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vNM–Stable Sets for Totally Balanced Games

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Abstract

This paper continues the treatise of vNM–Stable Sets for totally balanced (cooperative) games with a continuum of players. We generalize and extend the results obtained by a series of presentations of this topic (Part I,II,III,IV,V see [5], [6], [7], [8], [9]).

Thus, the coalitional function is generated by r "production factors" (non atomic measures). Some $q < r$ factors are normalized (i.e. "probabilities") establishing the core of the game. The remaining factors are represented by non–normalized non atomic measures. Hence, these factors are available in excess and the representing measure is not in the core of the game. As previously, we switch freely between interpretations, e.g., the defining measures can also be seen as initial distributions in a pure exchange market (a "glove game").

The requirements maintained in our previous presentations, i.e., orthogonality and the existence of just one carrier "across the corners" of the market, are completely dropped. All factors are being distributed all across the market and the core does not necessarily consist of orthogonal distributions. That is, no more can we distinguish well defined "corners" of the market or production process. Nevertheless, we consider the regime of a production factor, i.e., the carrier, the coalitions with positive measure, the distribution of quantities of this factor as a "cartel"; i.e., a large group of players commanding solely an indispensable domain of the market or production process.

Within this context we study "standard" vNM–Stable Sets. The result is an existence theorem of essentially the same nature as in the previous context: for each factor pick a specified imputation, absolutely continuous w.r.t this factor with density bounded by 1, then the convex hull of these imputations constitutes a vNM–Stable Set.

We interprete this as a solution concept which, other then the concepts of the "equivalence theorems" (notably the core), establish an influence of cartels not only by their power to achieve certain gains but also by their ability to prevent others from achieving anything without cooperation.

1 Introduction and Notation

We consider nonatomic and finite totally balanced games represented as minima of finitely many measures. In the context of finitely many players this class is equivalent to either the class of market games or the class of linear production games. (cf. SHAPLEY-SHUBIK[12]).

Our notation is taken from [5], [6], [7], [8] and [9]. For simplicity we refer to the setup within this series by the "Quasi Orthogonal" case, or for short by **QuO**. The orthogonal case has been treated comprehensively by EINY ET AL.[3] and by ROSENMÜLLER AND SHITOVITZ[10] and [11].

Presently, we consider a (cooperative) ***game*** with a continuum of players, i.e., a triple $(\boldsymbol{I}, \underline{\underline{\mathbf{F}}}_0, \boldsymbol{v})$ where $\boldsymbol{I} \in \mathbb{R}_+$ is an interval reflecting the ***players***, $\underline{\underline{\mathbf{F}}}_0$ is the $\sigma-$field of (Borel) measurable sets in $\boldsymbol{I}$ (the ***coalitions***), and $\boldsymbol{v}$ (the ***coalitional function***) is a mapping $\boldsymbol{v} : \underline{\underline{\mathbf{F}}}_0 \to \mathbb{R}_+$ which is absolutely continuous w.r.t. Lebesgue measure $\boldsymbol{\lambda}$. We focus on "linear production games", that is, $\boldsymbol{v}$ is described by finitely many (absolutely continuous) measures $\boldsymbol{\lambda}^\rho \ (\rho \in \mathbf{R})$ via

$$v(S) := \min\{\boldsymbol{\lambda}^\rho(S) \mid \rho \in \mathbf{R}\} \quad (S \in \underline{\underline{\mathbf{F}}}_0), \tag{1.1}$$

for short

$$\boldsymbol{v} = \bigwedge_{r \in \mathbf{R}} \boldsymbol{\lambda}^\rho . \tag{1.2}$$

$\boldsymbol{C}^\rho$ denotes the carrier of $\boldsymbol{\lambda}^\rho$ $(\rho \in \boldsymbol{R})$. We assume that $\boldsymbol{v}$ is normalized to $\boldsymbol{v}(\boldsymbol{I}) = 1$, more precisely, there is some $\mathbf{Q} \subseteq \mathbf{R}$, $\emptyset \neq \mathbf{Q} \neq \mathbf{R}$ such that

$$1 = \boldsymbol{v}(\boldsymbol{I}) = \boldsymbol{\lambda}^\rho(\boldsymbol{I}) < \boldsymbol{\lambda}^\sigma(\boldsymbol{I}) \quad (\rho \in \mathbf{Q},\ \sigma \in \mathbf{R} \setminus \mathbf{Q}) . \tag{1.3}$$

The measures generating $\boldsymbol{v}$ are seen as ***production factors*** or ***commodities*** (as $\boldsymbol{v}$ can be interpreted as a production game or a market game). The normalized measures $\boldsymbol{\lambda}^\rho \ (\rho \in \mathbf{Q})$ constitute the ***core*** of the game, the production factors represented by $\boldsymbol{\lambda}^\rho \ (\rho \in \mathbf{R} \setminus \boldsymbol{Q})$ are thought of to be available in abundance.

As in **QuO**, the solution concept we want to discuss is the ***vNM–Stable Set*** (VON NEUMANN-MORGENSTERN [13]). For completeness we repeat the definition

Definition 1.1. *1. Let $(I, \underline{\underline{\mathbf{F}}}_0, \boldsymbol{v})$ be a game. An **imputation** is a measure $\boldsymbol{\xi}$ with $\boldsymbol{\xi}(\boldsymbol{I}) = \boldsymbol{v}(\boldsymbol{I})$. The set of imputations is denoted by $\mathcal{J}(\boldsymbol{v})$. The **core** is the set of imputations*

$$\mathcal{C}(\boldsymbol{v}) := \{\boldsymbol{\xi} \in \mathcal{J}(\boldsymbol{v}) \mid \boldsymbol{\xi} \geq \boldsymbol{v}\}$$

*2. An imputation $\boldsymbol{\xi}$ **dominates** an imputation $\boldsymbol{\eta}$ w.r.t $S \in \underline{\underline{\mathbf{F}}}_0$ if $\boldsymbol{\xi}$ is **effective** for S, i.e.,*

$$\boldsymbol{\lambda}(S) > 0 \ \textit{ and } \ \boldsymbol{\xi}(S) \leq \boldsymbol{v}(S) \tag{1.4}$$

and if

$$\boldsymbol{\xi}(T) > \boldsymbol{\eta}(T) \quad (T \in \underline{\underline{\mathbf{F}}}_0,\ T \subseteq S, \boldsymbol{\lambda}(T) > 0) \tag{1.5}$$

holds true. That is, every subcoalition of S (almost every player in S) strictly improves its payoff at $\boldsymbol{\xi}$ versus $\boldsymbol{\eta}$. We write $\boldsymbol{\xi}\, \mathrm{dom}_S\, \boldsymbol{\eta}$ to indicate domination. It is standard to use $\boldsymbol{\xi}\, \mathrm{dom}\, \boldsymbol{\eta}$ whenever $\boldsymbol{\xi}\, \mathrm{dom}_S\, \boldsymbol{\eta}$ holds true for some *coalition $S \in \underline{\underline{\mathbf{F}}}_0$.*

Definition 1.2. *Let $\boldsymbol{v}$ be a game. A set $\mathcal{S} \subseteq \mathfrak{J}(\boldsymbol{v})$ is called a **vNM–Stable Set** if*

- *there is no pair $\boldsymbol{\xi}, \boldsymbol{\mu} \in \mathcal{S}$ such that $\boldsymbol{\xi}\, \mathrm{dom}\, \boldsymbol{\mu}$ holds true (*"internal stability"*).*
- *for every $\boldsymbol{\eta} \in \mathfrak{J}(\boldsymbol{v}) \setminus \mathcal{S}$ there exists $\boldsymbol{\xi} \in \mathcal{S}$ such that $\boldsymbol{\xi}\, \mathrm{dom}\, \boldsymbol{\eta}$ is satisfied (*"external stability"*).*

We distinguish between the ***uniform*** model and the ***continuous model***. For the present we focus on the uniform model, that is, we assume the densities of the measures $\boldsymbol{\lambda}^\rho$ to be piecewise constant, that is, to be stepfunctions.

To this end we fix some $\mathbf{t} \in \mathbb{N}$ and $\mathbf{T} := \{1, \ldots, \mathbf{t}\}$ as well as a ***partition*** of $\boldsymbol{I}$

$$\underline{\underline{\mathbf{D}}} = \underline{\underline{\mathbf{D}}}_{\mathbf{T}} = \{\boldsymbol{D}^\tau\}_{\tau \in \mathbf{T}} \tag{1.6}$$

into a family of disjoint (measurable) subsets of $\boldsymbol{I}$, i.e., $\boldsymbol{I} = \bigcup_{\{\tau \in \mathbf{T}\}} \boldsymbol{D}^\tau$. The elements of the partition are assumed to be of equal Lebesgue measure, say

$$\boldsymbol{\lambda}(\boldsymbol{D}^\tau) = \frac{1}{t} \quad (\tau \in \mathbf{T})$$

for some $t \in \mathbb{N}$ (hence the term "uniform"). The total mass of $\boldsymbol{I}$ is assumed to exceed 1, that is

$$\boldsymbol{\lambda}(\boldsymbol{I}) = \sum_{\tau \in \mathbf{T}} \boldsymbol{\lambda}(\boldsymbol{D}^\tau) = \sum_{\tau \in \mathbf{T}} \frac{1}{t} = \frac{\mathbf{t}}{t} > 1\ ;$$

(naturally, we choose t to be a divisor of $\mathbf{t}$).

The carriers are assumed to be compatible with the partition, that is, for a suitable $\mathbf{T}^\rho \subseteq \mathbf{T}$ we have

$$\boldsymbol{C}^\rho = \bigcup_{\tau \in \mathbf{T}^\rho} \boldsymbol{D}^\tau \quad (\rho \in \mathbf{R})\ . \tag{1.7}$$

For convenience we introduce the σ–algebra $\underline{\underline{\mathbf{F}}} = \underline{\underline{\mathbf{F}}}_{\mathbf{T}}$ generated by the atoms ("blocks") of $\underline{\underline{\mathbf{D}}}$. Then our assumptions result in $\boldsymbol{I} \in \underline{\underline{\mathbf{F}}}$ and $\boldsymbol{C}^\rho \in \underline{\underline{\mathbf{F}}}$ $(\rho \in \mathbf{R})$.

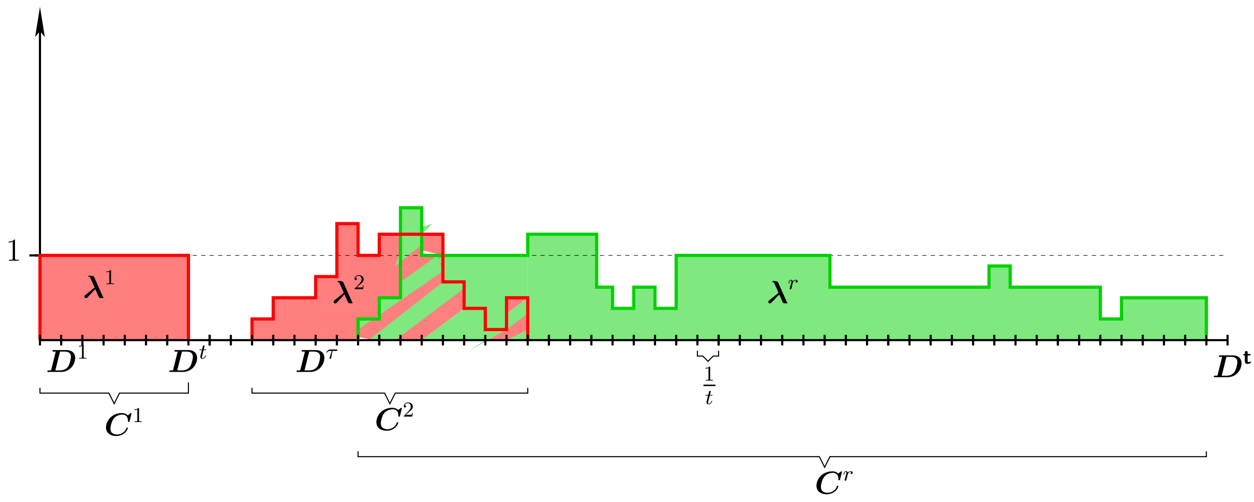

Figure 1.1: The Uniform Model

The description of a uniform model is completed by defining the piece-wise constant density of a measure $\boldsymbol{\lambda}^\rho$ by some vector $\boldsymbol{c}^\rho = \{c^\rho_\tau\}_{\tau\in\mathbf{T}} \in \mathbb{R}^{\mathbf{t}}_+$ via

$$\dot{\boldsymbol{\lambda}}^\rho = \sum_{\tau\in\mathbf{T}} c^\rho_\tau \mathbb{1}_{\boldsymbol{D}^\tau} . \tag{1.8}$$

Here $\mathbb{1}_S$ denotes the indicator function of a coalition S; the function $\dot{\boldsymbol{\lambda}}^\rho$ is of course measurable w.r.t. $\underline{\underline{\mathbf{F}}}$. Generally, a vector $\boldsymbol{x} \in \mathbb{R}^{\mathbf{t}}_+$ generates a function $\boldsymbol{\vartheta}^{\boldsymbol{x}}$ measurable w.r.t. $\underline{\underline{\mathbf{F}}}$ via

$$\boldsymbol{\vartheta}^{\boldsymbol{x}} = \sum_{\tau\in\mathbf{T}} x_\tau \mathbb{1}_{\boldsymbol{D}^\tau} \tag{1.9}$$

and $\boldsymbol{\vartheta}^{\boldsymbol{x}}$ is an imputation whenever $\boldsymbol{x}$ is a ***pre–imputation***, i.e., whenever

$$\begin{aligned} \int \boldsymbol{\vartheta}^{\boldsymbol{x}} d\boldsymbol{\lambda} \;=\; \sum_{\tau\in\mathbf{T}} \boldsymbol{\lambda}(\boldsymbol{D}_\tau) x_\tau \;=\; \frac{1}{t}\sum_{\tau\in\mathbf{T}} x_\tau \;=\; 1 \,, \\ \text{i.e.,} \qquad \sum_{\tau\in\mathbf{T}} x_\tau \;=\; t \end{aligned} \tag{1.10}$$

holds true. The normalization of $\boldsymbol{v}$ implies for some element $\boldsymbol{\lambda}^\rho$ ($\rho \in \mathbf{Q}$) in the core $\mathcal{C}(\boldsymbol{v})$

$$\begin{aligned} \boldsymbol{\lambda}^\rho(\boldsymbol{I}) \;=\; \sum_{\tau\in\mathbf{T}} \boldsymbol{\lambda}^\rho(\boldsymbol{D}_\tau) c^\rho_\tau \;=\; \frac{1}{t}\sum_{\tau\in\mathbf{T}} c^\rho_\tau \;=\; 1 \;=\; \boldsymbol{v}(\boldsymbol{I}) \;, \\ \text{i.e.} \qquad \sum_{\tau\in\mathbf{T}} c^\rho_\tau \;=\; t \end{aligned} \tag{1.11}$$

Figure 1.1 indicates the basic features of this model. $\boldsymbol{\lambda}^1$ and $\boldsymbol{\lambda}^2$ are core elements while $\boldsymbol{\lambda}^r$ is not.

The discrete nature of the densities $\overset{\bullet}{\boldsymbol{\lambda}}^{\rho}$ carries some implications for the establishment of dominance based on discrete analogues of concepts like coalitions, imputations, etc. We refer to these analogues as "pre–concepts", compare *Part 1*, i.e., [5] for the details. Thus, we have "pre–coalitions", "pre–imputations", "the pre–game", the "pre–core", etc.

