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A Note on Dynamic Consistency in Ambiguous Persuasion*

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Abstract

Beauchêne, Li, and Li (2019) show that ambiguous persuasion leads to new interim equilibria with higher ex ante value for the Sender compared to the standard Bayesian persuasion. However, in their equilibrium the strategy of the Receiver is in general not ex ante optimal. This note, defines rectangular beliefs over the full state space in the same setting as Beauchêne et al. (2019) and shows that given rectangular beliefs the Receiver behaves dynamically consistent. Hence, the interim equilibrium of Beauchêne et al. (2019) is an ante equilibrium, as well.

Key words and phrases: Bayesian Persuasion, Ambiguity Aversion, Dynamic Consistency
JEL subject classification: C73, D81

1 Introduction

Beauchêne, Li, and Li (2019), henceforth BLL, introduce ambiguity in the standard Bayesian persuasion setting of Kamenica and Gentzkow (2011) by allowing the Sender to choose a set of communication devices. Each communication device can generate a signal realization or message that reveals information about an unknown (risky) state $\omega \in \Omega$. Sender and Receiver only observe the signal realization without knowing which communication device generated the signal realization. Therefore ambiguity about the communication device induces ambiguity about the risky state ω. BLL characterize conditions under which the interim equilibrium under ambiguous persuasion leads to an higher ex ante expected payoff of the Sender as Bayesian persuasion. They focus on interim equilibria since ambiguity leads to dynamically inconsistent behavior of the Receiver. But they use the ex ante expected utility of the interim equilibrium to calculate the value of ambiguous persuasion and to compare it with the value of Bayesian persuasion, even if the interim equilibrium

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is not an ex ante equilibrium. Furthermore, they define rectangularity for beliefs over Ω and show that their definition of rectangularity can only be satisfied if ambiguity reduces to risk. Therefore, they claim that there is no gain of ambiguous persuasion compared to Bayesian persuasion if the players behave dynamically consistent.[1]

This note shows, that defining rectangularity over the full state space allows for rectangular ambiguous beliefs. These beliefs take the dependence of the ambiguous signal and the ex ante risky state into account. Therefore, the Sender can gain from an ambiguous strategy even if dynamic consistency is satisfied.

2 Model of Beauchêne, Li, and Li (2019)

We follow the model of BLL. The persuasion game consists of two players, a Sender (he) and a Receiver (she). The utility of both players depends of the state of the world $\omega \in \Omega$ and an action $a \in A$ chosen by the Receiver. We denote with $u(a, \omega)$ and $\nu(a, \omega)$ the utility of Receiver and Sender, respectively. Ω and A are compact subsets of the Euclidean space. Ex ante the state ω is unknown and both players have the same prior state belief $p_0 \in \Delta\Omega$ i.e. ex ante there is no ambiguity about the state.[2] The Sender tries to persuade the Receiver by choosing a signal that reveals information about the state. A signal consists of a finite set of signal realizations or messages M and a set of communication devices $\Pi = \{\pi_k\}_{k \in K}$.[3] Each communication device is a distribution over the set of messages for each $\omega \in \Omega$, i.e. $\pi_k(\cdot|\omega) \in \Delta M$ for all $\omega \in \Omega$. We assume that π_k have common support for all $k \in K$. The only difference to the standard Bayesian persuasion setting is the fact that the Sender chooses a set of communication devices instead of one communication device. Which of the communication devices generates the observed message is ambiguous to the Receiver and the Sender. After observing a message m, the Receiver updates his prior state belief using Bayes' rule. Since she does not know which communication device generated the message, she updates p_0 with respect to each communication device π_k. This leads to the following set of posterior state beliefs given the message $m \in M$

$$P_m = \left\{ p_m^{\pi_k}(\cdot) \in \Delta\Omega : p_m^{\pi_k}(\cdot) = \frac{p_0(\cdot)\pi_k(m|\cdot)}{\int_{\omega \in \Omega} p_0(\omega)\pi_k(m|\omega)\,\mathrm{d}\omega}, \pi_k \in \Pi \right\}.$$

Sender and Receiver have maxmin preferences à la Gilboa and Schmeidler (1989) i.e. they maximize their worst case expected utility. BLL assume that the Receiver maximizes her

[1] See Proposition 5 of Beauchêne et al. (2019).

