The Interest Rate and the Growth Rate - Steady-State-Efficiency in OLG-Models

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by

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Abstract

A general model of intertemporal consumption choice is developed, following Samuelson`s 1958 OLG-approach. The efficiency properties of the model are discussed with and without the introduction of durable goods, of productive capital, and fiat money. It is shown that the criterion of golden rule efficiency is not reasonable, if transition periods are taken into account. Moreover, the introduction of an infinitely lived institution, which grows at the steady state rate, will definitely prevent the interest from falling beyond the growth rate. Hence, the main arguments against intertemporal efficiency of the market mechanism in OLG-models turn out to be invalid.

Introduction

In his famous 1958 paper on the "biological" determination of the interest rate, Paul Samuelson showed that the market mechanism can readily fail to meet the golden-rule-condition for maximum lifetime consumption (Samuelson 1958). This result has generally been taken as a proof of market failure, although apparently no market deficiencies such as externalities, asymmetric information etc. are involved. Samuelson himself called it a paradox, which, according to Diamond (1965; p.1129) ultimately arises from the comparison of steady states. Similarly, Starrett (1972, pp. 282) argued that "golden rule efficiency ignores problems of transition". This is well known from common Solow-Swan-type growth models, where in general the interest rate is equal to the growth rate and hence the golden rule is met, unless some rate of time preference is employed. In Samuelson´s pure consumption-loans model, however, the golden rule is violated although no time preference is assumed at all.

Starrett (1972, pp 285) argued that, in contrast to the production models, in the pure-consumption model inefficiency only arises if the interest rate is lower than the growth rate. On the other hand, an interest rate being above the growth rate could simply reflect the transition costs of switching to the golden rule path and, therefore, could not be viewed as intertemporally inefficient. He seems to believe, however, that the former case is readily relevant and, hence, the market mechanism could not generally be trusted, even if a broader concept of intertemporal efficiency (i.e. including transition costs) is applied.[1]

In the sequel, the problem is discussed within the framework of a three-generations OLG-model which is both a generalization and an extension of the model in Samuelson (1958).

[1] See Starrett, 1972, pp 285. He concludes: „Thus, competition may be an inefficient method of distribution, and the reasons seem to be exactly the opposite of those fort he production model." (ibid, p. 287)

Starting with the pure consumption-loan case, durable goods, productive real capital, and fiat money are introduced into the model. By extending Starrett`s argument, it is shown that, with the interest rate exceeding the growth rate, the living generations always gain from deviating from the golden rule-path, even if the latter has already been realized. Moreover, it is shown that, with at least one infinitely lived institution which is growing at the steady state rate, it is virtually impossible for the interest rate to fall short of the growth rate in the steady state. Therefore, the market mechanism does much better with respect to intertemporal consumption choice than it is usually deduced from the golden rule-criterion. The paper finishes with a suggestive interpretation of the fundamental relation between the growth rate and the interest rate.

I. A Simple Textbook OLG-Model

We start with a generalized version of Samuelson`s original OLG-model. Let $\mathbf{w} = (w_1;w_2;w_3)$ be the vector of incomes and $\mathbf{c} = (c_1;c_2;c_3)$ the vector of consumption of three generations of consumers G_j respectively, where G_1 is the number of the youngest, G_2 is the number of the middle aged, and G_3 is the number of the older generation. All individuals live for three periods and seek to optimise their lifetime consumption according to a well behaved intertemporal utility function $U(c_1;c_2;c_3)$. There is no real capital, no durable good, and no money available to store one´s income for a following period. The only way of postponing ones consumption is the lending of real goods to other individuals, and the only way to bring ones consumption forward is to borrow real goods from them. Note that the elder generation G_3 cannot even do this, for they will not be alive anymore in the next period and hence will neither save nor obtain any credit. It is assumed that all individuals are purely self-interested and that there are no heritages. Therefore the vector of individual savings (in terms of real

goods) is **s** = (s_1;s_2;0). With q = (1+i) denoting the interest factor (and i being the interest rate), this leads to the following set of microeconomic budget constraints:[2]

$$(1)\ c_1 = w_1 - s_1$$
$$(2)\ c_2 = w_2 - s_2 + s_1 * q$$
$$(3)\ c_3 = w_3 + s_2 * q$$