For example, a ***pre–coalition*** is a nonnegative vector $\boldsymbol{a} = (a_\tau)_{\tau\in\mathbf{T}} \in \mathbb{R}^{\mathbf{t}}_{+}$. A pre–coalition serves to construct coalitions with a corresponding Lebesgue measure. Indeed, let $\vec{\boldsymbol{\lambda}}$ be the vector–valued measure defined by

$$\vec{\boldsymbol{\lambda}}(\star) = \{\boldsymbol{\lambda}(\star \cap \boldsymbol{D}^{\tau})\}_{\tau\in\mathbf{T}} \ . \tag{1.12}$$

Then, on one hand, for some $T \in \underline{\underline{\mathbf{F}}}$, the vector/pre–coalition

$$\boldsymbol{a} \ = \ (a_\tau)_{\tau\in\mathbf{T}} \ := \{\boldsymbol{\lambda}(T \cap \boldsymbol{D}^{\tau})\}_{\tau\in\mathbf{T}} \ = \ \vec{\boldsymbol{\lambda}}(T) \tag{1.13}$$

evaluates the coalition T such that we have

$$\boldsymbol{\lambda}(T) = \boldsymbol{e}\vec{\boldsymbol{\lambda}}(T) = \boldsymbol{e}\boldsymbol{a} \ . \tag{1.14}$$

(with $\boldsymbol{e} \ = \ (1,\ldots,1)$). Also we have

$$a_\tau = \boldsymbol{\lambda}(T \cap D^{\tau}) = \frac{1}{t}\frac{\boldsymbol{\lambda}(T|D^{\tau})}{\boldsymbol{\lambda}(D^{\tau})} \quad (\tau \in \mathbf{T}) \ .$$

Generally, if T is not necessarily measurable w.r.t. $\underline{\underline{\mathbf{F}}}$, then

$$\boldsymbol{\lambda}(T|\underline{\underline{\mathbf{F}}})(\star) = t\sum_{\tau\in\mathbf{T}} a_\tau \mathbb{1}_{D^{\tau}}(\star) \ ,$$

hence $\boldsymbol{a}$ – up to some re–normalization – describes the conditional distribution of T w.r.t. $\underline{\underline{\mathbf{F}}}$.

On the other hand, let $\boldsymbol{a} \in \mathbb{R}^{\mathbf{t}}_{+}$ be a pre–coalition. Choose $\varepsilon > 0$ sufficiently small such that $\varepsilon a_\tau \leq \boldsymbol{\lambda}(D^{\tau})$ $(\tau \in \mathbf{T})$ holds true. Then we can construct (via Ljapounoffs Theorem) a coalition $T^{\varepsilon\boldsymbol{a}}$ satisfying $\boldsymbol{\lambda}(T^{\varepsilon\boldsymbol{a}} \cap \boldsymbol{D}^{\tau}) = \varepsilon a_\tau$ $(\tau \in \mathbf{T})$ that is, we have analogous to (1.13),

$$\vec{\boldsymbol{\lambda}}(T^{\varepsilon\boldsymbol{a}}) = \varepsilon\boldsymbol{a} \ , \quad \boldsymbol{\lambda}(T^{\varepsilon\boldsymbol{a}}) = \varepsilon\boldsymbol{e}\boldsymbol{a} \ . \tag{1.15}$$

and also

$$\boldsymbol{\lambda}(T^{\varepsilon\boldsymbol{a}}|\underline{\underline{\mathbf{F}}})(\star) = t\sum_{\tau\in\mathbf{T}} \varepsilon a_\tau \mathbb{1}_{D^{\tau}}(\star) \ .$$

Analogously, consider now the measures $\boldsymbol{\lambda}^{\rho}$ $(\rho \in \mathbf{R})$. Recall that $\boldsymbol{c}^{\rho} \ = \ \{c^{\rho}_{\tau}\}_{\tau\in\mathbf{T}}$ is the vector defined by the density of $\boldsymbol{\lambda}^{\rho}$ according to (1.8). Using the analogue notation $\vec{\boldsymbol{\lambda}}^{\rho}$ for the derived vectorvalued measures we obtain

$$\vec{\boldsymbol{\lambda}}^{\rho}(\star) = \{\boldsymbol{\lambda}^{\rho}(\star \cap \boldsymbol{D}^{\tau})\}_{\tau\in\mathbf{T}} = \{c^{\rho}_{\tau}\boldsymbol{\lambda}(\star \cap \boldsymbol{D}^{\tau})\}_{\tau\in\mathbf{T}} \tag{1.16}$$

and hence

$$(1.17)\quad \vec{\boldsymbol{\lambda}}^{\rho}(T^{\varepsilon \boldsymbol{a}}) = \{c^{\rho}_{\tau}\boldsymbol{\lambda}(T^{\varepsilon \boldsymbol{a}} \cap \boldsymbol{D}^{\tau})\}_{\tau \in \mathbf{T}} = \{c^{\rho}_{\tau}\varepsilon a_{\tau}\}_{\tau \in \mathbf{T}} \;;\quad \boldsymbol{\lambda}^{\rho}(T^{\varepsilon \boldsymbol{a}}) = \varepsilon \boldsymbol{c}^{\rho}\boldsymbol{a}.$$

Hence $\boldsymbol{c}^{\rho}$ is also interpretet as a linear functional on pre–coalitions, i.e., a ***pre–measure***. By (1.11) we have $\boldsymbol{c}^{\rho}(1,\ldots,1) = t = t\boldsymbol{\lambda}^{\rho}(\boldsymbol{I})$ whenever $\boldsymbol{\lambda}^{\rho} \in \mathcal{C}(\boldsymbol{v})$.

Naturally, we define the ***pre–game*** as the (non–linear) functional on pre–coalitions $v : \mathbb{R}^{\mathbf{t}}_{+} \to \mathbb{R}$ given by

$$(1.18)\qquad v(\boldsymbol{a}) = min_{\rho \in \mathbf{R}}\boldsymbol{c}^{\rho}\boldsymbol{a} \;;$$

such that $v(1,\ldots,1) = t$. Then clearly for some coalition $T^{\varepsilon \boldsymbol{a}}$

$$(1.19)\qquad \boldsymbol{v}(T^{\varepsilon \boldsymbol{a}}) \;=\; v(\varepsilon \boldsymbol{a}) \;=\; \varepsilon v(\boldsymbol{a}) \;.$$

Proceeding we mention the set of ***pre–imputations*** is

$$(1.20)\qquad \boldsymbol{J}(v) \;=\; \left\{ \boldsymbol{x} \in \mathbb{R}^{\mathbf{t}}_{+} \;\middle|\; \sum_{\tau \in \mathbf{T}} x_{\tau} = t \right\}$$

and the ***pre–core***

$$(1.21)\qquad \boldsymbol{C}(v) \;:=\; \{\boldsymbol{x} \in \boldsymbol{J}(v) \,|\, \boldsymbol{x} \geq v\}$$

in view of (1.11).

2 ε–Relevant Coalitions

As in the orthogonal and semi–orthogonal case we can exhibit a finite set of pre–coalitions that are necessary and sufficient to be considered for the establishment of dominance relations. This is the family of "relevant vectors" or "relevant pre–coalitions".

Technically, the generalization of the procedure presented in **QuO** is straightforward, hence we will restrict ourselves to a descriptive treatment and refer to **QuO** for the details.

Consider the linear functional induced on pre–coalitions by a measure $\boldsymbol{\lambda}^\rho$ via its density, i.e.

$$\boldsymbol{c}^\rho \; : \; \mathbb{R}^{\mathbf{t}} \to \mathbb{R} \; , \quad \boldsymbol{c}^\rho(\boldsymbol{a}) \; := \; \sum_{\tau \in \mathbf{T}} c^\rho_\tau a_\tau \; = \; \boldsymbol{c}^\rho \boldsymbol{a} \quad . \tag{2.1}$$

The functional $\boldsymbol{c}^\rho(\bullet)$ reflects the action of $\boldsymbol{\lambda}^\rho$ on coalitions. E.g., for some pre–coalition $\boldsymbol{a}$ and some $\varepsilon > 0$ admitting a corresponding coalition $T^{\varepsilon \boldsymbol{a}}$ (see (1.17)), we have

$$\boldsymbol{\lambda}^\rho(T^{\varepsilon \boldsymbol{a}}) \; = \; \boldsymbol{c}^\rho(\varepsilon \boldsymbol{a}) \; = \; \varepsilon \boldsymbol{c}^\rho(\boldsymbol{a}) \; = \; \varepsilon \boldsymbol{c}^\rho \boldsymbol{a} \; . \tag{2.2}$$

Consequently, the pre–game v appears as a (non linear) functional on pre–coalitions via

$$v(\boldsymbol{a}) \; = \; \min_{\rho \in \mathbf{R}} \boldsymbol{c}^\rho(\boldsymbol{a}) \; .$$

and hence for some coalition $T^{\varepsilon \boldsymbol{a}}$

$$\varepsilon v(\boldsymbol{a}) \; = \; \min_{\rho \in \mathbf{R}} \varepsilon \boldsymbol{c}^\rho(\boldsymbol{a}) \; = \; \min_{\rho \in \mathbf{R}} \varepsilon \boldsymbol{c}^\rho(\boldsymbol{a}) \; = \; \min_{\rho \in \mathbf{R}} \boldsymbol{\lambda}^\rho(T^{\varepsilon \boldsymbol{a})} \; = \; \boldsymbol{v}(T^{\varepsilon \boldsymbol{a}}) \; .$$

Definition 2.1. *1. Let*

$$\boldsymbol{A} \; : \; \left\{ \boldsymbol{a} \in \mathbb{R}^{\mathbf{t}}_+ \;\middle|\; \boldsymbol{c}^\rho(\boldsymbol{a}) \geq 1 \quad (\rho \in \mathbf{R}) \right\} \; . \tag{2.3}$$

$\boldsymbol{A}^e$ denotes the set of extremal points of $\boldsymbol{A}$. The elements of $\boldsymbol{A}^e$ are called the ***relevant vectors*** *or* ***relevant pre–coalitions****.*

2. For $\varepsilon > 0$ a coalition T is called ***ε–relevant*** *if there is a relevant vector $\boldsymbol{a} \in \boldsymbol{A}^e$ such that $\vec{\boldsymbol{\lambda}}(T) = \varepsilon \boldsymbol{a}$. We then write $T = T^{\varepsilon \boldsymbol{a}}$ and sometimes speak of an* ***ε–$\boldsymbol{a}$–relevant*** *coalition.*

Now, as ε varies in the above set–up, it is sufficient to consider relevant vectors. That is, for some pre–coalition $\boldsymbol{a}$ the pre–game evaluated is

$$\min\{\boldsymbol{c}^\rho(\boldsymbol{a}) \mid \rho \in \mathbf{R}\} \; = \; v(\boldsymbol{a}).$$

and if we normalize $\boldsymbol{a}$ by introducing

$$\boldsymbol{a}^0 \; := \; \frac{\boldsymbol{a}}{v(\boldsymbol{a})} \; = \; \frac{\vec{\boldsymbol{\lambda}}(T)}{\boldsymbol{v}(T)},$$

then

$$v(\boldsymbol{a}^0) = \min\{\boldsymbol{c}^\rho(\boldsymbol{a}^0) \,\Big|\, \rho \in \mathbf{R}\} = \frac{1}{v(\boldsymbol{a})} v(\boldsymbol{a}) = 1 \ .$$

Thus, $\boldsymbol{a}^0$ is located on the boundary of $\boldsymbol{A}$. Then, for some $\varepsilon > 0$ sufficiently small and $\varepsilon' \ := \ \varepsilon \frac{v(\boldsymbol{a}^0)}{v(\boldsymbol{a})}$

$\varepsilon' := \varepsilon \frac{v(\boldsymbol{a}^0)}{v(\boldsymbol{a})} = \varepsilon \frac{1}{v(\boldsymbol{a})} \leq \varepsilon$ we have

$$\boldsymbol{v}(T^{\varepsilon' \boldsymbol{a}^0}) = \varepsilon' v(\boldsymbol{a}^0) = \varepsilon \frac{v(\boldsymbol{a}^0)}{v(\boldsymbol{a})} v(\boldsymbol{a}) = \varepsilon \boldsymbol{v}(\boldsymbol{a}^0)$$

$$\boldsymbol{v}(T^{\varepsilon' \boldsymbol{a}^0}) = \varepsilon' v(\boldsymbol{a}^0) = \varepsilon \frac{v(\boldsymbol{a}^0)}{v(\boldsymbol{a})} v(\boldsymbol{a}) = \varepsilon \boldsymbol{v}(\boldsymbol{a}^0)$$

with an obvious choice of $T^{\varepsilon' \boldsymbol{a}^0}$. That is, for discussing dominance relations, we can restrict ourselves on the boundary of $\boldsymbol{A}$. Generally, though $\boldsymbol{a}^0$ is located on the boundary of $\boldsymbol{A}$, it is not necessarily an extremal point of $\boldsymbol{A}$. However, we can find a convex combination of vectors in $\boldsymbol{A}^{\boldsymbol{e}}$ that represents $\boldsymbol{a}^0$. The main theorem with respect to ε–relevance ("The Inheritance Theorem") stipulates that this geometric procedure can be simulated within the realms of coalitions such that all dominance relations prevail. We copy the details from *Part I* [5] as the proof of these facts does not hinge on quasi–orthogonality.

In what follows, we will - without making this a formal definition - sometimes refer to a set of coefficients $\boldsymbol{\alpha}_{\bullet} \ = \ \{\alpha_\rho\}_{\rho \in \mathbf{R}}$ as ***"convex"*** if

$$\alpha_\rho \geq 0, \ (\rho \in \mathbf{R}) \text{ and } \sum_{\rho \in \mathbf{R}} \alpha_\rho = 1,$$

holds true.

Now, the first Lemma is of a purely geometric nature, more or less reflecting our above considerations.

Lemma 2.2. *For any $\widehat{\boldsymbol{a}} \in \boldsymbol{A}$ there exists $\bar{\boldsymbol{a}} \in \boldsymbol{A}$ such that*

1. $\bar{\boldsymbol{a}} \leq \widehat{\boldsymbol{a}}$
2. *There is a set $\boldsymbol{E} \subseteq \boldsymbol{A}^e$, say*

$$\boldsymbol{E} \ = \ \left\{\boldsymbol{a}^{(k)} \,\middle|\, k \in \mathbf{K}\right\} \subseteq \boldsymbol{A}^e \quad \textit{with} \quad \mathbf{K} \ := \ \{1, \ldots, K\}$$

of relevant vectors as well as a set of "convex" coefficients $\{\gamma_k\}_{k \in \mathbf{K}}$ such that

$$\bar{\boldsymbol{a}} \ = \ \sum_{k \in \mathbf{K}} \gamma_k \boldsymbol{a}^{(k)} \tag{2.4}$$

holds true.

3. *If* $\min\{\boldsymbol{c}^\rho\widehat{\boldsymbol{a}} \mid \rho \in \mathbf{R}_0\} = 1$*, then there is some* $\overline{\rho} \in \mathbf{R}$ *such that for* $k \in \boldsymbol{K}$

$$\boldsymbol{c}^{\overline{\rho}}(\widehat{\boldsymbol{a}}) \;=\; \boldsymbol{c}^{\overline{\rho}}(\bar{\boldsymbol{a}}) \;=\; \boldsymbol{c}^{\overline{\rho}}(\boldsymbol{a}^{(k)}) \;=\; 1$$

holds true.

The ***proof*** is the one of Lemma 3.2 in *Part I* of **QuO**.

We now reformulate the main result of this section .

Theorem 2.3 (**The Inheritance Theorem**). *Let* $\boldsymbol{\vartheta}, \boldsymbol{\eta}$ *be imputations and let* S *be a coalition such that* $\boldsymbol{\vartheta} \operatorname{dom}_S \boldsymbol{\eta}$*. Then there is* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$ *there is a relevant vector* $\boldsymbol{a}^\star \in \boldsymbol{A}^e$ *and an* ε*–relevant coalition* $T \;=\; T^{\boldsymbol{a}^\star\varepsilon} \subseteq S$ *satisfying*

$$\vec{\boldsymbol{\lambda}}(T) = \varepsilon \boldsymbol{a}^\star \qquad \text{and} \qquad \boldsymbol{\vartheta} \operatorname{dom}_T \boldsymbol{\eta}\ .$$

In other words, with respect to domination it is sufficient and necessary to consider ε*–relevant coalitions only.*

The ***proof*** is again taken from the one of *Theorem 3.3* of *Part I* of **QuO** (i.e. [5]) as there is no particular reference to the quasi–orthogonal game in that proof.

The following definitions are copied from our previous conventions.