[2] Our definition of belief differs from the one of BLL. To avoid confusion we use the term state belief whenever we refer to beliefs in the sense of BLL.

[3] Please note that we deviate from the model of BLL by defining Π as the set of communication devices. BLL define Π as the convex hull of the set of communication devices. Since Sender and Receiver have maxmin preferences, the minimization problems over $\{\pi_k\}$ or $\mathrm{co}\big(\{\pi_k\}\big)$ coincide.

interim worst case expected utility given that message m was observed

$$U(a, P_m) = \min_{p_m \in P_m} \mathbb{E}_{p_m}(u(a, \omega)). \tag{1}$$

As usual in the persuasion literature, we assume that the Receiver chooses the Sender preferred action if she has multiple maximizers. We denote with $\hat{a}(P_m)$ the (Sender preferred) best response of the Receiver after observing message m. The Sender chooses the signal (M, Π) that maximizes his worst case expected utility

$$\sup_{(M,\Pi)} \min_{\pi \in \Pi} \mathbb{E}_{p_0}\Big[\mathbb{E}_\pi\big[\nu(\hat{a}(P_m), \omega)|\omega\big]\Big].$$

BLL explore an interim equilibrium. Since the choice of the Sender is realized at the ex ante stage, the interim and ex ante mazimization problem of the Sender coincide. However, the interim best response of the Receiver is in general not ex ante optimal. Intuitively, ex ante the Receiver can hedge against ambiguity by playing any strategy that is constant with respect to the signal realization. The following example from BLL shows that ambiguity can lead to an higher expected payoff for the Sender. Furthermore, we show that the interim equilibrium strategy of the Receiver is not ex ante ante optimal.

Example 1 *Assume that the Sender is a brand name drug producer. The Receiver is a physician who can choose between prescribing the brand name drug* ($a = a_B$) *or an generic competitor of it* ($a = a_G$)*. The Sender always prefers that the Receiver prescribes the brand name drug. The Receiver preferences depends of the state which reflects the effectiveness of the generic drug. If the generic drug is effective* ($\omega = \omega_e$) *the Receiver prefers the generic drug if not* ($\omega = \omega_i$) *she prefers the brand name drug. The payoffs of Sender and Receiver are given by the following table.*

	ω_e	ω_i
a_B	$(1, 2)$	$(1, 2)$
a_G	$(0, 3)$	$(0, -1)$

Figure 1: Payoffs (S, R)

Sender and Receiver have a common ex ante state belief $p_0 = \mathbb{P}(\omega = \omega_i) < \frac{1}{4}$.[4] *BLL show that the optimal Bayesian persuasion signal is such that the set of messages consists of two messages* $M = \{i, e\}$ *and the communication device is given by*

$$\pi(e|\omega_e) = \frac{1 - 4p_0}{1 - p_0} = 1 - \pi(i|\omega_e)$$

[4]Please note, that we deviate from the Illustrating Example of BLL (page 317) by assuming $u_H = 3$, $u_L = -1$ and $c = 1$, which is consistent with the payoffs in Example 2 of BLL.

$$\pi(e|\omega_i) = 0 = 1 - \pi(i|\omega_i).$$

Then, the ex ante expected payoff of the Sender given the optimal Bayesian persuasion signal is $\mathbb{P}(m = i) \cdot 1 + \mathbb{P}(m = e) \cdot 0 = 4p_0 < 1$.

