Pareto meets Olson –

A Note on Pareto-optimality and Group Size in Linear Public Goods Games

by

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Abstract: In this paper I examine the relationship between Pareto-optimality and group size in linear public goods games or experiments. In particular, I use the standard setting of homogeneous linear public goods experiments and apply a recently developed tool to identify all Pareto-optimal allocations in such settings. It turns out that under any conceivable circumstances, *ceteris paribus*, small groups have a higher Pareto-ratio (Pareto-optimal allocations over total allocations) than large groups. Hence, if Pareto-optimality of an allocation is a property that makes such allocations acceptable and maintainable, small groups will find is easier to provide Pareto-optimal amounts of a public good than large groups. This is a novel reasoning for Mancur Olson's claim, in particular, with respect to what he has termed inclusive goods and inclusive groups.

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Introduction

In his seminal contribution, Mancur Olson (1971, p. 35) concludes that "*the larger the group, the farther it will fall short of providing an optimal amount of a collective good*". To put this differently, "*sufficiently small groups can provide themselves with some amount of a collective good through the voluntary and rational action of one or more of their members [and] in this they are distinguished from really large groups*" (Olson 1971, pp. 32-33). Olson's conclusion is essentially based on cost and benefit considerations. He shows, both analytically and graphically, that the collective good may be provided to some extent, if "*the gain to the individual exceeds the total costs of providing the collective good to the group*" (Olson 1971, p. 33). In this context it is worth emphasizing that the term 'gain' should be interpreted widely and may not only encompass incentives such as monetary and social ones, but also erotic, psychological, moral, etc. incentives (see Olson 1971, 61, fn. 17).

The main purpose of the present paper is to reconfirm Olson's statement by using a novel alternative reasoning for his claim that small groups may find it easier to provide themselves with an optimal amount of collective or public goods than large groups do. Moreover, the analysis is conducted within the framework of linear public goods games or experiments, which emerged only after Olson's work as a popular tool for analyzing human subject behavior with respect to providing public goods.

The paper is organized as follows. In the next section a typical linear public goods game is introduced and analyzed with respect to the purpose of this paper. Based on this analysis, I then discuss the connection between Pareto-optimality and group size in section three and derive a novel reasoning for Olson's statement. The final section summarizes and concludes.

Linear Public Goods Games

The literature on linear public goods games has been surveyed by Chaudhuri (2011), Zelmer (2003), and Ledyard (1995), among others. In a generalized homogenous standard setting, each human subject in the experiment faces an identical linear payoff function of the following form,

$$U_i = \alpha (B_i - x_i) + \beta \left(\sum_{i=1}^{s} x_i \right) , \qquad (1)$$

where the index i denotes the i-th subject, with $i = 1, \ldots, s$, U_i denotes individual payoff, B_i represents the given individual endowment or budget in each round, x_i denotes individual contribution to the public good, $(B_i - x_i)$ is the individual quantity of the private good and individual contributions to the public good summarized over all s subjects represent the quantity of the public good, which is consumed in a nonrival manner by all subjects and α and β are parameters of the model. Hence, in terms of Olson linear public goods games typically deal with inclusive goods and inclusive groups.

Moreover, U_i, B_i and x_i are usually measured in terms of tokens and right after the experiment subjects receive their payoff in local cash (e.g. Euro, Dollar) by applying a predetermined exchange rate between tokens and local cash. Also, the group size s and parameters α and β are selected by the experimenter in a way that a prisoner's dilemma situation arises, which is the case whenever the following condition holds:

$$1/s < MPCR < 1 \qquad (2)$$

where MPCR is the marginal per capita return of a contribution to the public good (e.g. see Croson 2007, p. 200). In general, the MPCR is the marginal incentive to contribute to the public good (e.g. see Ledyard 1995, p. 149). In the case of equation (1) the MPCR, therefore, amounts to: β/α.

At this point it is worth emphasizing that the following analysis holds for any parameter set that simultaneously fits equations (1) and (2), and where subjects take their decisions voluntarily, cannot communicate with each other and spend their budgets in a discrete manner (see Hokamp and Pickhardt 2011, for further details). However, these conditions prevail in any standard linear public goods experiment. For illustrative purposes alone assume that $B_i = 2$ for all i, $\alpha = 1$, $\beta = 0.4$, $s = 3$ and that subjects must spend their budgets token by token as in most linear public goods experiments. Hence, in the present case, subjects may choose between three alternatives: contributing their two tokens to the public good and keeping nothing as their private good, which is denoted as full-contribution (*FC*), contributing one token to the public good and keeping the other one as their private good, which is denoted as partial contribution (*PC*) or contributing nothing to the public good and keeping both tokens as their private good, which is denoted as non-contribution (*NC*).