The macroeconomic constraint is

$$(4)\ s_1 * G_1 + s_2 * G_2 = 0$$

Adding equations (1) to (3) yields the overall budget constraint for individual lifetime consumption:

$$(5)\ (c_3 - w_3) + q * (c_2 - w_2) + q^2 * (c_1 - w_1) = \sum_{j=1}^{3} (c_j - w_j) * q^{3-j} = 0$$

We generalize Samuelson`s original utility function U(**c**) = c_1*c_2*c_3 as follows:

$$(6)\ U(\mathbf{c}) = \prod_{j=1}^{3} c_j^{\alpha_j}$$

The α_j can be interpreted as consumption weights of the respective life periods.[3] In the Samuelson case with $\alpha_j = 1\ \forall$ j, there is no explicit time preference, and hence the optimisation problem results only from diminishing marginal utility of consumption in the respective periods.

[2] In the more general case of m Generations and hence with a lifespan of m equations (1) to (3) could be written as $\sum_{j=1}^{m} (c_j - w_j + s_j - s_{j-1} * q) = 0$ with m = 3 and $s_0 = s_m = 0$.

[3] By rewriting utility function (6) as $\ln U = \alpha_1 * \ln c_1 + \alpha_2 * \ln c_2 + \alpha_3 * \ln c_3$ it follows that time preference is positive if $\alpha_j < \alpha_{j-1}$ and vice versa.

After some manipulation of terms, maximizing (6) with respect to (5) yields:

$$(7)\ c_j = \frac{\alpha_j}{\sum_{j=1}^{3} \alpha_j} * q^{j-2} * \sum_{j=1}^{3} (w_j * q^{2-j})$$

Because the sum on the right hand side of (7) is constant, optimal individual consumption rises at the rate $q * \alpha_j / \alpha_{j-1}$ with rising age j. Note that α_j / α_{j-1} could thoroughly vary in the respective lifetime periods. In the Samuelson case of $\alpha_j = 1\ \forall$ j, optimal individual consumption rises simply at the interest factor q.

Still following Samuelson and many other authors, we compare the market result to the consumption path, which a benevolent dictator would choose. If n is the rate of population growth and $g = (1+n) = G_{j-1} / G_j$ is the corresponding growth-factor, one has to maximise (6) with respect to:

$$(8) \sum_{j=1}^{3} G_j * c_j - \sum_{j=1}^{3} G_j * w_j = \sum_{j=1}^{3} (c_j - w_j) * g^{3-j} = 0$$

Replacing g by q in the right-hand version of the macroeconomic constraint (8) would immediately transform it into its microeconomic pendant (5). Hence, the market solution apparently equals the optimal centralised consumption plan if - and only if – the growth factor g and the interest factor q are equal. This condition is apparently equivalent to Phelp´s golden rule of accumulation (Phelps 1965). However, in contrast to the latter, it also holds if we assume $\alpha_j \neq 1$ for any j, i.e. if any rate of time preference is implied. Moreover, as Samuelson

has shown, the market mechanism regularly fails to meet the golden rule even if all $\alpha_j = 1$, and, hence, no time preference is assumed.[4]

His example of **w** = (1;1;0) with g = 1 and $\alpha_j = 1 \ \forall$ j shows easily why. From equations (1), (2), and (4) it follows that

$$(9)\ g = \frac{G_1}{G_2} = -\frac{s_2}{s_1} \quad .$$

This inserted into (5) yields

$$(10)\ q * g = \frac{(c_3 - w_3)}{(c_1 - w_1)},$$

where c_3 and c_1 according to (7) are functions of q respectively. Solving (10) by numerical methods yields **c** = (1.434; 0.434; 0.132), $q \approx 0.303$ and an interest rate of $i \approx -0.697$.[5] The problem here is not that the interest rate is negative, but that it falls short of the growth rate, which is zero in this example. Therefore, the privately chosen consumption pattern, where c_j increases with rate q in each period of lifetime, is clearly inefficient in the light of the golden rule criterion (which would suggest a flat intertemporal consumption path because of n = 0). Moreover, an interest rate i = 0 ,which would meet the golden rule, is not at all feasible in this example. The members of the middle aged generation G_2 simply do not encounter enough demand of loans by the members of the young generation G_1. They therefore cannot postpone more of their consumption than c_3= 0.1315 (instead of the golden rule quantity c_3= 0.666) to

[4] As a matter of fact, it would be sufficient that all α_j are equal.

