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A note on ”Necessary and sufficient conditions for the perfect finite horizon folk theorem” [Econometrica, 63 (2): 425-430, 1995.]

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A note on "Necessary and sufficient conditions for the perfect finite horizon folk theorem" [Econometrica, 63 (2): 425-430, 1995.]

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Abstract: Smith (1995) presented a necessary and sufficient condition for the finite-horizon perfect folk theorem. In the proof of this result, the author constructed a family of five-phase strategy profiles to approach feasible and individually rational payoff vectors of the stage-game. These strategy profiles are not subgame perfect Nash equilibria of the finitely repeated game. I illustrate this fact with a counter-example. However, the characterization of attainable payoff vectors by Smith remains true. I provide an alternative proof.

Keywords: Finitely Repeated Games, Subgame Perfect Nash Equilibrium, Folk Theorem, Discount Factor.

JEL classification: C72, C73.

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1 Introduction

Benoit and Krishna (1984) proved a finite-horizon perfect folk theorem under two sufficient conditions on the stage-game. The first condition is the full-dimensionality defined in Fudenberg and Maskin (1986). A stage-game meets the full-dimensionality condition if the dimension of the set of feasible payoff vectors equals the number of players. The second condition of Benoit and Krishna (1984) is that each player receives at least two distinct payoffs at stage-game Nash equilibria. Smith (1995) generalized the result of Benoit and Krishna (1984) and provided a necessary and sufficient condition for the finite-horizon perfect folk theorem. Smith's condition is that the stage-game has recursively distinct Nash payoffs. This basically means that there exists a time horizon T such that each player receives at least two distinct payoffs at subgame perfect Nash equilibria of the T-fold repeated game.

In the proof of this result, and under the assumption that the stage-game has recursively distinct Nash payoffs, Smith constructed a family of five-phase strategy profiles to approximate feasible payoff vectors that dominate the effective minimax payoff vector of the stage-game. These strategy profiles are not subgame perfect Nash equilibria of the finitely repeated game. I illustrate this fact with a counter-example. However, the characterization of attainable payoff vectors by Smith remains true. I provide an alternative proof.

This note is organized as follows. Section 2 provides a counter-example and discusses the failure of Smith's (1995) proof. Section 3 recalls the model and formally states the finite-horizon perfect folk theorem of Smith (1995) and Section 4 provides an alternative proof the later result.

2 The counter-example

2.1 The stage-game

Consider the three-player stage-game G whose payoff matrix is given in Table 1. In the game G, player 1 chooses lines (a_1^1 or a_1^2), player 2 chooses columns (a_2^1 or a_2^2) and player 3 chooses matrices (a_3^1 or a_3^2).

The pure action profiles (a_1^2, a_2^1, a_3^2) and (a_1^1, a_2^2, a_3^2) are Nash equilibria of the stage-game G and each player receives distinct payoffs at those action profiles. Therefore, this game has recursively distinct Nash payoffs, see Definition 1. Players 1 and 2 have the

	a_3^1		a_3^2	
	a_2^1	a_2^2	a_2^1	a_2^2
a_1^1	0 0 0	2 2 0	2 2 2	3 3 3
a_1^2	0 0 0	1 1 0	2 2 1	2 2 2

Table 1: Payoff matrix of the stage-game G.

same utility function and are therefore equivalent.[3] The pure effective minimax payoff of player 1 (respectively player 2) equals 1 and is uniquely provided by the action profile $w^1 = w^2 = (a_1^2, a_2^2, a_3^1)$.[4]

The payoff vector $u = (\frac{3}{2}, \frac{3}{2}, \frac{3}{2})$ is feasible and strictly dominates the effective minimax payoff vector $\widetilde{\mu} = (1, 1, 1)$. The payoff vector u is therefore approachable by means of subgame perfect Nash equilibria of the finitely-repeated game with discounting; see Theorem 1.

In the proof of Theorem 1 of Smith (1995) which is stated in page 9 of this note, to approach the feasible payoff vector u, the author uses a five-phase strategy. I recall it below and show that it is not a subgame perfect Nash equilibrium profile.

2.2 The five-phase strategy of Smith

The strategy profile used by Smith (1995) employs the concept of payoff asymmetry family that I briefly recall below.

2.2.1 The payoff asymmetry family

The concept of payoff asymmetry family is introduced by Abreu et al. (1994). Such a family allows to suitably reward effective punishers after a punishment phase. In our example, the payoff vectors $x^1 = x^2 = (1.3, 1.3, 1.3)$ and $x^3 = (1.4, 1.4, 1.2)$ form a payoff asymmetry family relatively to u. Indeed, the payoff family $\{x^1, x^2, x^3\}$ meets the following requirements:

(A1) $x^i >> \widetilde{\mu}$ for all $i \in \{1, 2, 3\}$, [strict individually rationality]

[3]Player i is equivalent to player j in the game G if the utility function of player i is a positive affine transformation of the utility function of player j.

[4]The mixed effective minimax payoff of both players 1 and 2 also equals 1 and is uniquely provided by the pure action profile w^1.

(A2) $x_i^i < u_i$ for all $i \in \{1, 2, 3\}$, [target payoff domination]

and

(A3′) $x_i^i < x_i^j$ for all $i, j \in \{1, 2, 3\}$, $i \nsim j$.[5] [payoff asymmetry]

I should notice that (A3′) is an adjusted version of the original requirement (3) in Abreu et al. (1994) where the game meets the NEU (non-equivalent utility) property.

2.2.2 Length of phases

Let β^i be the best payoff vector of player i in the game G.
Let ω^i be worst payoff vector of player i in the game G.

Choose q such that for all i, $\omega_i^i + qx_i^i > \beta_i^i + 1$. Take $q = 4$.

