

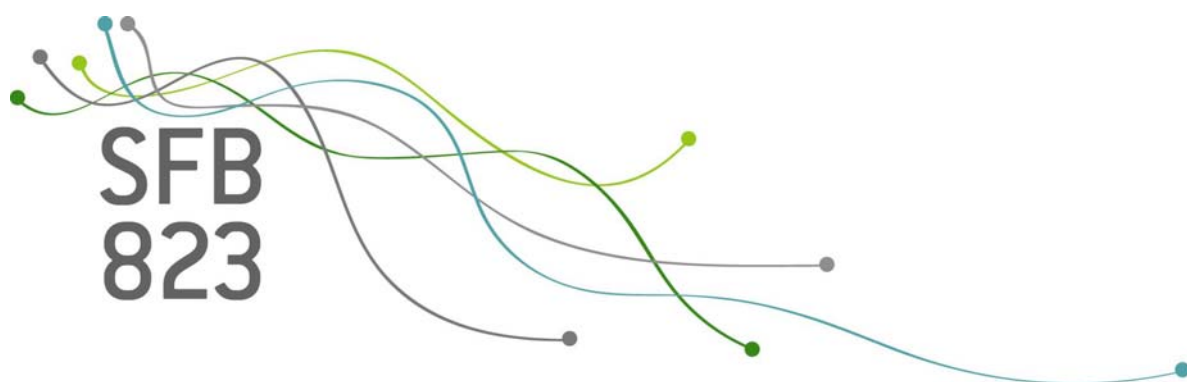
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Discussion Paper

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Sampling Distributions of Optimal Portfolio Weights and Characteristics in Low and Large Dimensions

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Abstract

Optimal portfolio selection problems are determined by the (unknown) parameters of the data generating process. If an investor want to realise the position suggested by the optimal portfolios he/she needs to estimate the unknown parameters and to account the parameter uncertainty into the decision process. Most often, the parameters of interest are the population mean vector and the population covariance matrix of the asset return distribution. In this paper we characterise the exact sampling distribution of the estimated optimal portfolio weights and their characteristics by deriving their sampling distribution which is present in terms of a stochastic representation. This approach possesses several advantages, like (i) it determines the sampling distribution of the estimated optimal portfolio weights by expressions which could be used to draw samples from this distribution efficiently; (ii) the application of the derived stochastic representation provides an easy way to obtain the asymptotic approximation of the sampling distribution. The later property is used to show that the high-dimensional asymptotic distribution of optimal portfolio weights is a multivariate normal and to determine its parameters. Moreover, a consistent estimator of optimal portfolio weights and their characteristics is derived under the high-dimensional settings. Via an extensive simulation study, we investigate the finite-sample performance of the derived asymptotic approximation and study its robustness to the violation of the model assumptions used in the derivation of the theoretical results.

Keywords: sampling distribution; optimal portfolio; parameter uncertainty; stochastic representation; high-dimensional asymptotics

1 Introduction

The solution to the optimal portfolio selection problems are determined by the parameters of the data generating process. In many cases, the optimal portfolio weights and their characteristics, like the portfolio mean, the portfolio variance, the value-at-risk (VaR), the conditional VaR (CVaR), etc, can be computed by only using the mean vector and the covariance matrix of the asset return distribution. More precisely, these relationships are summarized by the following five quantities:

$$V_{GMV} = \frac{1}{\mathbf{1}^\top \Sigma^{-1} \mathbf{1}}, \quad \mathbf{w}_{GMV} = \frac{\Sigma^{-1} \mathbf{1}}{\mathbf{1}^\top \Sigma^{-1} \mathbf{1}}, \quad R_{GMV} = \frac{\boldsymbol{\mu}^\top \Sigma^{-1} \mathbf{1}}{\mathbf{1}^\top \Sigma^{-1} \mathbf{1}}, \quad s = \boldsymbol{\mu}^\top \mathbf{Q} \boldsymbol{\mu}, \quad \mathbf{v} = \frac{\mathbf{Q} \boldsymbol{\mu}}{\boldsymbol{\mu}^\top \mathbf{Q} \boldsymbol{\mu}}, \quad (1.1)$$

where $\boldsymbol{\mu} = E(\mathbf{x})$ and $\Sigma = Var(\mathbf{x})$ are the mean vector and the covariance matrix of the p -dimensional asset return vector $\mathbf{x}$ and

$$\mathbf{Q} = \Sigma^{-1} - \frac{\Sigma^{-1} \mathbf{1} \mathbf{1}^\top \Sigma^{-1}}{\mathbf{1}^\top \Sigma^{-1} \mathbf{1}}. \quad (1.2)$$

The five quantities in (1.1) have an interesting financial interpretation. The p -dimensional vector $\mathbf{w}_{GMV}$ is the weights of the global minimum variance (GMV) portfolio, i.e. of the portfolio with the smallest variance, while R_{GMV} and V_{GMV} are the expected return and the variance of the GMV portfolio. The quantity s is the slope parameter of the efficient frontier, the set of all optimal portfolios following Markowitz's approach. This parameter together with R_{GMV} and V_{GMV} fully determine the location and the shape of the efficient frontier which is a parabola in the mean-variance space. Finally, the p -dimensional vector $\mathbf{v}$ is the weights of the so-called self-financing portfolio (cf. Korkie and Turtle [43]), i.e. the sum of its weights is equal to zero that is $\mathbf{1}^\top \mathbf{v} = 0$.

The five quantities in (1.1) determine the structure of many optimal portfolios, like the GMV portfolio, the mean-variance (MV) portfolio, the expected maximum exponential utility (EU) portfolio, the tangency (T) portfolio, the optimal portfolio that maximizes the Sharpe ratio (SR), the minimum VaR (MVar) portfolio, and the minimum CVaR (MCCVaR) portfolio, maximum value-of-return (MVoR) portfolio, maximum conditional value-of-return (MCCVoR) portfolio among others (see, e.g., Markowitz [45], Ingersoll [37], Jobson and Korkie [40], Alexander and Baptista [3], Alexander and Baptista [4], Okhrin and Schmid [49], Kan and Zhou [42], Frahm and Memmel [31], Bodnar et al. [21], Adcock [1], Woodgate and Siegel [57], Bodnar et al. [17], Bodnar et al. [11], Simaan et al. [54], Bodnar et al. [16], Bodnar et al. [9]). On the other hand, the quantities (1.1) cannot be directly used to compute the weights and the characteristics of these portfolios, since both $\boldsymbol{\mu}$ and Σ are unobservable parameters in practice.

As a result, an investor determines the optimal portfolios by replacing $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ in (1.1) with the corresponding sample estimators given by

$$\hat{\boldsymbol{\mu}} = \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i \quad \text{and} \quad \hat{\boldsymbol{\Sigma}} = \frac{1}{n-1} \sum_{i=1}^n (\mathbf{x}_i - \hat{\boldsymbol{\mu}})(\mathbf{x}_i - \hat{\boldsymbol{\mu}})^\top \quad (1.3)$$

given a sample of asset returns $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$. This approach leads to the sample or the so-called plug-in estimators of the optimal portfolios which are based on the corresponding sample estimators of (1.1) expressed as

$$\hat{V}_{GMV} = \frac{1}{\mathbf{1}^\top \hat{\boldsymbol{\Sigma}}^{-1} \mathbf{1}}, \quad \hat{\mathbf{w}}_{GMV} = \frac{\hat{\boldsymbol{\Sigma}}^{-1} \mathbf{1}}{\mathbf{1}^\top \hat{\boldsymbol{\Sigma}}^{-1} \mathbf{1}}, \quad \hat{R}_{GMV} = \frac{\hat{\boldsymbol{\mu}}^\top \hat{\boldsymbol{\Sigma}}^{-1} \mathbf{1}}{\mathbf{1}^\top \hat{\boldsymbol{\Sigma}}^{-1} \mathbf{1}}, \quad \hat{s} = \hat{\boldsymbol{\mu}}^\top \hat{\mathbf{Q}} \hat{\boldsymbol{\mu}}, \quad \hat{\mathbf{v}} = \frac{\hat{\mathbf{Q}} \hat{\boldsymbol{\mu}}}{\hat{\boldsymbol{\mu}}^\top \hat{\mathbf{Q}} \hat{\boldsymbol{\mu}}}, \quad (1.4)$$

with

$$\hat{\mathbf{Q}} = \hat{\boldsymbol{\Sigma}}^{-1} - \frac{\hat{\boldsymbol{\Sigma}}^{-1} \mathbf{1} \mathbf{1}^\top \hat{\boldsymbol{\Sigma}}^{-1}}{\mathbf{1}^\top \hat{\boldsymbol{\Sigma}}^{-1} \mathbf{1}}. \quad (1.5)$$

as well as to the sample (plug-in) estimators of the optimal portfolio weights.

The notion of the sampling distribution in portfolio allocation has recently been given large attention. Investors and researchers realize that the uncertainty introduced by using historical data needs to be integrated into the optimal portfolio decision process as well as to be properly assessed. The sampling distribution of the mean-variance portfolio was investigated as early as Jobson and Korkie [40], Britten-Jones [22], Okhrin and Schmid [49] where the distributions of estimated optimal portfolio weights were derived under the assumption of an independent sample of asset returns taken from a multivariate normal distribution. Moreover, both the asymptotic and finite-sample distributions of the estimated efficient frontier, the set of all mean-variance optimal portfolios, were obtained by Jobson [39], Bodnar and Schmid [19], Kan and Smith [41], and Bodnar and Schmid [20] among others, while Siegel and Woodgate [53] and Bodnar and Bodnar [7] presented its improved estimators and proposed a test of its existence. Some of these results were later extended to the high-dimensional setting in Frahm and Memmel [31], Glombeck [33], Bodnar et al. [17], Bodnar et al. [16].