Definition 2.4. 1. *Let* $\boldsymbol{x}$ *be a pre–imputation and let* $\boldsymbol{y} \in \mathbb{R}^{\mathbf{t}}$*. Also, let* $\boldsymbol{a}$ *be a pre–coalition. We shall say that* $\boldsymbol{x}$ ***dominates*** $\boldsymbol{y}$ ***via*** $\boldsymbol{a}$ *if*

$$\boldsymbol{x}\boldsymbol{a} \leq v(\boldsymbol{a}) \quad \text{and} \quad x_\tau > y_\tau \;\; \text{for all} \;\; \tau \;\; \text{with} \;\; a_\tau > 0\ . \tag{2.5}$$

We write $\boldsymbol{x} \operatorname{dom}_{\boldsymbol{a}} \boldsymbol{y}$ *to indicate domination, omitting the* $\boldsymbol{a}$ *if the quantification is general.*

2. *Let* v *be a pre–game. A set* $\mathbb{H}$ *of pre–imputations is called* ***vNM–Stable*** *if*

   - *there is no pair* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{H}$ *such that* $\boldsymbol{x} \operatorname{dom} \boldsymbol{y}$ *holds true (*"internal stability"*).*
   - *for every pre–imputation* $\boldsymbol{y} \notin \mathbb{H}$ *there exists* $\boldsymbol{x} \in \mathbb{H}$ *such that* $\boldsymbol{x} \operatorname{dom} \boldsymbol{y}$ *holds true (*"external stability"*).*

It is rather obvious that a version of the "Inheritance Theorem" holds true within the domain of pre–imputations as well, that is, in order to formulate relations of dominance it is sufficient to consider relevant pre–coalitions, i.e., extremals of $\boldsymbol{A}$.

Definition 2.5. *For any nonnegative measurable function* $\boldsymbol{\vartheta}$ *let*

$$m_\tau \;:=\; \operatorname{ess\,inf}_{\boldsymbol{D}^\tau} \quad (\tau \in \mathbf{T}) \quad \text{and} \quad \boldsymbol{m} \;:=\; (m_\tau)_{\tau \in \mathbf{T}}\ . \tag{2.6}$$

$\boldsymbol{m}$ *is the vector of (essential)* ***minima*** *of* $\boldsymbol{\vartheta}$*.*

Lemma 2.6. *Let $\boldsymbol{x}$ be a pre–imputation and let $\boldsymbol{\vartheta}$ be a (nonnegative) measurable function. Let $\boldsymbol{m}$ denote the vector of minima of $\boldsymbol{\vartheta}$. If, for some pre–coalition $\boldsymbol{a}$ we have $\boldsymbol{x}\,\mathrm{dom}_{\boldsymbol{a}}\,\boldsymbol{m}$, then there is $\varepsilon_0 > 0$ such that for all ε with $0 < \varepsilon < \varepsilon_0$ there is an ε–relevant coalition $T^\varepsilon = T^{\varepsilon \boldsymbol{a}}$ such that*

$$\boldsymbol{\vartheta}^{\boldsymbol{x}}\,\mathrm{dom}_{T^{\varepsilon\boldsymbol{a}}}\,\boldsymbol{\vartheta} \tag{2.7}$$

holds true.

Proof: Same as in [5], the proof is rather obvious and does not hinge on the framework of **QuO**.

q.e.d.

Corollary 2.7. *Let $\mathbb{H}$ be a vNM–Stable set of pre–imputations. Then*

$$\mathcal{H} \;:=\; \{\boldsymbol{\vartheta}^{\boldsymbol{x}} \mid \boldsymbol{x} \in \mathbb{H}\}$$

is a vNM–Stable set.

Proof:

1$^{\text{st}}$STEP :

Let $\boldsymbol{\eta} \notin \mathcal{H}$. Let $\boldsymbol{m}$ be the vector of minima of $\boldsymbol{\eta}$. Now, if $\boldsymbol{m}$ is an imputation, then $\boldsymbol{\eta}$ is measurable, actually $\boldsymbol{\eta} = \boldsymbol{\vartheta}^{\boldsymbol{m}}$. Then necessarily $\boldsymbol{m} \notin \mathbb{H}$. If $\boldsymbol{m}$ is not an imputation, then all the more $\boldsymbol{m} \notin \mathbb{H}$.

In any case there is $\boldsymbol{x} \in \mathbb{H}$ and $\boldsymbol{a}^\star$ such that $\boldsymbol{x}\,\mathrm{dom}_{\boldsymbol{a}^\star}\,\boldsymbol{m}$. According to Lemma 2.6 we know that, for suitable $\varepsilon > 0$ we have $\boldsymbol{\vartheta}^{\boldsymbol{x}}\,\mathrm{dom}_{T^{\varepsilon\boldsymbol{a}^\star}}\,\boldsymbol{\eta}$, hence $\mathcal{H}$ is externally stable.

2$^{\text{nd}}$STEP : Let $\boldsymbol{\vartheta}^{\boldsymbol{x}}, \boldsymbol{\vartheta}^{\boldsymbol{y}} \in \mathcal{H}$ and assume for some coalition T that $\boldsymbol{\vartheta}^{\boldsymbol{x}} dom_T \boldsymbol{\vartheta}^{\boldsymbol{y}}$ holds true. By the Inheritance Theorem we can assume that, for suitable $\boldsymbol{a}^\star \in \boldsymbol{A}^e$ and $\varepsilon > 0$, we have

$$\boldsymbol{\vartheta}^{\boldsymbol{x}} dom_{T^{\varepsilon\boldsymbol{a}^\star}} \boldsymbol{\vartheta}^{\boldsymbol{y}}$$

holds true. But as $\boldsymbol{\vartheta}^{\boldsymbol{x}}, \boldsymbol{\vartheta}^{\boldsymbol{y}}$ are both $\underline{\underline{\mathbf{F}}}$–measurable it is seen at once that this implies $\boldsymbol{x}\,\mathrm{dom}_{\boldsymbol{a}^\star}\,\boldsymbol{y}$, contradicting the internal stability of $\mathbb{H}$. That is $\mathcal{H}$ is internally stable, hence a vNM–Stable set.

q.e.d.

Now, introduce the convex closed set

$$\mathbb{C} \;:=\; \left\{ \boldsymbol{x} \in \mathbb{R}^{\mathbf{T}} \;\middle|\; \exists \boldsymbol{\alpha} \;=\; \{\alpha_\rho\}_{\rho \in \mathbf{R}} \text{ “convex” s.t. } \boldsymbol{x} \geq \sum_{\rho \in \mathbf{R}} \alpha_\rho \boldsymbol{c}^\rho \right\} . \tag{2.8}$$

Playing between vectors $\boldsymbol{x} \in \mathbb{C}$ and convex combinations of measures $\boldsymbol{\lambda}^\rho$ is suggested by the above expositions concerning relevant vectors and coalitions.

Remark 2.8. We will hencforth require that the family of pre–measures/linear functionals/vectors $\{\boldsymbol{c}^\rho\}_{\rho\in\mathbf{R}}$ is ***non–degenerate***. Essentially this means that certain determinants resulting from submatrices taken from that family are nonsingular. It can be seen that the set of families satisfying n.d. is open and dense and hence this requirement is satisfied "almost everywhere" in a topological sense.

To be more precise let us, in a first approach, observe that w.l.o.g. the vectors $\{\boldsymbol{c}^\rho\}_{\rho\in\mathbf{R}}$ are the extremals of $\mathbb{C}$. For, if for some $\sigma\in\mathbf{R}$

$$\boldsymbol{c}^\sigma \;=\; \sum_{\rho\in\mathbf{R}\setminus\{\sigma\}} \alpha_r \boldsymbol{c}^\rho$$

with suitable "convex" coefficients $\{\alpha_\rho\}_{\rho\in\mathbf{R}\setminus\{\sigma\}}$, then for any $\boldsymbol{a}\in\mathbb{R}^{\mathbf{T}}$

$$\boldsymbol{c}^\sigma\boldsymbol{a} \;=\; \sum_{\rho\in\mathbf{R}\setminus\{\sigma\}} \alpha_\rho \boldsymbol{c}^\rho\boldsymbol{a} \;\geq\; \min\{\boldsymbol{c}^\rho\boldsymbol{a}\mid \rho\in\mathbf{R}\setminus\{\sigma\}\}\ ,$$

and hence

$$v(\boldsymbol{a}) \;=\; \min_{\rho\in\mathbf{R}} \boldsymbol{c}^\rho(\boldsymbol{a}) \;=\; \min_{\rho\in\mathbf{R}\setminus\{\sigma\}} \boldsymbol{c}^\rho(\boldsymbol{a})\ .$$

Hence, $\boldsymbol{c}^\sigma$ can be omitted from our considerations.

Changing our viewpoint slightly and focussing on ***non–degeneracy*** we ask for the vectors of the family $\{\boldsymbol{c}^\rho\}_{\rho\in\mathbf{R}}$ to be *linearly independent*. This condition will be satisfied by an open and dense set of such families and results from our requirement about the family being non–degenerate.

For a second example, consider a relevant pre–coalition, i.e., an extremal $\boldsymbol{a}^\otimes\in\boldsymbol{A}^e$. Let

(2.9) $$\mathbf{T}^\otimes \;:=\; \{\tau\in\mathbf{T}\mid a_\tau>0\}\ \text{ and }\ \mathbf{R}^\otimes \;:=\; \{\rho\in\mathbf{R}\mid \boldsymbol{\lambda}^\rho\boldsymbol{a} \;=\; 1\ \}\ .$$

We call $\mathbf{T}^\otimes$ and $\mathbf{R}^\otimes$ the ***characteristics*** of $\boldsymbol{a}^\otimes$. To justify this notion, we want these two sets to characterize $\boldsymbol{a}^\otimes$ as the unique solution of the linear system of equations in variables $\{a_\tau\}_{\tau\in\mathbf{T}}$

(2.10) $$\begin{array}{rcll} \boldsymbol{c}^\rho\boldsymbol{a} & = & 1 & \rho\in\mathbf{R}^\otimes \\ a_\tau & = & 0 & \tau\in\mathbf{T}\setminus\mathbf{T}^\otimes\ . \end{array}$$

Again, this to be satisfied for all extremals of $\boldsymbol{A}^e$ results from $\{\boldsymbol{c}^\rho\}_{\rho\in\mathbf{R}}$ to be non–degenerate. Clearly, we are asking for a certain set of determinats to be non–sigular and this will be satisfied by an open and dense set of families $\{\boldsymbol{c}^\rho\}_{\rho\in\mathbf{R}}$.

In particular, n.d. implies that, for any pair of characteristics $|\mathbf{T}^\otimes|,|\mathbf{R}^\otimes|$ of some relevant pre–coalition $\boldsymbol{a}^\otimes$, we have $|\mathbf{T}^\otimes| \;=\; |\mathbf{R}^\otimes|>0$. Accordingly, we call (5.6) the ***characterizing system*** of $\boldsymbol{a}^\otimes$. Essentially, (2.10) reflects a system of $|\mathbf{T}^\otimes| \;=\; |\mathbf{R}^\otimes|$ equations in $|\mathbf{T}^\otimes| \;=\; |\mathbf{R}^\otimes|$ variables a_τ $(\tau\in\mathbf{T}^\otimes)$.

So, from now on we will always assume that the family $\{\boldsymbol{c}^\rho\}_{\rho\in\mathbf{R}}$ is non–degenerate in any context that appears suitable.

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Finally consider the set $\boldsymbol{A}$ and its extremals (which induce ε–relevant coalitions) as well as the set $\mathbb{C}$ as defined in (2.8). By definition of $\boldsymbol{A}$ we have $\boldsymbol{x}\boldsymbol{a} \geq 1$ for all $\boldsymbol{x} \in \mathbb{C}$ and hence $\boldsymbol{A}$ constitutes the ***normals*** to $\mathbb{C}$. Normalizing these normals means to restrict the discussion to the boundary of $\partial\boldsymbol{A}$ of $\boldsymbol{A}$ or essentially to $\boldsymbol{A}^e$. Hence, some $\boldsymbol{a} \in \partial\boldsymbol{A}$ constitutes a supporting hyperplane at $\mathbb{C}$

$$\{x \in \mathbb{C} \,|\, \boldsymbol{x}\boldsymbol{a} \;=\; 1\}$$

In particular, as a consequence of non–degeneracy, any relevant vector, that is an extremal of $\boldsymbol{A}$, constitutes a $((\mathbf{t}-1)$–dimensional) supporting hyperplane, i.e., a ***facet*** of $\mathbb{C}$. Then we have

Lemma 2.9.

$$\mathbb{C} = \left\{\boldsymbol{x} \in \mathbb{R}^{\mathbf{t}}_{+} \,\middle|\, \boldsymbol{x}\boldsymbol{a} \geq 1 (\boldsymbol{a} \in \boldsymbol{A})\right\} = \left\{\boldsymbol{x} \in \mathbb{R}^{\mathbf{t}}_{+} \,\middle|\, \boldsymbol{x}\boldsymbol{a}^{\otimes} \geq 1 (\boldsymbol{a}^{\otimes} \in \boldsymbol{A}^e)\right\} \tag{2.11}$$

as well as

$$\boldsymbol{A} = \left\{\boldsymbol{a} \in \mathbb{R}^{\mathbf{t}}_{+} \,\middle|\, \boldsymbol{x}\boldsymbol{a} \geq 1 \;(\boldsymbol{x} \in \mathbb{C})\right\} = \left\{\boldsymbol{a} \in \mathbb{R}^{\mathbf{t}}_{+} \,\middle|\, \boldsymbol{c}^{\rho}\boldsymbol{a} \geq 1 \;(\rho \in \mathbf{R})\right\} \tag{2.12}$$

meaning that $\boldsymbol{A}$ and $\mathbb{C}$ are related via some kind of duality relationship.

3 Reduction of The Relevant Pre–Coalitions

In view of ***non–degeneracy*** an extremal $\boldsymbol{a}^{\otimes} \in \boldsymbol{A}^{e}$ is uniquely defined by its ***characteristics***, that is the sets

$$\mathbf{T}^{\otimes} := \{\tau \in \mathbf{T} \,|\, a_{\tau} > 0\} \text{ and } \mathbf{R}^{\otimes} := \{\rho \in \mathbf{R} \,|\, \boldsymbol{\lambda}^{\rho}\boldsymbol{a} = 1\} \tag{3.1}$$

with $|\mathbf{T}^{\otimes}| = |\mathbf{R}^{\otimes}| > 0$. These two sets characterize the vector $\boldsymbol{a}^{\otimes}$ as the unique solution of the characterizing system, i.e., the linear system of equations in variables $\{a_{\tau}\}_{\tau\in\mathbf{T}}$

$$\begin{array}{rcll} \boldsymbol{c}^{\rho}\boldsymbol{a} & = & 1 & \rho \in \mathbf{R}^{\otimes} \\ a_{\tau} & = & 0 & \tau \in \mathbf{T}\setminus\mathbf{T}^{\otimes} \end{array} \tag{3.2}$$

Essentially (3.2) reflects a system of $|\mathbf{R}^{\otimes}|$ equations in $|\mathbf{T}^{\otimes}| = |\mathbf{R}^{\otimes}|$ variables a_{τ} $(\tau \in \mathbf{T}^{\otimes})$.

Consider a relevant pre–coalition $\boldsymbol{a}^{\otimes} \in \boldsymbol{A}^{e}$. As an extremal of $\boldsymbol{A}$, $\boldsymbol{a}^{\otimes}$ is connected to neighboring extremals via a set of edges. In particular, one can attempt to aproach a neighboring extremal by omitting one of the equations of the first type in (3.2), that is relaxing one of the constraints defined by the linear functionals involved in $\mathbf{R}^{\otimes}$. More precisely:

Theorem 3.1. *Let $\boldsymbol{a}^{\otimes} \in \boldsymbol{A}^{e}$ be a relevant pre–coalition and let $\mathbf{T}^{\otimes}$, $\mathbf{R}^{\otimes}$ be the characteristics with $|\mathbf{T}^{\otimes}| = |\mathbf{R}^{\otimes}| =: s \geq 2$. Let $\pi \in \mathbf{R}^{\otimes}$. Then there exists a unique $\boldsymbol{a}^{\star} = \boldsymbol{a}^{\star\pi} \in \boldsymbol{A}^{e}$ with characteristics $\mathbf{T}^{\star}$, $\mathbf{R}^{\star}$ such that he interval*

$$[\boldsymbol{a}^{\otimes}, \boldsymbol{a}^{\star}] := \{t\boldsymbol{a}^{\otimes} + (1-t)\boldsymbol{a}^{\star} \,|\, 0 \leq t \leq 1\} \tag{3.3}$$

is an edge of $\boldsymbol{A}$ and one of the following alternatives hold true:

1. *Either there exists $\tau \in \mathbf{T}^{\otimes}$ such that*

$$\mathbf{T}^{\star} = \mathbf{T}^{\otimes} \setminus \{\tau\} , \quad \mathbf{R}^{\star} = \mathbf{R}^{\otimes} \setminus \{\pi\} . \tag{3.4}$$

2. *Or else there exists some $\sigma \notin \mathbf{R}^{\otimes}$ such that*

$$\mathbf{T}^{\star} = \mathbf{T}^{\otimes} , \quad \mathbf{R}^{\star} = (\mathbf{R}^{\otimes} \setminus \{\pi\}) \cup \{\sigma\} . \tag{3.5}$$

Proof:

1^st^STEP :

Recall the characterizing system

$$\begin{array}{rcll} \boldsymbol{c}^{\rho}\boldsymbol{a} & = & 1 & (\rho \in \mathbf{R}^{\otimes}) \\ a_{\tau} & = & 0 & (\tau \notin \mathbf{T}^{\otimes}) . \end{array} \tag{3.6}$$

Consider the linear subspace (straight line) obtained by omitting $\boldsymbol{c}^\pi$ from the system, i.e.,

$$\mathbb{L}_{\mathsf{T}^\otimes}^{\mathsf{R}^\otimes \setminus \{\pi\}} := \left\{ \boldsymbol{a} \in \mathbb{R}^{\mathsf{t}} \,\middle|\, \begin{array}{rcl} \boldsymbol{c}^\rho \boldsymbol{a} & = & 1 \quad (\rho \in \mathsf{R}^\otimes \setminus \{\pi\}) \\ a_\tau & = & 0 \quad (\tau \notin \mathsf{T}^\otimes) \end{array} \right\} . \tag{3.7}$$

As $\boldsymbol{a}^\otimes$ is an extremal of $\boldsymbol{A}$, one half of this line intersects $\boldsymbol{A}$ and the other half runs outside of $\boldsymbol{A}$. The first halfline will hit the boundary of of $\boldsymbol{A}$ at some extremal point $\boldsymbol{a}^\star$ of $\boldsymbol{A}$.