Given q, choose r such that for all j with $j \nsim i$,

$q\omega_j^j + rx_j^i > \beta_j^j + rx_j^j + (q-1)u_j + 1.$ Take $r = 86$.

Take $t_h(q+r) = 3(q+r)$.

2.2.3 Smith's strategy

Let a be the outcome of a public randomization device that has an average payoff of $u = (\frac{3}{2}, \frac{3}{2}, \frac{3}{2})$.

Let $T \geq t_h(r+q)$ and σ be the strategy profile of the finitely-repeated game $G(T)$ described by the following five phases.

1. MAIN PATH: Play a until period $T - t_h(r+q)$. [If any player i deviates early, start 3; if some player deviates late, start 5.[6]]

2. GOOD RECURSIVE PHASE: Play the stage-game Nash equilibrium profile (a_1^1, a_2^2, a_3^2) till the end of the finitely-repeated game $G(T)$.

[5] The notation $i \nsim j$ means that player i is not equivalent to player j in the game G.
[6] A deviation is called late if it occurs during the final $q + r + t_h(r+q)$ periods of the repeated game; all others are called early deviations.

3. MINIMAX PHASE: Play w^i for q periods. [If player j (with $j \nsim i$) deviates, start 4.] Set $j \leftarrow i$.

4. REWARD PHASE: Play x^j for r periods. [If i deviates early, restart 3; if some player deviates late, start 5.]

5. BAD RECURSIVE NASH PHASE: Play the stage-game Nash equilibrium (a_1^2, a_2^1, a_3^2) until the end of the game.

2.2.4 A profitable deviation from σ

For all $k \geq 0$, let $T(k) = k + r + q + t_h(r + q)$. Let σ_1' be a strategy of player 1 in which player 1 deviates from a in the first period of the repeated game as well as at the beginning of each REWARD PHASE and plays her stage-game best response in each period of the MINIMAX PHASE. This deviation is profitable for large k. Indeed, if player 1 does not deviate from σ, she gets at most an expected payoff of $A(k) = \frac{1}{T(k)}\{\beta^1 + \frac{3(k+r+q-1)}{2} + 3t_h(r + q)\}$.

If she deviates and plays σ_1', she gets at least $B(k) = \frac{1}{T(k)}\{2(k - \lceil\frac{k-1}{q+1}\rceil - 2)\}$ where $\lceil\frac{k-1}{q+1}\rceil$ is the smallest integer greater than or equal to $\frac{k-1}{q+1}$.

As k goes to ∞, $A(k)$ goes to $\frac{3}{2}$ and $B(k)$ goes to $\frac{8}{5}$.

This means that for sufficiently long time horizon T and sufficiently high discount factor δ, the strategy profile σ is not a Nash equilibrium of the finitely-repeated game $G(\delta, T)$ and therefore not a subgame perfect Nash equilibrium of $G(\delta, T)$.

2.3 Intuition behind the failure of Smith's proof

Denote by w^i the profile of stage-game mixed actions at which player i receives her effective minimax payoff.[7]

If the utility function of player i in the stage-game G is equivalent to that of another player, say player j, then the effective minimax payoff of player i might be strictly greater than her minimax payoff and player i might even have a strict incentive to deviate from

[7] The strategy profile defined in page 12 has a slightly different interpretation. Indeed at that profile, a player whose utility function is equivalent to that of player i might have incentive to deviate. If she does so, she receives at most her stage-game effective minimax payoff.

w^i. Indeed, it might be the case that only player j plays a stage-game best response at the profile w^i. In that case, it is not convenient to use the five-phase strategy profile of Smith (1995) to approximate a payoff vector in which player i receives strictly less than her best response payoff at w^i.

Indeed, during the third phase of the five-phase strategy of Smith (1995), player i is minimaxed using the stage-game action profile w^i where she might not be at a stage-game best response. In addition, during this phase, deviations by any player who is equivalent to player i (including player i) are ignored. As in the counter-example above, player i might find it profitable to deviate from an ongoing path (either from the MAIN PATH or from the REWARD PHASE) to push her fellow players to start the MINIMAX PHASE where she is punished.

This failure of is not minor in the sense that for any specification of the action profile to be used in the MINIMAX PHASE where $i = 1$, at least one player will find it strictly profitable to deviate from the five-phase strategy of Smith (1995).

Denote a MINIMAX PHASE where $i = 1$ by MP(1).

Indeed, if for a given specification $\overline{w}^1$ of the stage-game profile to be repeatedly played in the phase MP(1) the strategy profile σ is a subgame perfect Nash equilibrium of the finitely-repeated game $G(T)$, then at $\overline{w}^1$ player 3 has to play a_3^1 with strictly positive probability. Otherwise the punishment payoff of player 1 in the minimax phase MP(1) will be strictly greater than player 1's entry in the target payoff vector $u = (\frac{3}{2}, \frac{3}{2}, \frac{3}{2})$. Given the choice of player 3 in $\overline{w}^1$, player 2 has to play a_2^2 with probability 1 at the profile $\overline{w}^1$. Otherwise she will find it strictly profitable to deviate from σ and repeatedly play her best response at $\overline{w}^1$ during the phase MP(1), as she will not be punished if she does so. Given that player 2 plays a_2^2 with probability 1 in the profile $\overline{w}^1$, player 1 has to play a_1^1 with probability 1 in the profile $\overline{w}^1$. Otherwise she will find it strictly profitable to deviate and to play a_1^1 with probability 1 in each round of the phase MP(1). Therefore, only convex sums of payoff vectors $(2, 2, 0)$ and $(3, 3, 3)$ are possible payoff to the profile of actions $\overline{w}^1$. This implies that player 1 receives an average payoff greater than or equal to 2 in each round of the minimax phase MP(1), which is strictly greater than her entry in the target equilibrium payoff u. This contrasts the idea of punishment behind a minimax phase, which is to deter deviations. A player should not find it profitable to start a minimax phase.