[5] See also Samuelson (1958), p. 478.

their seniority, despite of the negative interest rate. As Kotlikoff (2006) has shown, the situation would be even worse in a model with only two generations and **w** = (1;0). In such a world, lacking any durable goods and any sympathy, the elderly would have to grovel for food, while their kids would just "pelt them with candy wrappers" and "experience no qualm in watching their parents starve" (Kotlikoff, 2006, p.1). A central planner could, in contrast, command a more equal distribution of lifetime consumption and hence, according to the respective population growth, maximize the long-term-utility of everyone.

These examples are not at all far-fetched. On the contrary, the market solution would only by chance coincide with the golden rule-solution in a pure consumption-loan OLG-model (Starrett 1972). The general condition for this to apply is easily derived. From equation (7) we know that

$$(11)\; c_3 = q^2 * (\alpha_3 / \alpha_1) * c_1$$

By inserting (11) into (10) and requiring that q = g we find

$$(12)\; w_3^* = \frac{(2*\alpha_3 + \alpha_2) * w_1 * q^2 + (\alpha_3 - \alpha_1) * w_2 * q}{(2*\alpha_1 + \alpha_2)}$$

Hence the Samuelsonian OLG-model meets the golden rule condition only if condition (12) is met.[6] Otherwise, substantial inefficiencies seem inevitable unless a central planner intervenes in the market. As there is no reason why $w_3 = w_3^*$ should apply, apart from pure coincidence,

[6] If $\alpha_1 = \alpha_3$ holds equation (12) reduces to $w_3 = q^2 * w_1$. Interestingly, the interest rate then depends on neither the consumption weight α_2 nor the wage w_2 of the middle generation. Instead, just the relation α_1 / α_3 matters.

there seems to occur a fundamental kind of market failure. In particular, in the light of Samuelson´s results a pay-as-you-go pension system appears to be highly superior to a funded pension scheme, because the former could easily achieve the golden-rule-condition, while the latter does regularly not. [7]

II. Extending the model

Samuelson`s assumptions are quite narrow. It makes sense to relax them, in order to give private agents some more options for adjustment.

A first attempt could be to allow for the existence of a durable good which lasts for one period. Equation (4) would then change in $s_1 * G_1 + s_2 * G_2 = D$, where D denotes total investment in real goods per period. On the one hand, this would instantly prevent the interest rate from becoming negative, because with i < 0 everyone would prefer to invest in the durable good instead of lending. On the other hand, however, this could not prevent the interest rate from falling short of the population-growth, if the latter were positive. For with a positive interest rate, D would immediately become zero, and we were left with Samuelson´s original model again. Moreover, the existence of a durable good would even increase the danger of inefficiency, because with n < 0 it is impossible for i to equal n. Paradoxically, it follows that the additional option given to private agents seems to worsen the market failure instead of curing it.

Alternatively, following Diamond (1965), one could introduce productive real capital into the model. Equation (4) is then replaced by

[7] Note, however, that with w = (0;1;0) the problem disappears and hence an interesting and important class of OLG-models, where the young as well as the elder people have no income at all, is not affected by the Samuelson impossibility verdict.

$$(4i)\ s_1 * G_1 + s_2 * G_2 = K_{t+1}$$

where K_{t+1} denotes the amount of real capital which is employed for production in the following period.[8] Let us apply the Cobb-Douglas-production function

$$(13)\ Y = \bar{a} * L^{\phi} * K^{1-\phi} = \sum_{j=1}^{3} (G_j * w_j) + K * q$$

In contrast to Diamond, we assume that the capital good cannot be used for an infinite time, but wears out after one period. Then the right-hand side of (13) denotes the resulting factor income distribution with gross capital returns q = (1+i), where it is assumed that all members of a generation G_j are workers who earn wages w_j, the latter now being endogen. L denotes the sum of all generations` labour input shares, the latter being weighted by their relative marginal productivity:[9]

$$(14)\ L \equiv \left(\frac{w_1}{w_2} * \frac{G_1}{G_2} + 1 + \frac{w_3}{w_2} * \frac{G_3}{G_2} \right) * G_2 \equiv z * G_2$$

From (4i), (13), and (14), capital intensity k can be calculated as

$$(15)\ k \equiv \frac{K}{L} = \frac{G_{2;t-1} * \left(s_2 + s_1 * \frac{G_{1;t-1}}{G_{2;t-1}} \right)}{G_2 * z} = \frac{\left(\frac{s_2}{g} + s_1 \right)}{z}$$

[8] One could think of corn which is invested in the form of seed instead of being consumed as bread.
[9] G_2 and w_2 are effectively used as numeraires in (14).

