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Computing the Substantial-Gain-Loss-Ratio

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Abstract The Substantial-Gain-Loss-Ratio (SGLR) was developed to overcome some drawbacks of the Gain-Loss-Ratio (GLR) as proposed by Bernardo and Ledoit (2000). This is achieved by slightly changing the condition for a Good-Deal, i. e. on the most extreme but at the same time very small part of the state space.

As an empirical performance measure the SGLR can naturally handle outliers and is not easily manipulated. Additionally, the robustness of performance is illuminated via so-called β-diagrams.

In the present paper we propose an algorithm for the computation of the SGLR in empirical applications and discuss its potential usage for theoretical models as well. Finally, we present two exemplary applications of an SGLR-analysis on historic returns.

Keywords Substantial Gain-Loss-Ratio · Gain-Loss-Ratio · Performance Measure

1 Introduction

The Substantial-Gain-Loss-Ratio (SGLR) was derived to overcome certain theoretical and practical problems of the Gain-Loss-Ratio (GLR) as in [1].[1] The SGLR has all the properties of an acceptability index and further desirable

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[1] For a discussion of the theoretical drawbacks of the GLR, see [2].

features.[2] Calculations of the SGLR for different values of its parameter βs, which can be plotted as the so-called β-diagram, allow to investigate the distribution of the performance. In particular, it provides an insight into the amount of (over)performance which can be attributed solely to extreme events.

Concerning the calculation, [5] only considers the trivial case of a risk-neutral benchmark investor, i.e. an investor whose preferences can be described by a constant stochastic discount factor (SDF). Additional empirical considerations are left out entirely. Unfortunately, calculating the SGLR of an asset for arbitrary benchmark stochastic discount factors is not straightforward and generally means solving a nonlinear optimization problem with several nonlinear constraints.

However, in this paper we discuss an algorithm that allows to calculate the SGLR, using historic data. Therefore we focus on assets and SDFs the probability law of which is given by empirical distribution functions. Nevertheless, we further derive upper bounds for the SGLR of assets and SDFs with certain continuous distributions.

The paper proceeds as follows. Section 2 provides the necessary definitions and theoretical statements. Section 3 introduces the algorithm and explains its usage either for empirical applications or for the determination of an upper bound in specific theoretical models. Section 4 discusses two empirical applications and Section 5 concludes. Longer proofs are deferred to the Appendix.

2 Theoretical Foundations

2.1 The Setup

We are interested in the performance of a particular asset with payout X given a benchmark investor with SDF M, which are defined on a common probability space $(\Omega, \mathcal{F}, P)$. We assume that X and M are square-integrable and that, w.l.o.g., the price of X fulfills $p(X) = 0$. Then we can define the SGLR (see [5]) as follows: For given $\beta > 0$, we define the set of SDFs which are "close" to M by

$$\mathrm{SDF}_\beta^+(M) \ := \ \left\{ M' \in L_0^+(\Omega, \mathcal{F}, P) \ : \ \mathbb{E}M' = 1, \, \mathrm{Var}M' \leq \mathrm{Var}M + \beta, \, P(M' \neq M) \leq \beta \right\},$$

where $L_0^+(\Omega, \mathcal{F}, P)$ denotes the set of $(\Omega, \mathcal{F})$-measurable functions, which are positive P-a.s.. Observe that the condition $P(M' \neq M) \leq \beta$ heavily depends on the richness of the underlying probability space. This is why we assume that $(\Omega, \mathcal{F}, P)$ admits at least a random variable U which is uniformly distributed on $[0, 1]$ and independent of (M, X).

The SGLR is then defined by

$$\mathrm{SGLR}_\beta^M(X) \ := \ \inf_{M' \in \mathrm{SDF}_\beta^+(M)} \frac{\mathbb{E}(M'X)^+}{\mathbb{E}(M'X)^-} \ ,$$

[2] For a detailed discussion see [5].

where $(\cdot)^+$ denotes the *positive part*, i. e. $max(0,\cdot)$ and $(\cdot)^-$ denotes the *negative part* i. e. $-(min(0,\cdot))$.