The above reasoning teaches that the incentives of any player who is not at her stage-game best response at the profile $\overline{w}^1$ have to be controlled during a minimax phase. Note that this reasoning is not possible in case the stage-game meets the NEU (non-equivalent utility) property of Abreu et al. (1994) or the full dimensionality property of Fudenberg and Maskin (1986). Under those conditions, no player is equivalent to another and therefore any stage-game profile at which player i plays a stage-game best response and receives her minimax payoff is suitable for a minimax phase, see Benoit and Krishna (1984) and Smith (1993) for the finite-horizon perfect folk theorem under those properties.

The methods of Benoit and Krishna (1984) and Smith (1993) do not easily extend to games where some players have equivalent utility functions. But still, the finite-horizon perfect folk theorem for games that possibly violate the NEU condition as stated in Smith (1995) holds. This note provides a clear proof.

In the next section I recall Smith (1995)'s model and state his finite-horizon perfect folk theorem. I provide the proof of the latter theorem in Section 4.

3 Smith's model

3.1 The stage-game

Let $G = \langle A_i, \pi_i; i = 1, ..., n\rangle$ be a finite normal form n-player game, where A_i is player i's finite set of actions, and $\pi_i : A = \times_{i=1}^{n} A_i \to \mathbb{R}$ is player i's utility function. Let M_i be player i's mixed action set and let $M = \times_{i=1}^{n} M_i$. For any profile of actions $a \in A$, set $\pi(a) = (\pi_1(a), \ldots, \pi_n(a))$. For any profile of mixed actions $\mu = (\mu_1, \ldots, \mu_n) \in M$, denote by $\pi(\mu) = (\pi_1(\mu), \ldots, \pi_n(\mu))$ the vector of expected payoffs of players.

Let $\mathcal{J} = \{1, \ldots, n\}$ be the set of players. Let $\mathcal{J}(i)$ be the set of players whose utility function is a positive affine transformation of π_i. Let

$$\widetilde{\mu}_i = \min_{\mu\in M} \max_{j\in\mathcal{J}(i)} \max_{\mu'_j} \pi_i(\mu'_j, \mu_{-j})$$

be the effective minimax payoff of player i. Normalize the utilities functions of players such that $\widetilde{\mu}_i = 0$ for all i. Let $F = co\{\pi(\mu) : \mu \in M\}$ be convex hull of the set of expected payoff vectors. Let $F^* = \{u \in F : u_i > 0, \text{ for all } i\}$ be the feasible and strictly rational set.

Given a subset of players $\mathcal{J}' = \{j_1, \ldots, j_m\} \subset \mathcal{J}$ and their mixed actions profile

$$a_{\mathcal{J}'} = (a_{j_1}, \ldots, a_{j_m}) \in M_{j_1} \times M_{j_2} \times \cdots \times M_{j_m} \equiv M_{\mathcal{J}'}, \tag{1}$$

let $G(a_{\mathcal{J}'})$ be the induced $(n-m)-$player game for players $\mathcal{J}\backslash\mathcal{J}'$ obtained from G when the actions of players $\mathcal{J}'$ are fixed to $a_{\mathcal{J}'}$.

Define a Nash decomposition of the game G as an increasing sequence of $h \geq 0$ nonempty subset of players from $\mathcal{J}$, namely

$$\{\varnothing = \mathcal{J}_0 \subsetneq \mathcal{J}_1 \subsetneq \cdots \subsetneq \mathcal{J}_h \subseteq \mathcal{J}\}, \tag{2}$$

so that for $g = 1, \ldots, h$, actions $e_{\mathcal{J}_{g-1}}, f_{\mathcal{J}_{g-1}} \in M_{\mathcal{J}_{g-1}}$ exist with a pair of Nash payoff vectors $y(e_{\mathcal{J}_{g-1}})$ of $G(e_{\mathcal{J}_{g-1}})$ and $y(f_{\mathcal{J}_{g-1}})$ of $G(f_{\mathcal{J}_{g-1}})$ different exactly for players in $\mathcal{J}_g\backslash\mathcal{J}_{g-1}$, ie

$$y(e_{\mathcal{J}_{g-1}})_i \neq y(f_{\mathcal{J}_{g-1}})_i \tag{3}$$

for all $i \in \mathcal{J}_g\backslash\mathcal{J}_{g-1}$.

Definition 1 *The game G has recursively distinct Nash payoffs if there is a Nash decomposition with $\mathcal{J}_h = \mathcal{J}$.*

3.2 The finitely-repeated game

Let $G(\delta, T)$ be the $T-$fold repeated game in which players discount the futur with the parameter $\delta < 1$. Smith (1995) assumed that the monitoring structure is perfect so that each player can condition her current action on the past actions of all players.

A strategy behavioral strategy of player i in the repeated game $G(\delta, T)$ is a $T-$tuple $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{i,T})$ where for all $t \in \{1, \ldots, T\}$ and past history $h^t \in A^{t-1}$ (with $A^0 = \varnothing$), $\alpha_{it}(h^t)$ is the (possibly mixed) action that player i intends to play at time t if she observes h^t. The objective function of player i in the finitely-repeated game $G(\delta, T)$ is the expected discounted sum of her stage-game payoffs:

$$\pi_{iT}^{\delta}(\alpha) := \tfrac{1-\delta}{1-\delta^T} \textstyle\sum_{t=1}^{T} \delta^{t-1} \pi_{it}(\alpha)$$

where $\pi_{it}(\alpha)$ is player i's expected payoff at period t with the strategy profile $\alpha = (\alpha_1, \ldots, \alpha_n)$. The strategy profile α is a Nash equilibrium of the finitely-repeated game $G(\delta, T)$ if for all player i, α_i maximizes the objective function $\pi_{iT}^{\delta}(\cdot, \alpha_{-i})$ of player i.