Partial differentiation of (13) yields

$$(16)\ y_L = \phi * a * k^{1-\phi} = w_2$$

and

$$(17)\ y_K = (1-\phi) * a * k^{-\phi} = q$$

respectively.[10] Employing (17), capital intensity k is also calculated from the production side and then set equal to (15):

$$(18)\ k = \left(\frac{(1-\phi) * a}{q} \right)^{\frac{1}{\phi}} = \frac{\left(\frac{s_2}{g} + s_1 \right)}{z}$$

Regarding that savings s_1 and s_2 depend on the interest rate, q must in general be calculated by numerical methods. As can be easily seen from (17), the resulting interest rate may well be negative (i.e. q < 1), if savings are high enough, thereby forcing capital intensity to exceed the critical value $k^* = \left((1-\phi) * a\right)^{1/\phi}$. For example, a part of the crop could be rotting in the stores.[11] Much more important is, however, that q will again regularly differ from g, and hence the golden rule is still violated by the market mechanism. A most simple example is the Samuelsonian case of g = 1, $\alpha_j = 1\ \forall$ j , $w_1 = w_2 > 0$ and $w_3 = 0$. Assuming a = 1.53 and $\phi = 0.8$ we end up with **w** = (1;1;0), **c** = (0.826; 0.561; 0.381) and q = 0.679. Although the

[10] From (14) and the right-hand side of (13) it follows that $\sum_{j=1}^{3} G_j * w_j = L * w_2$.

[11] A nowadays more relevant example are dwells whose rental fees fall short of the deviations.

resulting interest rate i = -0.321 exceeds its corresponding value without real capital (which was -0.697) it still falls short of the growth rate and is therefore clearly inefficient.

Another possible way out was already sketched – though not actually modelled - by Samuelson himself, namely the introduction of money (Samuelson 1958, p. 481). The basic idea is that, with total money M being constant (and also a constant velocity of circulation v), the rate of inflation (or deflation) should be inversely related to the economy`s growth rate n. Fiat money therefore yields a real interest rate which is equal to the growth rate. If so, q could apparently never fall short of g, for then everyone would prefer hording money instead of lending. In contrast to the durable-good-case, this holds also true if the economy shrinks and, therefore, i = n < 0. The reason is, that, with money being available, individuals are no longer restricted to barter.[12] Instead, they can earn, buy, borrow and lend in terms of money. Denoting nominal terms as capitals, the set of individual budget constraints (1) to (3) changes to

$$(1i)\ C_1 = W_1 - S_1$$

$$(2i)\ C_2 = W_2 - S_2 + S_1 * \frac{p_2}{p_1} * q$$

$$(3i)\ C_3 = W_3 + S_2 * \frac{p_3}{p_2} * q$$

In the steady state, the inflation factor p_t/p_{t-1} is constant and perfectly foreseen by everyone, and q is, therefore, still the *real* interest factor. By dividing equations (1i) to (3i) by the price

[12] The workers of one epoch are given "a claim on workers of a later epoch, even though no real *quid pro quo* (other than money) is possible." (Samuelson, 1958, p. 482; italics in the original). This is what Samuelson called the social contrivance of money.

level p_t[13], we arrive exactly at the "real" budget constraints (1) to (3) again. The macroeconomic condition (4), however, changes now to the *nominally* defined constraint

$$(4\text{ii})\ S_1 * G_1 + S_2 * G_2 = H\ .$$

Written in real terms, this is

$$(4iii)\ s_1 * G_1 + s_2 * G_2 = h\ ,$$

where H denotes the aggregate amount of hoarding and h = H/p. In order to get a measure of the amount of real hoarding that is constant in the steady state, h must be related to any other growing macroeconomic aggregate, e.g. to total wages $L*w_2$. By making use of (14), the real hoarding share of total wages can then be reckoned as

$$(19)\ \omega \equiv \frac{h}{L * w_2} = \frac{s_1 * g + s_2}{z * w_2}$$

which is clearly constant for any $q = g = \bar{g}$.