Following Pickhardt (2005, p. 142; 2003, p. 188), Table 1 shows the set of feasible allocations, subject to equation (1) and the parameter set. In particular, in Table 1 *Allo.* denotes the allocation, s_{FC} (s_{PC}, s_{NC}) denotes the number of full-contributors (partial-contributors, non-contributors), U_{FC} (U_{PC}, U_{NC}) denotes the individual payoff of a full-contributor (partial-contributor, non-contributor), $s_{FC}\times U_{FC}$ ($s_{FC}\times U_{FC}$, $s_{FC}\times U_{FC}$) denotes the payoff of the group of full-contributors (partial-contributors, non-contributors) which for brevity is not displayed, X (= $\sum_i x_i$) denotes the total quantity of the public good, W denotes the group payoff or welfare level and CA denotes the number of clone allocations.

Table 1: Set of Feasible Allocations with *n* = 3

Allo.	$s_{FC}\times U_{FC}$	$s_{PC}\times U_{PC}$	$s_{NC}\times U_{NC}$	X	W	CA
1	–	–	3×2	0	6	0
2	–	1×1.4	2×2.4	1	6.2	2
3	**1×0.8**	**–**	**2×2.8**	**2**	**6.4**	**2**
4	–	2×1.8	1×2.8	2	6.4	2
5	**1×1.2**	**1×2.2**	**1×3.2**	**3**	**6.6**	**5**
6	–	3×2.2	–	3	6.6	0
7	**1×1.6**	**2×2.6**	**–**	**4**	**6.8**	**2**
8	**2×1.6**	**–**	**1×3.6**	**4**	**6.8**	**2**
9	**2×2**	**1×3**	**–**	**5**	**7**	**2**
10	**3×2.4**	**–**	**–**	**6**	**7.2**	**0**

Note: Allo. denotes the allocation, s_{FC} (s_{PC}, s_{NC}) denotes the number of full-contributors (partial-contributors, non-contributors), U_{FC} (U_{PC}, U_{NC}) denotes the individual payoff of a full-contributor (partial-contributor, non-contributor), $s_{FC}\times U_{FC}$ ($s_{FC}\times U_{FC}$, $s_{FC}\times U_{FC}$) denotes the payoff of the group of full-contributors (partial-contributors, non-contributors) which for brevity is not displayed, X (= $\sum_i x_i$) denotes the total quantity of the public good, W denotes the group payoff or welfare level and CA denotes the number of clone allocations. Figures set in bold denote a Pareto-optimal allocation.

For example, consider allocation three (*Allo.* 3) of Table 1. In this case one subject contributes his or her full budget to the public good (x_i = 2; alternative *FC*) and receives an individual payoff of 0.8 tokens, whereas the two remaining subjects both contribute nothing to the public good (x_i = 0; alternative *NC*) and receive an individual payoff of 2.8 tokens each. Likewise, in allocation six all three subjects choose alternative *PC* and receive an individual payoff of 2.2 tokens each, whereas in allocation ten they all choose alternative *FC* and get an individual payoff of 2.4 tokens each. Moreover, due to the prevailing prisoner dilemma deviation from contributing to the public good is always beneficial. For example, if allocation ten prevails a subject that switches *ceteris paribus* from alternative *FC* to *NC* will increase its individual payoff from 2.4 to 3.6 tokens in allocation eight. Hence, rational and selfish, individual payoff maximizing subjects will always choose not to contribute to the public good (*NC*) and allocation one represents the non-cooperative Nash-equilibrium. However, Table 1 also illustrates that pure altruists have an incentive to always contribute their entire endowment (*FC*) because each token contributed to the public good (*X*) increases group payoff or welfare (*W*) by 0.2 tokens.

With respect to the purpose of the present paper, another interesting feature of Table 1 is that it allows for identifying the entire set of Pareto-optimal allocations as a subset of the total number of allocations. In fact, Pareto-optimal allocations are denoted in bold in Table 1, whereas Pareto-inferior allocations are not. For example, allocation five is Pareto-optimal because there is no other allocation in the set of feasible allocations that makes at least one subject better off without making any other subject worse off. In contrast, if we consider an allocation that is not Pareto-optimal, such as allocation six, then there is at least one allocation in the set of feasible allocations that allows one subject to be better off, without making any other subject worse off, here allocation ten where all three subjects are better off (see Table 1).

