that $x_{is_i}(t)$ shrinks exponentially to zero. QED

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[13]For $\beta = \tau$ this result is probably well understood. For a related, but weaker statement about the iterated elimination of strictly dominated strategies see (Sandholm 2010, Theorem 7.4.2).

Lemma 7 *Let $\Gamma \in \mathcal{G}^*$, R a strategy selection, and $\tau \in \mathcal{T}^{PS}$. If R is a τ-CURB set then $\Theta(R)$ is asymptotically stable under $\dot{x} \in \tau(x) - x$.*

Proof: By the definition of a τ-CURB set and the upper hemi–continuity of τ we have that for any $x \in U$ where U is a sufficiently small neighborhood of $\Theta(R)$ it is true that for any $i \in I$ $s_i \in \tau_i(x)$ implies $s_i \in R_i$. Hence, for any $x \in U$ we must have that $\dot{x}_{is_i} = -x_{is_i}$ for all $i \in I$ and $s_i \notin R_i$. But then we must have that $||x(t) - \Theta(R)||_\infty$ shrinks exponentially to zero for all $x(0) \in U$. QED

A corollary to Lemma 7 is the following.

Corollary 2 *Let $\Gamma \in \mathcal{G}^*$. A robust equilibrium point (Okada (1983)) is asymptotically stable under the refined best reply dynamics (3).*

This follows from the fact that a robust equilibrium point is a singleton persistent retract. Note that in games in $\mathcal{G}^*$ a robust equilibrium point must be a pure strategy profile.

Note that a game could well have asymptotically stable sets under $\dot{x} \in \sigma(x) - x$ which are proper subsets of persistent retracts, but are not faces. The unique Nash equilibrium of Matching Pennies is an example. It turns out, however, that there may even be faces which are proper subset of persistent retracts and yet are asymptotically stable under $\dot{x} \in \sigma(x) - x$.

Game 5: Consider the following 4-player game $\Gamma = (I, S, u)$ with $I = \{1, 2, 3, 4\}$, $S_1 = \{H_1, T_1\}, S_2 = \{H_2, T_2\}, S_3 = \{D, U\}$, and $S_4 = \{A, B\}$. The utility functions are given as follows.

Players 1 and 2 are playing matching pennies and do not care about other players' strategies. I.e. their payoffs are given by

	H_2	T_2
H_1	2,0	0,2
T_1	0,2	2,0

for any strategy pair of players 3 and 4. Player 3's strategy U is strictly dominated by D. I.e. $u_3(s_1, s_2, U, s_4) = 0$, while $u_3(s_1, s_2, D, s_4) = 1$ for all $s_1 \in S_1$, $s_2 \in S_2$, $s_4 \in S_4$. Player 4's payoffs are more interesting. Strategy A provides player 4 a payoff of 0 regardless of all other players' strategies. I.e. $u_4(s_1, s_2, s_3, A) = 0$ for all $s_1 \in S_1$, $s_2 \in S_2$, $s_3 \in S_3$.

Player 4's payoffs from strategy B are given as follows: $u_4(T_1, T_2, U, B) = 1$, $u_4(T_1, T_2, D, B) = 0$, $u_4(H_1, T_2, D, B) = -1$, $u_4(T_1, H_2, D, B) = -1$, $u_4(H_1, T_2, U, B) = 0$, $u_4(T_1, H_2, U, B) = 0$, and finally, $u_4(H_1, H_2, D, B) = u_4(H_1, H_2, U, B) = -2$.

Claim 1 *The unique minimal σ-CURB set in this game is the face spanned by the strategy selection $\{H_1, T_1\} \times \{H_2, T_2\} \times \{D\} \times \{A, B\}$.*

Proof: That player 1's part of a minimal σ-CURB set is $\{H_1, T_1\}$, and player 2's is $\{H_2, T_2\}$ is immediate from the matching pennies structure of their payoffs. That player 3's part

of a minimal σ-CURB set is $\{D\}$ is immediate from the fact that D strictly dominates U. It thus only remains to be shown that player 4's part must be $\{A, B\}$.