We start by calculating the SGLR for the case that the law of (M,X) is obtained from an empirical distribution, i.e.

$$P(M\in A, X\in B) \;=\; \frac{1}{T}\sum_{i=1}^{T}\delta_{m_i}(A)\delta_{x_i}(B) \tag{A1}$$

for some $T\in\mathbb{N}$ and sequences $(m_i)_{i=1}^T$, $(x_i)_{i=1}^T$ with $m_i>0$ for all $i=1,\ldots,T$.

Later on, in order to derive upper bounds for the SGLR in case that (M,X) has an arbitrary distribution, we assume that M and X are bounded a.s., i.e.

$$\text{there is } K<\infty \text{ such that } |M|+|X|\le K \text{ a.s.} \tag{A2}$$

Note that (A1) implies (A2).

2.2 Approximating the SGLR under (A1)

Using the stated definition of the SGLR does not imply a straightforward calculation procedure, since $\mathrm{SDF}^+_\beta(M)$ is a large set and the representation of its elements in a computational environment is not canonical. The following lemma shows that a drastic reduction and an easy representation can be obtained by considering "discrete" subsets of $\mathrm{SDF}^+_\beta(M)$. Namely, we assume that (A1) holds and define for $k\in\mathbb{N}$

$$d\mathrm{SDF}_{\beta,k}(M) \;:=\; \left\{m'\in(0,\infty)^{Tk} \,:\, m' \text{ satisfies } C_1^d,\, C_2^d \text{ and } C_3^d\right\},$$

where C_1^d, C_2^d, C_3^d are the discretized analogues of the conditions appearing in the definition of $\mathrm{SDF}^+_\beta(M)$, i.e.

$$C_1^d \;:\; \frac{\sum_{i=1}^{Tk} m'_i}{Tk}=1, \qquad C_2^d \;:\; \frac{\sum_{i=1}^{Tk}(m'_i-1)^2}{Tk} \;\le\; \frac{\sum_{i=1}^{T}(m_i-1)^2}{T}+\beta$$

$$C_3^d \;:\; \left|\{i\in\mathbb{N} : m'_i\ne m_{(i \bmod T)}\}\right| \;\le\; Tk\beta.$$

Using $d\mathrm{SDF}_{\beta,k}(M)$, we also define the discretized SGLR by

$$d\mathrm{SGLR}^M_{\beta,k}(X) \;:=\; \inf_{m'\in d\mathrm{SDF}_{\beta,k}} \frac{\sum_{i=1}^{Tk}(m'_i x_{(i \bmod T)})^+}{\sum_{i=1}^{Tk}(m'_i x_{(\mathrm{i \bmod T})})^-}.$$

Proposition 1 *Assume* (A1)*. Then*

$$\lim_{k\to\infty} d\mathrm{SGLR}^M_{\beta,k}(X) \;=\; \mathrm{SGLR}^M_\beta(X).$$

A main step in the proof of Proposition 1 (given in the Appendix) is provided by the following result, which states that the infimum in the definition of the SGLR is indeed attained.

Lemma 1 *Assume* (A2)*. Then (upon possibly extending the underlying probability space) there is* $M^* \in \mathrm{SDF}_\beta(M)$ *such that*

$$\mathrm{SGLR}_\beta^M(X) \;=\; \frac{\mathbb{E}(M^*X)^+}{\mathbb{E}(M^*X)^-}.$$

Assume (A1) *in addition. Then the following holds:*

1. M^* *is constant a.s. on each of the sets* $\{M^* \neq M, X = x_i\}$*,* $i = 1, \dots, T$*. In particular, there is a* $\sigma(M, X, U)$*-measurable version of* M^**.*
2. *For each* $k \in \mathbb{N}$ *there is* $(m_i^*)_{i=1,\dots,Tk} \in d\mathrm{SDF}_{\beta,k}(M)$ *such that*

$$d\mathrm{SGLR}_\beta^M(X) \;=\; \frac{\sum_{i=1}^{Tk}(m_i^* x_{(i \bmod T)})^+}{\sum_{i=1}^{Tk}(m_i^* x_{(i \bmod T)})^-}$$

Now a calculation can be done by performing an optimization for each combination possible for selecting $\lfloor Tk\beta \rfloor$ individuals with changed SDF in the dSGLR formula. Unfortunately the number of those combinations increases rapidly for large $Tk\beta$. The next definition helps to describe combinations that can be ruled out a priori.