Furthermore, they construct an ambiguous persuasion signal that leads to an higher expected payoff of the Sender. Let $M = \{e, i\}$ *as before. The set of communication devices* $\Pi = \{\pi, \pi'\}$ *is given by a communication device that always reveals the true state and a communication device that do the opposite, i.e.*

$$\pi(i|\omega_i) = 1 = 1 - \pi(e|\omega_i) \qquad \pi(i|\omega_e) = 0 = 1 - \pi(e|\omega_e)$$
$$\pi'(i|\omega_i) = 0 = 1 - \pi'(e|\omega_i) \qquad \pi'(i|\omega_e) = 1 = 1 - \pi'(e|\omega_e).$$

Given this ambiguous communication device the interim state beliefs are $P_m = \{(0, 1), (1, 0)\}$ *for* $m \in \{e, i\}$. *Due to the maxmin preferences the worst case interim belief for both messages always give probability* 1 *to the ineffective state* ω_i. *Therefore, the Receiver choose the brand name drug with probability* 1. *Then, the ex ante expected payoff of the Sender is* 1, *which is greater as the ex ante expected payoff under optimal Bayesian persuasion.*

However, the ex ante expected payoff of the Receiver is given by

$$\min_{\pi \in \Pi} \sum_{m \in \{e,i\}} \big(\pi(m|\omega_e) + \pi(m|\omega_i)\big) \mathbb{E}_{p_m^\pi}\big(u(a_m, \omega)\big)$$

where a_m *denotes her action after observing the messages* m. *If she chooses the brand name drug independently of the signal that she will observe, her ex ante expected payoff equals*

$$2 \cdot \mathbb{P}(\omega = \omega_e) + 2 \cdot \mathbb{P}(\omega = \omega_i) = 2.$$

Her expected payoff if she always choose the generic drug is

$$3 \cdot \mathbb{P}(\omega = \omega_e) + 1 \cdot \mathbb{P}(\omega = \omega_i) = 3 - 4p_0.$$

Since $p_0 < \frac{1}{4}$, *the interim optimal strategy of always prescribing the brand name drug is not ex ante optimal and the Receiver behaves dynamically inconsistent.*

3 Dynamic Consistent Belief Formation Process

One way to rule out dynamically inconsistent behavior in dynamic settings with multiple priors are rectangular belief sets as defined e.g. by Epstein and Schneider (2003) and Pahlke (2018). Both paper define rectangular beliefs over the full state space. In this section we show that defining beliefs over an adequate state space, allows the definition of non-singleton rectangular belief sets. Then given a rectangular belief set the Receiver behaves dynamically consistent and the interim equilibrium of BLL is an ex ante equilibrium and therefore a Perfect Bayesian Equilibrium.

In the ambiguous persuasion setting M is part of the strategy of the Sender. The next proposition shows that the Sender chooses without loss of generality $M \subset A \cup b(A)$ where $b(A)$ is a duplicated set of A such that there exists a bijection $b(\cdot)$ between A and $b(A)$. Given this result, we can define rectangular ex ante beliefs over $\Omega \times (A \cup b(A))$.

Proposition 1 *Let $(M, \Pi) \in \operatorname{argsup} \min_{\pi \in \Pi} \mathbb{E}_{p_0} [\mathbb{E}_{\pi} [\nu(\hat{a}(P_m), \omega)|\omega]]$. Then, there exist (M', Π') with $M' \subset A \cup b(A)$ where $b(\cdot) : A \mapsto b(A)$ is a bijection between A and $b(A)$ and $\Pi' = \{\pi'_1, \pi'_2\}$ such that (M', Π') generates the same value for the Sender as (M, Π).*

The intuition of the result is as follows. Kamenica and Gentzkow (2011) show that for Bayesian persuasion it is without loss of generality that $M \subset A$. BLL show that ambiguous persuasion increases the value for the Sender compared to Bayesian persuasion only if the Sender uses a signal with synonyms. Synonyms are messages that copy the meaning of another message i.e. they induce the same posterior state belief set or best response of the Receiver. To allow the use of synonyms we have to duplicate the message space. Therefore $M \subset A \cup b(A)$.