Obviously, for h > 0 it must be true that $q = p_t/p_{t-1} = g$, due to arbitrage in giving loans or holding money. Therefore, real hoarding h can be calculated by replacing q by g in equation (7). The resulting intertemporal consumption pattern **c** is then, of course, the golden rule consumption pattern. For example, in the Samuelsonian example with **w** = (1; 1; 0) and g = 1, the introduction of fiat money $M = \overline{M}$ leads to $\bar{\omega} = 0{,}5$ and $c_j = 0{,}666$ for $\forall$j .

[13] The subscripts in equations (1i) to (3i) stand for the periods of individual lifetime t here. They are no longer identical for the respective cohorts j, because nominal terms may change in time due to changing prices.

Nevertheless, also this solution is far from being perfect. First of all, if $q > p_t/p_{t-1} = g$, hoarding will obviously stop and money is solely used as a medium of transaction. This leads back to the original model without money, and **c** again ceases to meet the golden rule condition. Moreover, it is doubtful if society would in reality accept permanent inflation or deflation, which only aims at the abstract target of golden rule efficiency. They would presumably rather stick to a stable money policy and install instead a pay-as-you-go pension scheme in order to escape the obvious limits of private capital markets in terms of efficiency.

III. Solving the paradox

It seems that we are left with one of the many paradoxes in capital theory. None of the common causes of market failure applies here, and nevertheless decentralized decision making obviously fails to be efficient. Although this result has baffled many writers on the subject – including Samuelson himself –, it has lastly been accepted by most of them, although more or less unwillingly.[14] Others have at least broken some bricks out of the seemingly insurmountable wall (Diamond 1965; Starrett 1972), mainly by pointing to the limits of a pure steady state analysis.

A first breakthrough was achieved by Starrett, who argued that inefficiency occurs only if the interest rate falls short of the growth rate, but *not* if it exceeds it. With the help of our model it can be demonstrated why this is true.[15] We keep assuming all $\alpha_j = 1$ and hence time preference being zero, for – unlike in common growth theory - this is *not* the crucial point here. With $q \neq g$ and the market consumption pattern (7) initially being realized, by moving

[14] Early commentators were Cass/Yaari (1966) and Meckling (1960).
[15] Starrett presents a more general proof but uses a model with just two periods of lifetime.

to the golden rule consumption pattern, lifetime utility U(**c**) could be elevated for all individuals *in the long run*. The new – and permanent – vector of consumption would then be

$$(7i)\ \mathrm{c}_{\mathrm{j}}^{*} = \frac{\alpha_{\mathrm{j}}}{\sum_{j=1}^{3}\alpha_{\mathrm{j}}} * g^{j-2} * \sum_{j=1}^{3}(w_j * g^{2-j})$$

where q is simply replaced by g in comparison to (7). However, with q > g, there must be some losers in the transition period. For on the one hand, with q exceeding g, consumption tends to be shifted in favour of the young generation G_1.[16] On the other hand, however, the change of the consumption pattern does not change the total amount of possible consumption within the transition period. Therefore, someone has to suffer a loss in that period, namely the oldest generation G_3 ,whose consumption per capita declines.[17] They will not at all benefit from the change in the consumption pattern, because they will already be dead when the golden rule path begins to deploy its advantages. Therefore, if the interest rate exceeds the growth rate, the resulting market consumption pattern can definitely *not* be called inefficient in a strictly Paretian sense (Starrett, 1972, pp 281).

Starretts argument can even be extended as follows:

Proposition I: Given the assumptions above, if the golden rule consumption pattern is already established, at time t all of the *then living* cohorts G_j^t would either benefit or at least remain unaffected from a switchback to the market solution, if the competitive q exceeds g.

[16] For j = 1 the first deviation of (7) with respect to q is clearly negative, which means that G_1 would be privileged by the golden rule compared to a market solution with q > g.

[17] The first deviation of (7) with respect to q is clearly positive for j = 3, i.e. they would gain from rising q and therefore suffer from returning to the golden rule. For j = 2 it can be shown that the opposite is true, but due to the argument above they could definitely not compensate the elderly without suffering a loss themselves.