There are two possibilities: Either $\boldsymbol{a}^\star$ satisfies a further equation $a_\tau = 0$ for some $\tau \in \mathsf{T}^\oplus$. Or else $\boldsymbol{a}^\star$ satisfies a further equation $\boldsymbol{c}^\sigma \boldsymbol{a} = 1$.

1stSTEP :

With respect to the first alternative we have

$$\mathsf{T}^\star = \mathsf{T}^\otimes \setminus \{\tau\} , \quad \mathsf{R}^\star = (\mathsf{R}^\otimes \setminus \{\pi\}) . \tag{3.8}$$

and $\boldsymbol{a}^\otimes$ is the unique solution of the (characteristic) system

$$\begin{array}{rcl} \boldsymbol{c}^\rho \boldsymbol{a} & = & 1 \quad (\rho \in \mathsf{R}^\star) \\ a_\tau & = & 0 \quad (\tau \notin \mathsf{T}^\star) . \end{array} \tag{3.9}$$

2ndSTEP :

In the second alternative $\boldsymbol{a}^\star$ is the solution of the system

$$\begin{array}{rcl} \boldsymbol{c}^\rho \boldsymbol{a} & = & 1 \quad (\rho \in (\mathsf{R}^\otimes \setminus \{\pi\}) \cup \{\sigma\}) \\ a_\tau & = & 0 \quad (\tau \notin \mathsf{T}^\otimes) . \end{array} \tag{3.10}$$

Then we have

$$\mathsf{T}^\star = \mathsf{T}^\otimes , \quad \mathsf{R}^\star = (\mathsf{R}^\otimes \setminus \{\pi\}) \cup \{\sigma\}. \tag{3.11}$$

q.e.d.

Theorem 3.2. *Let $\boldsymbol{a}^\otimes \in \boldsymbol{A}^e$ be a relevant vector and let $\mathsf{T}^\otimes$, $\mathsf{R}^\otimes$ be the characteristics with $|\mathsf{T}^\otimes| = |\mathsf{R}^\otimes| =: s$. Let $\overline{\tau} \notin \mathsf{T}^\otimes$. Then there exists a unique $\boldsymbol{a}^\star = \boldsymbol{a}^{\star\overline{\tau}} \in \boldsymbol{A}^e$ with characteristics $\mathsf{T}^\star$, $\mathsf{R}^\star$ such that he interval*

$$[\boldsymbol{a}^\otimes, \boldsymbol{a}^\star] := \{ t\boldsymbol{a}^\otimes + (1-t)\boldsymbol{a}^\star \mid 0 \le t \le 1 \} \tag{3.12}$$

is an edge of $\boldsymbol{A}$ and one of the following alternatives hold true:

1. *Either there exists $\sigma \notin \mathsf{R}^\otimes$ such that*

$$\mathsf{T}^\star = \mathsf{T}^\otimes \cup \{\overline{\tau}\} , \quad \mathsf{R}^\star = \mathsf{R}^\otimes \cup \{\sigma\} . \tag{3.13}$$

2. *Or else there exists some* $\widehat{\tau} \in \mathbf{T}^{\otimes}$ *such that*

$$\mathbf{T}^{\star} = (\mathbf{T}^{\otimes} \setminus \{\widehat{\tau}\}) \cup \{\overline{\tau}\}\ , \quad \mathbf{R}^{\star} = \mathbf{R}^{\otimes}\ . \tag{3.14}$$

Proof: The extremal $\boldsymbol{a}^{\otimes}$ is located with the linear subspace/straight line obtained by adding $\overline{\tau}$ to the system, i.e.,(omitting the equation $a_{\overline{\tau}} = 0$)

$$\mathbb{L}^{\mathbf{R}^{\otimes}}_{\mathbf{T}^{\otimes}\cup\{\overline{\tau}\}} := \left\{ \boldsymbol{a} \in \mathbb{R}^{\mathbf{t}} \,\middle|\, \begin{array}{rcll} \boldsymbol{c}^{\rho}\boldsymbol{a} & = & 1 & (\rho \in \mathbf{R}^{\otimes}) \\ a_{\tau} & = & 0 & (\tau \notin (\mathbf{T}^{\otimes} \cup \{\overline{\tau}\})) \end{array} \right\} . \tag{3.15}$$

The intersection of this line with $\boldsymbol{A}$ constitues an edge of $\boldsymbol{A}$, the second vertex of which is $\boldsymbol{a}^{\star}$ as described.

q.e.d.

Corollary 3.3. *Let* $\boldsymbol{a}^{\otimes} \in \boldsymbol{A}^{e}$ *be a relevant pre–coalition and let* $\mathbf{T}^{\otimes}$, $\mathbf{R}^{\otimes}$ *be the characteristics. Then* $\boldsymbol{a}^{\otimes}$ *is located on edges*

$$[\boldsymbol{a}^{\otimes}, \boldsymbol{a}^{\star}] := \{t\boldsymbol{a}^{\otimes} + (1-t)\boldsymbol{a}^{\star} \mid 0 \leq t \leq 1\} \tag{3.16}$$

with neighboring vertices $\boldsymbol{a}^{\star\rho}$ $(\rho \in \mathbf{R}^{\otimes})$ *and* $\boldsymbol{a}^{\star\tau}(\tau \notin \mathbf{T}^{\otimes})$ *according to Theorem 3.1 and Theorem 3.2. Moreover,*

1. *If* $s = |\mathbf{T}^{\otimes}| = |\mathbf{R}^{\otimes}| = 1$, *say* $\mathbf{R}^{\otimes} = \{\pi\}$ *and* $\mathbf{T}^{\otimes} = \{\widehat{\tau}\}$, *then* $\boldsymbol{a}^{\otimes} = \boldsymbol{e}^{\widehat{\tau}}$ *is a basis vector of* $\mathbb{R}^{\mathbf{t}}$ *contained in just one* $\boldsymbol{A}_1^{\pi}$. . *In this case there are exactly* $\mathbf{t} - 1$ *adjacent edges* $[\boldsymbol{a}^{\otimes}, \boldsymbol{a}^{\star\overline{\tau}}]$ *for* $\overline{\tau} \in \mathbf{T} \setminus \{\widehat{\tau}\}$ *with either* $\mathbf{T}^{\star} = (\mathbf{T}^{\otimes} \setminus \{\widehat{\tau}\}) \cup \{\overline{\tau}\}\ , \quad \mathbf{R}^{\star} = \mathbf{R}^{\otimes}$ *according to* (3.13) *or else* $\mathbf{T}^{\star} = (\mathbf{T}^{\otimes} \setminus \{\widehat{\tau}\}) \cup \{\overline{\tau}\}\ , \quad \mathbf{R}^{\star} = \mathbf{R}^{\otimes}$ *according to* (3.14). *That is, there is a departure edge for each* $\boldsymbol{a}^{\star\overline{\tau}}$ *for each* $\overline{\tau} \neq \widehat{\tau}$.

2. *Is* $s \geq 2$, *then there are exactly* $\mathbf{t}$ *adjacent edges* $[\boldsymbol{a}^{\otimes}, \boldsymbol{a}^{\star}]$. *These are either obtained by "departing" in direction of some* $\overline{\tau}$ *as in Theorem 3.2 or by leaving some* $\boldsymbol{A}_1^{\pi}$ *as in 3.1.*

3. *Any extremal* $\boldsymbol{a}^{\otimes}$ *of* $\boldsymbol{A}$ *has at most* r *positive coordinates.*

Definition 3.4. *Let* $\boldsymbol{a}^{\ominus} \in \boldsymbol{A}^{e}$ *be a relevant pre–coalition and let* $\mathbf{T}^{\ominus}$, $\mathbf{R}^{\ominus}$ *be the characteristics. Let* $\tau \in \mathbf{T}^{\ominus}, \pi \in \mathbf{R}^{\ominus}$.

1. *We say that* $\boldsymbol{a}^{\ominus}$ *is **reduced** to* $\boldsymbol{a}^{\oplus}$ *via* $(\tau, \pi) \in \mathbf{T}^{\ominus} \times \mathbf{R}^{\ominus}$ *if* (3.4) *is valid. That is, for the second edge* $\boldsymbol{a}^{\oplus}$ *described by* (3.3) *and the corresponding characteristics we have*

$$\mathbf{T}^{\oplus} = \mathbf{T}^{\ominus} \setminus \{\tau\}\ , \quad \mathbf{R}^{\oplus} = \mathbf{R}^{\ominus} \setminus \{\pi\}\ . \tag{3.17}$$

We write

$$\boldsymbol{a}^{\ominus} \overset{\tau,\pi}{\Rrightarrow} \boldsymbol{a}^{\oplus}\ .$$

*and refer to the operation indicated as a "process of reduction" or just **reduction**.*

2. *A relevant pre–coalition* $\boldsymbol{a}^{\oplus} \in \boldsymbol{A}^{e}$ *is said to be* ***irreducible*** *if there is no* $(\tau, \pi, \boldsymbol{a}^{\star}) \in \mathbf{T}^{\oplus} \times \mathbf{R}^{\oplus} \times \boldsymbol{A}^{e}$ *such that*

$$\boldsymbol{a}^{\oplus} \overset{\tau,\pi}{\Rightarrow} \boldsymbol{a}^{\star}$$

holds true.

Lemma 3.5. *Let* $\boldsymbol{a}^{\ominus} \in \boldsymbol{A}^{e}$ *be a relevant pre–coalition and let* $\mathbf{T}^{\ominus}$, $\mathbf{R}^{\ominus}$ *be the characteristics. Then there exists an irreducible relevant pre–coalition* $\boldsymbol{a}^{\oplus} \in \boldsymbol{A}^{e}$ *with characteristics* $\mathbf{T}^{\oplus} \subseteq \mathbf{T}^{\ominus}$, $\mathbf{R}^{\oplus} \subseteq \mathbf{R}^{\ominus}$.

Proof: Obvious, reduce $\boldsymbol{a}^{\ominus}$ a finite number of steps if necessary.

q.e.d.

Example 3.6. This examples exhibits the situation in the quasi orthogonal case and shows that we do have an appropriate generalization at hand.

We have to slightly adapt our notation as follows. In the context of **QuO** the production factors are enumerated by $\{0, 1, \ldots, r\}$ and $\{1, \ldots, r\}$ enumerates the (orthogonal) elements of the core. Adapting this to our present version means that we use $\mathbf{R} = \{0, 1, \ldots, r\}$ and $\boldsymbol{Q} = \{1, \ldots, r\}$. With this convention we discuss the versions of relevant vectors $\boldsymbol{a}^{\odot}, \boldsymbol{a}^{\oplus}$ and $\boldsymbol{a}^{\ominus}$ that appear in that context of **QuO**.

1$^{\text{st}}$STEP :

First of all, for some $\sigma \in \mathbf{Q}$ consider some relevant vector (of the "second kind") $\boldsymbol{a}^{0\oplus\sigma}$ determined by equations involving

$$\boldsymbol{c}^{\rho} \ (\rho \in (\mathbf{Q} \setminus \{\sigma\}) \cup \{0\}) \ = \ \mathbf{R} \setminus \{\sigma\}$$

Such a vector is determined by some "undercutting" sequence

$$\text{(3.18)} \qquad \widehat{\boldsymbol{\tau}} \ = \ (\widehat{\tau}_1, \ldots, \widehat{\tau}_r) \quad \text{with} \quad c^0_{\widehat{\tau}_\rho} \ = \ l^{\rho}_0 = \min_{\boldsymbol{C}^{\rho}} c^0_{\rho} \quad (\rho \in \mathbf{Q} \setminus \{\sigma\}) \ .$$

That is, each $\widehat{\tau}_\rho$ takes the minimal value of the density $c^0_{\bullet}$ over the carrier $\boldsymbol{C}^{\rho}$ $(\rho \in \mathbf{Q} \setminus \{\sigma\})$. Hence, the characteristics are

$$\text{(3.19)} \qquad \mathbf{T}^{\oplus} \ = \ \{\widehat{\tau}_1, \ldots, \widehat{\tau}_r\} \ = \ \{\widehat{\tau}_\rho\}_{\rho \in \mathbf{Q}} \quad \text{and} \quad \mathbf{R}^{\oplus} \ = \ \mathbf{R} \setminus \{\sigma\} \ .$$

By assumptions in **QuO**, the "undercutting" sequence satisfies

$$\text{(3.20)} \qquad \sum_{\tau \in \mathbf{T}^{\oplus}} c^0_{\tau} = \boldsymbol{c}^0(\mathbf{T}^{\oplus}) = \sum_{\rho \in \mathbf{Q}} l^{\rho}_0 = \sum_{\rho=1}^{r} l^{\rho}_0 < 1 \ ,$$

and the shape of $\boldsymbol{a}^{0\oplus\sigma}$ is indicated by

$$\boldsymbol{a}^{0\oplus\sigma} \ = \ (\ldots 1 \ldots, \ldots 1, \ldots, \ldots \frac{1 - \sum_{\rho \in \mathbf{Q} \setminus \{\sigma\}} c^0_{\widehat{\tau}_\rho}}{c^0_{\widehat{\tau}_\sigma}} \ldots, \ldots 1 \ldots, \ldots 1 \ldots) \ .$$

Thus, the characteristic equations defining this vector are

$$\begin{aligned} \boldsymbol{c}^{\rho}\boldsymbol{a}^{0\oplus\sigma} &= 1 \quad (\rho \in \mathbf{Q} \setminus \{\sigma\}) \ , \\ \boldsymbol{c}^{0}\boldsymbol{a}^{0\oplus\sigma} &= 1 \\ a_{\tau} &= 0 \quad (\tau \notin \mathbf{T}^{\oplus}) \ . \end{aligned} \tag{3.21}$$

Moreover we have

$$\boldsymbol{c}^{\sigma}\boldsymbol{a}^{0\oplus\sigma} = \frac{1 - \sum_{\rho\in\mathbf{Q}\setminus\{\sigma\}} c^{0}_{\widehat{\tau}_{\rho}}}{c^{0}_{\widehat{\tau}_{\sigma}}} = \frac{1 - \sum_{\tau\in\mathbf{T}^{\oplus}\setminus\{\widehat{\tau}_{\sigma}\}} c^{0}_{\tau}}{c^{0}_{\widehat{\tau}_{\sigma}}} > 1 \ .$$

It is not hard to see that $\boldsymbol{a}^{0\oplus\sigma}$ is irreducible.