Proof of Proposition 1. Corollary 1 of BLL shows that there exists π_1 and π_2 such that $(M, \{\pi_1, \pi_2\})$ generates the same value as (M, Π). Hence, we have to show that (M', Π') generates the same value as $(M, \{\pi_1, \pi_2\})$. We first look at the case where the Sender does not use synonyms.

i) Sender does not use synonyms.

   Since $(M, \{\pi_1, \pi_2\})$ does not use synonyms, there does not exist $m, m' \in M$ with $m \neq m'$ such that $\hat{a}(P_m) = \hat{a}(P_{m'})$. Remember, that p_m^{π} denotes the posterior state belief of the Receiver given the message m and the communication device π. Furthermore, $\hat{a}(P_m)$ denotes Receivers best response given message $m \in M$ and the communication devices $\{\pi_1, \pi_2\}$. Since $(M, \{\pi_1, \pi_2\})$ does not use synonyms, there exists at most one $m \in M$ for each $a \in A$ such that $a = \hat{a}(P_m)$. We define $\bar{\pi}_i(\cdot|\omega) \in \Delta M'$ with $M' \subset A$ such that

$$\bar{\pi}_i(a|\omega) = \begin{cases} \pi_i(m|\omega) & \text{if } \exists m \in M \text{ with } a = \hat{a}(P_m) \\ 0 & \text{otherwise} \end{cases} .$$

   Then, the posterior state belief $p_m^{\pi_i}$ equals the posterior state belief $p_a^{\bar{\pi}_i}$ if $a = \hat{a}(P_m)$. Therefore, $(M, \{\pi_1, \pi_2\})$ and $(M', \{\bar{\pi}_1, \bar{\pi}_2\})$ generate the same set of posterior state beliefs and the same best response of the Receiver. Since the best response does not change, the value of the Sender is the same for both signals.

i) Sender uses synonyms.

If $(M, \{\pi_1, \pi_2\})$ uses synonyms we can split M in M_1 and M_2 such that there exist a bijection between M_1 and M_2 and $M_1 \cup M_2 = M$. Then $(M_1, \{\hat{\pi}_1, \hat{\pi}_2\})$ with

$$\hat{\pi}_i(m|\omega) = \frac{\pi_i(m|\omega)}{\sum_{m \in M_1} \pi_i(m|\omega)}$$

defines a signal that does not use synonyms. Hence, as in case 1 there exists $(M_1', \{\bar{\pi}_1, \bar{\pi}_2\})$ with $M_1' \subset A$ that generates the same value as $(M_1, \{\hat{\pi}_1, \hat{\pi}_2\})$. Similar one can define the restriction of π_i to M_2 and find $(M_2', \{\tilde{\pi}_1, \tilde{\pi}_1\})$ with $M_2' \subset b(A)$, that generates the same value as M_2 and the restriction of π_i to M_2. Then, $(M', \{\pi_1', \pi_2'\})$ with $M' = M_1' \cup M_2'$ and

$$\pi_i'(a|\omega) = \begin{cases} \bar{\pi}_i(a|\omega) \sum_{m \in M_1} \pi_i(m|\omega) & \text{if } a \in A \\ \tilde{\pi}_i(a|\omega) \sum_{m \in M_2} \pi_i(m|\omega) & \text{if } a \in b(A) \end{cases}$$

generates the same value as $(M, \{\pi_1, \pi_2\})$.