The sample mean vector and the sample covariance matrix given by (1.3) have been used extensively in previous research (see, e.g., Britten-Jones [22], Memmel and Kempf [47], Okhrin and Schmid [50]) for estimating the asset return vector and its covariance matrix. These estimators appear to be consistent and the estimated optimal portfolios which involved them have desirable asymptotic properties when the portfolio dimension is considerably smaller than the sample size. However, they cannot be longer used when a high-dimensional portfolio is constructed due to their pure performance when the portfolio dimension is comparable to the

sample size. One of the issues lies in that the quantities (1.4) depend on the inverse covariance matrix whereas the sample inverse covariance matrix is not a consistent estimator in the high-dimensional settings (see, e.g., Bodnar et al. [10]). To cope with these limitations a number of improved estimators have been considered in the literature (cf., Efron and Morris [29], Jagannathan and Ma [38], Golosnoy and Okhrin [34], Frahm and Memmel [31], DeMiguel et al. [27], Rubio et al. [52], Yao et al. [58]).

We contribute to the existent literature by deriving the joint sampling distribution of the estimated five quantities in (1.4) which solely determine the structure of optimal portfolios. These results are then used to establish a unified approach for characterizing the sampling distributions of the estimated weights and the corresponding estimated characteristics of optimal portfolios. The goal is achieved by presenting the joint distribution of $(\hat{V}_{GMV}, \hat{\mathbf{w}}_{GMV}^\top, \hat{R}_{GMV}, \hat{s}, \hat{\mathbf{v}}^\top)^\top$ in terms of a very useful stochastic representation. A stochastic representation is a computationally efficient tool in statistics and econometrics to characterize the distribution of a random variable/vector which is widely used in both conventional and Bayesian statistics. While it plays a special role in the theory of elliptical distributions (c.f., Gupta et al. [36]), the stochastic representation is also a very popular method to generate random variables/vectors in computational statistics (see, e.g., Givens and Hoeting [32]). The applications of stochastic representations in the determination of the posterior distributions of estimated optimal portfolios can be found in Bodnar et al. [12] and Bauder et al. [6]. Finally, Zellner and Ando [59] among others argued that the direct Monte Carlo approach based on stochastic representations is a computationally efficient method to calculate Bayesian estimation. In the present paper, we employed the derived stochastic representation for $(\hat{V}_{GMV}, \hat{\mathbf{w}}_{GMV}^\top, \hat{R}_{GMV}, \hat{s}, \hat{\mathbf{v}}^\top)^\top$ in the derivation of their high-dimensional asymptotic distribution as well as in obtaining the high-dimensional asymptotic distribution of estimated optimal portfolios.

The rest of the paper is organized as follows. In Section 2, we derive the finite-sample joint distribution of $(\hat{V}_{GMV}, \hat{\mathbf{w}}_{GMV}^\top, \hat{R}_{GMV}, \hat{s}, \hat{\mathbf{v}}^\top)^\top$. This result is then used to establish the sampling distributions of the estimated optimal portfolio weights and their estimated characteristics in Section 3. Section 4 presents the asymptotic distributions of the estimated weights derived under the large-dimensional asymptotics. The results of the finite-sample performance of the asymptotic distributions and the robustness analysis to the distributional assumptions imposed on the data-generating process is investigated in Section 5, while final remarks are given in Section 6. The technical derivations are moved to the appendix (Section 7).

2 Exact sampling distribution of $\hat{V}_{GMV}$, $\hat{\mathbf{w}}_{GMV}$, $\hat{R}_{GMV}$, $\hat{s}$, and $\hat{\mathbf{v}}$

Throughout the paper we assume that the p -dimensional vectors of asset returns $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are independent and normally distributed with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, i.e. $\mathbf{x}_i \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ for $i = 1, \dots, n$. While Fama [30] argued that the distribution of monthly asset returns can be well approximated by the normal distribution, Tu and Zhou [55] found no significant impact of heavy tails on the performance of optimal portfolios.

The stochastic representation of $\hat{V}_{GMV}$, $\hat{\boldsymbol{\theta}}$, $\hat{R}_{GMV}$, $\hat{s}$, and $\hat{\boldsymbol{\eta}}$ is derived in a more general case, namely by considering linear combinations of $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\eta}}$ expressed as

$$\hat{\boldsymbol{\theta}} = \mathbf{L}\hat{\mathbf{w}}_{GMV} \quad \text{and} \quad \hat{\boldsymbol{\eta}} = \mathbf{L}\hat{\mathbf{v}},$$

where $\mathbf{L}$ is a $k \times p$ matrix of constant with $k < p - 1$ and $\text{rank}(\mathbf{L}) = k$. In the same way, we define the population counterparts of $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\eta}}$ given by

$$\boldsymbol{\theta} = \mathbf{L}\mathbf{w}_{GMV} \quad \text{and} \quad \boldsymbol{\eta} = \mathbf{L}\mathbf{v}.$$

Since $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$ are independently distributed (cf. Rencher [51]), the conditional distribution of $(\hat{V}_{GMV}, \hat{\boldsymbol{\theta}}^\top, \hat{R}_{GMV}, \hat{s}, \hat{\boldsymbol{\eta}}^\top)^\top$ under the condition $\hat{\boldsymbol{\mu}} = \tilde{\boldsymbol{\mu}}$ is equal to the distribution of $(\hat{V}_{GMV}, \hat{\boldsymbol{\theta}}^\top, \hat{R}_{GMV}, \hat{s}, \hat{\boldsymbol{\eta}}^\top)^\top$ with

$$\tilde{R}_{GMV} = \frac{\tilde{\boldsymbol{\mu}}^\top \hat{\boldsymbol{\Sigma}}^{-1} \mathbf{1}}{\mathbf{1}^\top \hat{\boldsymbol{\Sigma}}^{-1} \mathbf{1}}, \quad \tilde{s} = \tilde{\boldsymbol{\mu}}^\top \hat{\mathbf{Q}} \tilde{\boldsymbol{\mu}}, \quad \text{and} \quad \tilde{\boldsymbol{\eta}} = \frac{\mathbf{L} \hat{\mathbf{Q}} \tilde{\boldsymbol{\mu}}}{\tilde{\boldsymbol{\mu}}^\top \hat{\mathbf{Q}} \tilde{\boldsymbol{\mu}}}, \quad (2.1)$$

while their population counterparts we denote by:

$$\check{R}_{GMV} = \frac{\tilde{\boldsymbol{\mu}}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}}, \quad \check{s} = \tilde{\boldsymbol{\mu}}^\top \mathbf{Q} \tilde{\boldsymbol{\mu}}, \quad \text{and} \quad \check{\boldsymbol{\eta}} = \frac{\mathbf{L} \mathbf{Q} \tilde{\boldsymbol{\mu}}}{\tilde{\boldsymbol{\mu}}^\top \mathbf{Q} \tilde{\boldsymbol{\mu}}}. \quad (2.2)$$

Let the symbol $\stackrel{d}{=}$ denote the equality in distribution. In Theorem 2.1 we present a joint stochastic representation of $\hat{V}_{GMV}$, $\hat{\boldsymbol{\theta}}$, $\hat{R}_{GMV}$, $\hat{s}$, and $\hat{\boldsymbol{\eta}}$ which will be used in the next section to characterize the distribution of portfolio weights on the efficient frontier. The proof is given in the appendix.

Theorem 2.1. *Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ be independent and normally distributed with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, i.e. $\mathbf{x}_i \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ for $i = 1, \dots, n$ with $n > p$. Define $\mathbf{M} = (\mathbf{L}^\top, \tilde{\boldsymbol{\mu}}, \mathbf{1})^\top$ and assume that $\text{rank}(\mathbf{M}) = k + 2$. Let $\boldsymbol{\Sigma}$ be positive definite. Then, a joint stochastic representation of $\hat{V}_{GMV}$, $\hat{R}_{GMV}$, $\hat{\boldsymbol{\theta}}$, $\hat{s}$, and $\hat{\boldsymbol{\eta}}$ is given by*

$$(i) \hat{V}_{GMV} \stackrel{d}{=} \frac{V_{GMV}}{n-1} \xi_1;$$

$$(ii) \hat{R}_{GMV} \stackrel{d}{=} R_{GMV} + \sqrt{V_{GMV}} \left(\frac{z_1}{\sqrt{n}} + \sqrt{f} \frac{t_1}{\sqrt{n-p+1}} \right);$$