Finally, it is worth noting that allocations one to ten in Table 1 represent the set of Pareto-distinguishable allocations. For each of these allocations the permutations can be calculated from, $s! / (s_{FC}!\ s_{PC}!\ s_{NC}!)$, see Hokamp and Pickhardt (2011). In Table 1 column *CA* denotes the number of clone allocations, which is the permutation for this allocation minus one arbitrarily selected master allocation. The master allocation is shown in Table 1 as the Pareto-distinguishable allocation and the number of clone allocations then indicates that this master allocation has a number of Pareto-non-distinguishable clone allocations. The total number of allocations can be calculated by adding all master allocations plus their clone allocations, here: 10 + 17 = 27, according to Table 1. The total number of Pareto-optimal allocations is calculated by adding the allocations denoted in bold in the same manner, which gives: 6 + 13 = 19. This allows for calculating what Hokamp and Pickhardt (2011) call the Pareto-ratio, e.g. the number of Pareto-optimal allocations over the number of allocations, here: 19/27 ≈ 0.70.

In fact, we now have discussed linear public goods games in sufficient detail to move on to analyzing the relationship between Pareto-optimality and group size in the following section.

Pareto-optimality and Group Size

To proceed, in a first step group size is increased *ceteris paribus* from three to five subjects. Again, this size is used for simplicity alone and with a view to keep the Table 2 readable. Inspection of Table 2 and a comparison with Table 1 reveals the following: i) the number of Pareto-distinguishable allocations is now 21 instead of 10, ii) the total number of allocations is up to 21 + 222 = 243, iii) the number of Pareto-optimal allocations is up to 6 + 45 = 51, iv) and the Pareto-ratio is down to 51/243 ≈ 0.21.

Table 2: Set of Feasible Allocations with s = 5

Allo.	$s_{FC}\times U_{FC}$	$s_{PC}\times U_{PC}$	$s_{NC}\times U_{NC}$	*X*	*W*	*CA*
1	–	–	5×2	0	10	0
2	–	1×1.4	4×2.4	1	11	4
3	–	2×1.8	3×2.8	2	12	9
4	1×0.8	–	4×2.8	2	12	4
5	–	3×2.2	2×3.2	3	13	9
6	1×1.2	1×2.2	3×3.2	3	13	19
7	–	4×2.6	1×3.6	4	14	4
8	1×1.6	2×2.6	2×3.6	4	14	29
9	2×1.6	–	3×3.6	4	14	9
10	–	5×3	–	5	15	0
11	1×2	3×3	1×4	5	15	19
12	2×2	1×3	2×4	5	15	29
13	1×2.4	4×3.4	–	6	16	4
14	2×2.4	2×3.4	1×4.4	6	16	29
15	**3×2.4**	**–**	**2×4.4**	**6**	**16**	**9**
16	2×2.8	3×3.8	–	7	17	9
17	**3×2.8**	**1×3.8**	**1×4.8**	**7**	**17**	**19**
18	**3×3.2**	**2×4.2**	**–**	**8**	**18**	**9**
19	**4×3.2**	**–**	**1×5.2**	**8**	**18**	**4**
20	**4×3.6**	**1×4.6**	**–**	**9**	**19**	**4**
21	**5×4**	**–**	**–**	**10**	**20**	**0**

Note: Allo. denotes the allocation, s_{FC} (s_{PC}, s_{NC}) denotes the number of full-contributors (partial-contributors, non-contributors), U_{FC} (U_{PC}, U_{NC}) denotes the individual payoff of a full-contributor (partial-contributor, non-contributor), $s_{FC}\times U_{FC}$ ($s_{FC}\times U_{FC}$, $s_{FC}\times U_{FC}$) denotes the payoff of the group of full-contributors (partial-contributors, non-contributors) which for brevity is not displayed, X denotes the quantity of the public good, W denotes the group payoff or welfare level and CA denotes the number of clone allocations. Figures set in bold denote a Pareto-optimal allocation.