■

Proposition 1 shows that without loss of generality we can assume that $M \subset A \cup b(A)$. Then, a strategy of the Sender (M, Π) is completely characterized by Π if we assume that he chooses $\pi_k \in \Pi$ with common finite support on $A \cup b(A)$, i.e. $\text{supp}(\pi_k(\cdot|\omega)) = \text{supp}(\pi_j(\cdot|\omega)) \subset A \cup b(A)$ for all $\omega \in \Omega$ and $j, k \in K$ and $\text{supp}(\pi_j(\cdot|\omega))$ is finite. For the rest of the paper we will use the term strategy of the Sender for such a Π. Furthermore we denote with $\text{supp}(\Pi)$ the support of $\pi_k \in \Pi$ for all $k \in K$.

3.1 Rectangular Beliefs

Given the results from the previous section we can define beliefs over the full state space $\Omega \times (A \cup b(A))$.

Definition 1 *For a strategy Π of the Sender we define the* ***set of ex ante beliefs*** *of the Receiver as*

$$\Phi_\Pi^0 = \Bigg\{ \rho^k \in \Delta(\Omega \times (A \cup b(A))) : \exists \pi_k \in \Pi \ s.t.$$
$$\rho^k(\omega, m) = \begin{cases} p_0(\omega)\pi_k(m|\omega) & \text{if } m \in supp(\Pi) \\ 0 & \text{otherwise} \end{cases} \forall k \in K \Bigg\}.$$

At the interim stage the Receiver observes a message $m \in \text{supp}(\Pi)$. The information structure of the game can be represented by the following partitions

$$\mathcal{F}_0 = \Omega \times (A \cup b(A),)$$
$$\mathcal{F}_1 = \Big\{ \{\Omega \times m\}_{m \in A \cup b(A)} \Big\}.$$

Then, given an observation $\hat{m} \in \operatorname{supp}(\Pi)$ the Receiver updates his ex ante belief set prior-by-prior using Bayes' rule, i.e. she updates each prior belief in Φ^0_Π with Bayes' rule

$$\rho^k|_{\hat{m}} = \rho^k((\omega, m)|\hat{m}) = \frac{p_0(\omega)\pi_k(m|\omega)}{\int_{\omega' \in \Omega} p_0(\omega')\pi_k(m|\omega')\,\mathrm{d}\omega}$$

if $m = \hat{m}$ and 0 otherwise. Then, the set of updated beliefs given $\hat{m}$ is

$$\mathrm{Bay}(\Phi^0_\Pi|\hat{m}) = \{\rho^k|_{\hat{m}} \text{ with } \rho^k \in \Phi^0_\Pi\}.$$

Please note that $\rho^k((\omega, m)|\hat{m}) = 0$ for $\hat{m} \notin \operatorname{supp}(\Pi)$. Furthermore, $\rho^k((\omega, \hat{m})|\hat{m}) = p^{\pi_k}_{\hat{m}}(\omega)$ for all ω. Therefore, Receivers maximization problem at the interim stage given our definition of beliefs is the same as in BLL.

To define rectangularity we need updated beliefs and marginal beliefs of observing $m \in A \cup b(A)$. The marginal belief of observing m under ρ^k is

$$\rho^k(\Omega, m) = \int_{\omega \in \Omega} \rho^k(\omega, m)\,\mathrm{d}\omega = \int_{\omega \in \Omega} p_0(\omega)\pi_k(m|\omega)\,\mathrm{d}\omega$$

Now we can define rectangularity. Intuitively, using rectangular ex ante beliefs allows each player to combine the common ex ante information p_0 and their conjecture about the strategy of the opponent taking their knowledge about the information structure of the game into account.