The strategy profile α is a subgame perfect Nash equilibrium of the finitely-repeated game $G(\delta, T)$ if after any history h^t, the restriction $\alpha_{|h^t}$ of α to the history h^t is a Nash equilibrium of the remaining game.

Let

$$V(\delta,T)=\{\pi^{\delta}_{T}(\alpha)=(\pi^{\delta}_{1T}(\alpha),\cdots,\pi^{\delta}_{nT}(\alpha))\mid \alpha \text{ is a subgame perfect Nash equilibrium of } G(\delta,T)\}$$

be the set of subgame perfect Nash equilibrium payoff vectors of the finitely-repeated game $G(\delta, T)$.

Theorem 1 (See Smith (1995)) *Suppose that the stage-game G has recursively distinct Nash payoffs. Then for the finitely-repeated game $G(\delta, T)$, $\forall u \in F^*$ and $\forall \varepsilon > 0$, $\exists T_0 < \infty$ and $\delta_0 < 1$ so that $T \geq T_0$ and $\delta \in [\delta_0, 1] \;\Rightarrow\; \exists v \in V(\delta, T)$ with $\|u - v\| < \varepsilon$.*

4 A proof of Smith's folk theorem

I follow Smith (1995) and assume that players condition their choices on the outcome of a publicly observed exogenous continuous random variable. For simplicity, I also assume that the discount factor equals 1. The later assumption is without loss of generality as it does not change the incentives of players if those are strict.

The main ingredient of the proof of Theorem 1 is a multi-level reward path function whose existence is guaranteed by the recursively distinct Nash payoffs condition, see Lemma 1. The multi-level reward path function allows to independently leverage the behavior of players near the end of the finitely-repeated game, no matter if there are or not players who have equivalent utility functions. In addition, and backwardly, this multi-level reward path function allows to leverage the behavior of players at any stage of the finitely-repeated game.

Gossner (1995) used similar method to prove a finite-horizon perfect folk theorem with unobservable mixed strategies. The advantage of Lemma 1 is that it does not require the dimension of the set of feasible payoff vectors to equal the number of players neither each player to have at least two distinct payoffs at Nash equilibria of the stage-game.

Denote by $G(T)$ the T-fold finitely repeated game $G(\delta, T)$ where the discount factor δ equals 1. In the game $G(T)$, the utility of player i at the behavioral strategy α is

$$\pi^T_i(\alpha) := \lim_{\delta\to 1} \pi^{\delta}_{iT}(\alpha)$$

which is equal to the payoff average $\frac{1}{T}\sum_{t=1}^{T}\pi_{iT}(\alpha)$. Let

$$V(1,T) := \{\pi^T(\alpha) = (\pi^T_1(\alpha),\cdots,\pi^T_n(\alpha)) \mid \alpha \text{ is a SPNE of } G(T)\}$$

be the set of subgame perfect Nash equilibrium payoff vectors of the finitely repeated game $G(T)$ and let

$$AP = \{u \in F \mid \forall \varepsilon > 0, \exists T_0 < \infty \text{ so that } T \geq T_0 \Rightarrow \exists v \in V(1,T) \text{ with } \|u - v\| < \varepsilon\}$$

be the set of feasible payoff vectors that are approachable by means of subgame perfect Nash equilibria of finite repetitions of the stage-game G. To prove Theorem 1, we will show that $F^* \subseteq AP$.

Lemma 1 *Suppose that the stage-game G has recursively distinct Nash payoffs. Then there exists $\phi > 0$ such that for all $p \geq 0$, there exists $r_p > 0$ and*

$$\theta^p : \{0,1\}^n \cup \{(-1,\cdots,-1)\} \to M^{r_p} := M \times \cdots \times M$$

such that for all $\gamma \in \{0,1\}^n \cup \{(-1,\cdots,-1)\}$, $\theta^p(\gamma)$ is a play path generated by a subgame perfect Nash equilibrium of the repeated game $G(r_p)$. Furthermore, for all $i \in N$, $\gamma, \gamma' \in \{0,1\}^n$ we have

$$\pi_i^{r_p}[\theta^p(1,\gamma_{-i})] - \pi_i^{r_p}[\theta^p(0,\gamma_{-i})] \geq \phi \tag{4}$$

$$\pi_i^{r_p}[\theta^p(\gamma)] - \pi_i^{r_p}[\theta^p(-1,\cdots,-1)] \geq \phi \tag{5}$$

$$|\pi_i^{r_p}[\theta^p(\gamma)] - \pi_i^{r_p}[\theta^p(\gamma_{\mathcal{J}(i)}, \gamma'_{\mathcal{J}\setminus\mathcal{J}(i)})]| < \frac{1}{2^p}. \tag{6}$$

This lemma says that, if the stage-game G has recursively distinct Nash payoffs, then we can (almost-) independently leverage the behavior of each player near the end of the game. This lemma also allows to construct credible punishment schemes and to approximate any feasible payoff vector that dominates the effective minimax payoff vector by means of SPNE of the finitely repeated game.