Definition 2 For $A \subseteq \mathbb{N}$ and $i \in \mathbb{N}$ we define

$$d^A(m_i, x_i) := \big\{(m_j, x_j) : (j \in A) \wedge \big((0 < x_i < x_j \wedge m_i < m_j) \vee (0 > x_i > x_j \wedge m_i > m_j)\big)\big\}$$

the set of (m_i, x_i) *dominating summands of* A. We say that j *dominates* i if $j \in d^A(m_i, x_i)$.
Further we define for $h \in \mathbb{N}$

$$D^A(h) := \{I \subseteq A \,:\, |I| = h \;\text{ and }\; d^{A\setminus I}(m_i, x_i) = \emptyset \;\text{ for all } i \in I\}$$

as the *index-set of non-dominated h-collections of A*.

The following lemma states that for the calculation of the dSGLR only non-dominated $\lfloor Tk\beta \rfloor$-collections must be considered.

Lemma 2 *Assume* (A2)*. Let* $(m_i^*)_{i=1}^{Tk} \in d\mathrm{SDF}_{\beta,k}(M)$ *be the optimal element, given by Lemma 1. Then*

$$\{i \leq Tk \,:\, m_i^* \neq m_{(i \bmod T)}\} \in D^{\{1,\cdots Tk\}}(\lfloor Tk\beta \rfloor)$$

Therefore we have shown that only non-dominated (x_i, m_i) must be considered as potential changed summands in the dSGLR formula. This results in a strongly reduced computation time in empirical applications.

In the following section we introduce an algorithm that is based on this observations and discuss its application. We close this section by showing that discrete approximations can be used to find theoretical bounds for the SGLR.

2.3 Upper and lower bounds for the SGLR of an asset with absolutely continuous probability distribution

Assuming (A2), we can give a lower bound for $\mathrm{SGLR}_\beta^M(X)$ which is by no means optimal but can efficiently be computed if the distribution of (M,X) is known. Given $\beta > 0$, let $x_\beta^\pm := \sup\{t : P(X^\pm \geq t) \geq \beta\}$ and define $A^\pm := \{X^\pm > 0, X^\pm \geq x_\beta^\pm\}$. Finally, let x_0 be such that $X^- \leq x_0$ a.s. and m_0 such that $M \leq m_0$. Then set

$$M^* := M\mathbb{1}_{(A^+\cup A^-)^c} + (m_0)^{1/2} \cdot \frac{-x_0}{X}\mathbb{1}_{A^-} + 0 \cdot \mathbb{1}_{A^+} \ .$$

Lemma 3 *Assume* (A2). *It holds that*

$$\mathrm{SGLR}_\beta^M(X) \ \geq \ \frac{\mathbb{E}(M^*X)^+}{\mathbb{E}(M^*X)^-}.$$

In the same way, we can obtain a first upper bound. Define $x_{\beta/2}^\pm$ as before and let $A^\pm = \{X^\pm > 0, X^\pm \geq x_{\beta/2}^\pm\}$. Assume moreover that $P(A^\pm)$ indeed equals $\beta/2$. This is not fulfilled only if $X^\pm = 0$ on a set of mass larger than $1-\beta/2$. In such a case, decrease β until this assumption is met. Define

$$M' \ := \ M\mathbb{1}_{(A^+\cup A^-)^c} + (M+\delta)\mathbb{1}_{A^-} + (M-1)^+ \cdot \mathbb{1}_{A^+},$$

where $\delta \in [0,1]$ is chosen in such a way that $\mathbb{E}M' = 1$.