(iii)

$$\begin{aligned} \hat{\boldsymbol{\theta}} \stackrel{d}{=} & \boldsymbol{\theta} + \sqrt{V_{GMV}} \left(\frac{s\boldsymbol{\eta} + \mathbf{z}_2/\sqrt{n}}{\sqrt{f}} \frac{t_1}{\sqrt{n-p+1}} \right. \\ & \left. + \left(\mathbf{L}\mathbf{Q}\mathbf{L}^\top - \frac{(s\boldsymbol{\eta} + \mathbf{z}_2/\sqrt{n})(s\boldsymbol{\eta} + \mathbf{z}_2/\sqrt{n})^\top}{f} \right)^{1/2} \sqrt{1 + \frac{t_1^2}{n-p+1}} \frac{\mathbf{t}_2}{\sqrt{n-p+2}} \right); \end{aligned}$$

$$(iv) \hat{s} \stackrel{d}{=} (n-1) \left(1 + \frac{t_1^2}{n-p+1} \right) \frac{f}{\xi_2} \text{ with}$$

$$f = \frac{\xi_3}{n} + \left(s\boldsymbol{\eta} + \frac{\mathbf{z}_2}{\sqrt{n}} \right)^\top (\mathbf{L}\mathbf{Q}\mathbf{L}^\top)^{-1} \left(s\boldsymbol{\eta} + \frac{\mathbf{z}_2}{\sqrt{n}} \right); \quad (2.3)$$

(v)

$$\begin{aligned} \hat{\boldsymbol{\eta}} \stackrel{d}{=} & \frac{s\boldsymbol{\eta} + \mathbf{z}_2/\sqrt{n}}{f} + \frac{1}{\sqrt{f}} \left(\mathbf{L}\mathbf{Q}\mathbf{L}^\top - \frac{(s\boldsymbol{\eta} + \mathbf{z}_2/\sqrt{n})(s\boldsymbol{\eta} + \mathbf{z}_2/\sqrt{n})^\top}{f} \right)^{1/2} \\ & \times \left(\frac{1}{\sqrt{1 + \frac{t_1^2}{n-p+1}}} \frac{\mathbf{t}_2}{\sqrt{n-p+2}} \frac{t_1}{\sqrt{n-p+1}} + \left(\mathbf{I}_k + f \frac{\mathbf{t}_2 \mathbf{t}_2^\top}{n-p+2} \right)^{1/2} \frac{\mathbf{t}_3}{\sqrt{n-p+3}} \right) \end{aligned}$$

where $\xi_1 \sim \chi_{n-p}^2$, $\xi_2 \sim \chi_{n-p+2}^2$, $\xi_3 \sim \chi_{p-k-1;n\boldsymbol{\mu}^\top \mathbf{A}\boldsymbol{\mu}}^2$, $z_1 \sim \mathcal{N}(0, 1)$, $\mathbf{z}_2 \sim \mathcal{N}_k(\mathbf{0}, \mathbf{L}\mathbf{Q}\mathbf{L}^\top)$, $t_1 \sim t(n-p+1)$, $\mathbf{t}_2 \sim t_k(n-p+2)$, and $\mathbf{t}_3 \sim t_k(n-p+3)$ are mutually independent with

$$\mathbf{A} = \mathbf{Q} - \mathbf{Q}\mathbf{L}^\top (\mathbf{L}\mathbf{Q}\mathbf{L}^\top)^{-1} \mathbf{L}\mathbf{Q}. \quad (2.4)$$

The results of Theorem 2.1 provides a simple way how observations from the sample distribution of $\hat{V}_{GMV}$, $\hat{R}_{GMV}$, $\hat{\boldsymbol{\theta}}$, $\hat{s}$, and $\hat{\boldsymbol{\eta}}$ can be drawn. It is remarkable that in a single simulation run, random variables from well-known distributions should be simulated only. Moreover, the total dimension of independently simulated variables is equal to $(3k+5)$, which is considerably small when the direct simulation will be used that are based by drawing a $p \times p$ matrix from a Wishart distribution and a p -dimensional vector from a normal distribution. To this end, we point out that both the square roots in (iii) and (v) can be computed analytically which will further facilitate to speed up the simulation study. This observation is based on the following two equalities

$$(\mathbf{D} - \mathbf{b}\mathbf{b}^\top)^{1/2} = \mathbf{D}^{1/2} (\mathbf{I} - c\mathbf{D}^{-1/2}\mathbf{b}\mathbf{b}^\top\mathbf{D}^{-1/2}) \quad (2.5)$$

where $\mathbf{D}^{1/2}$ is a square root of $\mathbf{D}$ and $c = (1 - \sqrt{1 - \mathbf{b}^\top \mathbf{D}^{-1} \mathbf{b}}) / \mathbf{b}^\top \mathbf{D}^{-1} \mathbf{b}$ and

$$(\mathbf{I} + \mathbf{d}\mathbf{d}^\top)^{1/2} = \mathbf{I} + a\mathbf{d}\mathbf{d}^\top \quad (2.6)$$

where $a = (\sqrt{1 + \mathbf{d}^\top \mathbf{d}} - 1) / \mathbf{d}^\top \mathbf{d}$. Hence, it holds that

$$\begin{aligned} & \left(\mathbf{L}\mathbf{Q}\mathbf{L}^\top - \frac{(s\boldsymbol{\eta} + \mathbf{z}_2/\sqrt{n})(s\boldsymbol{\eta} + \mathbf{z}_2/\sqrt{n})^\top}{f} \right)^{1/2} \\ &= (\mathbf{L}\mathbf{Q}\mathbf{L}^\top)^{1/2} \left(\mathbf{I}_k - \frac{1 - \sqrt{\frac{\xi_3}{nf}}}{f - \frac{\xi_3}{n}} (\mathbf{L}\mathbf{Q}\mathbf{L}^\top)^{-1/2} (s\boldsymbol{\eta} + \mathbf{z}_2/\sqrt{n})(s\boldsymbol{\eta} + \mathbf{z}_2/\sqrt{n})^\top (\mathbf{L}\mathbf{Q}\mathbf{L}^\top)^{-1/2} \right) \end{aligned} \quad (2.7)$$

and

$$\left(\mathbf{I}_k + f \frac{\mathbf{t}_2 \mathbf{t}_2^\top}{n - p + 2} \right)^{1/2} = \mathbf{I}_k + \left(\sqrt{1 + f \frac{\mathbf{t}_2^\top \mathbf{t}_2}{n - p + 2}} - 1 \right) \frac{\mathbf{t}_2 \mathbf{t}_2^\top}{\mathbf{t}_2^\top \mathbf{t}_2}. \quad (2.8)$$

As a result, the inverse matrices and square roots of matrices in stochastic representations given in Theorem 2.1 are function of population quantities only. Thus, independently of the length of the generated sample they all should be computed only once. This is not longer true when simulations are based on generating the realizations of the sample covariance matrix and the sample mean vector. Putting all these together, an efficient algorithm is obtained which allow us to generate samples of arbitrary large size from the sample distribution of $\hat{V}_{GMV}$, $\hat{R}_{GMV}$, $\hat{\boldsymbol{\theta}}$, $\hat{s}$, and $\hat{\boldsymbol{\eta}}$ in a relatively small amount of time. Another important application of findings of Theorem 2.1 leads to an efficient way for sampling from the sample distribution of the optimal portfolio weights and their estimated characteristics which will be discussed in detail in the next section. These results will be used to assessed the finite-sample properties of the estimated optimal portfolio weights.

3 Exact sampling distribution of optimal portfolio weights

The weights of the optimal portfolios that belong to the efficient frontier have the following structure

$$\mathbf{w}_g = \mathbf{w}_{GMV} + g(R_{GMV}, V_{GMV}, s)\mathbf{v} \quad (3.1)$$

with their k linear combinations expressed as

$$\mathbf{L}\mathbf{w}_g = \boldsymbol{\theta} + g(R_{GMV}, V_{GMV}, s)\boldsymbol{\eta}, \quad (3.2)$$

where the function $g(R_{GMV}, V_{GMV}, s)$ determines a specific type of an optimal portfolio. It is remarkable that this function depends on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ only over the three quantities R_{GMV} , V_{GMV} , and s which fully determine the whole efficient frontier in the mean-variance space. By considering the general form of (3.2) we are able to cover a number of well-known optimal portfolios: the global minimum variance (GMV) portfolio, the mean-variance (MV) portfolio, the expected maximum exponential utility (EU) portfolio, the tangency (T) portfolio, the optimal portfolio that maximizes the Sharpe ratio (SR), the minimum value-at-risk (MVaR) portfolio, and the minimum conditional value-at-risk (MCVaR) portfolio, the maximum value-of-return (MVoR) portfolio, the maximum conditional value-of-return (MVoR) portfolio, among others. The specific choices of $g(\cdot, \cdot, \cdot)$ for each of these optimal portfolios are provided in Table 1.

Portfolio	$g(R_{GMV}, V_{GMV}, s)$	Additional quantities
GMV	0	
MV	$R_{GMV} - \mu_0$	$\mu_0 \in \mathbb{R}$ -target expected return
EU	γs	$\gamma > 0$ is the risk-aversion coefficient
T	$V_{GMV}s/(R_{GMV} - r_f)$	r_f is the risk-free return
SR	$V_{GMV}s/R_{GMV}$	
MVaR	$s\sqrt{V_{GMV}/(z_\alpha^2 - s)}$	$z_\alpha = \Phi^{-1}(\alpha)$
MCVaR	$s\sqrt{V_{GMV}/(k_\alpha^2 - s)}$	$k_\alpha = \exp\{-z_\alpha^2/2\}/(2\pi(1 - \alpha))$
MVoR	$\frac{(R_{GMV}+v_0)s + \sqrt{z_\alpha^2 s((R_{GMV}+v_0)^2 + (s - z_\alpha^2)V_{GMV})}}{z_\alpha^2 - s}$	$v_0 > 0$ is the target value-at-risk
MVoR	$\frac{(R_{GMV}+k_0)s + \sqrt{k_\alpha^2 s((R_{GMV}+k_0)^2 + (s - k_\alpha^2)V_{GMV})}}{k_\alpha^2 - s}$	k_0 is the target conditional value-at-risk

Table 1: Choice of the function g for several optimal portfolios. The symbol $\Phi(\cdot)$ denotes the distribution function of the standard normal distribution and $\Phi^{-1}(\cdot)$ stands for its inverse.