Assume that the finitely repeated game will last with a reward phase where players are rewarded with respect to their behavior in the earlier stage of the repeated game, that players are informed that the reward path to be used is a SPNE path $\theta^p(\gamma)$ of the repeated game $G(r_p)$. Furthermore, assume that γ is initialized to the value $(1,\cdots,1)$ and that each player has the possibility to update her entry in the vector γ each time where a player whose utility function is not equivalent to her deviates. Inequality (4) says that, given the profile γ_{-i} of players of the block $\mathcal{J}\setminus\{i\}$, player i strictly prefers the path $\theta^p(1,\gamma_{-i})$ to the path $\theta^p(0,\gamma_{-i})$. Inequality (6) ensures that the incentives of players of different equivalence classes are almost independent for sufficiently large p. The strategy constructed in the proof of Theorem 1 does not allow a player to strategically improve her payoff by giving to players whose utilities' function are equivalent to her a chance to update their entries in the vector γ.

Consider for instance the stage-game whose payoff matrix is given by Table 1. In that game, player 1 and player 2 have the same utility function and are therefore equivalent. Figure 3 below displays the relative position of the payoff vectors $\pi^{r_p}[\theta^p(\gamma)]$ where $\gamma \in \{0,1\}^3 \cup \{(-1,-1,-1)$. The path $\theta^p(-1,-1,-1)$ will allow to deter deviations that occurs near the end of the game.

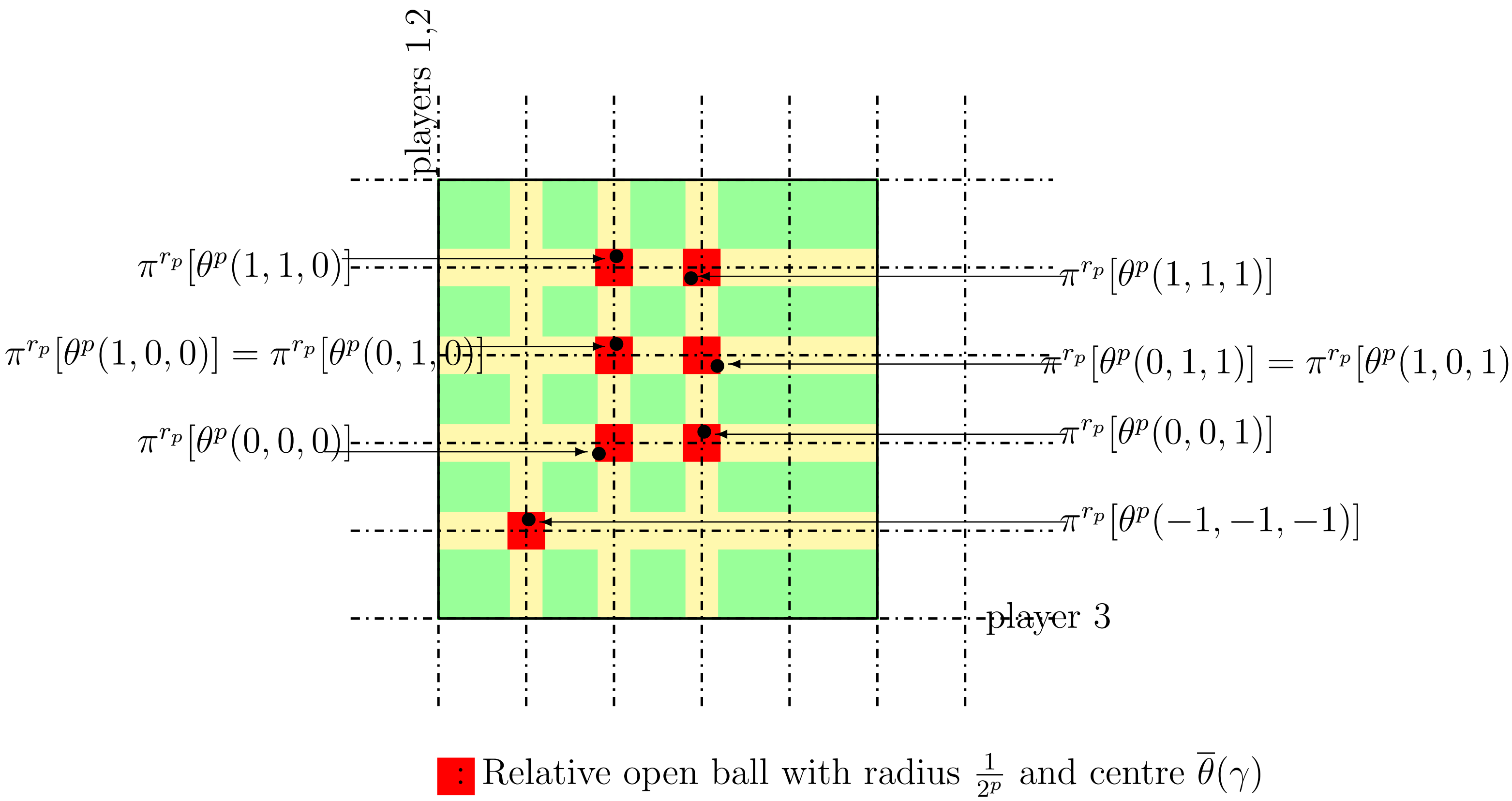


Figure 3: An example of relative position of the payoff vectors $\pi^{r_p}[\theta^p(\gamma)]$.

A detailed proof of Lemma 1 is presented in Section 5.

Proof of Smith (1995)'s folk theorem.

Let u be a feasible payoff vector that lies in the relative interior of F^*, and let a be the outcome of a public randomization device that has an expected payoff vector of u.