Lemma 4 *Assume* (A2). *It holds that*

$$\mathrm{SGLR}_\beta^M(X) \ \leq \ \frac{\mathbb{E}(M'X)^+}{\mathbb{E}(M'X)^-}$$

An asymptotic upper bound is now obtained by using discrete approximations.

Lemma 5 *Assume* (A2). *If* $(M_T, X_T) \to (M,X)$ *in law and in* L^2*, then*

$$\liminf_{T\to\infty} \mathrm{SGLR}_\beta^{M_T}(X_T) \ \geq \ \mathrm{SGLR}_\beta^M(X). \tag{1}$$

Note that the L^2-convergence holds e.g. if (M_T, X_T), $T \geq 0$, and (M,X) satisfy (A2) with a uniform bound K. This holds in particular if F_{M_T,X_T} is an empirical distribution function coming from realizations of (M,X).

3 Calculation Procedure

In the last section we have set the basis for the computation of the SGLR. Furthermore we have showed how the calculation can be drastically simplified by the exclusion of dominated SDFs from the optimization set. In this section we introduce a specific algorithm to calculate the dSGLR.

Firstly we note that $D^A(n)$ can be recursively calculated.

Lemma 6 *Assume* (A1) *. Then*

$$D^A(n) = \bigcup_{i \in A} \{B \cup \{i\} : B \in D^A(n-1) \wedge \nexists j \in A \setminus (B \cup \{i\}) \text{ s.t. } j \text{ dominates } i\}$$

We assume that we have observed a sample of payouts and corresponding SDFs $((x_i, m_i))_{1 \leq i \leq T}$ and that a value for $0 < \beta < 1$ and $k \in \mathbb{N}$ is given. Upfront we need to load the data, multiply it k times and normalize m, s.t. it sums up to Tk. Then the following steps can be performed:

1. Compute $D^{\{1,\cdots Tk\}}(\lfloor \beta Tk \rfloor)$ recursively.
2. Calculate $\inf_{m' \in dSDF_{\beta,k}(I)} \frac{\sum_{i=1}^{Tk} (m'_i x_{(\text{i mod T})})^+}{\sum_{i=1}^{Tk} (m'_i x_{(\text{i mod T})})^-}$ for each $I \in D^{\{1,\cdots Tk\}}(\lfloor \beta Tk \rfloor)$ via Lagrange optimization, where $dSDF_{\beta,k}(I)$ is $dSDF_{\beta,k}$ restricted to SDF-changes on I.
3. Return the minimum of the calculated values of step two.

In step 1 the performance may be improved by a presorting of (m, x). Noting that combinations I, J with $\{m_i x_i | i \in I\} = \{m_j x_j | j \in J\}$ result in the same optimized value in step 2 and precluding such redundant combinations reduces the number of calculations further.

A possible Matlab implementation using GPU-parallelisation for the first step and CPU-parallelisation for the multiple performance of the second step can be downloaded on the first author's homepage.[3] The algorithm can either be used to find an upper bound of an SGLR in a theoretical context or to perform an evaluation based on historical data, which is discussed in the subsequent section.

4 Empirical Application

In this section we show how the SGLR can be applied in an empirical context. Therefore we present two scenarios using two different paradigms for the construction of the SDF, i.e. a market-based and a consumption-based approach. In both cases we assume that there is a German investor who considers investing in the S&P500 index.[4]

4.1 CAPM-DAX-based performance evaluation of the S&P500

In the first example we calculate the historical performance of monthly returns of the S&P500 index from the viewpoint of an investor who exclusively holds the DAX30 index. As SDF for the investor we use the one implied by the

[3] https://janvoelzke.de/SGLR.shtml.

[4] Index data for DAX and S&P500 were provided by Thomson Reuters.