2^nd STEP : Next we consider relevant vectors of the "third kind" which will turn out to be reducible. The characteristic equations now involve all $\boldsymbol{c}^{\rho}$ $(\rho \in \mathbf{R})$. However, again there is a specific coordinate σ which we choose to be $\sigma = r$. Then the generic relevant vector $\boldsymbol{a}^{0\ominus r}$ of the"third kind" is given by a sequence

$$\widehat{\boldsymbol{\tau}} = (\widehat{\tau}_1, \ldots, \widehat{\tau}_r, \overline{\tau}_r) \quad \text{with} \quad \widehat{\tau}_{\rho} = l^{\rho}_{0} \quad (\rho \in \mathbf{Q}) \quad \text{and} \quad \overline{\tau}_r \in \boldsymbol{C}^{r}.$$

Accordingly we put

$$\mathbf{T}^{\ominus} := \{\widehat{\tau}_1, \ldots, \widehat{\tau}_r, \overline{\tau}_r\}, \quad \mathbf{R}^{\ominus} = \mathbf{R} = \{0, 1, \ldots, r\} \ . \tag{3.22}$$

Now assumptions are

$$\boldsymbol{c}^{0}(\mathbf{T}^{\ominus} \setminus \{\overline{\tau}_r\}) = \sum_{\rho=1}^{r} c^{0}_{\widehat{\tau}_{\rho}} = \sum_{\rho=1}^{r} l^{\rho}_{0} < 1 < \sum_{\rho=1}^{r-1} c^{0}_{\widehat{\tau}_{\rho}} + c^{0}_{\overline{\tau}_r} = \boldsymbol{c}^{0}(\mathbf{T}^{\ominus}) \ . \tag{3.23}$$

Each $\widehat{\tau}_{\rho}$ takes the minimal value of the density $\boldsymbol{c}^{0}_{\bullet}$ over the carrier $\boldsymbol{C}^{\rho}$, but $c^{0}_{\overline{\tau}_r}$ is "large" by comparison. Writing $l^{\star}_{r} := \sum_{\rho\in\mathbf{R}-\{r\}} c^{0}_{\widehat{\tau}_{\rho}}$, the shape of $\boldsymbol{a}^{0\ominus r}$ is as follows.

$$\begin{aligned} \boldsymbol{a}^{0\ominus r} &= (\ldots 1 \ldots ; \ldots 1, \ldots ; \\ &\quad \ldots, \frac{l^{\star}_{r} + c^{0}_{\overline{\tau}_r} - 1}{c^{0}_{\overline{\tau}_r} - c^{0}_{\widehat{\tau}_r}}, \ldots, \frac{1 - (l^{\star}_{r} + c^{0}_{\widehat{\tau}_r})}{c^{0}_{\overline{\tau}_r} - c^{0}_{\widehat{\tau}_r}}, \ldots) \\ &=: (\ldots 1 \ldots ; \ldots 1, \ldots ; \ldots, \alpha, \ldots, \beta, \ldots) \ . \end{aligned} \tag{3.24}$$

Here α and β are determined by

$$\begin{aligned} \alpha + \beta &= 1 \\ \alpha c^{0}_{\widehat{\tau}_r} + \beta c^{0}_{\overline{\tau}_r} &= 1 - l^{\star}_{r} \ . \end{aligned} \tag{3.25}$$

Clearly the characteristics of $\boldsymbol{a}^{0\ominus r}$ are

$$\begin{aligned} \boldsymbol{c}^{\rho}\boldsymbol{a}^{0\ominus r} &= 1 \quad (\rho \in \mathbf{R}^{\ominus} = \mathbf{R}) \ , \\ a_{\tau} &= 0 \quad (\tau \notin \mathbf{T}^{\ominus}) \ . \end{aligned} \tag{3.26}$$

Now consider the linear subspace/straight line described by omitting $\boldsymbol{c}^r$ from the above system, that is,

$$(3.27)\quad \mathbb{L}_{\mathbf{T}^\pi}^{\mathbf{R}^\ominus\setminus\{r\}} := \left\{ \boldsymbol{a}\in\mathbb{R}^{\mathbf{t}} \;\middle|\; \begin{array}{rcll} \boldsymbol{c}^\rho \boldsymbol{a} &=& 1 & (\rho\in\mathbf{R}^\ominus\setminus\{r\})\;, \\ a_\tau &=& 0 & (\tau\notin\mathbf{T}^\ominus \;:=\; \{\widehat{\tau}_1,\ldots,\widehat{\tau}_r,\overline{\tau}_r\}). \end{array} \right\}.$$

We parametrize this line via $t\to\boldsymbol{a}^t$ with

$$(3.28)\quad \boldsymbol{a}^t \;:=\; (\ldots 1\ldots;\ldots 1,\ldots;\ldots,\frac{t}{c^0_{\widehat{\tau}_r}},\ldots,\frac{1-(l^\star_r+t)}{c^0_{\overline{\tau}_r}}\ldots)\;\;(t\in\mathbb{R})\;.$$

For $t=\alpha h_{\widehat{\tau}_r}$ this parametrization yields after some computation

$$(3.29)\qquad \boldsymbol{a}^{\alpha h_{\widehat{\tau}_r}} = (\ldots 1\ldots;\ldots 1,\ldots;\ldots,\alpha,\ldots,\beta,\ldots) \;=\; \boldsymbol{a}^{0\ominus r}\;,$$

and for $t=h_{\widehat{\tau}_r}$ we obtain immediately

$$(3.30)\quad \boldsymbol{a}^{h_{\widehat{\tau}_r}} \;=\; (\ldots 1\ldots;\ldots 1,\ldots;\ldots,1,\ldots,\frac{1-(l^\star_r+h_{\overline{\tau}_r})}{h_{\overline{\tau}_r}}\ldots) \;=\; \boldsymbol{a}^{0\oplus r}\;,$$

which is the relevant vector $\boldsymbol{a}^{0\oplus r}$ that appeared in the $1^{st}STEP$. Thus we see that one can depart from $\boldsymbol{a}^{0\ominus r}$ along the edge

$$[\boldsymbol{a}^{0\ominus r},\boldsymbol{a}^{0\oplus r}]$$

reaching $\boldsymbol{a}^{0\oplus r}$, or, in other words, we have a reduction

$$\boldsymbol{a}^{0\ominus r}\overset{r,0}{\Rightarrow}\boldsymbol{a}^{0\oplus r}.$$

$3^{\mathbf{rd}}$STEP : Without entering into the details we mention that relevant vectors $\boldsymbol{a}^\odot$ of the "first kind"are irreducible and determined by a characteristic system involving all $\boldsymbol{c}^\rho$ $(\rho\in\mathbf{Q})$.

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4 Truncating the Excess Supply Measures

We denote the irreducible extremals of $\boldsymbol{A}$ by

$$(4.1)\qquad \boldsymbol{A}^{\oplus} \;:=\; \left\{\boldsymbol{a}^{\oplus} \in \boldsymbol{A}^{e} \,\middle|\, \boldsymbol{a}^{\oplus} \text{ is irreducible } \right\} .$$

Also we define

$$(4.2)\quad \boldsymbol{H} \;:=\; \left\{\boldsymbol{x} \in \boldsymbol{J}(v) \,\middle|\, \boldsymbol{x}\boldsymbol{a}^{\oplus} \geq v(\boldsymbol{a}^{\oplus}) = 1 \;(\boldsymbol{a}^{\oplus} \in \boldsymbol{A}^{\oplus})\right\} \;\supseteq\; \boldsymbol{J}(\boldsymbol{v}) \cap \mathbb{C} .$$

The candidates for the construction of vNM–Stable Sets will be certain imputations in $\boldsymbol{H}$, more precisely:

Definition 4.1. *Let $\rho \in \mathbf{R}$. A pre–imputation $\bar{\boldsymbol{x}}^{\rho}$ satisfying*

$$(4.3)\qquad \bar{\boldsymbol{x}}^{\rho} \in \boldsymbol{H} \;, \quad \bar{\boldsymbol{x}}^{\rho} \leq \boldsymbol{c}^{\rho}$$

*is said to be a **truncation** of $\boldsymbol{c}^{\rho}$. The imputation corresponding to $\bar{\boldsymbol{x}}^{\rho}$ is the measure*

$$(4.4)\qquad \bar{\boldsymbol{\xi}}^{\rho} = \dot{\boldsymbol{\xi}}^{\bar{\boldsymbol{x}}^{\rho}} := \sum_{\tau \in \mathbf{T}} \bar{x}^{\rho}_{\tau} \boldsymbol{\lambda}_{|\boldsymbol{D}^{\tau}} \qquad \textit{with density} \quad \dot{\bar{\boldsymbol{\xi}}}^{\rho} = \sum_{\tau \in \mathbf{T}} \bar{x}^{\rho}_{\tau} \mathbb{1}_{\boldsymbol{D}^{\tau}} \; .$$

*$\bar{\boldsymbol{\xi}}^{\rho}$ is a **truncation measure** induced by $\boldsymbol{\lambda}^{\rho}$.*

Later on, if $\{\bar{\boldsymbol{x}}^{\rho}\}_{\rho \in \mathbf{R}}$ is a family of truncations, then the convex hull

$$(4.5)\qquad \mathbb{H} \;:=\; \mathbf{ConvH}\, \{\bar{\boldsymbol{x}}^{\rho}\}_{\rho \in \mathbf{R}}$$

will be considered to be a *candidate* for the construction of a vNM–Stable Set.

Clearly, whenever $\rho \in \mathbf{Q}$, that is, whenever $\boldsymbol{c}^{\rho}$ is a pre–core element, then (4.3) implies $\bar{\boldsymbol{x}}^{\rho} = \boldsymbol{c}^{\rho}$. Hence, the only truncation of a (pre–) core element is that (pre–) core element itself. In order to exhibit truncations we, therefore, can restrict our focus on $\rho \in \mathbf{R}\backslash\mathbf{Q}$. Clearly, for $\rho \in \mathbf{R}\backslash\mathbf{Q}$ and some truncation $\bar{\boldsymbol{x}}^{\rho}$ we obtain $\bar{\boldsymbol{x}}^{\rho}\boldsymbol{a}^{\oplus} = \boldsymbol{c}^{\rho}\boldsymbol{a}^{\oplus} = 1$ whenever $\boldsymbol{c}^{\rho}\boldsymbol{a}^{\oplus} = 1$.

Analogously to the situation in $\boldsymbol{A}$, we introduce for some $\sigma \in \mathbf{R}$, the ***characteristics*** of $\boldsymbol{c}^{\sigma}$. As we do not want to explicitely introduce an enumeration of $\boldsymbol{A}$ we obtain for the characteristics:

$$\mathbf{T}^{\sigma} \;:=\; \{\tau \in \mathbf{T} \,|\, c^{\sigma}_{\tau} > 0\}, \quad \boldsymbol{A}^{\sigma} \;:=\; \{\boldsymbol{a}^{\star} \in \boldsymbol{A}^{e} \,|\, \boldsymbol{c}^{\sigma}\boldsymbol{a}^{\star} = 1\} .$$

Then, in view of non–degeneracy, we have

$$|\mathbf{T}^{\sigma}| \;=\; |\boldsymbol{A}^{\sigma}| \;=\; |\boldsymbol{C}^{\sigma}| > 0$$

and $\boldsymbol{c}^{\sigma}$ is the unique solutin of its ***characteristic system*** which is the linear system of equations in variables $\{x_{\tau}\}_{\tau \in \mathbf{T}}$ given by

$$(4.6)\qquad \begin{array}{rcll} \boldsymbol{x}\boldsymbol{a}^{\star} & = & 1 & \boldsymbol{a}^{\star} \in \boldsymbol{A}^{\sigma} \\ x_{\tau} & = & 0 & \tau \in \mathbf{T} \setminus \mathbf{T}^{\sigma} . \end{array}$$

Next, if we introduce

$$\boldsymbol{A}^{\oplus\sigma} \;:=\; \{\boldsymbol{a}^{\oplus} \in \boldsymbol{A}^{\sigma} \,\big|\, \boldsymbol{a}^{\oplus} \text{ is irreducible } \}$$

then a truncation $\bar{\boldsymbol{x}}^{\sigma}$ has to satisfy $\bar{\boldsymbol{x}}^{\sigma}\boldsymbol{a}^{\oplus} = 1$ for $\boldsymbol{a}^{\oplus} \in \boldsymbol{A}^{\oplus\sigma}$. Based on these prerequisites, the following outlines a procedure for the construction of candidates.

Definition 4.2. *We define*

$$(4.7)\qquad \begin{aligned} \overset{\vee}{\boldsymbol{E}}{}^{\sigma} \;&:=\; \left\{\tau \in \mathbf{R} \,\middle|\, \exists \boldsymbol{a}^{\oplus} \in \boldsymbol{A}^{\oplus},\ \textit{s.t.}\ \tau \in \mathbf{T}^{\oplus}, \sigma \in \mathbf{R}^{\oplus}\right\} \\ \overset{\wedge}{\boldsymbol{E}}{}^{\sigma} \;&:=\; \boldsymbol{C}^{\sigma} \setminus \overset{\vee}{\boldsymbol{E}}{}^{\sigma}\ . \end{aligned}$$

The following pre–imputations $\{\bar{\boldsymbol{x}}^{\rho}\}_{\rho\in\mathbf{R}}$ provide the candidates for the extremals of some vNM–Stable Set $\mathbb{H}$.

Definition 4.3. *Let $\sigma \in \mathbf{R}\backslash\mathbf{Q}$. A pre–imputation $\bar{\boldsymbol{x}}^{\sigma}$ is said to be a* ***standard truncation vector*** *corresponding to $\boldsymbol{c}^{\sigma}$ if the following holds true.*

1.

$$(4.8)\qquad \overline{x}^{\sigma}_{\tau} := c^{\sigma}_{\tau} \quad (\tau \in \overset{\vee}{\boldsymbol{E}}{}^{\sigma})$$

2.

$$(4.9)\qquad \overline{x}^{\sigma}_{\tau} \leq c^{\sigma}_{\tau} \quad (\tau \in \overset{\wedge}{\boldsymbol{E}}{}^{\sigma})$$

3.

$$(4.10)\qquad \overline{x}^{\sigma}_{\tau} = 0 \quad (\tau \in \mathbf{T} \setminus (\overset{\vee}{\boldsymbol{E}}{}^{\sigma} \cup \overset{\wedge}{\boldsymbol{E}}{}^{\sigma}))$$

such that $\bar{\boldsymbol{x}}^{\sigma}(\mathbb{C}^{\sigma}) \;=\; \bar{\boldsymbol{x}}^{\sigma}(\boldsymbol{I}) \;=\; t\ .$

The ***standard truncation measure*** $\bar{\boldsymbol{\xi}}^{\sigma}$ *induced by $\boldsymbol{\lambda}^{\sigma}$ is the corresponding imputation generated as in Definition 4.2. We call $\bar{\boldsymbol{x}}^{\sigma}$ a* ***standard truncation vector*** *corresponding to $\boldsymbol{c}^{\sigma}$. The imputation generated is the measure*

$$(4.11)\qquad \overline{\boldsymbol{\xi}}^{\sigma} = \sum_{\tau\in\mathbf{T}} \overline{x}^{\sigma}_{\tau}\boldsymbol{\lambda}_{|\boldsymbol{D}^{\tau}} \quad \textit{with density} \quad \overset{\bullet}{\overline{\boldsymbol{\xi}}}{}^{\sigma t} = \overset{\bullet}{\boldsymbol{\xi}}{}^{\bar{\boldsymbol{x}}^{\sigma}} := \sum_{\tau\in\mathbf{T}^{(t)}} \overline{x}^{\sigma}_{\tau}\mathbb{1}_{\boldsymbol{D}^{\tau}}\ ,$$

Generally a standard truncation vector/measure $\bar{\boldsymbol{x}}^{\sigma}/\overline{\boldsymbol{\xi}}^{\sigma}$ results in a (pre–) ***imputation***, only if $\bar{\boldsymbol{x}}^{\sigma}(\overset{\wedge}{\boldsymbol{E}}{}^{\sigma}) \;\leq\; t$ holds true.

Theorem 4.4. *Let $\sigma \in \mathbf{R} \setminus \mathbf{Q}$ and and let $\bar{\boldsymbol{x}}^{\sigma}$ be a standard truncation. Let $\boldsymbol{a}^{\otimes} \in \boldsymbol{A}^{\sigma}$ be a relevant precoalition.*

1. *If* $\boldsymbol{a}^{\otimes} = \boldsymbol{a}^{\oplus}$ *is irreducible (i.e.,* $\boldsymbol{a}^{\oplus} \in \boldsymbol{A}^{\oplus\sigma}$*), then*

$$\bar{\boldsymbol{x}}^{\pi}\boldsymbol{a}^{\oplus} = 1 = v(\boldsymbol{a}^{\oplus}) \ . \tag{4.12}$$

2. *If* $\boldsymbol{a}^{\otimes} = \boldsymbol{a}^{\ominus}$ *is reducible, (i.e.,* $\boldsymbol{a}^{\oplus} \in \boldsymbol{A}^{\sigma} \setminus \boldsymbol{A}^{\oplus\sigma}$*), then there is* $\boldsymbol{a}^{\oplus}$ *with characteristics* $\mathbf{T}^{\oplus} \subseteq \mathbf{T}^{\ominus}$, $\mathbf{R}^{\oplus} \subseteq \mathbf{R}^{\ominus}$.