Obtain ϕ, r_1 and θ^1 with $p = 1$ from the Lemma 1. Let $q_1 > 0$ and $q_2 > 0$ such that

$$0 < q_1\pi_i(w^i) + q_2 r_1 \pi_i^{r_1}[\theta^1(1,\cdots,1)] < \frac{q_1 + q_2 r_1}{2} u_i \tag{7}$$

and

$$-2\rho + \frac{q_1}{2}u_i > 0 \; for \; all \; i \in N. \tag{8}$$

Given q_1, q_2 and r_1, choose r such that

$$-2(q_1 + q_2 r_1)\rho + r\phi > 0. \tag{9}$$

Given q_1 q_2, r_1 and r, choose $p_0 > 0$ such that

$$\frac{q_2 r_1}{2}u_i - \frac{r}{2^{p_0}} > u_i - \frac{r}{2^{p_0}} > 0 \tag{10}$$

Apply the Lemma 1 to p_0 and obtain r_{p_0} and θ^{p_0}. Update $q_1 \leftarrow r_{p_0} q_1; q_2 \leftarrow r_{p_0} q_2 r_1$; $r \leftarrow r_{p_0} r$. The quantities ϕ, θ^1, q_1, q_2, r, r_1 and θ^{p_0} are such that

$$0 < q_1 \pi_i(w^i) + q_2 \pi_i^{r_1}[\theta^1(1, \cdots, 1)] < \frac{q_1 + q_2}{2}u_i \tag{11}$$

$$-2(q_1 + q_2)\rho + r\phi > 0 \tag{12}$$

$$-2\rho + \frac{q_1 + q_2}{2}u_i - \frac{r}{2^{p_0}} > 0 \tag{13}$$

and

$$u_i - \frac{r}{2^{p_0}} > 0 \; for \; all \; i \in N. \tag{14}$$

The $T-$period equilibrium outcome sequence is

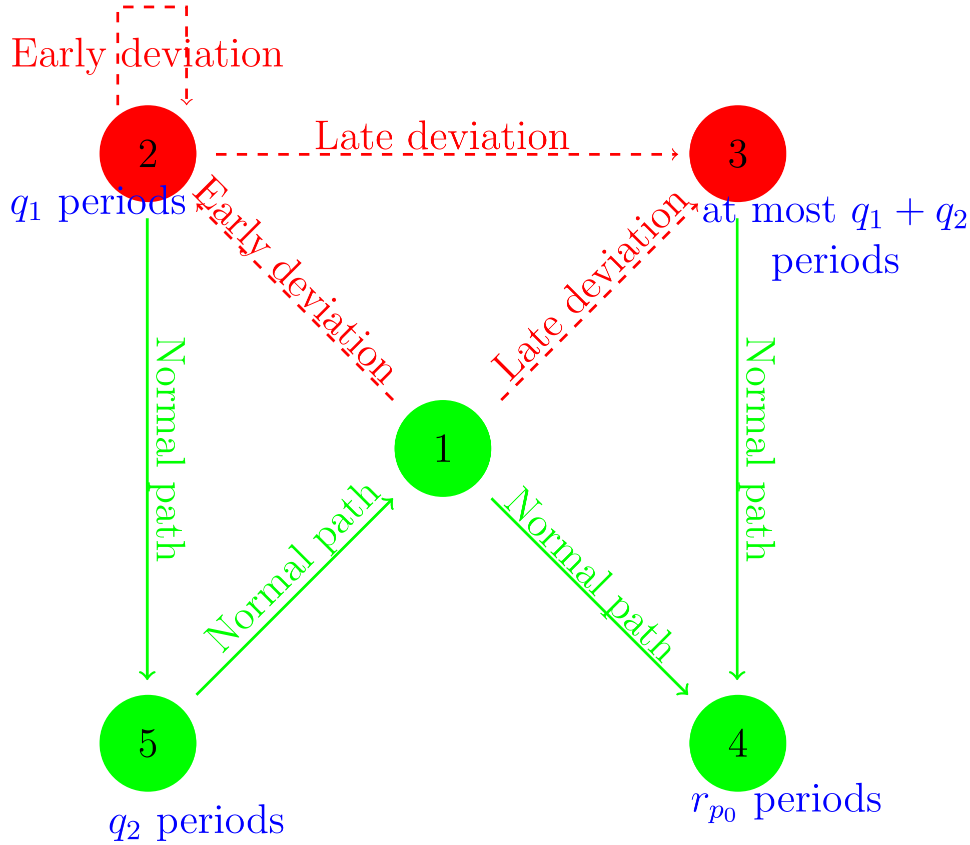
$$a, \ldots, a; \theta^{p_0}(1, \cdots, 1)$$

where a is played for $T - r$ periods and the path $\theta^{p_0}(1, \cdots, 1)$ is of length r.
Now I describe the subgame perfect Nash equilibrium σ of the finitely-repeated game that supports the equilibrium path. For all $i \in \mathcal{J}$, let w^i be a stage-game action profile such that

$$\max_{j \in \mathcal{J}(i)} \max_{m_j \in M_j} u_i(m_j, w^i_{-j}) = 0.$$

At the action profile w^i, no player of the block $\mathcal{J}(i)$ has to be at a best response. Playing a best response to the action profile w^i, a player of the block $\mathcal{J}(i)$ receives at most her effective minimax payoff.

Set $\gamma = (1, \cdots, 1)$. From now on, call a deviation late if it occurs during the final $q_1 + q_2 + r$ periods of the finitely-repeated game $G(T)$; all others are called early deviations. The strategy profile σ involves 5 phases and can be graphed as follows:



1. MAIN PATH: Play a until period $T-r$. [If any player i deviates early, start 2; if some player deviates late, start 3.] Go to 4.

2. MINIMAX PHASE $P(i)$: During this phase, each player $j \in \mathcal{J}(i)$ has to play her action w_j^i while each player of the block $N \backslash \mathcal{J}(i)$ can play whatever action she wants. This phase last for q_1 periods. [If any player $j \in \mathcal{J}(i)$ deviates early, restart 2.; if any player $j \in \mathcal{J}(i)$ deviates late, start 3.]
At the end of this phase, for all player $j \notin \mathcal{J}(i)$, set $\gamma_j = 0$ if there is at least one period of the MINIMAX PHASE where player j played an action different to w_j^i and set $\gamma_j = 1$ otherwise. Go to 5.