In fact, a closer inspection of Tables 1 and 2 shows the driving force behind the drop of the Pareto-ratio. Given the parameter values and equations (1) and (2), a Pareto-optimal allocation strictly requires that no more than two subjects choose a deviation from the alternative full contribution, s_{FC}. Put differently, the sum of the number of subjects having chosen the alternatives partial contribution, s_{PC}, and/or non-contribution, s_{NC}, must not exceed two, that is, $s_{PC} + s_{NC} \leq 2$. Inspection of Table 1 and 2 shows that this condition is met by all

Pareto-optimal allocations, which implies that this condition is independent from the group size s. However, since the group size must necessarily meet the condition, $s = s_{FC} + s_{PC} + s_{NC}$, it immediately follows that an increase or decrease in the group size, s, requires an identical increase or decrease of the number of full contributors, s_{FC}. Thus, in Table 1, with $s = 3$, at least one full contributor is required for a Pareto-optimal allocation, $s_{FC} \geq 1$, whereas in Table 2, with $s = 5$, at least three full contributors are required for a Pareto-optimal allocation, $s_{FC} \geq 3$. Again, inspection of Tables 1 and 2 confirms this result. Moreover, at this point it is worth noting that Hokamp and Pickhardt (2011) have derived and proved these conditions for the general case. Using their result four, Table 3 shows relevant values for higher group sizes.

Table 3: Group Size, Pareto-ratio and Full Contributors

Group Size	3	5	10	50	75	100
Pareto-ratio	19/27	51/243	$201/3^{10}$	$5{,}001/3^{50}$	$11{,}251/3^{75}$	$20{,}001/3^{100}$
	(≈0.70)	(≈0.21)	(≈0.003)	(≈6.96^{-21})	(≈1.85^{-32})	(≈3.88^{-44})
$s_{PC} + s_{NC}$	≤2	≤2	≤2	≤2	≤2	≤2
s_{FC}	≥1	≥3	≥8	≥48	≥73	≥98

Note: s_{FC} (s_{PC}, s_{NC}) denotes the number of full-contributors (partial-contributors, non-contributors).

To summarize, Tables 1 through 3 demonstrate that the share of Pareto-optimal allocations among the overall set of feasible allocations is *ceteris paribus* much higher in small groups than in large groups. Moreover, other things being equal, in small groups a much lower share of the group members has to fully contribute to reach a Pareto-optimal allocation than in large groups. In fact, if the group size gets very large, almost all group members have to fully contribute to the public good to reach a Pareto-optimal allocation.

To put this differently, in small groups, the share of subjects, who are motivated according to Olson by an individual gain (of social, erotic, psychological, or moral nature) that exceeds the total costs of providing the collective good to the group, can be substantially smaller than in large groups.[1] As there is no good reason to assume that the distribution of these behavioral (Olson) types in a group depends on the group size, small groups have a much larger chance

[1] Note that monetary incentives are excluded here because this would imply that for one or more group members the MPCR is higher than one, which would violate condition (2). However, Brandts and Schram (2001) and others did run such experiments. In contrast, motivations of a social, erotic, psychological, or moral nature are compatible with condition (2) and numerous experiments have shown that human subjects indeed contribute to public goods due to these and similar motivations.

to reach a Pareto-optimal provision level of public goods because they need a substantially lower share of group members who show such an Olson-type behavior pattern.

Hence, on purely technical grounds, small groups will find it much easier to agree on a Pareto-optimal provision of public goods. To this extent, the preceding analysis represents a novel reasoning for Mancur Olson's (1971, p. 35) claim that "*the larger the group, the farther it will fall short of providing an optimal amount of a collective good*".

Concluding Remarks

By using the popular framework of linear public goods games and a new tool for identifying all Pareto-optimal allocations in such games, this note has shown that *ceteris paribus* there is a negative relation between the Pareto-ratio and the group size in these games. This finding supports the claim of Mancur Olson as noted above.

Further, it must be emphasized that the analysis explicitly applies to what Olson has termed inclusive goods and inclusive groups. Therefore, arguments put forward by early critics of Olson`s analysis, for example by Chamberlin (1974, p. 712), who argued that Olson's analysis would not hold for inclusive goods and groups, but just for exclusive ones, may be clearly rejected.

Finally, by making use of the tools presented here and in Hokamp and Pickhardt (2011), it would now be possible to experimentally test whether human subjects do find it easier to agree on a Pareto-optimal allocation in linear public goods games, if they have the relevant information about the Pareto-optimality of the prevailing allocation. Thus, testing Olson's assertion with human subjects in laboratory environments seems to be a promising task for future research.

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