Proof: Assume, for instance, that all $\alpha_j = 1$, g = 1 and **w** = (1; 1; 2). Then the golden rule consumption pattern is $\mathbf{c}^*$ = (1,333; 1,333; 1,333), while the competitive consumption pattern would be **c** = (0,887; 1,276; 1,837), with q = 1,439 > g. Starting with **c*** in period t, society could then decide to switch to the competitive **c** in the *following* period t+1. That would leave the welfare of G_3^t unchanged, for they will not live any more in t+1. Cohort G_2^t would clearly be better off with the switch, for their consumption in t+1 rises from 1,333 to 1,837. Cohort G_1^t would suffer a small consumption loss of 1,333 – 1,276 in t+1, but gain an increase of 1,837 – 1,333 in t+2. The latter must exceed the former, for the switch to **c** will definitely be at the expense of cohort G_1^{t+1}, who have not yet been born in period t. Hence, because of the unchanged total sum $c_1+c_2+c_3$ which is available per period, the sum of c_2 and c_3 must go beyond the sum of $c_2^* + c_3^*$. It might indeed be possible that $c_2^{\alpha_2} * c_3^{\alpha_3} < c_2^{*\alpha_2} * c_3^{*\alpha_3}$, but then it follows from $c_2 + c_3 > c_2^* + c_3^*$, that cohort G_2^t could compensate cohort G_1^t, such that the latter is lastly better off with **c** rather than with $\mathbf{c}^*$, q.e.d..[18]

However, this argument does not hold for the opposite case where q < g. Shifting consumption towards the golden rule path would now benefit the elder and the middle generation in the transition period, while the young ones would suffer a loss *in that period*.[19] But, in contrast to the above case, the members of the seemingly loosing generation are at the very beginning of their life and will, therefore, reap all the fruits of the golden rule path in the following periods. Because the latter is, by definition, the best lifetime consumption plan they could achieve, nobody finally loses. Hence, an interest rate being below the growth rate is unequivocally inefficient.

[18] The maximum utility which cohort G_2^t could gain is therefore not $c_2^{\alpha_2} * c_3^{\alpha_3}$ but $(c_2 + c_3 - c_3^*)^{\alpha_2} * c_3^{\alpha_3}$

[19] This can easily be demonstrated by just reversing the arguments above. In the Samuelson case **c** = **(**c_1; c_2; c_3**)** would change from (1.434; 0.434; 0.132) to (0.667; 0,667; 0.667).

But we can go even further by stating another proposition:

Proposition II: With at least one infinitely living institution, which grows at the economies growth rate, it is generally *impossible* for the interest rate to be below the growth rate in a steady state.

Proof: As is well known from the theory of public finance, with q < g, the state`s debt ratio d = D/Y can be held constant in a steady state, even with an enduring public primary deficit b = -S/Y. What is true for the state, is also true for any other infinitely lived institution, growing at rate g. From that it follows, that there are no limits for debt taking and, therefore, the demand for debt must rise until the interest rate at least equals the growth rate, q.e.d.

For a simple illustration, we write the total debt of any institution[20] in year t as

$$(20)\ D_t = D_{t-1} + (-S_t) + i * D_{t-1} = q * D_{t-1} - S_t$$

Dividing (20) by Y, and requiring that the debt ratio d is constant over time, finally yields[21]

$$(21)\ b = \left(1 - \frac{q}{g}\right) * \bar{d}$$

Equation (21) clearly shows that the institution could maintain a permanent primary deficit quota b > 0 , without thereby raising the total debt ratio, if q is lower than g. In other words, with q < g in the steady state, there are no limits to lending, and the consumption ratio C/Y =

[20] One could think of foundations, of enterprises, or of family dynasties where individual debt and assets are inherited ever and anon.
[21] Remember that $g = (1+n) = Y_t/Y_{t-1}$.

1 + b could be raised infinitely by simply raising the debt ratio d. These paradisiacal conditions would, in turn, raise credit demand ad infinitum, until finally the interest rate at least equals the growth rate and the party ends.[22]

If we add a corresponding infinitely lived institution to the "monetary" version of our model,[23] equation (4iii) changes to

$$(4iv)\; s_1 * G_1 + s_2 * G_2 = \frac{B}{p}$$

where B/p is the real worth of bonds which the institution (call it a bank) has emitted. Now the individuals can save both in terms of consumption-credits to others and in terms of bonds, thereby lending to the bank. B/p can then be calculated in the same way as real hoarding H/p in the "monetary" model from section II (and will of course generate the same results). Accordingly, B will also regularly be positive for q = g and zero for q > g. In contrast to the hoarding approach, however, this mechanism does not require inflation or deflation, but is perfectly compatible with a stable price level.