3. *If* $\boldsymbol{a}^{\otimes} = \boldsymbol{a}^{\ominus}$ *is reducible, then*

$$\bar{\boldsymbol{x}}^{\sigma t}\boldsymbol{a}^{\ominus} \leq 1 \ = \ v(\boldsymbol{a}^{\sigma\ominus}) \ . \tag{4.13}$$

Proof:

1stSTEP :

Let $\boldsymbol{a}^{\oplus} \in \boldsymbol{A}^{\oplus\sigma}$ be an irreducible relevant vector. Then, for $\tau \in \mathbf{T}^{\oplus}$ we have $\tau \in \overset{\vee}{\boldsymbol{E}}{}^{\sigma}$ and hence $\overline{x}^{\sigma}_{\tau} = c^{\sigma}_{\tau}$ by (4.8). Consequently

$$\bar{\boldsymbol{x}}^{\sigma}\boldsymbol{a}^{\oplus} = \boldsymbol{c}^{\sigma}\boldsymbol{a}^{\oplus} = v(\boldsymbol{a}^{\oplus}) = 1 \ .$$

2ndSTEP : Next let $\boldsymbol{a}^{\ominus} \in \boldsymbol{A}^{\sigma} \setminus \boldsymbol{A}^{\oplus\pi}$ be a reducible relevant vector. Consider a sequence of reducible vectors starting with $\boldsymbol{a}^{\ominus}$ and ending in some irreducible $\boldsymbol{a}^{\oplus}$, each one being obtained by reducing the previous element of the sequence. Then, clearly $\mathbf{T}^{\oplus} \subseteq \mathbf{T}^{\ominus}$, $\mathbf{R}^{\oplus} \subseteq \mathbf{R}^{\ominus}$.

3rdSTEP : Moreover, in view of (4.8) and (4.9) we have $\overline{x}^{\sigma}_{\tau} \leq \boldsymbol{c}^{\sigma}_{\tau}$, for all $\tau \in \mathbf{T}$, hence, as $\sigma \in \mathbf{R}^{\ominus}$

$$\bar{\boldsymbol{x}}^{\sigma}\boldsymbol{a}^{\ominus} \leq \boldsymbol{c}^{\sigma}\boldsymbol{a}^{\ominus} \ = \ 1$$

q.e.d.

Theorem 4.5. *Assume that, for some* $\sigma \in \mathbf{R}$*,*

$$\bar{\boldsymbol{x}}^{\sigma}(\overset{\wedge}{\boldsymbol{E}}{}^{\sigma}) \ \leq \ t \ .$$

Then there exists an imputation $\bar{\boldsymbol{x}}^{\sigma}$ *such that*

1.

$$\bar{\boldsymbol{x}}^{\sigma} \leq \boldsymbol{c}^{\sigma} \ , \tag{4.14}$$

2.

$$\bar{\boldsymbol{x}}^{\sigma} \in \boldsymbol{H} \ . \tag{4.15}$$

3. *Whenever* $\sigma \in \boldsymbol{Q}$*, (i.e.,* $\boldsymbol{c}^{\sigma} \in \mathbb{C}$ *or equivalently,* $\boldsymbol{\lambda}^{\sigma}$ *is a core element), then uniquely*

$$\bar{\boldsymbol{x}}^{\sigma} = \boldsymbol{c}^{\sigma} \tag{4.16}$$

Proof:

The last item is obvious as any pre–core element $\boldsymbol{c}^\rho$ is an imputation and satisfies *all* conditions $\boldsymbol{c}^\rho \boldsymbol{a}^\star \leq 1$ for $\boldsymbol{a}^\star \in \boldsymbol{A}^e$ and not just the ones for $\boldsymbol{H}$. Therefore, we restrict ourselves on $\sigma \in \mathbf{R} \setminus \mathbf{Q}$. Then our claim is a consequence of Theorem 4.4.

q.e.d.

Example 4.6. Again we consider the quasi orthogonal case **QuO** and compare our present generalization to the previous set up. The notation is as in Example 3.6.

For $\sigma = 0 \in \mathbf{R} \setminus \mathbf{Q}$, i.e., $\boldsymbol{c}^\sigma = \boldsymbol{c}^0$ we will exhibit the shape of a truncated vector $\bar{\boldsymbol{x}}^0$.

1$^{\text{st}}$STEP :

First of all we determine $\overset{\vee}{\mathbf{E}}^0 = \overset{\vee}{\mathbf{E}}^\sigma$. Consider some relevant vector of the "second kind" $\boldsymbol{a}^{0\oplus\sigma}$ as discussed in the first step of Example 3.6, with shape

$$\boldsymbol{a}^{0\oplus\sigma} = (\ldots 1 \ldots, \ldots 1, \ldots, \ldots \frac{1 - \sum_{\rho \in \mathbf{Q}\setminus\{\sigma\}} c^0_{\widehat{\tau}_\rho}}{c^0_{\widehat{\tau}_\sigma}} \ldots, \ldots 1 \ldots, \ldots 1 \ldots) \,.$$

As $\boldsymbol{a}^{0\oplus\sigma}$ is irreducible we have $\mathbf{T}^\oplus \subseteq \overset{\vee}{\mathbf{E}}^0$. As the undercutting sequence $\mathbf{T}^\oplus$ as well as σ can be varied we obtain

$$\overset{\vee}{\mathbf{E}}^0 = \bigcup_{\mathbf{T}^\oplus undercutting} \mathbf{T}^\oplus \,.$$

Hence, whenever $\bar{\boldsymbol{x}}^0$ is a truncated vector w.r.t. $\boldsymbol{c}^0$, then $\bar{\boldsymbol{x}}^0{}_{|\overset{\vee}{\mathbf{E}}^\sigma} = \boldsymbol{c}^0{}_{|\overset{\vee}{\mathbf{E}}^0}$; i.e., $\bar{\boldsymbol{x}}^0$coinides with $\boldsymbol{c}^0$ on every undercutting sequence.

2$^{\text{nd}}$STEP : Next, again for $\boldsymbol{c}^\sigma = \boldsymbol{c}^0$, we consider relevant vectors of the "third kind" which turned out to be reducible in the $2^{nd}STEP$ of Example 3.6. There we have found a reduction

$$\boldsymbol{a}^{0\ominus r} \overset{r,0}{\Rightarrow} \boldsymbol{a}^{0\oplus r}.$$

As $\boldsymbol{a}^{0\oplus r}$ is irreducible, we conclude that

$$\text{(4.17)} \quad \widehat{\tau}_1, \ldots, \widehat{\tau}_r \in \overset{\vee}{\boldsymbol{E}}^0,\ \ \overline{\tau}_r \in \overset{\wedge}{\boldsymbol{E}}^0\ ,\ \text{ i.e. } \ \mathbf{T}^\oplus = \mathbf{T}^\ominus \setminus \{\overline{\tau}_r\} \subseteq \overset{\vee}{\boldsymbol{E}}^0,\ \ \{\overline{\tau}_r\} \subseteq \overset{\wedge}{\boldsymbol{E}}^0.$$

as well as

$$\text{(4.18)} \qquad \mathbf{R}^\oplus = \{0, 1, \ldots, (r-1)\} \subseteq \mathbf{R}^\ominus = \mathbf{R} \,.$$

Combining we have

$$\text{(4.19)} \qquad \overset{\vee}{\boldsymbol{E}}^0 = \bigcup_{\mathbf{T}^\oplus \text{ undercutting}} \mathbf{T}^\oplus \,, \quad \overset{\wedge}{\boldsymbol{E}}^0 = \mathbf{T} \setminus \overset{\vee}{\boldsymbol{E}}^0 \,.$$

and $\bar{\boldsymbol{x}}^0_{|\overset{\vee}{\boldsymbol{E}}^0} = \boldsymbol{c}^0_{|\overset{\vee}{\boldsymbol{E}}^0}$.

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Recall the situation as presented in *Part IV* of **QuO**, in particular *Theorem 3.3* and the presentation in *Section 4*. There, if $\bar{\boldsymbol{x}}^\sigma(\overset{\wedge}{\boldsymbol{E}}^\sigma) \leq t$ holds true for $\sigma = 0 \in \mathbf{R} \setminus \mathbf{Q}$, then we know that a standard truncation $\bar{\boldsymbol{x}}^\sigma = \bar{\boldsymbol{x}}^0$ is an extremal of $\boldsymbol{H}$. Within this context, if $\bar{\boldsymbol{x}}^\sigma(\overset{\wedge}{\boldsymbol{E}}^\sigma) > t$, then the core is stable, that is, for the construction of (candidate for) a vNM–Stable Set one can dispose of $\boldsymbol{c}^0$ or $\boldsymbol{\lambda}^0$ respectively.

In the present general context, therefore, we would like to claim that $\bar{\boldsymbol{x}}^\sigma(\overset{\wedge}{\boldsymbol{E}}^\sigma) > t$, implies as well that $\boldsymbol{c}^\sigma$ can be omitted from the construction. This, however, is presently not within our range.

5 The Facets of $\mathbb{C}$ and External Stability

The the extremals of

$$\boldsymbol{A} = \left\{\boldsymbol{a} \in \mathbb{R}^{\mathbf{t}}_{+} \,\middle|\, \boldsymbol{c}^\rho \boldsymbol{a} \geq 1 \quad (\rho \in \mathbf{R})\right\}$$

are the relevant pre–coalitions which by the Inheritance Theorem (Theorem 2.3) provide the ε–relevant coalitions that are necessary and sufficient regarding dominance.

The elements of $\boldsymbol{A}$ constitute the normals to the comprehensive convex hull of the $\{\boldsymbol{c}^\rho\}_{\rho \in \mathbf{R}}$ which is

$$\text{(5.1)} \qquad \mathbb{C} = \left\{\boldsymbol{x} \in \mathbb{R}^{\mathbf{T}} \,\middle|\, \exists \boldsymbol{\alpha}_{\bullet} \text{ “convex” } \quad \text{s.t.} \quad \boldsymbol{x} \geq \sum_{\rho \in \mathbf{R}} \alpha_\rho \boldsymbol{c}^\rho \right\}$$

as introduced in (2.8). Hence,

$$\text{(5.2)} \quad \mathbb{C} = \left\{\boldsymbol{x} \in \mathbb{R}^{\mathbf{t}}_{+} \,\middle|\, \boldsymbol{x}\boldsymbol{a}^{\otimes} \geq 1 \ (\boldsymbol{a} \in \boldsymbol{A})\right\} = \left\{\boldsymbol{x} \in \mathbb{R}^{\mathbf{t}}_{+} \,\middle|\, \boldsymbol{x}\boldsymbol{a}^{\star} \geq 1 \ (\boldsymbol{a}^{\otimes} \in \boldsymbol{A}^e)\right\}.$$

In view of Remark 2.8 the vectors $\boldsymbol{c}^\rho$ are the extremals of $\mathbb{C}$.

Now, let us focus on the facets of $\mathbb{C}$. For $\boldsymbol{a}^{\otimes} \in \boldsymbol{A}^e$ denote the corresponding $\mathbb{C}$–supporting hyperplane by

$$\text{(5.3)} \qquad \boldsymbol{H}_{\boldsymbol{a}^{\otimes}} := \left\{\boldsymbol{x} \in \mathbf{R}^{\mathbf{t}} \,\middle|\, \boldsymbol{x}\boldsymbol{a}^{\otimes} = 1\right\}.$$

As $\boldsymbol{a}^{\otimes}$ is extremal and in view of non–degeneracy, $\mathbb{C} \cap \boldsymbol{H}_{\boldsymbol{a}^{\otimes}}$ is maximal in dimension and located within the boundary of $\mathbb{C}$. Thus we have

Definition 5.1. *Let $\boldsymbol{a}^{\otimes} \in \boldsymbol{A}^e$. Then the unique ($(\mathbf{t}-1)$–dimensional)* ***facet*** *corresponding to $\boldsymbol{a}^{\otimes}$ is*

$$\text{(5.4)} \qquad \mathbf{F}_{\mathbf{a}^{\otimes}} := \mathbb{C} \cap \boldsymbol{H}_{\boldsymbol{a}^{\otimes}}.$$

Remark 5.2. We imagine $\mathbb{C}$ to be very high dimensional while the number r of the spanning vectors $\{\boldsymbol{c}^\rho\}_{\rho\in\mathbf{R}}$ is relatively small. Therefore, the facets of $\mathbb{C}$ are generically not compact. Rather they appear as the sum of a halfspace and the convex hull of the extremals involved. Figure 5.1 describes a possible structure of $\mathbb{C}$ and $\boldsymbol{A}$ – in 2 dimensions, so far from being representative. Figure 5.2 indicates the same structure in 3 dimensions ... which we have to imagine in much higher dimensions.

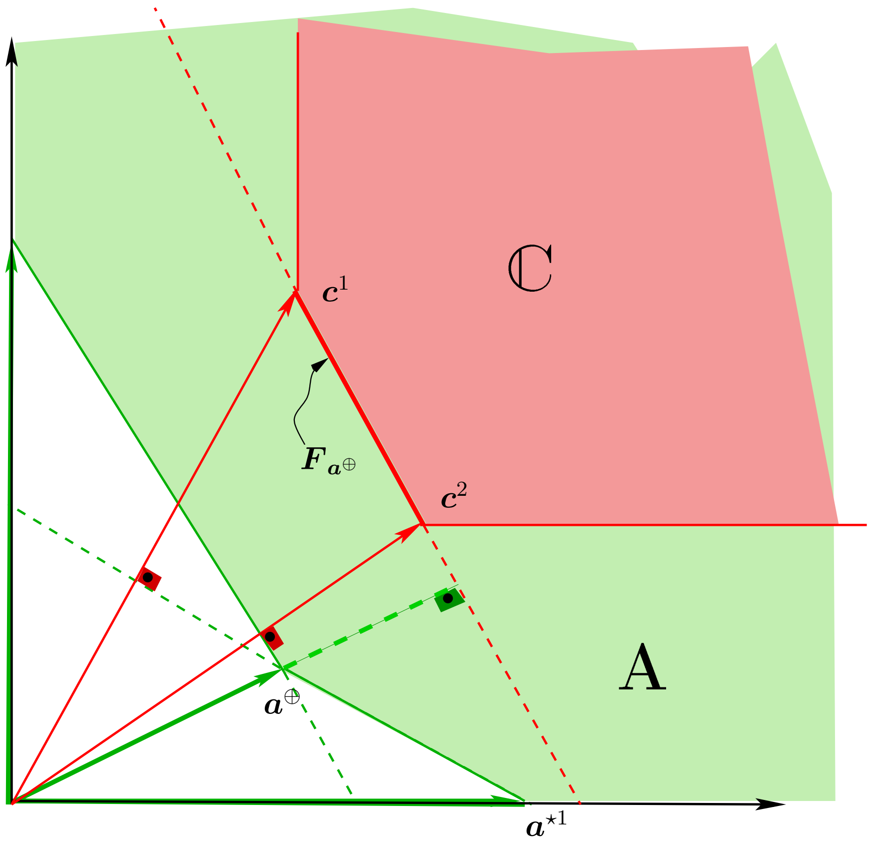

Figure 5.1: The shape of $\mathbb{C}$ and $\boldsymbol{A}$ – Two Dimensions

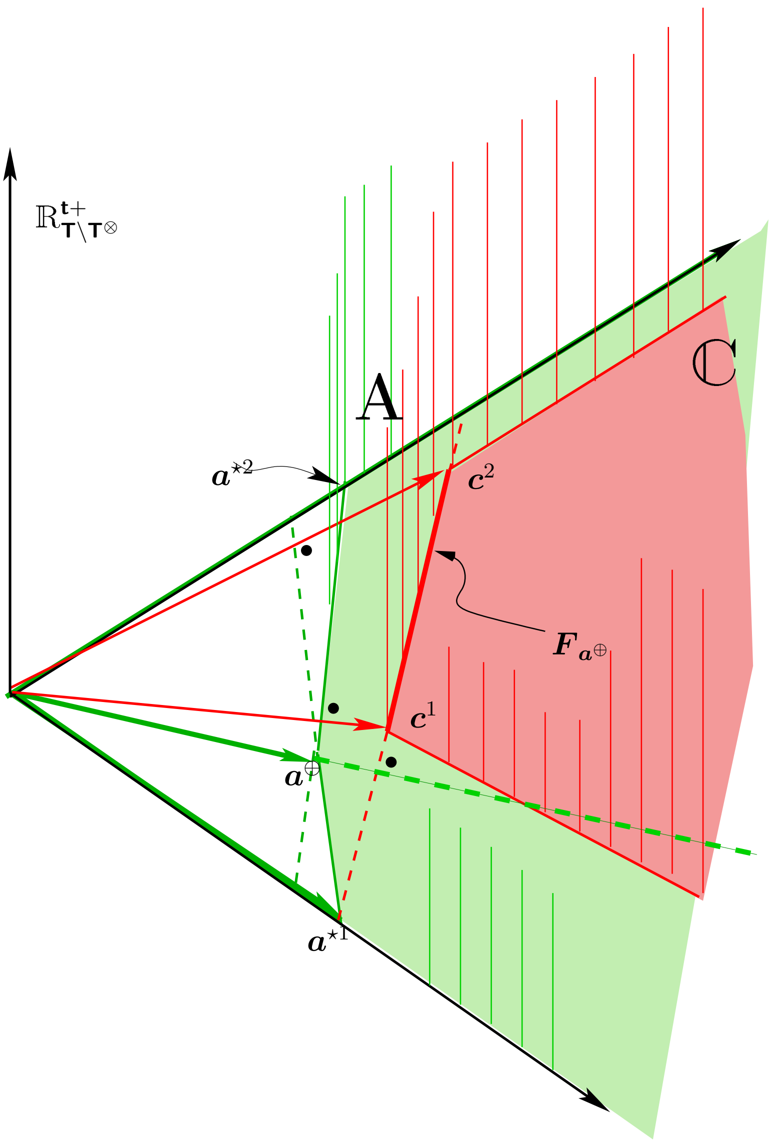

Figure 5.2: The shape of $\mathbb{C}$ and $\boldsymbol{A}$ – Higher Dimensions

More precisely, let $\mathbf{T}^{\otimes}$ and $\mathbf{R}^{\otimes}$ be the characteristics of $\boldsymbol{a}^{\otimes}$, then $\mathbf{F}^{\oplus}_{\boldsymbol{a}}$ can be written