Obviously A is the unique best reply against (H_1, H_2, D). Thus A is included in player 4's part of the minimal σ-CURB set. Furthermore, a short calculation shows that $u_4(x_1, x_2, x_3, B) > u_4(x_1, x_2, x_3, A) = 0$ if and only if $x_3(U) > \frac{x_1(H_1) + x_2(H_2)}{1 - x_1(H_1)x_2(H_2)}$. The strategy profiles satisfying this condition form an open set. It contains the strategy profile (T_1, T_2, D, A) in its boundary. Therefore B is included in player 4's part of the minimal σ-CURB set as well. QED

Claim 2 *In this game, the face spanned by the strategy selection $\{H_1, T_1\} \times \{H_2, T_2\} \times \{D\} \times \{A\}$ is asymptotically stable under $\dot{x} \in \sigma(x) - x$.*

Proof: Consider a solution, ζ, to $\dot{x} \in \sigma(x) - x$ through some initial point x^0 close to this face. By the matching pennies structure of player 1 and 2's payoffs we must have that $\zeta_{1H_1}(t, x^0) \to \frac{1}{2}$ as $t \to \infty$. Also $\zeta_{2H_2}(t, x^0) \to \frac{1}{2}$ as $t \to \infty$. Furthermore we must have $\zeta_{3D}(t, x^0) \to 1$ as $t \to \infty$ by the fact that D strictly dominates U. In fact we must have that $\zeta_{3D}(t, x^0) > \zeta_{3D}(t', x^0)$ if $t > t'$.

Note that this game has only one Nash equilibrium $(\frac{1}{2}H_1 + \frac{1}{2}T_1, \frac{1}{2}H_2 + \frac{1}{2}T_2, D, A)$. Given the above observations, it must be true that $\zeta_{4A}(t, x^0) \to 1$ as $t \to \infty$ for all $x^0 \in \Theta$.

The only thing left to show is that the face spanned by $\{H_1, T_1\} \times \{H_2, T_2\} \times \{D\} \times \{A\}$ is also Lyapunov stable. In order to show this, we must prove that for any neighborhood V of the face there is another neighborhood $U \subset V$ such that any solution to $\dot{x} \in \sigma(x) - x$ with initial state in U must stay in V for all $t \geq 0$.

Let V be a neighborhood such that for any $x \in V$ we have $x_3(U) < \epsilon$ for some $\epsilon > 0$. Let $E(\epsilon) \subset \Theta$ denote the ϵ-box around state (T_1, T_2, D, A). I.e. $x \in E$ if and only if $x_1(H_1) \leq \epsilon$, $x_2(H_2) \leq \epsilon$, $x_3(U) \leq \epsilon$, and $x_4(B) \leq \epsilon$.

For $x^0 \in V \setminus E(2\epsilon)$, i.e. x^0 is a state outside the 2ϵ-box around (T_1, T_2, D, A). Given the matching pennies structure of the game between players 1 and 2, it is easy to see that $\zeta(t, x^0) \notin E(\epsilon)$ for any $t \geq 0$. Thus, by the argument in the proof of claim 1, B is never best against $\zeta(t, x^0)$ for any $t \geq 0$, and thus $\zeta(t, x^0)_4(B)$, will shrink to 0 as $t \to \infty$.

For $x^0 \in E(2\epsilon)$ things are different. For some such initial states x^0, $\zeta(t, x^0)$ can go through $E(\epsilon)$ for some time. For part of this time, indeed, player 4's strategy B could be best and could grow. However, there is an upper bound on this time $\zeta(t, x^0)$ spends within $E(\epsilon)$ which depends on ϵ. For ϵ small enough, the direction $\zeta(t, x^0)$ takes for players 1 and 2 must be towards T_1, H_2 (irrespective of what players 3 and 4 do). Given that T_2 is thus not best anywhere for player 2 in this ϵ-box around (T_1, T_2, D, A) means that T_2 will shrink (or H_2 grow). In fact we must have $\zeta(t, x^0)_2(T_2) = x_2^0(T_2)e^{-t}$. In the worst case, we, thus, have that $\zeta(T, x^0) \notin E(\epsilon)$ for all $t \geq T$ if $T = -\ln(1 - \epsilon)$. Thus, $-\ln(1 - \epsilon)$ is the longest time possible $\zeta(t, x^0)$ stays within $E(\epsilon)$ for any $x^0 \in E(2\epsilon)$. Part of this time, at most all of it, B could be (uniquely) best for player 4. Thus, $\zeta(t, x^0)_4(B)$ could grow for up to this amount of time. Given $x^0 \in V$ and, thus, $x_4^0(B) \leq \epsilon$ we have that $\zeta(t, x^0)_4(B) \leq 1 - (1 - \epsilon)e^{\ln(1-\epsilon)} = 1 - (1 - \epsilon)^2 < 2\epsilon$. As $\epsilon \to 0$ this expression tends to 0 as well. This implies that for any neighborhood V of the face spanned by