Definition 2 *The pasting of ex ante belief $\bar{\rho} \in \Phi^0_\Pi$ and a collection of updated beliefs $(\rho|_{\hat{m}})_{\hat{m}} \in \bigtimes_{\hat{m} \in supp(\Pi)} Bay(\Phi^0_\Pi|\hat{m})$ is defined as*[5]

$$\begin{aligned}\bar{\rho} \circ (\rho|_{\hat{m}})_{\hat{m}}(\omega, m) &:= \int_{\hat{m} \in A \cup b(A)} \bar{\rho}(\Omega, \hat{m})\rho(\omega, m|\hat{m})\,\mathrm{d}\hat{m} \\ &= \Big(\int_{\omega' \in \Omega} p_0(\omega')\bar{\pi}(m|\omega')\,\mathrm{d}\omega\Big)\frac{p_0(\omega)\pi(m|\omega)}{\int_{\omega' \in \Omega} p_0(\omega')\pi(m|\omega')\,\mathrm{d}\omega}.\end{aligned}$$

The set of ex ante beliefs is called ***rectangular*** *(or stable under pasting) if it contains all pasting of an arbitrary ex ante belief $\bar{\rho} \in \Phi^0_\Pi$ and arbitrary interim beliefs $(\rho|_{\hat{m}})_{\hat{m}}$, i.e.*

$$\bar{\rho} \circ (\rho|_{\hat{m}})_{\hat{m}}(\cdot) \in \Phi^0_\Pi$$

for all $\bar{\rho} \in \Phi^0_\Pi$ and $(\rho|_{\hat{m}})_{\hat{m}} \in \bigtimes_{\hat{m} \in supp(\Pi)} Bay(\Phi^0_\Pi|\hat{m})$.

If Φ^0_Π is not rectangular one can always construct the smallest set which is rectangular and contains Φ^0_Π. We call this set the rectangular hull and denote it with $\mathrm{rect}(\Phi^0_\Pi)$. Furthermore, simple calculations show that $\mathrm{Bay}(\Phi^0_\Pi|\hat{m}) = \mathrm{Bay}(\mathrm{rect}(\Phi^0_\Pi)|\hat{m})$. The same holds for

[5]Please note, that the pasting is always well defined due to the common support assumption. Furthermore, the second equality follows since $\rho(\omega, m|\hat{m}) = 0$ if $m \neq \hat{m}$.

the set of marginal beliefs under Φ_Π^0 and under $\text{rect}(\Phi_\Pi^0)$. For a more detailed explanation of the construction and the properties of the rectangular hull please see Pahlke (2018).

3.2 Dynamic Consistency

Finally, we show that rectangularity implies dynamically consistent behavior of the Receiver.

Proposition 2 *Assume that the Receiver has rectangular ex ante beliefs. Then, the interim equilibrium of BLL is an ex ante equilibrium, as well.*

Proof. The Sender does not observe any information between his ex ante and interim choice. Therefore, his ex ante maximization problem is the same as his interim maximization problem. We only have to show that the Receivers interim best response of BLL is an interim and ex ante best response given rectangular beliefs. Remember that $p_{\hat{m}}^{\pi_k}(\cdot) = \rho^k((\cdot,\hat{m})|\hat{m})$ for all $\hat{m} \in \text{supp}(\Pi)$ and that the set of Bayesian updates given Φ_Π^0 or $\text{rect}(\Phi_\Pi^0)$ are the same. Therefore, the interim best response given the state beliefs of BLL is an interim best response given rectangular beliefs, as well. Furthermore, we can rewrite the interim expected utility of the Receiver after observing the message $\hat{m}$ of Equation 1 as

$$\min_{\rho(\cdot|\hat{m})\in \text{Bay}(\text{rect}(\Phi_\Pi^0)|\hat{m})} \mathbb{E}_{\rho(\cdot|\hat{m})}(u(a,\omega)).$$

His ex ante expected utility is

$$\min_{\rho\in\text{rect}(\Phi_\Pi^0)} \int_{\hat{m}\in A\cup b(A)} \rho(\Omega,\hat{m})\mathbb{E}_{\rho(\cdot|\hat{m})}(u(a,\omega))\,\mathrm{d}\hat{m}.$$