CAPM when considering the DAX30 as the market portfolio and a monthly risk-free rate of $r_f = 0.14\%$.[5] We set x_t as the monthly returns of the S&P500 and $m_t := a + bR_t^{DAX}$, where R_t^{DAX} are the monthly gross returns of the DAX and $a := \frac{1}{R_f} - bE(R^{DAX})$ resp. $b = \frac{E(R^{DAX}) - R_f}{R_f Var(R^{DAX})}$.[6]

Using monthly data from the last 10 years we obtain dSGLRs as depicted in form of a β-diagram in Figure 1 on the left. The horizontal line marks the GLR of a risk-neutral investor as in [1]. We can see that for the specific investor the S&P500 index is less attractive than for the risk-neutral investor, which can be explained by the positive correlation of the two indices. The β-diagram does not vary much, i.e. the substantial attractiveness resembles the original one. The result indicates that the performance is robust and due neither to outliers in the return sample nor to single extreme values in the SDF.

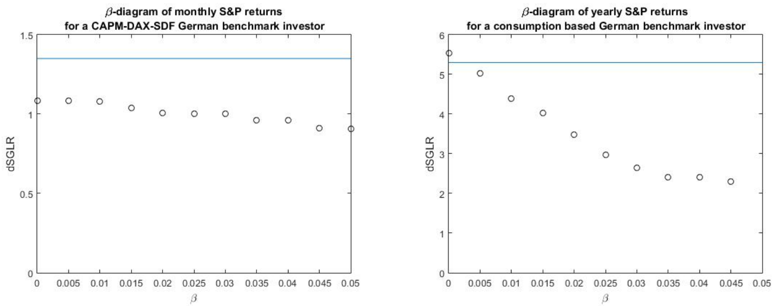

Fig. 1 Performance of S&P500 returns from different perspectives.

4.2 (German) Consumption-based performance evaluation of the S&P500

As a second example, we look at the evaluation of the yearly performance of the S&P500 index (return + dividends) from the perspective of a German investor. This time we use a consumption-based approach, i.e. the benchmark SDF is based on consumption data. The investor multiplies each payout with his marginal utility of consumption growth for the evaluation. Following [3, p. 13 and p. 24] , we specify the SDF as $m_t = c(\frac{c_t}{c_{t-1}})^{-\gamma}$, with $\gamma = 50$.[7] We use yearly data from 1971 to 2014.[8]

The result is depicted in Figure 1 on the right side. Again the horizontal line marks the risk-neutral GLR as in [1]. The SGLR for the specified investor

[5] For the model parameters $R_f, E(R^{DAX})$ and $Var(R^{DAX})$ we use empirical values, based on data starting in 1972. E.g. the risk-free rate is implied by the average normalized monthly returns of the 3-month T-bill for the last 43 years.

[6] Cp. [3, p. 139].

[7] A risk aversion parameter of 50 should not be considered as realistic, but is a value that is necessary to meet stylized asset pricing facts in that model. Cp. [3, p. 24].

[8] Consumption data is taken from [4, p. 8].

is even larger for small values of β indicating a very high attractiveness of the yearly performance of the S&P index when we take the whole sample distribution into account. However, the corresponding β-diagram is decreasing significantly, which indicates that the performance is partly driven by extreme values in some years.[9]

For the GLR this would imply strong changes over different choices of sample years.

5 Conclusion

We have introduced an algorithm for the calculation of the SGLR in empirical applications. Necessary theory has been derived. We have further showed how the procedure can be used for the approximation of an upper bound of the SGLR in theoretical models. Finally we have discussed two exemplary applications of an SGLR performance analysis based on historical data.