$\{H_1, T_1\} \times \{H_2, T_2\} \times \{D\} \times \{A\}$ there is another neighborhood U such that $x_3(U) \leq \epsilon$ for some $\epsilon > 0$ (small enough) and all $x \in V$ such that $\zeta(t, x^0) \in V$ for all $t \geq 0$ and all $x^0 \in U$. QED

Note that $\sigma = \beta$ in this game. Therefore the converse of Lemma 7 is not true for the best reply dynamics as well. The difference in the seemingly discrepant conclusions of Claim 1 and 2 is driven by the following observation. B is the unique best reply for player 4 in a cone shaped set of opponent strategy profiles with apex (T_1, T_2, D). Hence B is included in each σ−CURB set. Still, the face spanned by the smaller strategy selection $\{H_1, T_1\} \times \{H_2, T_2\} \times \{D\} \times \{A\}$ is asymptotically stable: The trajectories starting in this cone leave it quickly, the more quickly the closer the starting point is to the (smaller) face. Thus there is a vanishing amount of time in which B can grow.

Lemma 8 *Let $\Gamma \in \mathcal{G}^*$ and R a strategy selection. If $\Theta(R)$ is asymptotically stable under the τ-best reply dynamics $\dot{x} \in \tau(x) - x$, for $\tau \in \mathcal{T}^{PS}$, then R is a τ-prep set. Furthermore, if for all strategy selections R', which are proper subsets of R, $\Theta(R')$ is not asymptotically stable under $\dot{x} \in \tau(x) - x$ (i.e. $\Theta(R)$ is a minimal asymptotically stable set) then R is a tight τ-prep set.*

Proof: As $\tau \in \mathcal{T}^{PS}$ for every $x \in \Theta$ there is a $T_i(x)$ such that $\tau_i(x) = \Delta(T_i(x))$ for all $i \in I$. Now suppose $\Theta(R)$ is asymptotically stable under $\dot{x} \in \tau(x) - x$ and R is not a τ-prep set. Then there is an $x \in \Theta(R)$ and a player $i \in I$ such that $T_i(x) \cap R_i = \emptyset$. But then, by the upper-hemi continuity of τ, there is a neighborhood of x such that $T_i(x') \subset T_i(x)$ for all x' in this neighborhood. Thus a solution to $\dot{x} \in \tau(x) - x$ starting from a state in the interior of this neighborhood must spend a finite amount of time within this neighborhood. In this finite amount of time all strategies in R_i must shrink. Thus $\Theta(R)$ is not asymptotically stable, which is a contradiction, proving the first part of this Lemma. To prove the second part, suppose R is not a tight τ-prep set and asymptotically stable. Then there is a player $i \in I$ and $s_i \in R_i$ such $s_i \notin T_i(x)$ for any $x \in \Theta(R)$. But then R', derived from R by letting $R'_i = R_i \setminus \{s_i\}$ and $R'_j = R_j$ for all $j \neq i$, is also asymptotically stable under $\dot{x} \in \tau(x) - x$. QED

Not every τ-prep set is asymptotically stable under $\dot{x} \in \tau(x) - x$. To see this, for $\tau = \sigma$, consider the following game.

	D	E	F
A	1,1	1,1	2,1
B	1,1	0,0	3,1
C	1,2	1,3	1,1

Game 6: This game is taken from Samuelson (1992).

It is a symmetric game. Note that strategies C and F are weakly dominated. The unique persistent retract of this game is spanned by $\{A, B\} \times \{D, E\}$. There are three singleton σ-prep sets. These are $\{(A, D)\}$, $\{(A, E)\}$, and $\{(B, D)\}$. None of these are asymptotically stable under the refined best reply dynamics. Thus, none of these are MASF. To see this

note that $\dot{x} \in \sigma(x) - x$ has a solution starting at (A,D) that gradually takes play towards (A,E).

We are finally in a position to prove our main Theorem.