$$
\begin{aligned}
&\mathbf{F}^{\otimes}_{\boldsymbol{a}} = \\
(5.5)\qquad &\left\{ \boldsymbol{x} = \boldsymbol{x}^1 + \boldsymbol{x}^2 \,\middle|\, \exists \boldsymbol{\alpha_\bullet} \text{ “convex”} : \boldsymbol{x}^1 = \sum_{\rho \in \mathbf{R}^{\otimes}} \alpha_\rho \boldsymbol{c}^\rho ,\ \boldsymbol{x}^2 \in \mathbb{R}^{\mathbf{t}+}_{\mathbf{T}\setminus\mathbf{T}^{\otimes}} \right\} \\
&= \ \mathbf{ConvH}\left\{ \boldsymbol{c}^\rho \,\middle|\, \rho \in \mathbf{R}^{\otimes} \right\} + \mathbb{R}^{\mathbf{t}+}_{\mathbf{T}\setminus\mathbf{T}^{\otimes}} \ .
\end{aligned}
$$

For $\rho \in \mathbf{R}^{\otimes}$ we have $\boldsymbol{c}^\rho \in \mathbb{C} \cap \boldsymbol{H}_{\boldsymbol{a}^{\otimes}}$, Thus the vectors $\{\boldsymbol{c}^\rho\}_{\rho \in \mathbf{R}^{\otimes}}$ are the extremals of the facet $\mathbf{F}^{\ominus}_{\boldsymbol{a}} = \mathbb{C} \cap \boldsymbol{H}_{\boldsymbol{a}^{\otimes}}$.

In turn the vectors $\{\boldsymbol{c}^\rho\}_{\rho \in \mathbf{R}^{\otimes}}$ determine $\boldsymbol{a}^{\otimes}$ via the characterizing system

$$
(5.6)\qquad
\begin{aligned}
\boldsymbol{c}^\rho \boldsymbol{a} &= 1 \quad \rho \in \mathbf{R}^{\star} \\
a_\tau &= 0 \quad \tau \in \mathbf{T} \setminus \mathbf{T}^{\star} \ .
\end{aligned}
$$

Somewhat more suggestively written, $\widehat{\boldsymbol{x}} \in \mathbf{F}_{\boldsymbol{a}^{\otimes}}$ admits of a representation

$$\widehat{\boldsymbol{x}} \;=\; \widehat{\boldsymbol{x}}^{\mathsf{T}^{\otimes}} + \widehat{\boldsymbol{x}}^{\mathsf{T}\setminus\mathsf{T}^{\otimes}} \;=\; \sum_{\rho\in\mathsf{R}^{\otimes}} \alpha_\rho \boldsymbol{c}^{\rho} + \widehat{\boldsymbol{x}}^{\mathsf{T}\setminus\mathsf{T}^{\otimes}}, \tag{5.7}$$

with a suitable "conex" set $\boldsymbol{\alpha}_{\bullet} = \{\alpha_\rho\}_{\rho\in\mathsf{R}^{\otimes}}$ of coefficients and nonnegative $\widehat{\boldsymbol{x}}^{\mathsf{T}\setminus\mathsf{T}^{\otimes}} \in \mathbb{R}^{+}_{\mathsf{T}\setminus\mathsf{T}^{\otimes}}$.

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Now we combine the geometrical picture of facets with the concept of dominance. As a prerequesit, we consider a version involving the vectors $\{\boldsymbol{c}^{\rho}\}_{\rho\in\mathsf{R}}$ – which are generally not imputations – in some kind of a dominance relation.

Theorem 5.3. *1. Let $\boldsymbol{x}^0 \notin \mathbb{C}$. Then there exists some $\boldsymbol{a}^{\otimes} \in \boldsymbol{A}^e$ with characteristics $\mathsf{T}^{\otimes}$ and $\mathsf{R}^{\otimes}$ as well as $\widehat{\boldsymbol{x}} \in \mathbb{C} \cap \boldsymbol{H}_{\boldsymbol{a}^{\otimes}}$ such that*

$$\begin{array}{rcl} \widehat{\boldsymbol{x}}_{\mid \mathsf{T}^{\otimes}} & > & \boldsymbol{x}^0_{\mid \mathsf{T}^{\otimes}} , \\ \widehat{\boldsymbol{x}}\boldsymbol{a}^{\otimes} & = & 1 \;=\; \boldsymbol{v}(\boldsymbol{a}^{\otimes}) , \end{array} \tag{5.8}$$

Moreover there is a "convex" combination $\boldsymbol{\alpha}_{\bullet} = \{\alpha_\rho\}_{\rho\in\mathsf{R}^{\otimes}}$ such that

$$\widehat{\boldsymbol{x}}_{\mid \mathsf{T}^{\otimes}} \;=\; \sum_{\rho\in\mathsf{R}^{\otimes}} \alpha_\rho \boldsymbol{c}^{\rho}_{\mid \mathsf{T}^{\otimes}} \tag{5.9}$$

holds true.

2. Let $\boldsymbol{x}^0 \in \mathbb{C}$. Then there exists some $\boldsymbol{a}^{\otimes} \in \boldsymbol{A}^e$ with characteristics $\mathsf{T}^{\otimes}$ and $\mathsf{R}^{\otimes}$ as well as $\widehat{\boldsymbol{x}} \in \mathbb{C} \cap \boldsymbol{H}_{\boldsymbol{a}^{\otimes}}$ such that

$$\widehat{\boldsymbol{x}} \leq \boldsymbol{x}^0 . \tag{5.10}$$

$\widehat{\boldsymbol{x}}$ is a convex combination of the $\boldsymbol{c}^{\rho}$ ($r \in \mathsf{R}^{\otimes}$) exactly if $\widehat{\boldsymbol{x}}^{\mathsf{T}\setminus\mathsf{T}^{\otimes}} = 0$.

Proof:

1$^{\mathrm{st}}$STEP : First of all consider the case that $\boldsymbol{x}^0 \notin \mathbf{C}$. As $\mathbb{C}$ is a closed convex polyhedron we can find some $\boldsymbol{x}^{00} \notin \mathbb{C}$ such that $\boldsymbol{x}^0 < \boldsymbol{x}^{00}$. Then, minimizing the distance from $\boldsymbol{x}^{00}$ to $\mathbb{C}$, find some $\widehat{\boldsymbol{x}}$ in $\partial\mathbb{C}$ such that $\widehat{\boldsymbol{x}} - \boldsymbol{x}^{00}$ is a multiple of a normal, say $\boldsymbol{a}^{\otimes}$, hence nonnegative.

W.l.o.g we can assume that $\boldsymbol{a}^{\otimes}$ is extremal, i.e., a relevant pre–coalition. Moreover, as $\widehat{\boldsymbol{x}}$ is an element of the face $\mathbb{C} \cap \boldsymbol{H}_{\boldsymbol{a}^{\otimes}}$ corresponding to $\boldsymbol{a}^{\otimes}$, we have for a suitable "convex" $\boldsymbol{\alpha}_{\bullet}$:

$$\widehat{\boldsymbol{x}}_{\mid \mathsf{T}^{\otimes}} = \sum_{\rho\in\mathsf{R}^{\otimes}} \alpha_\rho \boldsymbol{c}^{\rho}_{\mid \mathsf{T}^{\otimes}} .$$

That is, we have

$$\widehat{\boldsymbol{x}} \geq \boldsymbol{x}^{00} > \boldsymbol{x}^0 \text{ , hence } \widehat{\boldsymbol{x}}_{\mid \mathsf{T}^{\otimes}} > \boldsymbol{x}^0_{\mid \mathsf{T}^{\otimes}} . \tag{5.11}$$

as well as

$$(5.12)\qquad \widehat{\boldsymbol{x}}\boldsymbol{a}^{\otimes} \;=\; \widehat{\boldsymbol{x}}_{|\mathbf{T}^{\otimes}}\boldsymbol{a}^{\otimes} = \sum_{\rho\in\mathbf{R}^{\otimes}} \alpha_\rho \boldsymbol{c}^{\rho}\boldsymbol{a}^{\otimes} \;=\; \sum_{\rho\in\mathbf{R}^{\otimes}} \alpha_\rho \;=\; 1 \;=\; \boldsymbol{v}(\boldsymbol{a}^{\otimes})\;.$$

Now (5.11) and (5.11) verify (5.8).

2ndSTEP : Now let $\boldsymbol{x}^0 \in \mathbb{C}$ with $\boldsymbol{x}^0 \notin \partial\mathbf{C}$. Choose $\widehat{\boldsymbol{x}} \in \partial\mathbb{C}$ such that $\widehat{\boldsymbol{x}} < \boldsymbol{x}^0$ holds true. Again, for some facet $\boldsymbol{F}_{\boldsymbol{a}^{\otimes}}$ of $\mathbb{C}$ we have $\widehat{\boldsymbol{x}} \in \boldsymbol{F}_{\boldsymbol{a}^{\otimes}}$; let the characteristics as usual be $\mathbf{T}^{\otimes}$, and $\mathbf{R}^{\otimes}$; thus

$$(5.13)\qquad \widehat{\boldsymbol{x}} \;=\; \sum_{\rho\in\mathbf{R}^{\otimes}} \alpha_\rho \boldsymbol{c}^{\rho} + \widehat{\boldsymbol{x}}^{\mathbf{T}\setminus\mathbf{T}^{\otimes}} < \boldsymbol{x}^0 \text{ and } \widehat{\boldsymbol{x}}_{\;|\;\mathbf{T}^{\otimes}} < \boldsymbol{x}^0_{\;|\;\mathbf{T}^{\otimes}}.$$

Now, similar to (5.12) we have

$$(5.14)\qquad \widehat{\boldsymbol{x}}\boldsymbol{a}^{\otimes} \;=\; \sum_{\rho\in\mathbf{R}^{\otimes}} \alpha_\rho \boldsymbol{c}^{\rho}\boldsymbol{a}^{\otimes} \;=\; \sum_{\rho\in\mathbf{R}^{\otimes}} \alpha_\rho \;=\; 1 \;=\; \boldsymbol{v}(\boldsymbol{a}^{\otimes})$$

verifying our second claim.

3rdSTEP : The case that $\boldsymbol{x}^0 \in \partial\mathbb{C}$ is the case can be omitted, as the treatment is obvious.

q.e.d.

Remark 5.4. Note again that formula (5.8) cannot be seen as a dominance relation, for $\widehat{\boldsymbol{x}}$ is not necessarily an imputation.

However, if we include the exact case in our discussion, that is, for the moment assume that $\boldsymbol{c}^{\rho}(\mathbf{T}) = r$ (i.e. $\boldsymbol{\lambda}^{\rho}(\boldsymbol{I}) = 1$) for $\rho \in \mathbf{R}$, then Theorem 5.3 indeed implies that the pre–core (i.e., the convexy hull of $\boldsymbol{c}^{\rho}$ ($r \in \mathbf{R}$)) is externally stable. For in this case all imputations outside of $\mathbb{C}$ are dominated while for $\boldsymbol{x}^0 \in \mathbb{C}$ we have necessarily $\boldsymbol{x}^0 \in \boldsymbol{C}(v)$ (the pre–core). And as $\boldsymbol{x}^0$ is an imputation, we have by (5.13)

$$t = \sum_{\tau\in\mathbf{T}} \boldsymbol{x}^0_\tau \le \sum_{\tau\in\mathbf{T}} \widehat{\boldsymbol{x}}_\tau = \sum_{\rho\in\mathbf{R}^{\otimes}} \alpha_\rho \sum_{\tau\in\mathbf{T}} \boldsymbol{c}^{\rho}_\tau + \sum_{\tau\in\mathbf{T}} \widehat{\boldsymbol{x}}^{\mathbf{T}\setminus\mathbf{T}^{\otimes}}_\tau = t + \sum_{\tau\in\mathbf{T}} \widehat{\boldsymbol{x}}^{\mathbf{T}\setminus\mathbf{T}^{\otimes}}_\tau$$

which implies necessarily $\boldsymbol{x}^0 = \widehat{\boldsymbol{x}}$ and $\widehat{\boldsymbol{x}}^{\mathbf{T}\setminus\mathbf{T}^{\otimes}} = 0$, thus

$$\boldsymbol{x}^0 \;=\; \sum_{\rho\in\mathbf{R}^{\otimes}} \alpha_\rho \boldsymbol{c}^{\rho} \in \boldsymbol{C}(v)\;.$$

Now we introduce the truncations $\bar{\boldsymbol{x}}^{\rho}$ and the corresponding measures $\bar{\boldsymbol{\xi}}^{\rho}$ and show that the dominance relations can as well be established.

◦ ∼∼∼∼∼∼ ◦

Definition 5.5. *Let $\{\bar{\boldsymbol{x}}^\rho\}_{\rho\in\mathbf{R}}$ be a family of truncation vectords and let $\{\bar{\boldsymbol{\xi}}^\rho\}_{\rho\in\mathbf{R}}$ be the corresponding family of truncation measures. Then*

$$(5.15) \qquad \mathbb{H} = \boldsymbol{ConvH}\{\bar{\boldsymbol{x}}^\rho\}_{\rho\in\mathbf{R}} \quad \text{and} \quad \mathcal{H} = \boldsymbol{ConvH}\{\bar{\boldsymbol{\xi}}^\rho\}_{\rho\in\mathbf{R}} \ ,$$

*are called a (candidates for) a **standard vNM–Stable Set**.*

We will eventually prove that $\mathbb{H}$ and $\mathcal{H}$ are indeed vNM–stable.

At this stage let us revive the interpretation we attach to the vNM–Stable Set of the above structure. This will be a short version of the elaborate discussion provided in **QuO**, more precisely in Section 1 of [6]; the reader is directed to the lengthy discussion provided in that context.

Following HART[4], we interprete some pair $(\boldsymbol{C}^\rho, \boldsymbol{\lambda}^\rho)$ as a cartel commanding factor ρ. Imagine that a vNM–Stable Set is related to a bargaining process of two stages: in one stage a normalized measure – an "internal" distribution of wealth within the cartel – is selected in each of the cartels and in another stage a relative distribution of wealth is determined between the various cartels which amounts to selecting a convex combination internal distribution. This constitutes an "external" procedure resulting in a set of imputations establishing a vNM–Stable Set.