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6 Appendix

Proof (of Lemma 1) There is a sequence $(M_n)_{n\in\mathbb{N}} \subset \mathrm{SDF}_\beta(M)$ such that $\lim_{n\to\infty} \mathbb{E}(M_nX)^+/\mathbb{E}(M_nX)^- = \mathrm{SGLR}_\beta^M(X)$. Since $\mathrm{Var}(M_n) \leq \mathrm{Var}(M) + \beta$, the sequence is bounded in L^2. Thus, the sequence $\mu_n := P((M, M_n, X) \in \cdot)$ is tight. Hence, it is weakly compact, i.e. there is a subsequence $(M, M_{k_n}, X)_{n\in\mathbb{N}}$ which converges in law to a triple of random variables $(\tilde{M}, M^*, \tilde{X})$ which satisfies $(\tilde{M}, \tilde{X}) \overset{law}{=} (M, X)$ as well as

$$\mathbb{E}M^* = \lim_{n\to\infty} \mathbb{E}M_n = 1, \quad \mathbb{E}(M^*\tilde{X})^\pm = \lim_{n\to\infty} \mathbb{E}(M_nX)^\pm,$$
$$\mathrm{Var}(M^*) = \lim_{n\to\infty} \mathrm{Var}(M_n) \leq \mathrm{Var}(M) + \beta.$$

[9] Here this is manly driven by the extreme SDF values in some years, due to the unrealistic choice of γ.

By the Portmanteau theorem,

$$P(\tilde{M} - M^* = 0) \geq \limsup_{n\to\infty} P(M_n - M^* = 0) \geq 1 - \beta.$$

Upon possibly extending the underlying probability space, a copy of M^* can be realized on the same probability space as M, hence becoming an element of $\mathrm{SDF}_\beta(M)$.

Now assume (A1). For any $M' \in \mathrm{SDF}^+_\beta(M)$, using that $P(X = x_j) = 1/T$ for each j,

$$\begin{aligned}\frac{\mathbb{E}(M'X)^+}{\mathbb{E}(M'X)^-} &= \frac{\sum_{j=1}^T x_j^+ \mathbb{E}[M' \mid X = x_j] P(X = x_j)}{\sum_{j=1}^T x_j^- \mathbb{E}[M' \mid X = x_j] P(X = x_j)} \\ &= \frac{\sum_{j=1}^T x_j^+ \mathbb{E}[M' \mid X = x_j]}{\sum_{j=1}^T x_j^- \mathbb{E}[M' \mid X = x_j]}. \end{aligned} \tag{2}$$

This shows that the Gain-Loss-Ratio depends on M' only through the values $\mathbb{E}[M'|X = x_j]$, $j = 1, \dots, T$. This can be further separated into (assuming that the conditions have nonzero probability)

$$\mathbb{E}[M'|X = x_j] \; = \; \mathbb{E}[M'|X = x_j, M' \neq M] + \mathbb{E}[M|X = x_j, M' = M].$$

In order to optimize M' in the sense of a minimal Gain-Loss-Ratio, one has to decrease its value at large positive x_j and / or increase its value at large negative x_j, with the constraints on expectation and variance. It follows from the formula of total variance that the variance of M' is minimal if it holds $M' = \mathbb{E}[M'|X = x_j, M' \neq M]$ on the sets $\{X = x_j, M' \neq M\}$, compared to the case where M' is not constant on these sets. By the above considerations, the Gain-Loss-Ratio remains the same irrespective of whether M' is constant or not. Hence, the optimal M' is constant on such sets.

Observe furthermore that the value in (2) depends only on the *probabilities* of the sets not on the explicit realisation of $\{X = x_j, M' \neq M\}$ as subsets of Ω. Thus, the following defines a (M, X, U)-measurable random variable which is an element of $\mathrm{SDF}^+_\beta(M)$ and has the same Gain-Loss-Ratio as M^*. Let $p_j := P(X = x_j, M^* \neq M)$, $m_j^* := \mathbb{E}[M^* \mid X = x_j, M^* \neq M)$ and define

$$M^{**} \; := \; M + \sum_{j=1}^T (m_j - M) \mathbb{1}_{[0,p_j]}(U) \mathbb{1}_{\{X = x_j\}}.$$

Turning to the last assertion, we may as before choose a sequence $\big((m_i^n)_{i=1}^{Tk}\big)_n$ such that the associated Gain-Loss-Ratios converge to the minimal one. Since each $m_i^n \geq 0$ and $\frac{1}{Tk}\sum_{i=1}^{Tk} m_i^n = 1$, the numbers m_i^n are uniformly bounded (by Tk), and hence there is a convergent subsequence, the limit of which we denote by $(m_i^*)_{i=1}^{Tk}$. It is then readily checked that $(m_i^*) \in d\mathrm{SDF}^+_\beta(M)$, and that

$$d\mathrm{SGLR}^M_{\beta,k}(X) \; = \; \frac{\sum_{i=1}^{Tk} m_i^* x^+_{(i \bmod T)}}{\sum_{i=1}^{Tk} m_i^* x^-_{(i \bmod T)}}.$$

□

Now we are in a position to prove Proposition 1.