One might ask who would finally collect the gains from costless lending in case of q < g. However, we need not be explicit on this point, simply because in the steady state $q \geq g$ must hold, which, according to (21,) implies that all profits from pure lending will finally vanish.

[22] This argument has already briefly been sketched in van Suntum (2004, pp.123)
[23] See section II above. We do not assume any hoarding here. Money is rather assumed to be used (and needed) just for transactions in both real goods and bonds.

One can easily combine the real-capital-version of the model from section II with the infinitely lived institution approach, assuming e.g. that the consumer goods producing enterprise does the additional lending B themselves. Then we get

$$(4v)\; s_1 * G_1 + s_2 * G_2 = \frac{B}{p} + K_{t+1}$$

With q = g, the equilibrium amount of K_{t+1} and hence also the amount of B/p is easily calculated by setting the marginal productivity of K equal to g in equation (17), for with B > 0 the rate of return, the interest rate, and the growth rate must all be the same.[24]

IV. Concluding remarks

We conclude that, while an interest rate above the growth rate is innocuous in terms of Paretian efficiency, the case of i falling permanently short of n is utterly impossible. This is not only true for theoretical steady state models, but seems to apply to the real economy as well. There has at least never been a longer period in any advanced economy during which the growth rate was below the interest rate. Hence, in the light of the preceding analysis, no severe - if any - market failure concerning decentralized intertemporal consumption choice remains.

Some remarks on the fundamental relation between the interest rate and the growth rate seem appropriate. Assume utility function (6) with n = 0, all $\alpha_j = 1$ and **w** = (1;1;1) respectively. Then everyone will be totally satisfied with the flat consumption path **c** = (1; 1; 1) which

[24] Capital intensity is then easily reckoned as $k = \left(\frac{(1-\phi) * a}{g} \right)^{\frac{1}{\phi}}$

would result from both the market process and the golden rule plan. Now let population growth n rise from 0 to 1. Why should anyone switch to the golden rule pattern $c^* = (0.583; 1.167; 2.333)$ although his personal income as well as the interest rate has remained completely unaltered?[25] The answer is given by the following Proposition:

Proposition III: Given the assumptions above, with any growth rate other than zero, the sum of lifetime consumption *per head* (i.e. $c_1 + c_2 + c_3$) could be raised without employing any additional resources, by shifting a greater share of total income to the generation with the lowest number of heads. This applies to both a growing and a declining population. There is, in principle, no limit for the possible lifetime consumption per capita if $|n|$ is sufficiently large.

Proof: The effect of a changing number of heads can easily be demonstrated by dividing the macroeconomic constraint (8) by G_2 to obtain[26]

$$(22)\ g * c_1 + c_2 + \frac{1}{g} * c_3 = g * w_1 + w_2 + \frac{1}{g} * w_3 = z * w_2$$

Equation (22) can be viewed as a resource-consumption function, where the left-hand factors g, 1, and 1/g are weights which indicate the relative resource requirements of c_1, c_2, and c_3 respectively. It is immediately clear from (22) that, in order to maximize the sum $c_1 + c_2 + c_3$, one has to shift total income to the generation with the smallest weight factor respectively, q.e.d.[27]

[25] The figures can easily be reckoned from (7) and (7i) respectively.
[26] The right hand part of (22) is derived by inserting equation (14).
[27] With g = 1 the division of consumption over time would be arbitrary. If $g > 1 \Rightarrow c_1 = c_2 = 0$; $c_3 = z * w_2 * g$; if $g < 1 \Rightarrow c_3 = c_2 = 0$; $c_1 = z * w_2/g$

With a logarithmic utility function like (6), the reasonable amount of such a consumption shift is of course limited, due to the rising marginal utility of decreasing consumption in the discriminated periods. Nevertheless the argument sheds some light on the seeming miracle that macroeconomic growth matters for utility maximization, even if all individual incomes remain the same. However, as was argued above, it need not at all be a central authority to make this hidden link actually work in order to prevent inefficiency, because the market mechanism does the job as well.

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