3. LATE DEVIATION: Each player can play whatever action she want till period $T-r$. At period $T-r$, set $\gamma = (-1, \cdots, -1)$. Go to 4.

4. END OF THE GAME: Follow $\frac{r}{r_{p_0}}$ times a SPNE that supports the equilibrium path $\theta^{p_0}(\gamma)$.

5. SPE PHASE: Follow $\frac{q_2}{r_1}$ times the SPNE of the game $G(r_1)$ whose play path is $\theta^1(1,\cdots,1)$. Go back to 1.

For sufficiently large time horizon T, the strategy profile σ is a subgame perfect Nash equilibrium of the finitely repeated game $G(T)$

In the following, call a player $j \notin \mathcal{J}(i)$ effective punisher if $\gamma_j = 1$ at the end of the MINIMAX PHASE $P(i)$. I prove the following:

A) It is strictly dominant for any player $j \notin \mathcal{J}(i)$ to be effective punisher during a MINIMAX PHASE $P(i)$

B) No early deviation from the MINIMAX PHASE is profitable

C) No early deviation from the MAIN PATH is profitable

D) No late deviation is profitable

A) It is strictly dominant to be effective punisher during a MINIMAX PHASE

If player $j \notin \mathcal{J}(i)$ is effective punisher, she receives at least:

1. $-(q_1+q_2)\rho$ during the MINIMAX PHASE and the SPE PHASE;
2. some payoff U_j till period $T-r$;
3. $r \cdot \pi_i^{r_{p_0}}[\theta^{p_0}(1,\gamma_{-j})]$ in the last r periods of the repeated game.

In total she receives at least $-(q_1+q_2)\rho + U_j + r \cdot \pi_i^{r_{p_0}}[\theta^{p_0}(1,\gamma_{-j})]$.

If player j is not effective punisher, she receives at most:

1. $(q_1+q_2)\rho$ during the MINIMAX PHASE and the SPE PHASE;
2. the same payoff U_j till period $T-r$;
3. $r \cdot \pi_i^{r_{p_0}}[\theta^{p_0}(0,\gamma_{-j})]$ in the last r periods of the repeated game.

In total $(q_1+q_2)\rho + U_j + r \cdot \pi_i^{r_{p_0}}[\theta^{p_0}(0,\gamma_{-j})]$ which is less than or equal to $(q_1+q_2)\rho + U_j + r \cdot \pi_i^{r_{p_0}}[\theta^{p_0}(1,\gamma_{-j})] - r\phi$, see inequality (4). As $-2(q_1+q_2)\rho + r\phi > 0$, it is strictly dominant for any player $j \notin \mathcal{J}(i)$ to be effective punisher.

B) No early deviation from the MINIMAX PHASE is profitable

If player $i \in \mathcal{J}(i)$ deviates early from the MINIMAX PHASE, she receives at most:

1. 0 in the deviation period;

2. $q_1\pi_i(w^i)+q_2\pi_i^{r_1}[\theta^1(1,\cdots,1)]$ in the new MINIMAX PHASE and the following SPE PHASE;

3. some payoff U_i till the end of the game.

If player i does not deviates, she receives at least:

1. $q_1\pi_i(w^i)+q_2\pi_i^{r_1}[\theta^1(1,\cdots,1)]+u_i$ till the end of the SPE PHASE;

2. the payoff $U_i-\frac{r}{2^{p_0}}$ till the end of the game.

As $u_i-\frac{r}{2^{p_0}}>0$, no early deviation from the MINIMAX PHASE is profitable.

C) No early deviation from the MAIN PATH is profitable

If player i deviates early from the MAIN PATH, she receives at most:

1. ρ in the deviation period;

2. $q_1\pi_i(w^i)+q_2\pi_i^{r_1}[\theta^1(1,\cdots,1)]$ in the MINIMAX PHASE and the SPE PHASE;

3. some payoff U_i till period $T-r$;

4. $r\cdot\pi_i^{r_{p_0}}[\theta^{p_0}(\gamma_{\mathcal{J}(i)},1,\cdots,1)]$ in phase 4.

In total $\rho+q_1\pi_i(w^i)+q_2\pi_i^{r_1}[\theta^1(1,\cdots,1)]+U_i+r\cdot\pi_i^{r_{p_0}}[\theta^{p_0}(\gamma_{\mathcal{J}(i)},1,\cdots,1)]$ which is strictly less than $\rho+\frac{q_1+q_2}{2}u_i+U_i+r\cdot\pi_i^{r_{p_0}}[\theta^{p_0}(\gamma_{\mathcal{J}(i)},1,\cdots,1)]$, see inequality (11).

If player i does not deviates, she receives at least:

1. $-\rho+(q_1+q_2)\cdot u_i$ till the end of the SPE PHASE;

2. the same payoff U_i till period $T-r$;

3. $r\cdot\pi_i^{r_{p_0}}[\theta^{p_0}(\gamma)]$ in phase 4.

In total $-\rho+(q_1+q_2)\cdot u_i+U_i+r\cdot\pi_i^{r_{p_0}}[\theta^{p_0}(\gamma)]$ which is strictly greater than $-\rho+(q_1+q_2)\cdot u_i+U_i+r\cdot\pi_i^{r_{p_0}}[\theta^{p_0}(\gamma_{\mathcal{J}(i)},1,\cdots,1)]-\frac{r}{2^{p_0}}$, see inequality (6).