Proof (of Proposition 1) Observe that $d\mathrm{SDF}^+_{\beta,k}(M) \subset \mathrm{SDF}^+_\beta(M)$ (with the obvious identifications), hence it suffices to show that

$$\limsup_{k\to\infty} d\mathrm{SGLR}^M_{\beta,k}(X) \;\le\; \mathrm{SGLR}^M_\beta(X) \;=\; \frac{\mathbb{E}(M^*X)^+}{\mathbb{E}(M^*X)^-},$$

where M^* is given by Lemma 1. Since (A1) holds, M^* is constant a.s. on each of the sets $\{M^* \neq M, X = x_i\}$, $i = 1, \dots, T$, i.e. there are m^*_i, $i = 1, \dots, T$, such that

$$\frac{\mathbb{E}(M^*X)^+}{\mathbb{E}(M^*X)^-} = \frac{\sum_{i=1}^T \left[m_i x_i^+ P(M^* = M, X = x_i) + m^*_i x_i^+ P(M^* \neq M, X = x_i)\right]}{\sum_{i=1}^T \left[m_i x_i^- P(M^* = M, X = x_i) + m^*_i x_i^- P(M^* \neq M, X = x_i)\right]}$$

Now the right hand side can easily be approximated by a sequence in $d\mathrm{SDF}_{\beta,k}$ of increasing fineness k. □

Proof (of Lemma 2) Assume the converse, i.e., $I := \{i \le Tk : m^*_i \neq m_{(i \bmod T)}\} \notin D^{\{1,\dots,Tk\}}(\lfloor Tk\beta \rfloor)$. Thus there is $i \in I$ and $j \in \{1, \dots, T_k\} \setminus I$ such that j dominates i. Then define for $1 \le r \le Tk$

$$m^{new}_r := \begin{cases} m^*_r & \text{if } r \neq i, j \\ m_j + (m^*_i - m_i) & \text{if } r = j \\ m_i & \text{if } r = i \end{cases}$$

Hence $m^{new} \in dSDF_{\beta,k}(m)$ and a case-by-case consideration shows that

$$\frac{\sum_{i=1}^{Tk} (m_i x_{(\text{i mod T})})^+}{\sum_{i=1}^{Tk} (m_i x_{(\text{i mod T})})^-} < \frac{\sum_{i=1}^{Tk} (m^{new}_i x_{(\text{i mod T})})^+}{\sum_{i=1}^{Tk} (m^{new}_i x_{(\text{i mod T})})^-}$$

which contradicts the optimality of m^*. □

Proof (of Lemma 3) In order to minimize the Gain-Loss-Ratio, we have to decrease M on the set $\{X > 0\}$, and to increase M on the set $\{X < 0\}$. The total mass of the set where we change M is restricted to β, hence we cannot do better than changing M on both a subset A^+ of $\{X > 0\}$ *and* a subset A^- of $\{X < 0\}$, both of which have mass β. Obviously, changing M has the most effect on the subsets as described above. Since M has to remain nonnegative, the optimal choice on A^+ is $M' = 0$. To justify the choice on A^-, we take into account the restriction on the variance; $\mathrm{Var}(M') \overset{!}{\le} \mathrm{Var}M + \beta$. Increasing the value of M "away" from its expectation by 1 on a set of mass β increases the variance by β. But one has to take into account that at the same time, on a different set, the value of M is brought closer to its expectation. This might reduce the variance by at most $(m_0 - 1)^{1/2}\beta$. This difference can be "invested" on A^+, too. Finally, since we increase uniformly on the set A^-, we have to bound X by $-x_0$. □

Proof (of Lemma 4) This follows from similar considerations as in the proof of Lemma 3.