Let $\hat{\mathbf{w}}_g$ denote the sample estimator of the optimal portfolio weights given in the general form as in (3.2) which is obtained by plugging the sample mean vector and the sample covariance matrix instead of the unknown population counterparts. Then, k linear combinations of the optimal portfolio weights are estimated by

$$\mathbf{L}\hat{\mathbf{w}}_g = \hat{\boldsymbol{\theta}} + g(\hat{R}_{GMV}, \hat{V}_{GMV}, \hat{s})\hat{\boldsymbol{\eta}}. \quad (3.3)$$

By Theorem 2.1 the exact sampling distribution of (3.3) is derived in terms of its stochastic representation. The results are summarized in Theorem 3.1 whose proof follows from Theorem 2.1.

Theorem 3.1. *Under the conditions of Theorem 2.1, it holds that*

$$\begin{aligned}
\mathbf{L}\hat{\mathbf{w}}_g &\stackrel{d}{=} \boldsymbol{\theta} + \left(\sqrt{\frac{V_{GMV}}{f}} \frac{t_1}{\sqrt{n-p+1}} + \frac{g(\hat{R}_{GMV}, \hat{V}_{GMV}, \hat{s})}{f} \right) (s\boldsymbol{\eta} + \mathbf{z}_2/\sqrt{n}) \\
&\quad + \left(\mathbf{L}\mathbf{Q}\mathbf{L}^\top - \frac{(s\boldsymbol{\eta} + \mathbf{z}_2/\sqrt{n})(s\boldsymbol{\eta} + \mathbf{z}_2/\sqrt{n})^\top}{f} \right)^{1/2} \left(\sqrt{V_{GMV}} \sqrt{1 + \frac{t_1^2}{n-p+1}} \right. \\
&\quad \left. + \frac{g(\hat{R}_{GMV}, \hat{V}_{GMV}, \hat{s})}{\sqrt{f}} \frac{t_1/\sqrt{n-p+1}}{\sqrt{1 + \frac{t_1^2}{n-p+1}}} \right) \frac{\mathbf{t}_2}{\sqrt{n-p+2}} + \frac{g(\hat{R}_{GMV}, \hat{V}_{GMV}, \hat{s})}{\sqrt{f}} \\
&\quad \times \left(\mathbf{I}_k + f \frac{\mathbf{t}_2 \mathbf{t}_2^\top}{n-p+2} \right)^{1/2} \frac{\mathbf{t}_3}{\sqrt{n-p+3}}
\end{aligned} \tag{3.4}$$

where the joint stochastic representation of $\hat{V}_{GMV}$, $\hat{R}_{GMV}$ and $\hat{s}$ is given in (i)-(v) of Theorem 2.1.

From findings of Theorem 3.1 we can derive a number of important results. First, they provide a complete characterization of the sampling distribution of the estimators for the optimal portfolio weights. This distribution can be assessed by drawing samples with independent observations from the derived stochastic representation of a relatively large size and then applying the well-established statistical methods for estimating the distribution function, the density, the moments, etc. Second, the obtained stochastic representation in Theorem 3.1 provide an efficient way for generating samples from the finite-sample distribution of $\mathbf{L}\hat{\mathbf{w}}_g$ following the discussion provided in Section 2 after Theorem 2.1 which is based by drawing independent realizations from the well-known univariate and multivariate distributions. To this end, we note that the two square roots in (3.4) should be computed as given in (2.7) and (2.7). As a result, the derived stochastic representation includes the inverse and the square roots of population matrices only and, hence, these objects should be computed only once during the whole simulation study. That is not longer the case when the observations from the sampling distribution of the estimated optimal portfolio weights is obtained by their corresponding definition, i.e. by generating independent sample from the Wishart and normal distributions. Third, for the chosen values of the population quantities used in the simulation study, we can constructed concentration sets of optimal portfolio weights. Fourth, an important probabilistic result about the sampling distribution of $\mathbf{L}\hat{\mathbf{w}}_g$ follows directly from the derived stochastic representation. Namely, that the finite-sample distribution of $\mathbf{L}\hat{\mathbf{w}}_g$ depends on the population mean vector $\boldsymbol{\mu}$ and the population covariance matrix $\boldsymbol{\Sigma}$ only over R_{GMV} , V_{GMV} , s , $\boldsymbol{\theta}$, $\boldsymbol{\eta}$, and $\mathbf{L}\mathbf{Q}\mathbf{L}$. Only these seven quantities will have to be fixed when the samples from the distribution of $\mathbf{L}\hat{\mathbf{w}}_g$ has to be drawn. In particular, in the case of a single linear combination, i.e. when $k = 1$, we only have

to fix six univariate quantities independently of the dimension p of the data-generating process.

In a similar way, we derive statistical inference for the estimated characteristics of optimal portfolio with weights $\hat{\mathbf{w}}_g$ as given (3.1). The expected return of the optimal portfolio with the weights (3.1) is given by

$$R_g = R_{GMV} + g(R_{GMV}, V_{GMV}, s), \quad (3.5)$$

while its variance is

$$V_g = V_{GMV} + \frac{g(R_{GMV}, V_{GMV}, s)^2}{s}. \quad (3.6)$$

Similarly, the VaR, the CVaR, the VoR, and the CVoR are computed by

$$VaR_g = -(R_{GMV} + g(R_{GMV}, V_{GMV}, s)) - z_\alpha \sqrt{V_{GMV} + \frac{g(R_{GMV}, V_{GMV}, s)^2}{s}}, \quad (3.7)$$

$$CVaR_g = -(R_{GMV} + g(R_{GMV}, V_{GMV}, s)) - k_\alpha \sqrt{V_{GMV} + \frac{g(R_{GMV}, V_{GMV}, s)^2}{s}}, \quad (3.8)$$

and by symmetry

$$VoR_g = (R_{GMV} + g(R_{GMV}, V_{GMV}, s)) - z_\alpha \sqrt{V_{GMV} + \frac{g(R_{GMV}, V_{GMV}, s)^2}{s}}, \quad (3.9)$$

$$CVoR_g = (R_{GMV} + g(R_{GMV}, V_{GMV}, s)) - k_\alpha \sqrt{V_{GMV} + \frac{g(R_{GMV}, V_{GMV}, s)^2}{s}}. \quad (3.10)$$

Inserting the sample mean vector and the sample covariance matrix in (3.5)-(3.10) instead of the population counterparts, we get the sample estimators of the optimal portfolio characteristics. The application of Theorem 2.1 leads to the statement about their (joint) sampling distribution which is presented in Theorem 3.2

Theorem 3.2. *Under the conditions of Theorem 2.1, the stochastic representation of the estimated characteristic of optimal portfolio are obtained as in (3.5)-(3.10) where R_{GMV} , V_{GMV} , and s are replaced by their sample counterparts $\hat{R}_{GMV}$, $\hat{V}_{GMV}$, and $\hat{s}$ with*

$$\begin{aligned} \hat{V}_{GMV} &\stackrel{d}{=} \frac{V_{GMV}}{n-1} \xi, \\ \hat{R}_{GMV} &\stackrel{d}{=} R_{GMV} + \sqrt{\frac{V_{GMV}}{n} \left(1 + \frac{p-1}{n-p+1} \psi\right)} z, \\ \hat{s} &\stackrel{d}{=} \frac{(n-1)(p-1)}{n(n-p+1)} \eta, \end{aligned}$$

where $\xi \sim \chi_{n-p}^2$, $\psi \sim F(p-1, n-p+1, ns)$, $z \sim N(0, 1)$ are mutually independent.

The proof of Theorem 3.2 is given in the appendix. It has to be noted that the joint distribution of all six estimators $(\hat{R}_g, \hat{V}_g, \widehat{VaR}_g, \widehat{CVaR}_g, \widehat{VoR}_g, \widehat{CVoR}_g)$ is completely determined by three mutually independent random variables ξ , ψ , and z with the standard marginal univariate distribution. Moreover, it depends on the unknown population mean vector and covariance matrix only over three univariate quantities R_{GMV} , V_{GMV} , and s which uniquely determine the whole efficient frontier in the mean-variance space. To this end, the stochastic representation derived for the estimated optimal portfolio characteristics appear to be simpler than the one obtained in Theorem 3.1 for the corresponding estimator of the optimal portfolio weights. Similarly, the independent realizations from the joint distribution of $(\hat{R}_g, \hat{V}_g, \widehat{VaR}_g, \widehat{CVaR}_g, \widehat{VoR}_g, \widehat{CVoR}_g)$ can be drawn efficiently by employing the results of Theorem 3.2.