We first show the following relation of ex ante and interim worst case expected utility. Let ρ^* denote the ex ante worst case belief given rectangular beliefs. Then,

$$\begin{aligned}&\int_{\hat{m}\in A\cup b(A)} \rho^*(\Omega,\hat{m})\mathbb{E}_{\rho^*(\cdot|\hat{m})}(u(a,\omega))\,\mathrm{d}\hat{m}\\ &\qquad= \int_{\hat{m}\in A\cup b(A)} \rho^*(\Omega,\hat{m}) \min_{\rho(\cdot|\hat{m})\in \text{Bay}(\text{rect}(\Phi_\Pi^0)|\hat{m})} \mathbb{E}_{\rho(\cdot|\hat{m})}(u(a,\omega))\,\mathrm{d}\hat{m} \qquad (2)\end{aligned}$$

To proof Equation 2 we first show that the left hand side is greater equal the right hand side.

$$\begin{aligned}&\int_{\hat{m}\in A\cup b(A)} \rho^*(\Omega,\hat{m}) \underbrace{\mathbb{E}_{\rho^*(\cdot|\hat{m})}(u(a,\omega))}_{\geq \min_{\rho(\cdot|\hat{m})\in \text{Bay}(\text{rect}(\Phi_\Pi^0)|\hat{m})} \mathbb{E}_{\rho(\cdot|\hat{m})}(u(a,\omega))} \mathrm{d}\hat{m}\\ &\geq \int_{\hat{m}\in A\cup b(A)} \rho^*(\Omega,\hat{m}) \min_{\rho(\cdot|\hat{m})\in \text{Bay}(\text{rect}(\Phi_\Pi^0)|\hat{m})} \mathbb{E}_{\rho(\cdot|\hat{m})}(u(a,\omega))\,\mathrm{d}\hat{m}\end{aligned}$$

To proof the other direction, let $\rho'(\cdot|\hat{m})$ be the worst case belief given that she observed $\hat{m}$. Then, due to rectangularity, there exist a $\bar{\rho}$ in $\mathrm{rect}(\Phi^0_\Pi)$ such that $\rho^* \circ (\rho'|_{\hat{m}})_{\hat{m}} = \bar{\rho}$. Furthermore rectangularity implies, that $\bar{\rho}(\cdot|\hat{m}) = \rho'(\cdot|\hat{m})$ and $\bar{\rho}(\Omega, \hat{m}) = \rho^*(\Omega, \hat{m})$ for all $\hat{m}$.

$$
\begin{aligned}
\int_{\hat{m}\in A\cup b(A)} \rho^*(\Omega, \hat{m}) \mathbb{E}_{\rho^*(\cdot|\hat{m})}(u(a,\omega))\,\mathrm{d}\hat{m} &\leq \int_{\hat{m}\in A\cup b(A)} \bar{\rho}(\Omega, \hat{m}) \mathbb{E}_{\bar{\rho}(\cdot|\hat{m})}(u(a,\omega))\,\mathrm{d}\hat{m} \\
&= \int_{\hat{m}\in A\cup b(A)} \rho^*(\Omega, \hat{m}) \mathbb{E}_{\rho'(\cdot|\hat{m})}(u(a,\omega))\,\mathrm{d}\hat{m} \\
&= \int_{\hat{m}\in A\cup b(A)} \rho^*(\Omega, \hat{m}) \min_{\rho(\cdot|\hat{m})\in \mathrm{Bay}(\mathrm{rect}(\Phi^0_\Pi)|\hat{m})} \mathbb{E}_{\rho(\cdot|\hat{m})}(u(a,\omega))\,\mathrm{d}\hat{m}
\end{aligned}
$$

Combining both directions proofs Equation 2. Finally we show that an interim best response of the Receiver is an ex ante best response, as well. We denote with $\hat{a}_{\hat{m}}$ the (Sender preferred) interim best response of the Receiver given message $\hat{m}$, i.e.