Proof (of Lemma 5) Choose a sequence $(M'_{T_n})_{n\in\mathbb{N}}$, such that M'_{T_n} is an element of $\mathrm{SDF}^+_\beta(M_{T_n})$ for each n and

$$\lim_{n\to\infty} \frac{\mathbb{E}(M'_{T_n}X_{T_n})^+}{\mathbb{E}(M'_{T_n}X_{T_n})^-} \;=\; \liminf_{T\to\infty} \mathrm{SGLR}^{M_T}_\beta(X_T) \;=:\; \underline{C}.$$

This can be done by a diagonal argument: for each fixed T there is a sequence in $\mathrm{SDF}^+_\beta(M_T)$ such that the associated Gain-Loss-Ratio converges to $\mathrm{SGLR}^{M_T}_\beta(X_T)$ and there is a subsequence of $\big(\mathrm{SGLR}^{M_T}_\beta(X_T)\big)_T$ that converges to $\underline{C}$.

The sequence $(M'_{T_n}, M_{T_n}, X_{T_n})_{n\in\mathbb{N}}$ is tight: (M_{T_n}, X_{T_n}) converge in L^2 and $\mathrm{Var}(M'_{T_n}) \le \mathrm{Var}(M_{T_n}) + \beta$, hence the sequence $(M'_{T_n})_{n\in\mathbb{N}}$ is bounded in L^2 as well. Thus, it is weakly compact, i.e. there are random variables $(\tilde{M}', \tilde{M}, \tilde{X})$ and a subsequence $(t_n)_{n\in\mathbb{N}}$ of $(T_n)_{n\in\mathbb{N}}$, such that

$$\lim_{n\to\infty} (M'_{t_n}, M_{t_n}, X_{t_n}) \;=\; (\tilde{M}', \tilde{M}, \tilde{X}) \qquad \text{in law.}$$

Since (M_T, X_T) converges in law to (M, X), it follows that $(\tilde{M}, \tilde{X})$ has the same law as (M, X). Moreover, the Portmanteau theorem for weak convergence (applied to the closed set $\{0\}$ and the open set $(-\infty, 0)$, resp.) yields that

$$P(\tilde{M}' - \tilde{M} = 0) \ge \limsup_{n\to\infty} P(M'_{t_n} - M_{t_n} = 0) \ge 1 - \beta \quad \text{and}$$
$$P(\tilde{M}' < 0) \le \liminf_{n\to\infty} P(M'_{t_n} < 0) = 0.$$

Moreover, the L^2 convergence implies

$$\mathrm{Var}(\tilde{M}') = \lim_{n\to\infty} \mathrm{Var} M'_{t_n} \le \limsup_{n\to\infty} \mathrm{Var}(M_{t_n}) + \beta = \mathrm{Var}(\tilde{M}) + \beta \quad \text{and}$$
$$\mathbb{E}\tilde{M}' = \lim_{n\to\infty} \mathbb{E}M'_{t_n} = 1.$$

Thus, $\tilde{M}' \in \mathrm{SDF}^+_\beta(\tilde{M})$[10]. Consequently, since $(\tilde{M}, \tilde{X})$ has the same law as (M, X),

$$C \le \frac{\mathbb{E}(\tilde{M}'\tilde{X})^+}{\mathbb{E}(\tilde{M}'\tilde{X})^-} = \lim_{n\to\infty} \frac{\mathbb{E}(M_{t_n}X_{t_n})^+}{\mathbb{E}(M_{t_n}X_{t_n})^-} = \underline{C}.$$

□

[10] To be precise, one has to assure that $\tilde{M}'$ is a measurable function of $\tilde{M}, \tilde{X}$ and U. This can be shown by considering simple functions (step functions) which approximate $\tilde{M}'$.