Thus we perceive to two rounds of the bargaining process: in one round ("external" from the view of the members of a cartel) (representatives of) all cartels bargain about a share in the distribution of wealth – that is, they agree upon a "convex" $\boldsymbol{\alpha}_\bullet$ representing these shares.

The "internal" round of discussions is then engineered within each cartel in order to establish a ("local") distribution of the wealth inside each cartel (normalized, i.e., formally a probability).

The order of organization of these two rounds is not being emphasized, but could be another important matter. Obviously for cartels on the "short side of the market" – the core elements – the internal distribution is being dictated by just this core element. For cartels on the long side, players have to agree about some imputation within this cartel which turns out to necessarily be a distribution absolutely continuous w.r.t. the factor distribution with a density bounded by one. In the Orthogonal Case this is a result of the Characterization Theorem [10],[11]). For the Quasy–Orthogonal case it follows from the constructions of the Truncated Standard Stable Sets in **QuO**.

The competitive concepts – the Core, the Competitive Equilibrium, the Shapley value etc. assign no wealth to cartels on the long side of the market, see e.g.BILLERA AND RAANAN [2] or AUMANN AND SHAPLEY [1] for the (continuous) Shapley value. The vNM–Stable Set however emphasizes the role of cooperation – not only by reflecting the "power to achieve" certain wealth during the bargaining process but also by incorporating the "power to prevent": a group of cartels cannot achieve anything without including each cartel on the long side of the market.

While the "power to achieve" is reflected by internal stability, the "power to prevent" is reflected by the concept of external stability. Indeed, the core as the set of undominated imputations is always internally stable. That is, internal stability and the power to achieve are closely connected. Indeed, that cartels of the long side should agree to a distribution which assigns them just nothing is – from the view of cooperative theory – not plausible.

Outside the core one may find imputations that cannot be dominated by elements of the core. But within a vNM–Stable Set we find always imputations that can be used to discredit them. Thus, vNM–Stable Sets result in reflecting the "preventive power" of cartels because they involve arguments of prevention via external stability.

Equivalence Theorems for large games between the Core, the Shapley Value, the Competitive Equilibrium etc indicate that competition forces out cooperation in a large market. As a consequence we would rather hold that these concepts are not appropriate for reflecting the results of cooperation.

Naturally, one has to ponder about the notion of a "cartel" being reflected by a pair $(\boldsymbol{C}^{\rho}, \boldsymbol{\lambda}^{\rho})$. For now, other than in the orthogonal case or even in **QuO**, players or rather small coalitions can be members of different cartels.

Theorem 5.6 (Extern Dominance – via Pre–Coalitions). *Let $\mathbb{H}$ be a standard candidate. Let $\boldsymbol{x}^0 \in \boldsymbol{J}(v)$ be a preimputation. If $\boldsymbol{x}^0 \notin \mathbb{H}$, then there is $\bar{\boldsymbol{x}} \in \mathbb{H}$ and a relevant irreducible pre–coalition $\boldsymbol{a}^{\otimes}$ such that $\bar{\boldsymbol{x}} \operatorname{dom}_{\boldsymbol{a}^{\oplus}} \boldsymbol{x}^0$ holds true. That is, $\mathbb{H}$ is externally stable.*

Proof:

1$^{\text{st}}$STEP :

First of all assume that $\boldsymbol{x}^0 \notin \mathbb{C}$ holds true.

Then, according to Theorem 5.3, choose $\boldsymbol{a}^{\otimes} \in \boldsymbol{A}^e$ with characteristics $\mathbf{T}^{\otimes}$ and $\mathbf{R}^{\otimes}$ as well as $\widehat{\boldsymbol{x}} \in \mathbb{C} \cap \boldsymbol{H}_{\boldsymbol{a}^{\otimes}}$ such that, with a suitable convex set of coefficients $\boldsymbol{\alpha}_{\bullet}$, we have

$$
\begin{aligned}
\widehat{\boldsymbol{x}}_{\mid \mathbf{T}^{\otimes}} &> \boldsymbol{x}^0_{\mid \mathbf{T}^{\otimes}} , \\
\widehat{\boldsymbol{x}}_{\mid \mathbf{T}^{\otimes}} &= \sum_{\rho \in \mathbf{R}^{\otimes}} \alpha_{\rho} \boldsymbol{c}^{\rho}_{\mid \mathbf{T}^{\otimes}} , \\
\widehat{\boldsymbol{x}} \boldsymbol{a}^{\otimes} &= 1 = \boldsymbol{v}(\boldsymbol{a}^{\otimes}) .
\end{aligned}
\tag{5.16}
$$

Define, with the same set of coefficients $\boldsymbol{\alpha}_{\bullet}$ as given by (5.16),

$$
\bar{\boldsymbol{x}} := \sum_{\rho \in \mathbf{R}^{\otimes}} \alpha_{\rho} \bar{\boldsymbol{x}}^{\rho}
\tag{5.17}
$$

By means of Lemma 3.5 we can find $\boldsymbol{a}^{\oplus} \in \boldsymbol{A}^e$ with characteristics $\mathbf{T}^{\oplus}$ and $\mathbf{R}^{\oplus}$ such that

$$
\mathbf{T}^{\oplus} \subseteq \mathbf{T}^{\ominus} \text{ and } \mathbf{R}^{\oplus} \subseteq \mathbf{R}^{\ominus} .
$$

By Definition 4.3, we have $\bar{\boldsymbol{x}}^{\rho}\ _{\mid\ \mathbf{T}^{\oplus}} = \boldsymbol{c}^{\rho}\ _{\mid\ \mathbf{T}^{\oplus}} (\rho \in \mathbf{R}^{\oplus})$ and consequently $\bar{\boldsymbol{x}}\ _{\mid\ \mathbf{T}^{\oplus}} = \widehat{\boldsymbol{x}}\ _{\mid\ \mathbf{T}^{\oplus}}$, thus

$$\bar{\boldsymbol{x}}\ _{\mid\ \mathbf{T}^{\oplus}} = \widehat{\boldsymbol{x}}\ _{\mid\ \mathbf{T}^{\oplus}} > \boldsymbol{x}^{0}\ _{\mid\ \mathbf{T}^{\oplus}}\ , \tag{5.18}$$

Moreover

$$\bar{\boldsymbol{x}}\boldsymbol{a}^{\oplus} \ = \ \sum_{\rho\in\mathbf{R}^{\otimes}} \alpha_{\rho}\bar{\boldsymbol{x}}^{\rho}\boldsymbol{a}^{\oplus} = \sum_{\rho\in\mathbf{R}^{\otimes}} \alpha_{\rho}\boldsymbol{c}^{\rho}\boldsymbol{a}^{\oplus} = \sum_{\rho\in\mathbf{R}^{\otimes}} \alpha_{\rho} \ = 1\ . \tag{5.19}$$

Now, (5.18) and (6.3) show that

$$\bar{\boldsymbol{x}}\,\mathrm{dom}_{\boldsymbol{a}^{\oplus}}\,\boldsymbol{x}^{0}$$

holds true indeed.

2^nd^STEP :

Now assume that that $\boldsymbol{x}^{0} \in \mathbb{C}$ holds true. Then we can apply the 2^{nd} *item* of Theorem 5.3. That is, we can pick some $\boldsymbol{a}^{\otimes} \in \boldsymbol{A}^{e}$ with characteristics $\mathbf{T}^{\otimes}$ and $\mathbf{R}^{\otimes}$ as well as $\widehat{\boldsymbol{x}} \in \mathbb{C} \cap \boldsymbol{H}_{\boldsymbol{a}^{\otimes}}$ such that

$$\widehat{\boldsymbol{x}} \leq \boldsymbol{x}^{0}\ . \tag{5.20}$$

Now, by Remark 5.2, in particular formula (5.7), $\widehat{\boldsymbol{x}} \in \partial\overline{\mathbb{C}}$ admits of a representation

$$\widehat{\boldsymbol{x}} \ = \ \sum_{\rho\in\mathbf{R}^{\otimes}} \alpha_{\rho}\boldsymbol{c}^{\rho} + \widehat{\boldsymbol{x}}^{\mathbf{T}\setminus\mathbf{T}^{\otimes}}\ .$$

As $\bar{\boldsymbol{x}}^{\rho}(\rho \in \mathbf{R}^{\otimes})$ is an imputation we have $\sum_{\tau\in\mathbf{T}} \overline{x}^{\rho}_{\tau} = t \ \ (\rho \in \mathbf{R}^{\otimes})$ and hence

$$\begin{aligned} t \ &= \ \sum_{\rho\in\mathbf{R}^{\otimes}} \alpha_{\rho}\left(\sum_{\tau\in\mathbf{T}} \overline{x}^{\rho}_{\tau}\right) \ \leq \ \sum_{\rho\in\mathbf{R}^{\otimes}} \alpha_{\rho}\left(\sum_{\tau\in\mathbf{T}} \boldsymbol{c}^{\rho}_{\tau}\right) \\ &\leq \ \sum_{\rho\in\mathbf{R}^{\otimes}} \alpha_{\rho}\left(\sum_{\tau\in\mathbf{T}} \boldsymbol{c}^{\rho}_{\tau}\right) + \sum_{\tau\in\mathbf{T}} \widehat{\boldsymbol{x}}^{\mathbf{T}\setminus\mathbf{T}^{\otimes}}_{\tau} \\ &= \ \sum_{\tau\in\mathbf{T}} \widehat{x}_{\tau} \ \leq \ \sum_{\tau\in\mathbf{T}} x^{0}_{\tau} \ = \ t \end{aligned}$$

hence necessarily $\widehat{\boldsymbol{x}}^{\mathbf{T}\setminus\mathbf{T}^{\otimes}} = 0$ and all inequalities are equations, that is,

$$\boldsymbol{x}^{0} = \sum_{\rho\in\mathbf{R}^{\otimes}} \alpha_{\rho}\bar{\boldsymbol{x}}^{\rho} \in \mathbb{H}\ ,$$

a contradiction.

q.e.d.

6 Internal stability

Finally we are going to deal with internal stability of $\mathbb{H}$. We start treating pre–concepts – as we know, this will eventually suffice to establish the corresponding properties of concepts regarding the continuous coalitional function $\boldsymbol{v}$. The first theorem – similarly to Theorem 5.3 – deals with the complete functionals $\boldsymbol{c}^\rho$ ($\rho \in \mathbf{R}$). In what follows, we do not speak of dominance in this context as $\boldsymbol{x}^\star$ and $\widehat{\boldsymbol{x}}$ are not necessarily imputations.

Theorem 6.1. *There is no triple* $\boldsymbol{x}^\star,\ \widehat{\boldsymbol{x}} \in \boldsymbol{ConvH}\{\boldsymbol{c}^\rho \mid \rho \in \mathbf{R}\},\ \boldsymbol{a}^\otimes in \boldsymbol{A}^e$, *such that* $\boldsymbol{x}^\star > \widehat{\boldsymbol{x}}$ *and,* $\boldsymbol{x}^\star \boldsymbol{a}^\otimes = 1$ *.*

Proof: Assume that,with suitable "convex" $\boldsymbol{\alpha_\bullet}$ and $\boldsymbol{\beta_\bullet}$, we have

$$\boldsymbol{x}^\star = \sum_{\rho \in \mathbf{R}} \alpha_\rho \boldsymbol{c}^\rho \quad , \quad \widehat{\boldsymbol{x}} = \sum_{\rho \in \mathbf{R}} \beta_\rho \boldsymbol{c}^\rho$$

as well as

$$\boldsymbol{x}^\star > \widehat{\boldsymbol{x}} \quad \text{and} \quad \boldsymbol{x}^\star \boldsymbol{a}^\otimes = 1 \ .$$

Then

$$1 = \sum_{\rho \in \mathbf{R}} \alpha_\rho \boldsymbol{c}^\rho \boldsymbol{a}^\otimes = \boldsymbol{x}^\star \boldsymbol{a}^\otimes > \widehat{\boldsymbol{x}} \boldsymbol{a}^\otimes = \sum_{\rho \in \mathbf{R}} \beta_\rho \boldsymbol{c}^\rho \boldsymbol{a}^\otimes \geq \sum_{\rho \in \mathbf{R}} \beta_\rho = 1$$

yields a contradiction.

q.e.d.

Theorem 6.2 (Internal Stability – Pre–Coalitions)**.**
A standard candidate $\mathbb{H}$ *is internally stable, hence vNM–Stable.*

Proof: Let $\bar{\boldsymbol{x}}, \widehat{\boldsymbol{x}} \in \mathbb{H}$ and let $\boldsymbol{a}^\otimes$ be a relevant pre–coalition with characteristics $\mathbf{T}^\otimes$, $\mathbf{R}^\otimes$ such that

$$\bar{\boldsymbol{x}} \operatorname{dom}_{\boldsymbol{a}^\otimes} \widehat{\boldsymbol{x}} \ .$$

Assume that, for suitable "convex" $\boldsymbol{\alpha_\bullet}$ and $\boldsymbol{\beta_\bullet}$, we have

$$\bar{\boldsymbol{x}} = \sum_{\rho \in \mathbf{R}} \alpha_\rho \bar{\boldsymbol{x}}^\rho \quad \text{and} \quad \widehat{\boldsymbol{x}} = \sum_{\rho \in \mathbf{R}} \beta_\rho \bar{\boldsymbol{x}}^\rho.$$

According to Lemma 3.5 we can find $\boldsymbol{a}^\oplus \in \boldsymbol{A}^e$ with characteristics $\mathbf{T}^\oplus$ and $\mathbf{R}^\oplus$ such that

$$\mathbf{T}^\oplus \subseteq \mathbf{T}^\ominus \text{ and } \mathbf{R}^\oplus \subseteq \mathbf{R}^\ominus \ .$$

By Definition 4.3, we have $\bar{\boldsymbol{x}}^\rho \big|_{\mathbf{T}^\oplus} = \boldsymbol{c}^\rho \big|_{\mathbf{T}^\oplus}$ ($\rho \in \mathbf{R}$) . Now

$$\bar{\boldsymbol{x}} \boldsymbol{a}^\oplus > \widehat{\boldsymbol{x}} \boldsymbol{a}^\oplus \ . \tag{6.1}$$

follows from dominance. On the other hand

$$\bar{\boldsymbol{x}} \boldsymbol{a}^\oplus = \sum_{\rho \in \mathbf{R}} \alpha_\rho \bar{\boldsymbol{x}}^\rho \boldsymbol{a}^\oplus = \sum_{\rho \in \mathbf{R}} \alpha_\rho \boldsymbol{c}^\rho \boldsymbol{a}^\oplus = \sum_{\rho \in \mathbf{R}} \alpha_\rho = 1 \tag{6.2}$$

as well as

$$\widehat{\boldsymbol{x}}\boldsymbol{a}^{\oplus} = \sum_{\rho\in\mathbf{R}} \beta_\rho \bar{\boldsymbol{x}}^\rho \boldsymbol{a}^{\oplus} = \sum_{\rho\in\mathbf{R}} \beta_\rho \boldsymbol{c}^\rho \boldsymbol{a}^{\oplus} = \sum_{\rho\in\mathbf{R}} \beta_\rho = 1 \,, \tag{6.3}$$

a contradiction. **q.e.d.**

Theorem 6.3. $\{\bar{\boldsymbol{x}}^\rho\}_{\rho\in\mathbf{R}}$ *be a family of truncations and let* $\left\{\bar{\boldsymbol{\xi}}^\rho\right\}_{\rho\in\mathbf{R}}$ *be the corresponding set of imputations for* $\boldsymbol{v}$*. Then*

$$\mathcal{H} = \boldsymbol{ConvH}\left\{\bar{\boldsymbol{\xi}}^\rho\right\}_{\rho\in\mathbf{R}} \tag{6.4}$$

is a vNM–Stable Set.

Proof: Follows from Theorem 5.6, Theorem 6.2, and Corollary 2.7.

q.e.d.

Remark 6.4. We call $\mathcal{H}$ a ***standard*** vNM–Stable set quite in accordance with the notation introduced in [10] and [11]. For, each $\bar{\boldsymbol{\xi}}^\rho$ is absolutely continuous with respect to $\boldsymbol{\lambda}^\rho$ with a density bounded by 1. $\mathcal{H}$ is also called ***truncated*** as its extremal elements are being generated by a simple truncation procedure applied to the $\boldsymbol{\lambda}^\rho$ ($\rho \in \mathbf{R}$).

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