Theorem 4 *Let $\Gamma \in \mathcal{G}^*$. If R is a strategy selection such that $\Theta(R)$ is a persistent retract (minimal σ-CURB set), then $\Theta(R)$ contains a MASF. Conversely, if R is a strategy selection such that $\Theta(R)$ is a MASF, then R is a tight σ-prep set.*

Proof: For the first part suppose that R is a strategy selection such that $\Theta(R)$ is a persistent retract (minimal σ-CURB set). Then that $\Theta(R)$ is asymptotically stable under $\dot{x} \in \sigma(x) - x$ follows from Lemma 6 and Lemma 7. Thus, it either is a MASF or it contains one. The second part follows from Lemma 8. $\qquad$ QED

If one is interested in (partially) characterizing not only MASFs but also its asymptotically stable subsets one approach could be to define an appropriate version of Balkenborg and Schlag's (2007) strict equilibrium sets based, however, on the refined best reply correspondence. We conjecture that these sets, appropriately defined, are asymptotically stable under $\dot{x} \in \sigma(x) - x$.

6 Conclusion

In this paper we endeavored to find the smallest faces of the set of mixed strategy profiles that can be justifiably called evolutionary stable. In order to do so we introduce generalizations of the best reply correspondence which satisfy four, we believe reasonable, criteria. The criteria are as follows. A generalized best reply correspondence must have a product structure, as we want players to choose independently. At least one best reply must always be available to players. It must be point-wise closed and convex. Closedness is more of a technical requirement, but convexity derives from the desire to have players randomize arbitrarily between their generalized best replys. Finally we require a generalized best reply correspondence to be upper hemi–continuous. This is an important technical requirement as it guarantees that such a generalized best reply correspondence has a fixed point and the differential inclusion based on it always has a solution. In terms of player behavior it translates to the requirement that if one were to perturb the current strategy profile of the opponents a little bit this player will not choose a new strategy which was not formerly in the set of generalized best-replies.

We define a generalized best reply dynamics as an appropriate differential inclusion based on the respective generalized best reply correspondence. We define a minimal asymptotically stable face (MASF) as a face of the set of mixed strategy profiles which is asymptotically stable under some such generalized deterministic evolutionary process with the additional property that it does not properly contain another face which is also asymptotically stable under some (possibly different) generalized best reply dynamics.

We show (Theorem 1) that the set of all generalized best reply correspondences (and, hence, dynamics) is, for most games, a complete lattice with a unique smallest element. We call this smallest element the refined best reply correspondence (and dynamics).

The main results in this paper offer a partial characterization of minimal asymptotically stable faces (MASFs). We show (Theorem 2) that every persistent retract (Kalai and Samet (1984)) is the same as a minimal CURB-set (Basu and Weibull (1991)) based on, however, the refined best reply correspondence, and contains a MASF (Theorem 4). A MASF can be a proper subface of a persistent retract, as we show by example, and must be (Theorem 4) a tight prep-set (Voorneveld (2004)) based on, again, the refined best reply correspondence. Thus, MASFs are sets somewhere "in-between" CURB and prep sets based on the refined best reply correspondence.

Our findings are, thus, reminiscent of Hurkens's (1995) findings that a stochastic best reply learning process based on semi-robust best-replies leads to play eventually leading to a persistent retract. Altogether this suggests that while it is difficult to justify Nash equilibrium behavior, either epistemically or through evolution or learning, yet alone any of its point-wise refinements or even set-wise refinements such as Kohlberg and Mertens's (1986) strategically stable sets, there are relatively small sets of strategy profiles which are justifiable through learning. Furthermore every persistent retract, or CURB set based on the refined best reply correspondence, contains such a set, while it is a necessary condition for a face to be a MASF that this face be spanned by a tight prep-set based, again, on the refined best reply correspondence.

We, thus, suggest that in applied game theory work MASFs, or at least appropriate variations of CURB sets, persistent retracts, and prep sets, which are as of now not used to a great extent[14], may be very apt choices for a solution concept in some contexts.

A On the generic equivalence of own-payoff equivalence and payoff equivalence

Adapting a notion of Brandenburger and Friedenberg (2007) for perfect information games, let a normal form game satisfy the Single Payoff Condition (SPC) if all own-payoff equivalent pure strategies are also payoff equivalent. Not every game satisfies the SPC: a player other than i might not be indifferent between player i's own-payoff equivalent strategies (as is the case in Game 3). Thus, our restriction that the game should have no own-payoff equivalent strategies for any player i is a stronger requirement than saying the game has to be in semi-reduced normal form (see e.g., page 147 in Ritzberger (2002)). However, games not satisfying the SPC are exceptional. Trivially, for generic normal form games there are neither own-payoff nor payoff equivalent strategies and hence the SPC holds. This is of little interest because most important classes of normal from games such as normal forms of extensive form games or of finitely repeated games are non-generic. Requiring genericity conflicts with imposing any additional structure on the class of games considered.[15] In this appendix we identify a condition on a class of normal form games which implies that the SPC holds generically **within this class**. This condition is shown

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[14]Notable exceptions are e.g., Kalai and Samet (1985), Balkenborg (1993), Blume (1994, 1996), Hurkens (1996), van Damme and Hurkens (1996), and Gordon (2006).