Another interesting financial application of the derived theoretical findings of Theorem 3.2 is present in the case of the EU portfolio whose sample expected return and sample variance possess the following stochastic representations:

$$\hat{R}_{EU} \stackrel{d}{=} \hat{R}_{GMV} + \gamma^{-1} \hat{s}, \quad (3.11)$$

$$\hat{V}_{EU} \stackrel{d}{=} \hat{V}_{GMV} + \gamma^{-2} \hat{s}. \quad (3.12)$$

As a result, it appears that $\hat{R}_{EU}$ and $\hat{V}_{EU}$ conditionally independent given the estimated slope parameter of the efficient frontier $\hat{s}$. Only in the limit case, when the risk aversion coefficient γ becomes infinity, i.e. the EU portfolio is located in the vertex of the efficient frontier and, thus, coincides with the GMV portfolio, the two estimated portfolio characteristics become unconditionally independent. In all other cases, the dependence between them is fully captured by the estimated geometry of the efficient frontier.

4 High-dimensional asymptotic distributions

The derived stochastic representations of Sections 3 and 4 are also very useful in the derivation of the asymptotic distributions of the estimators of optimal portfolio weights and their estimated characteristics. To this end, we note that the same approach can be used independently whether the dimension of the data generating process p is assumed to be fix or it is allow to grow together with the sample size that additionally can be used to analyze the structure of high-dimensional optimal portfolios. These two regimes have been intensively discussed in statistical literature. The former asymptotic regime, i.e. with fixed p , is called the "standard asymptotics" (see,

e.g., Le Cam and Yang [44]). Here, both the sample mean and the sample covariance matrix is proven to be consistent estimators for the corresponding population counterparts. challenges arise when p is comparable to n , i.e. both the dimension p and the sample size n tend to infinity while their ratio p/n tends to a positive constant $c \in [0, 1)$, the so-called concentration ratio. It is called the "large dimensional asymptotics" or "Kolmogorov asymptotics" (c.f., Bühlmann and Van De Geer [23], Cai and Shen [24]), while the case $c = 0$ corresponds to the standard asymptotics.

Although, there is a large amount of research done on the asymptotic behavior of functionals which include only the sample mean vector or only the sample covariance matrix under the high-dimensional asymptotics (see, e.g., Bai and Silverstein [5], Cai et al. [25], Wang et al. [56], Bodnar et al. [10], Bodnar et al. [14], Bodnar et al. [8]), the situation becomes more complicated when both the sample mean vector and the sample (inverse) covariance matrix are present in the expressions. The problem is still unsolved and attracts both the researchers and the practitioners. In this section, we show how the derived stochastic representations of Sections 2 and 3 can be employed in the derivation of the high-dimensional asymptotic distributions of the estimated optimal portfolios and their characteristics. The main advantage of the suggested approach based on the stochastic representations is that they clearly separate the deterministic quantities from the stochastic ones where the joint asymptotic distributions of the later can be determined.

Throughout this section we will impose the following technical conditions on the functions involving the population mean vector and the population covariance matrix:

(A1) There exist m and M such that

$$0 < m \leq \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \leq M < \infty \quad \text{and} \quad 0 < m \leq \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} \leq M < \infty \quad (4.1)$$

uniformly in p . Moreover, for a linear combination of optimal portfolio weights determined by the p -dimensional vector $\mathbf{1}$ it holds that

$$0 < m \leq \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} \leq M < \infty \quad (4.2)$$

uniformly in p .

The financial interpretation of Assumption (A1) is based on the fact that it ensures that the parameters of the efficient frontier R_{GMV} , V_{GMV} , and s as well as the components of k linear combinations of optimal portfolio weights $\mathbf{L}\mathbf{w}_g$ are all finite numbers. Mathematically, it

may happen depending on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ that some quantities of R_{GMV} , V_{GMV} , s , and $\mathbf{L}\mathbf{w}_g$ tend to infinity as p increases. In such cases, one should replace the constants m and M in (4.1) and (4.2) by $p^{-\kappa}m$ and $p^{-\kappa}M$ for some $\kappa > 0$. This approach would lead only to minor changes in the expressions of the derived asymptotic covariance matrices in this section where some terms might disappear (see, e.g., Bodnar et al. [13] for similar discussion).

To this end, by an abuse of notations we use the same notations for the functions involving the population mean vector $\boldsymbol{\mu}$ and the population covariance matrix $\boldsymbol{\Sigma}$ and their corresponding deterministic limits. For instance, $\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$ will also be used to denote the limit $\lim_{p \rightarrow \infty} \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$. The interpretation of the quantities becomes clear from the text where they are used.

4.1 High-dimensional asymptotic distribution of $\hat{V}_{GMV}$, $\hat{R}_{GMV}$, $\hat{\boldsymbol{\theta}}$, $\hat{s}$, and $\hat{\boldsymbol{\eta}}$

Before presenting the high-dimensional asymptotic results for the estimated optimal portfolio weights and their characteristic, we derive the asymptotic stochastic representation for the five quantities $\hat{V}_{GMV}$, $\hat{R}_{GMV}$, $\hat{\boldsymbol{\theta}}$, $\hat{s}$, and $\hat{\boldsymbol{\eta}}$. It is presented in Theorem 4.1 in terms of several independently normally distributed random variables/vectors. Such a presentation allows also to characterize the asymptotic dependence structure $\hat{V}_{GMV}$, $\hat{R}_{GMV}$, $\hat{\boldsymbol{\theta}}$, $\hat{s}$, and $\hat{\boldsymbol{\eta}}$ as well as to derive the expression of the asymptotic covariance matrix which is given after Theorem 4.1.

Theorem 4.1. *Under the conditions of Theorem 2.1 and Assumption (A1), it holds that*

$$\begin{aligned}
(i) \quad & \sqrt{n-p} \left(\hat{V}_{GMV} - \frac{1-p/n}{1-1/n} V_{GMV} \right) \xrightarrow{d} \sqrt{2}(1-c)V_{GMV}u_1, \\
(ii) \quad & \sqrt{n-p} \left(\hat{R}_{GMV} - R_{GMV} \right) \xrightarrow{d} \sqrt{V_{GMV}} (\sqrt{1-c}u_4 + \sqrt{s+c}u_5), \\
(iii) \quad & \sqrt{n-p} \left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta} \right) \xrightarrow{d} \sqrt{V_{GMV}} \left(\frac{su_5}{\sqrt{s+c}} \boldsymbol{\eta} + \left(\mathbf{LQL}^\top - \frac{s^2}{s+c} \boldsymbol{\eta} \boldsymbol{\eta}^\top \right)^{1/2} \mathbf{u}_6 \right), \\
(iv) \quad & \sqrt{n-p} \left(\hat{s} - \frac{(s+p/n)(1-1/n)}{1-p/n+2/n} \right) \\
& \xrightarrow{d} \frac{1}{1-c} \left(\sqrt{2(1-c)} (c + 2\boldsymbol{\mu}^\top \mathbf{A}\boldsymbol{\mu}) u_2 + 2s\sqrt{(1-c)} \boldsymbol{\eta}^\top (\mathbf{LQL}^\top)^{-1/2} \mathbf{u}_3 + \sqrt{2}(s+c)u_7 \right), \\
(v) \quad & \sqrt{n-p} \left(\hat{\boldsymbol{\eta}} - \frac{s}{s+p/n} \boldsymbol{\eta} \right) \xrightarrow{d} \frac{1}{\sqrt{s+c}} \left(\mathbf{LQL}^\top - \frac{s^2}{s+c} \boldsymbol{\eta} \boldsymbol{\eta}^\top \right)^{1/2} \mathbf{u}_8 \\
& + \frac{\sqrt{1-c}}{(s+c)} \left(\mathbf{LQL}^\top - 2\frac{s^2}{s+c} \boldsymbol{\eta} \boldsymbol{\eta}^\top \right) (\mathbf{LQL}^\top)^{-1/2} \mathbf{u}_3 - \frac{s\sqrt{2(1-c)}(c+2\boldsymbol{\mu}^\top \mathbf{A}\boldsymbol{\mu})u_2}{(s+c)^2} \boldsymbol{\eta}
\end{aligned}$$

for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$ where $u_1, u_2, \mathbf{u}_3, u_4, u_5, \mathbf{u}_6, u_7, \mathbf{u}_8$ are mutually independent, $u_1, u_2, u_4, u_5, u_7 \sim N(0, 1)$ and $\mathbf{u}_3, \mathbf{u}_6, \mathbf{u}_8 \sim N_k(\mathbf{0}, \mathbf{I}_k)$.

Several interesting results are summarized in the statement of Theorem 4.1 whose proof is given in the appendix. We observe that three quantities related to the estimators of the weights and of the characteristics of the GMV portfolio, the vertex point on the efficient frontier, are asymptotically independent of the estimated slope parameter of the efficient frontier $\hat{s}$ which determines the curvature of the efficient frontier as well as of the estimated weights of the self-financing portfolio $\hat{\boldsymbol{\eta}}$ which is related to the location of the selected optimal portfolio in the efficient frontier. Moreover, the sample variance of the GMV portfolio appears to be asymptotically independent of its estimated expected return $\hat{R}_{GMV}$ and the estimator of the weights $\hat{\boldsymbol{\theta}}$ following the finite-sample findings of Theorem 2.1. However, it is surprising that the covariance between $\hat{\boldsymbol{\theta}}$ and $\hat{R}_{GMV}$ is partly determined by the estimated self-financing portfolio $\hat{\boldsymbol{\eta}}$ due to the deterministic expression close to u_5 in the asymptotic stochastic representations of $\sqrt{n-p}(\hat{R}_{GMV} - R_{GMV})$ and $\sqrt{n-p}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$. Finally, the direct application of the derived stochastic representations in Theorem 4.1 leads to the expression of the asymptotic covariance matrix as given in Corollary 4.1.