$$
\min_{\rho(\cdot|\hat{m})\in \mathrm{Bay}(\Phi^0_\Pi|\hat{m})} \mathbb{E}_{\rho(\cdot|\hat{m})}(u(\hat{a}_{\hat{m}},\omega)) \geq \min_{\rho(\cdot|\hat{m})\in \mathrm{Bay}(\Phi^0_\Pi|\hat{m})} \mathbb{E}_{\rho(\cdot|\hat{m})}(u(a_{\hat{m}},\omega))
$$

for any arbitrary $a_{\hat{m}} \in A$ and all $\hat{m} \in \mathrm{supp}(\Pi)$. We have to show that $(\hat{a}_{\hat{m}})_{\hat{m}\in\mathrm{supp}(\Pi)}$ is ex ante optimal. Since $\rho(\Omega, \hat{m}) \geq 0$ for all $\hat{m} \in \mathrm{supp}(\Pi)$ and $\rho(\Omega, \hat{m}) = 0$ for all $\hat{m} \notin \mathrm{supp}(\Pi)$, Equation 2 implies

$$
\begin{aligned}
&\min_{\rho\in\mathrm{rect}(\Phi^0_\Pi)} \int_{\hat{m}\in A\cup b(A)} \rho(\Omega, \hat{m}) \mathbb{E}_{\rho(\cdot|\hat{m})}(u(a_{\hat{m}},\omega))\,\mathrm{d}\hat{m} \\
&\quad = \min_{\rho\in\mathrm{rect}(\Phi^0_\Pi)} \int_{\hat{m}\in A\cup b(A)} \rho(\Omega, \hat{m}) \min_{\rho'(\cdot|\hat{m})\in \mathrm{Bay}(\Phi^0_\Pi|\hat{m})} \mathbb{E}_{\rho(\cdot|\hat{m})}(u(a_{\hat{m}},\omega))\,\mathrm{d}\hat{m} \\
&\quad \leq \min_{\rho\in\mathrm{rect}(\Phi^0_\Pi)} \int_{\hat{m}\in A\cup b(A)} \rho(\Omega, \hat{m}) \min_{\rho(\cdot|\hat{m})\in \mathrm{Bay}(\Phi^0_\Pi|\hat{m})} \mathbb{E}_{\rho(\cdot|\hat{m})}(u(\hat{a}_{\hat{m}},\omega))\,\mathrm{d}\hat{m} \\
&\quad = \min_{\rho\in\mathrm{rect}(\Phi^0_\Pi)} \int_{\hat{m}\in A\cup b(A)} \rho(\Omega, \hat{m}) \mathbb{E}_{\rho(\cdot|\hat{m})}(u(\hat{a}_{\hat{m}},\omega))\,\mathrm{d}\hat{m}
\end{aligned}
$$

for any arbitrary $(a_{\hat{m}})_{\hat{m}\in\mathrm{supp}(\Pi)}$.

Hence, Receivers ex ante best response equals the interim best response and the interim equilibrium of Beauchêne, Li, and Li (2019) satisfies ex ante optimality. ■

4 Discussion

4.1 Endogenous Ambiguity in Cheap Talk

Kellner and Le Quement (2018) show that in a cheap talk setting an ambiguous strategy of the Sender can lead to an interim equilibrium that improves the ex ante expected payoff of Sender and Receiver. Similar to the Ambiguous persuasion setting, the equilibrium strategy of the Receiver is not ex ante optimal in the setting of Kellner and Le Quement (2018). However, similarly to the procedure described above defining beliefs and rectagularity over the full state space leads to a dynamically consistent equilibrium with the same strategies as in the interim equilibrium of Kellner and Le Quement (2018).

We have shown, that taking the full state space and the dependence of risky states and ambiguous signals into account, allows a definition of rectangular beliefs that leads to dynamically consistent behavior and still maintains the results of BLL and Kellner and Le Quement (2018). Therefore, ambiguity induces new equilibria in persuasion and cheap talk settings even if the players behave dynamically consistent.

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