[15]For an illuminating discussion on this point see Mertens (2004).

to be satisfied by the classes of normal forms of extensive form games, of finitely repeated games, and of cheap talk games. Thus, the restriction to games satisfying SPC and hence, after the identification of payoff equivalent strategies, to games in $\mathcal{G}^*$, is not a severe one.

Definition 4 *A normal form game satisfies the Single Payoff Condition (SPC) if the following holds for all players $i \in I$. Two strategies $s_i, s'_i \in S_i$ satisfy the equations*

$$u_i\left(s_i, s_{-i}\right) = u_i\left(s'_i, s_{-i}\right)$$

for all $s_{-i} \in S_{-i}$ only if also the equations

$$u_j\left(s_i, s_{-i}\right) = u_j\left(s'_i, s_{-i}\right)$$

holds for all $j \in I$ and all $s_{-i} \in S_{-i}$.

Definition 5 *For a given set of strategy combinations $\mathcal{S}$ consider a family of normal form games $\{\Gamma^\mu\}_{\mu \in O}$ given by utility functions*

$$u_i\left(s, \mu\right) \tag{5}$$

for $s \in \mathcal{S}$ and $i \in I$, which depend on a vector of parameters μ taken from a nonempty open set O in some Euclidian space $\mathbb{R}^k$. We call the family analytic if all $u_i\left(s, \mu\right)$ are analytic functions in μ for given $s \in S$.[16] We say that the family satisfies the Functional Single Payoff Condition if the following holds for all players $i \in I$. Two strategies $s_i, s'_i \in S_i$ satisfy the functional identities

$$u_i\left(s_i, s_{-i}, \mu\right) = u_i\left(s'_i, s_{-i}, \mu\right) \qquad \text{for all} \qquad \mu \in O$$

for all $s_{-i} \in S_{-i}$ only if also the functional identities

$$u_j\left(s_i, s_{-i}, \mu\right) = u_j\left(s'_i, s_{-i}, \mu\right) \qquad \text{for all} \qquad \mu \in O$$

hold for all $j \in I$ and all $s_{-i} \in S_{-i}$.

Proposition 3 *Suppose the analytic family of games $\{\Gamma^\mu\}_{\mu \in O}$ satisfies the Functional Single Payoff Condition. Then for generic $\mu \in O$ the game Γ^μ satisfies the Single Payoff Condition.*

Proof: Fix $i \in I$, $s_i, s'_i \in S_i$, $s_{-i} \in S_{-i}$ and $j \in I$ such that $u_j\left(s_i, s_{-i}, \mu\right)$ and $u_j\left(s'_i, s_{-i}, \mu\right)$ are distinct as functions in μ. Then the set of parameter values μ for which

$$u_j\left(s_i, s_{-i}, \mu\right) = u_j\left(s'_i, s_{-i}, \mu\right) \tag{6}$$

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[16]An analytic function is a function that is locally described by power series. The notion covers most functions arising in applications, in particular linear and rational functions or functions like e^x or $\ln(x)$. In our examples the functions are always linear.

is a closed lower dimensional analytic set because the function is analytic (see e.g., Gunning and Rossi (1965)). Because there are finitely many choices of $i \in I$, $s_i, s_i' \in S_i$ $s_{-i} \in S_{-i}$ and $j \in I$ to consider we find that for μ outside a lower dimensional analytic subset D of O the identity (6) for **some** $i \in I$, $s_i, s_i' \in S_i$ $s_{-i} \in S_{-i}$ and $j \in I$ implies the identity (6) for **all** $i \in I$, $s_i, s_i' \in S_i$ $s_{-i} \in S_{-i}$ and $j \in I$ and also $\mu \in O$. In particular, the SPC condition holds for all $\mu \notin D$. QED