Corollary 4.1. *Under the conditions of Theorem 2.1 and Assumption (A1), it holds that*

$$\sqrt{n-p} \begin{pmatrix} \hat{V}_{GMV} - \frac{1-p/n}{1-1/n} V_{GMV} \\ \hat{R}_{GMV} - R_{GMV} \\ \hat{\boldsymbol{\theta}} - \boldsymbol{\theta} \\ \hat{s} - \frac{(s+p/n)(1-1/n)}{1-p/n+2/n} \\ \hat{\boldsymbol{\eta}} - \frac{s}{s+p/n} \boldsymbol{\eta} \end{pmatrix} \rightarrow N_{2k+3}(\mathbf{0}, \boldsymbol{\Xi})$$

with

$$\boldsymbol{\Xi} = \begin{pmatrix} 2V_{GMV}^2(1-c)^2 & 0 & 0 & 0 & 0 \\ 0 & V_{GMV}(1+s) & V_{GMV}s\boldsymbol{\eta}^\top & 0 & 0 \\ 0 & V_{GMV}s\boldsymbol{\eta} & V_{GMV}\mathbf{L}\mathbf{Q}\mathbf{L}^\top & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{\Xi}_{s,s} & \boldsymbol{\Xi}_{s,\boldsymbol{\eta}}^\top \\ 0 & 0 & 0 & \boldsymbol{\Xi}_{s,\boldsymbol{\eta}} & \boldsymbol{\Xi}_{\boldsymbol{\eta},\boldsymbol{\eta}} \end{pmatrix}$$

for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$ where

$$\Xi_{s,s} = \frac{2(c+2s)}{(1-c)} + 2\frac{(s+c)^2}{(1-c)^2}, \quad (4.3)$$

$$\Xi_{\boldsymbol{\eta}, \boldsymbol{\eta}} = \frac{s+1}{(s+c)^2} \mathbf{L} \mathbf{Q} \mathbf{L}^\top - \frac{s^2(2c(1-c) + (s+c)^2)}{(s+c)^4} \boldsymbol{\eta} \boldsymbol{\eta}^\top, \quad (4.4)$$

$$\Xi_{s, \boldsymbol{\eta}} = \frac{2s(2c - s + 4\boldsymbol{\mu}^\top \mathbf{A} \boldsymbol{\mu})}{(s+c)^2} \boldsymbol{\eta}.$$

4.2 High-dimensional asymptotic distribution of optimal portfolio weights

The results of Theorem 4.1 are used to derived the high-dimensional asymptotic distribution of the estimated optimal portfolio weights $\hat{\mathbf{w}}_g$ as well as of the corresponding estimated characteristics of this portfolio given in Section 3.

Let

$$\hat{\boldsymbol{\lambda}} = (\hat{R}_{GMV}, \hat{V}_{GMV}, \hat{s})^\top \quad \text{and} \quad \boldsymbol{\lambda} = \left(R_{GMV}, (1-c)V_{GMV}, \frac{s+c}{1-c} \right)^\top \quad (4.5)$$

where the results of Theorem 4.1 show that

$$\begin{aligned} \hat{R}_{GMV} - R_{GMV} &= o_P(1), \\ \hat{V}_{GMV} - (1-c)V_{GMV} &= o_P(1), \\ \hat{s} - \frac{s+c}{1-c} &= o_P(1), \end{aligned}$$

where $o_P(1) \xrightarrow{a.s.} 0$ for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$.

Throughout this section it is assumed that the function $g(x, y, z)$ is differentiable with first order continuous derivatives and define

$$\begin{aligned} g_1(x_0, y_0, z_0) &= \left. \frac{\partial g(x, y, z)}{\partial x} \right|_{(x,y,z)=(x_0,y_0,z_0)}, \\ g_2(x_0, y_0, z_0) &= \left. \frac{\partial g(x, y, z)}{\partial y} \right|_{(x,y,z)=(x_0,y_0,z_0)}, \\ g_3(x_0, y_0, z_0) &= \left. \frac{\partial g(x, y, z)}{\partial z} \right|_{(x,y,z)=(x_0,y_0,z_0)}. \end{aligned}$$

The asymptotic distribution of $\mathbf{L} \hat{\mathbf{w}}_g$ is given in Theorem 4.2 with the proof presented in the appendix.

Theorem 4.2. Let $g(., ., .)$ be differentiable with first order continuous derivatives. Then, under the conditions of Theorem 2.1 and Assumption (A1), we get

$$\sqrt{n-p} \left(\mathbf{L} \hat{\mathbf{w}}_g - \left(\boldsymbol{\theta} + \frac{sg(\boldsymbol{\lambda})}{s+p/n} \boldsymbol{\eta} \right) \right) \xrightarrow{d} N_k(\mathbf{0}, \boldsymbol{\Omega}_{\mathbf{L},g}) \quad (4.6)$$

for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$ with

$$\begin{aligned} \boldsymbol{\Omega}_{\mathbf{L},g} = & \left(\left(\frac{1-c}{s+c} + g(\boldsymbol{\lambda}) \right) \frac{g(\boldsymbol{\lambda})}{s+c} + V_{GMV} \right) \mathbf{L} \mathbf{Q} \mathbf{L}^\top + s^2 \left\{ 2 \frac{(1-c)^2 V_{GMV}^2}{(s+c)^2} g_2(\boldsymbol{\lambda}) \right. \\ & + \left(\frac{g_3(\boldsymbol{\lambda})}{1-c} - \frac{g(\boldsymbol{\lambda})}{s+c} \right)^2 \frac{2(1-c)c}{(s+c)^2} + \frac{4(1-c)}{(s+c)^2} \left[g(\boldsymbol{\lambda}) \left(\frac{g_3(\boldsymbol{\lambda})}{1-c} - \frac{g(\boldsymbol{\lambda})}{s+c} \right) + s \left(\frac{g_3(\boldsymbol{\lambda})}{1-c} - \frac{g(\boldsymbol{\lambda})}{s+c} \right)^2 \right] \\ & \left. + \frac{V_{GMV}(1-c)}{(s+c)^2} g_1(\boldsymbol{\lambda})^2 + \frac{V_{GMV}}{(s+c)} g_1(\boldsymbol{\lambda}) + \frac{2}{1-c} g_3(\boldsymbol{\lambda})^2 - \frac{g(\boldsymbol{\lambda})^2}{(s+c)^2} \right\} \boldsymbol{\eta} \boldsymbol{\eta}^\top. \end{aligned} \quad (4.7)$$

In the special case of the EU portfolio we get $g(x, y, z) = \gamma^{-1}z$, $g_1(x, y, z) = g_2(x, y, z) = 0$, and

$$\frac{g_3(\boldsymbol{\lambda})}{1-c} - \frac{g(\boldsymbol{\lambda})}{s+c} = \frac{1}{1-c} \gamma^{-1} - \frac{\gamma^{-1}(s+c)}{(1-c)(s+c)} = 0.$$

As a results, the asymptotic covariance matrix of $\mathbf{L} \hat{\mathbf{w}}_{EU}$ is expressed as

$$\boldsymbol{\Omega}_{\mathbf{L},EU} = \left(\left(\frac{1-c}{s+c} + \gamma^{-1} \frac{s+c}{1-c} \right) \frac{\gamma^{-1}}{1-c} + V_{GMV} \right) \mathbf{L} \mathbf{Q} \mathbf{L}^\top + \frac{(1-2c)\gamma^{-2}s^2}{(1-c)^2} \boldsymbol{\eta} \boldsymbol{\eta}^\top. \quad (4.8)$$

In the same way, the high-dimensional asymptotic distribution of the estimated optimal portfolio characteristics is obtained. Following (3.5)-(3.10), $(R_g, V_g, VaR_g, CVaR_g, VoR_g, CVoR_g)$ are functions of R_{GMV} , V_{GMV} , and s only. On the other hand, Theorem 4.1 determines the joint high-dimensional asymptotic distribution of $\hat{R}_{GMV}$, $\hat{V}_{GMV}$, and $\hat{s}$ expressed as

$$\sqrt{n-p} \begin{pmatrix} \hat{R}_{GMV} - R_{GMV} \\ \hat{V}_{GMV} - \frac{1-p/n}{1-1/n} V_{GMV} \\ \hat{s} - \frac{(s+p/n)(1-1/n)}{1-p/n+2/n} \end{pmatrix} \rightarrow N_3(\mathbf{0}, \boldsymbol{\Xi}_{RVs})$$

for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$ with

$$\boldsymbol{\Xi}_{RVs} = \begin{pmatrix} V_{GMV}(1+s) & 0 & 0 \\ 0 & 2V_{GMV}^2(1-c)^2 & 0 \\ 0 & 0 & \frac{2(c+2s)}{(1-c)} + 2\frac{(s+c)^2}{(1-c)^2} \end{pmatrix},$$

which shows that $(\hat{R}_{GMV}, \hat{V}_{GMV}, \hat{s})$ are asymptotically independently distributed.