As $-2\rho+\frac{q_1+q_2}{2}u_i-\frac{r}{2^{p_0}}>0$, no early deviation from the MAIN PATH is profitable.

D) No late deviation is profitable

If from an ongoing path (MAIN PATH or MINIMAX PHASE) player i deviates late, then she receives at most:

1. $(q_1+q_2)\rho$ till the beginning of the END OF THE GAME;

2. $r \cdot \pi_i^{r_{p_0}}[\theta^{p_0}(-1, \cdots, -1)]$ in the END OF THE GAME.

If player i does not deviate, she receives at least:

1. $-(q_1 + q_2)\rho$ till the beginning of the END OF THE GAME;

2. $r \cdot \pi_i^{r_{p_0}}[\theta^{p_0}(\gamma)]$ in the END OF THE GAME, where $\gamma \in \{0, 1\}^n$.

As $r \cdot \pi_i^{r_{p_0}}[\theta^{p_0}(\gamma)] \geq r \cdot \pi_i^{r_{p_0}}[\theta^{p_0}(-1, \cdots, -1)] + r\phi$ and $r\phi > 2(q_1 + q_2)\rho$, no late deviation is profitable. This concludes the proof. ■

5 Proof of intermediate results

In this section I proceed to the proof of Lemma 1. I first show that under the recursively distinct Nash payoffs condition, each player has many continuation equilibrium payoffs, which is necessary for the construction of our multi-level reward path function.

Lemma 2 *Suppose that the stage-game G has recursively distinct Nash payoffs. Then there exists $T_0 > 0$ such that for all $T > T_0$, each player receives at least two distinct payoffs at SPNE of $G(T)$.*

Proof of Lemma 2.

Let $\{\varnothing = \mathcal{J}_0 \subsetneq \mathcal{J}_1 \subsetneq \cdots \subsetneq \mathcal{J}_h = \mathcal{J}\}$ be the Nash decomposition of G.
I show by induction that for all $l \leq h$ there exists $T_{0,l} > 0$ such that each player of $\mathcal{J}_l$ receives distinct payoffs at SPNE of $G(T)$ for all $T > T_{0,l}$.
Players of the block $\mathcal{J}_1$ receive distinct payoffs at Nash equilibria of G. Therefore, the property holds for $l = 1$. Let $l < h$ such that $T_{0,l}$ exists. Let $i \in \mathcal{J}_{l+1} \backslash \mathcal{J}_l$, and let μ be a Nash equilibrium profile of $G(\mu_{\mathcal{J}_l})$ in which player i receives a payoff that is different to her unique stage-game Nash equilibrium payoff. Let η_1 and η_2 be two SPNE play paths of $G(T_{0,l} + 1)$ where each player of $\mathcal{J}_l$ receives distinct payoffs. The path $\eta^i = (\mu, \eta_1, \eta_2 \cdots, \eta_1, \eta_2)$ is a SPNE play path. At η^i, player i receives a payoff that is different to her stage-game Nash equilibrium payoff which is also a SPNE payoff. The conjunction lemma (see Benoit and Krishna (1984)) guarantee the existence of $T_{0,l+1}$. ■

Proof of Lemma 1.

The set AP is non-empty and convex and therefore has a relative interior point x. Let $\phi > 0$ such that the relative ball $\widetilde{B}(x, 5\phi n)$ is included in AP. For all $\gamma \in \{-1, 0, 1\}^n$ and $j \in \mathcal{J}$, let

$$\theta_j(\gamma) = x_j - \phi|\mathcal{J}(j)| + 3\phi \textstyle\sum_{j' \in \mathcal{J}(j)} \gamma_{j'}$$

and

$$\theta(\gamma) = (\theta_1(\gamma), \cdots, \theta_n(\gamma)).$$

For all $\gamma \in \{0,1\}^n$, we have

$$\theta_i(1, \gamma_{-i}) - \theta_i(0, \gamma_{-i}) = 3\phi;$$

$$\theta_i(\gamma) - \theta_i(-1, \cdots, -1) \geq 3\phi$$

and

$$\|\theta(\gamma) - x\| < 5n\phi.$$

Furthermore, since each player receives distinct payoffs within the set AP and players within the same equivalence class $\mathcal{J}(i)$ have equal entry at the payoff vector $\theta(\gamma)$, we have that

$$\theta(\gamma) \in \widetilde{B}(x, 5\phi n) \subseteq AP.^{8}$$

Let $p \geq 0$ and $\varepsilon = \frac{1}{2}\min\{\phi, \frac{1}{2^p}\}$. For each $\gamma \in \{0,1\}^n \cup \{(-1, \cdots, -1)\}$, there exists $T_{0\gamma p} < \infty$ so that for all $T \geq T_{0\gamma p}$, there exists $\alpha^{\gamma p}$ a subgame perfect Nash equilibrium of the repeated game $G(T)$ such that $\|\pi_T(\alpha^{\gamma p}) - \theta(\gamma)\| < \varepsilon$.

Take $r_p = \max\{T_{0\gamma p} \mid \gamma \in \{0,1\}^n \cup \{(-1, \cdots, -1)\}\}$. For all $\gamma \in \{0,1\}^n \cup \{(-1, \cdots, -1)\}$ and $p \geq 0$, let $\theta^p(\gamma)$ be the SPNE play path generated by the SPNE $\alpha^{\gamma p}$ of the repeated game $G(r_p)$. ■

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[8]Indeed, from Lemma 2, each player receives distinct payoffs at subgame perfect Nash equilibria of finite repetitions of the stage-game G and, as corollary of the conjunction lemma (see Lemma 3.2 in Benoit and Krishna (1984)), each subgame perfect Nash equilibrium payoff vector of a finite repetition of the stage-game G with no discount belongs to AP.

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