Example 1 In a **cheap talk game** players first send simultaneously and independently public messages m_i from message spaces M_i. After all players have received the combination of messages

$$m = (m_1, \cdots, m_n) \in M = \times_{i \in I} M_i \tag{7}$$

they choose simultaneously and independently actions $a_i \in A_i$. A pure strategy in such a game consists of a message m_i and a function $f_i : M \to A_i$. The play of any strategy combination s will result in a combination of messages $m \in M$ and a combination of actions

$$a = (a_1, \cdots, a_n) \in A = \times_{i \in I} A_i, \tag{8}$$

where, in a cheap talk game, only the latter is payoff relevant. In this example the parameter space is $\mathbb{R}^{A \times I}$. For $\mu \in \mathbb{R}^{A \times I}$ we define the utility function by

$$u_i(s, \mu) = \mu_{a,i} \tag{9}$$

where a is the combination of actions induced by s. The utility function is then for each $s \in S$ the projection onto a particular component of the vector μ. The identity

$$u_i(s_i, s_{-i}, \mu) = u_i(s_i', s_{-i}, \mu) \tag{10}$$

can only hold for **all** μ if both functions project onto the same component of μ, i.e., if the play of both (s_i, s_{-i}) and (s_i', s_{-i}) results in the same combination of actions a, although in possibly different combinations of messages. (If (s_i, s_{-i}) and (s_i', s_{-i}) would result in different combinations of actions a and a' the equality would not hold in the game where all players get 1 after a and 0 after a'.) If this is the case then, by construction,

$$u_j(s_i, s_{-i}, \mu) = \mu_{a,j} = u_j(s_i', s_{-i}, \mu) \tag{11}$$

for all j and μ. Thus Proposition 3 applies and we conclude that the SPC holds generically in cheap talk games.

Example 2 In an **extensive game without chance moves** the play of any pure strategy combination results in a terminal node $t \in T$. In this case the parameter space for a given extensive form is $\mathbb{R}^{T \times I}$ and the utility function is $u_i(s, \mu) = \mu_{t,i}$ if s induces t. The same arguments as for cheap talk games imply that the SPC holds in generic extensive form games with no random moves. Notice, though, that almost no extensive game with the extensive form of a cheap talk game is itself a cheap talk game. Hence the previous result is not a special case of this one.

Example 3 In an extensive game with chance moves the parameter space remains as in the previous example, but the utility function becomes

$$u_i(s, \mu) = \sum_{t \in T} p_t \mu_{t,i} \tag{12}$$

where p_t is the probability with which terminal node t is reached when the pure strategy combination s is played. Clearly, the equation

$$u_i(s_i, s_{-i}, \mu) = \sum_{t \in T} p_t \mu_{t,i} = \sum_{t \in T} p'_t \mu_{t,i} = u_i(s'_i, s_{-i}, \mu) \tag{13}$$

can only hold for **all** $\mu \in R^{T \times I}$ if $p_t = p'_t$ for all $t \in T$. Thus the SPC holds for generic extensive form games even with chance moves.

Example 4 In a finitely repeated game with perfect monitoring, no discounting and $t \geq 0$ periods, the play of a pure strategy combination s results in a sequence $(a_1, a_2, \cdots, a_t)$ of combinations of actions in the stage game. The payoff to a player can be written as

$$\sum_{a \in A} k_{s,a} \mu_{a,i} \tag{14}$$

where the parameter $\mu_{a,i}$ is player i's payoff in the stage game from the combination of actions a and $k_{s,a}$ is the number of times a is played in the sequence $(a_1, a_2, \cdots, a_t)$. If for two strategy combinations $s = (s_i, s_{-i})$ and $s' = (s'_i, s_{-i})$

$$u_i(s, \mu) = \sum_{a \in A} k_{s,a} \mu_{a,i} = \sum_{a \in A} k_{s',a} \mu_{a,i} = u_i(s', \mu) \tag{15}$$

holds for all $\mu \in R^{A \times I}$ then $k_{s,a} = k_{s',a}$ for all $a \in A$ and, hence,

$$u_i(s, \mu) = \sum_{a \in A} k_{s,a} \mu_{a,i} = \sum_{a \in A} k_{s',a} \mu_{a,i} = u_i(s', \mu) \tag{16}$$

Again, the SPC holds generically in repeated games.

Example 5 Consider finally the class of normal form games which satisfy for every $i \in I$, every $s_{-i} \in S_{-i}$ and any $s_i \, s'_i \in S_i$ the equation

$$u_i(s_i, s_{-i}, \mu) = u_i(s'_i, s_{-i}, \mu) \tag{17}$$

If at least one player has two strategies, then this class does not satisfy the Functional Single Payoff Condition. Almost all games in this class violate the SPC.

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