Let $h_{g,i}(R_{GMV}, R_{GMV}, s)$ denote the i -th characteristic of the optimal portfolio with the weights $\mathbf{w}_g$ and let $h_{g,i}(\hat{\boldsymbol{\lambda}})$ stand for its sample estimated where $\boldsymbol{\lambda}$ is defined in (4.5). The j -th first order partial derivative of $h_{g,i}(\cdot)$ at $\boldsymbol{\lambda}$ we denote by $h_{g,i;j}(\boldsymbol{\lambda})$. Then we get the following result about the high-dimensional distribution of estimated optimal portfolio characteristic whose proof is obtained from the proof of Theorem 4.2.

Theorem 4.3. Let $h_{g,i}(\cdot, \cdot, \cdot)$, $i = 1, \dots, q$, be differentiable with first order continuous derivatives. Then, under the conditions of Theorem 2.1 and Assumption (A1), we get

$$\sqrt{n-p} \begin{pmatrix} h_{g,1}(\hat{\lambda}) - h_{g,1}(\lambda) \\ \vdots \\ h_{g,q}(\hat{\lambda}) - h_{g,q}(\lambda) \end{pmatrix} \rightarrow N_q(\mathbf{0}, \Xi_h)$$

for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$ with $\Xi_h = (\Xi_{h;ij})_{i,j=1,\dots,q}$ where

$$\Xi_{h;ij} = \sum_{l=1}^3 \Xi_{RVs;ll} h_{g,i;l}(\lambda) h_{g,j;l}(\lambda). \quad (4.9)$$

4.3 Interval estimation and high-dimensional test theory

The results of Theorems 4.2 and 4.3 indicate that both $\mathbf{L}\hat{\mathbf{w}}_g$ and $h_{g,i}(\hat{\lambda})$, $i = 1, \dots, q$, are not consistent estimators for $\mathbf{L}\hat{\mathbf{w}}_g$ and $h_{g,i}(R_{GMV}, V_{GMV}, s)$, $i = 1, \dots, q$, respectively. While the asymptotic bias of the sample estimator of linear combinations of optimal portfolio weights is $(\frac{s}{s+c}g(\lambda) - g(R_{GMV}, V_{GMV}, s))\boldsymbol{\eta}$, the asymptotic bias in the estimator of the i -th portfolio characteristic is $h_{g,i}(\lambda) - h_{g,i}(R_{GMV}, V_{GMV}, s)$.

On the other hand, the results of Theorem 4.1 already provide consistent estimators for V_{GMV} , R_{GMV} , $\boldsymbol{\theta}$, s , and $\boldsymbol{\eta}$. Namely, they are given by

$$\hat{V}_{GMV;c} = \frac{\hat{V}_{GMV}}{1 - p/n}, \quad (4.10)$$

$$\hat{R}_{GMV;c} = \hat{R}_{GMV}, \quad (4.11)$$

$$\hat{\boldsymbol{\theta}}_c = \hat{\boldsymbol{\theta}}, \quad (4.12)$$

$$\hat{s}_c = \frac{n-p}{n} \left(\hat{s} - \frac{p}{p+n} \right), \quad (4.13)$$

$$\hat{\boldsymbol{\eta}}_c = \frac{\hat{s}_c + p/n}{\hat{s}_c} \hat{\boldsymbol{\eta}}. \quad (4.14)$$

Combining these equalities, we derive consistent estimators for $\mathbf{L}\hat{\mathbf{w}}_g$ and $h_{g,i}(R_{GMV}, V_{GMV}, s)$ expressed as

$$\mathbf{L}\hat{\mathbf{w}}_{g;c} = \hat{\boldsymbol{\theta}} + g(\hat{R}_{GMV;c}, \hat{V}_{GMV;c}, \hat{s}_c) \hat{\boldsymbol{\eta}}_c \quad (4.15)$$

and

$$\hat{h}_{g,i;c} = h_{g,i}(\hat{R}_{GMV;c}, \hat{V}_{GMV;c}, \hat{s}_c). \quad (4.16)$$

In Theorem 4.4, the asymptotic covariance matrices of the consistent estimators of optimal portfolio weights and their characteristics are present.

Theorem 4.4. Let $\boldsymbol{\lambda} = (R_{GMV}, V_{GMV}, s)^\top$. Then, under the conditions of Theorems 4.2 and 4.3, it holds that

(a) $\sqrt{n-p}(\mathbf{L}\hat{\mathbf{w}}_{g;c} - \mathbf{L}\mathbf{w}_g) \xrightarrow{d} N_k(\mathbf{0}, \boldsymbol{\Omega}_{\mathbf{L},g,c})$ for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$ with

$$\begin{aligned} \boldsymbol{\Omega}_{\mathbf{L},g,c} = & \left(\left(\frac{1-c}{s+c} + \frac{s+c}{s} g(\boldsymbol{\lambda}_0) \right) \frac{g(\boldsymbol{\lambda}_0)}{s} + V_{GMV} \right) \mathbf{L} \mathbf{Q} \mathbf{L}^\top \\ & + s^2 \left\{ 2 \frac{(1-c)V_{GMV}^2}{s(s+c)} g_2(\boldsymbol{\lambda}_0) + \left(\frac{g_3(\boldsymbol{\lambda}_0)(s+c)}{s} - \frac{g(\boldsymbol{\lambda}_0)}{s} \right)^2 \frac{2(1-c)c}{(s+c)^2} \right. \\ & + \frac{4(1-c)}{(s+c)^2} \left[\frac{s+c}{s} g(\boldsymbol{\lambda}_0) \left(\frac{g_3(\boldsymbol{\lambda}_0)(s+c)}{s} - \frac{g(\boldsymbol{\lambda}_0)}{s} \right) + s \left(\frac{g_3(\boldsymbol{\lambda}_0)(s+c)}{s} - \frac{g(\boldsymbol{\lambda}_0)}{s} \right)^2 \right] \\ & \left. + \frac{V_{GMV}(1-c)}{s^2} g_1(\boldsymbol{\lambda}_0)^2 + \frac{V_{GMV}}{s} g_1(\boldsymbol{\lambda}_0) + \frac{2(1-c)(s+c)^2}{s^2} g_3(\boldsymbol{\lambda}_0)^2 - \frac{g(\boldsymbol{\lambda}_0)^2}{s^2} \right\} \boldsymbol{\eta} \boldsymbol{\eta}^\top; \end{aligned} \quad (4.17)$$

(b)

$$\sqrt{n-p} \begin{pmatrix} \hat{h}_{g,1,c} - h_{g,1}(\boldsymbol{\lambda}_0) \\ \vdots \\ \hat{h}_{g,q,c} - h_{g,q}(\boldsymbol{\lambda}_0) \end{pmatrix} \rightarrow N_q(\mathbf{0}, \boldsymbol{\Xi}_{h,c})$$

for $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$ with $\boldsymbol{\Xi}_{h,c} = (\Xi_{h,c;ij})_{i,j=1,\dots,q}$ where

$$\begin{aligned} \Xi_{h,c;ij} = & V_{GMV}(1+s)h_{g,i;1}(\boldsymbol{\lambda}_0)h_{g,j;1}(\boldsymbol{\lambda}_0) + 2V_{GMV}^2h_{g,i;2}(\boldsymbol{\lambda}_0)h_{g,j;2}(\boldsymbol{\lambda}_0) \\ & + (2s^2 + 4s + 2c)h_{g,i;3}(\boldsymbol{\lambda}_0)h_{g,j;3}(\boldsymbol{\lambda}_0). \end{aligned}$$

Since both $\boldsymbol{\Omega}_{\mathbf{L},g,c}$ and $\boldsymbol{\Xi}_{h,c}$ depend on unobservable quantities, we have to estimate them consistently under the high-dimensional asymptotic regime when confidence regions for the optimal portfolio weights and for the optimal portfolio characteristics are derived.

Consistent estimators for V_{GMV} , R_{GMV} , $\boldsymbol{\theta}$, s , and $\boldsymbol{\eta}$ are given in (4.10)-(4.14). Similarly, a consistent estimator for

$$\mathbf{L} \mathbf{Q} \mathbf{L}^\top = \mathbf{L} \left(\boldsymbol{\Sigma}^{-1} - \frac{\boldsymbol{\Sigma}^{-1} \mathbf{1} \mathbf{1}^\top \boldsymbol{\Sigma}^{-1}}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}} \right) \mathbf{L}^\top = \mathbf{L} \boldsymbol{\Sigma}^{-1} \mathbf{L}^\top - \frac{1}{V_{GMV}} \boldsymbol{\theta} \boldsymbol{\theta}^\top$$

is constructed. First, V_{GMV} and $\boldsymbol{\theta}$ are replaced by their consistent estimators $\hat{V}_{GMV;c}$ and $\hat{\boldsymbol{\theta}}_c$. Second, we use that a consistent estimator for $\mathbf{l}_i^\top \boldsymbol{\Sigma}^{-1} \mathbf{l}_j$ with deterministic vectors $\mathbf{l}_i$ and $\mathbf{l}_j$ satisfying Assumption (A1) is given by $(1-p/n) \mathbf{l}_i^\top \boldsymbol{\Sigma}^{-1} \mathbf{l}_j$ (c.f., Bodnar et al. [16, Lemma 5.3]). As a result, $\mathbf{L} \mathbf{Q} \mathbf{L}^\top$ is consistently estimated by $(1-p/n) \mathbf{L} \hat{\mathbf{Q}} \mathbf{L}^\top$ with $\hat{\mathbf{Q}}$ given in (1.5) and, hence, V_{GMV} , R_{GMV} , $\boldsymbol{\theta}$, s , $\boldsymbol{\eta}$, and $\mathbf{L} \mathbf{Q} \mathbf{L}^\top$ with their consistent estimators in (4.17) and (4.18),

we obtain consistent estimators for $\mathbf{\Omega}_{\mathbf{L},g,c}$ and $\mathbf{\Xi}_{h,c}$ denoted by $\hat{\mathbf{\Omega}}_{\mathbf{L},g,c}$ and $\hat{\mathbf{\Xi}}_{h,c}$. For instance, a consistent estimator for the covariance matrix of the estimated weights of the EU portfolio is given by:

$$\begin{aligned} \hat{\mathbf{\Omega}}_{\mathbf{L},EU,c} = & \left(\left(\frac{1-c_n}{\hat{s}_c + c_n} + (\hat{s}_c + c_n)\gamma^{-1} \right) \gamma^{-1} + \hat{V}_{GMV;c} \right) (1-c_n) \mathbf{L} \hat{\mathbf{Q}} \mathbf{L}^\top \\ & + \gamma^{-2} \left\{ \frac{2(1-c_n)c_n^3}{(\hat{s}_c + c_n)^2} + 4(1-c_n)c_n \frac{\hat{s}_c(\hat{s}_c + 2c_n)}{(\hat{s}_c + c_n)^2} + \frac{2(1-c_n)c_n^2(\hat{s}_c + c_n)^2}{\hat{s}_c^2} - \hat{s}_c^2 \right\} \hat{\boldsymbol{\eta}}_c \hat{\boldsymbol{\eta}}_c^\top, \end{aligned} \quad (4.18)$$

where $c_n = p/n$.

The suggested consistent estimators of $\mathbf{\Omega}_{\mathbf{L},g,c}$ and $\mathbf{\Xi}_{h,c}$ are then used to derived $(1 - \beta)$ asymptotic confidence intervals for the population optimal portfolio weights and their characteristics. In the case of k linear combination of the optimal portfolio weights $\mathbf{w}_g$ we get

$$C_{\mathbf{L},g;1-\beta} = \left\{ \boldsymbol{\omega} : (n-p) (\mathbf{L} \hat{\mathbf{w}}_{g;c} - \mathbf{L} \mathbf{w}_g)^\top \hat{\mathbf{\Omega}}_{\mathbf{L},g,c}^{-1} (\mathbf{L} \hat{\mathbf{w}}_{g;c} - \mathbf{L} \mathbf{w}_g) \leq \chi_{k;1-\beta}^2 \right\}, \quad (4.19)$$

where $\chi_{k;1-\beta}^2$ denotes the $(1 - \beta)$ quantile from the χ^2 -distribution with k degrees of freedom.

Finally, using the duality between the interval estimation and the test theory (c.f., Aitchison [2]) a test on the equality of k -linear combination of optimal portfolio weights to a preselected vector $\mathbf{r}$ can be derived. Namely, one has to reject the null hypothesis $H_0 \mathbf{L} \mathbf{w}_g = \mathbf{r}$ in favour to the alternative hypothesis $H_0 \mathbf{L} \mathbf{w}_g = \mathbf{r}$ at significance level β as soon as $\mathbf{r}$ does not belong to the confidence interval $C_{\mathbf{L},g;1-\beta}$ as given in (4.19). Similar results are also obtained in the case of optimal portfolio characteristics.

5 Finite-sample performance and robustness analysis

The finite sample performance of the derived high-dimensional asymptotic approximation of the sampling distribution of the estimated optimal portfolio weights is investigated via an extensive Monte Carlo study in this section. Additionally, we study the robustness of the obtained asymptotic distributions to the violation of the assumption of normality used in their derivation. The following two simulation scenarios will be considered in the simulation study:

Scenario 1 Multivariate normal distribution:

Sample of asset returns $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are generated independently from $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$;

Scenario 2 Multivariate t -distribution: Sample of asset returns $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are generated independently from multivariate t -distribution with degrees of freedom $d = 10$, location

parameter $\boldsymbol{\mu}$, scale matrix $\frac{d-2}{d}\boldsymbol{\Sigma}$. This choice of the scale matrix ensures that the covariance matrix of $\mathbf{x}_i$ is $\boldsymbol{\Sigma}$.

Scenario 1 corresponds to the assumption used in the derivation of the theoretical results of the paper, while **Scenario 2** violates this assumption by allowing heavy tails in the distribution of the asset returns. In both scenarios the components of $\boldsymbol{\mu}$ are generated from $U(-0.2, 0.2)$. The eigenvalues of the covariance matrix $\boldsymbol{\Sigma}$ are fixed such that 20% of them are equal to 0.2, 40% are equal to 1, and 40% are equal to 5, while its eigenvectors are simulated from the Haar distribution. Furthermore, we put $n = 1000$ and $c \in \{0.5, 0.9\}$. The results of the simulation study are illustrated in the case of five quantities $\hat{V}_{GMV}$, $\hat{\boldsymbol{\theta}}_{GMV}$, $\hat{R}_{GMV}$, $\hat{s}$, and $\hat{\eta}$, and the estimator for first weight of the EU portfolio with $\gamma = 20$ and $\mathbf{L} = (1, 0, 0, \dots, 0)$.

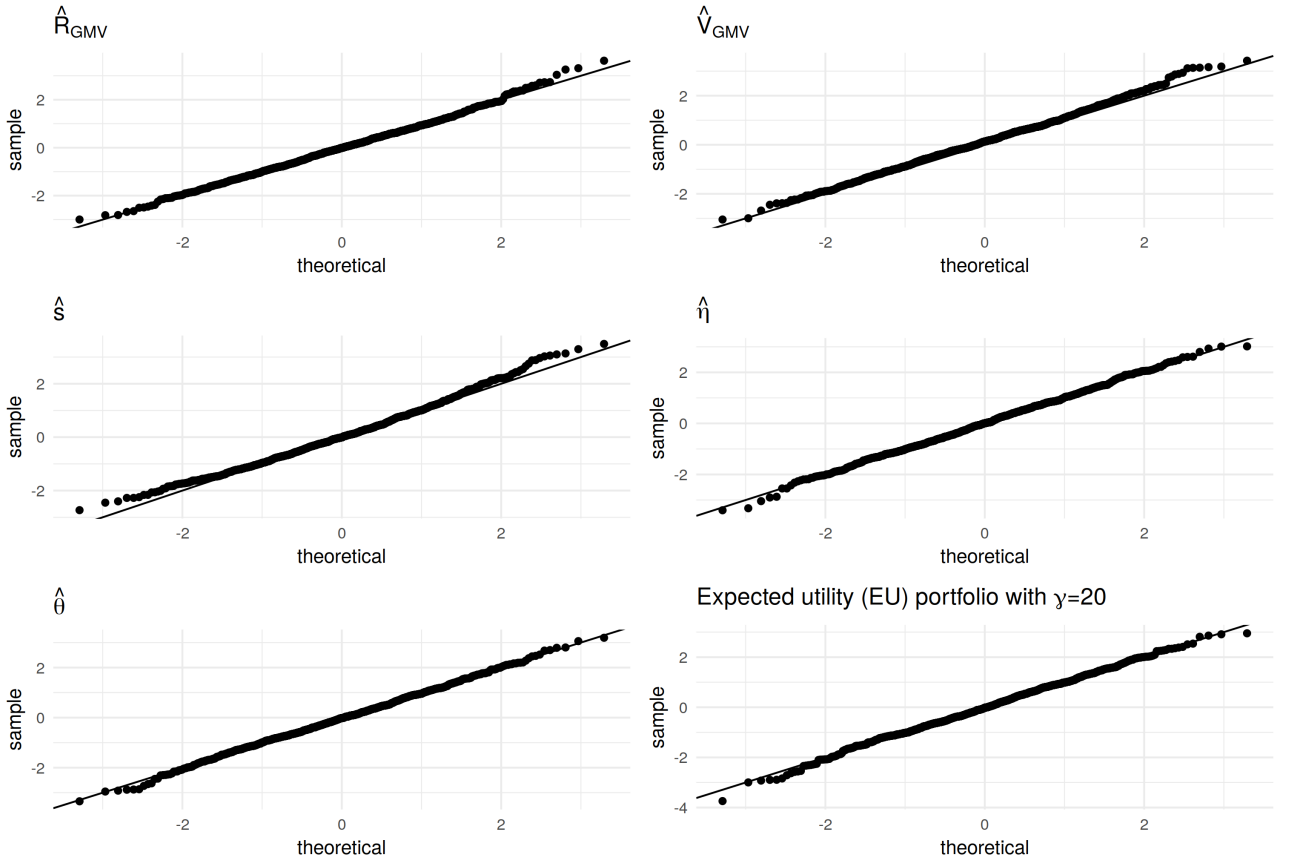


Figure 1: QQ-plots of the standardized quantities of $\hat{V}_{GMV}$, $\hat{\boldsymbol{\theta}}_{GMV}$, $\hat{R}_{GMV}$, $\hat{s}$, $\hat{\eta}$, and $\mathbf{L}\hat{\mathbf{w}}_{EU}$ in comparison to their high-dimensional asymptotic distribution. Data generating from **Scenario 1** with $c = 0.5$.

In Figures 1 to 4, the QQ-plots are shown for each of the six estimated quantities, where the theoretical quantities obtained from the high-dimensional asymptotic approximations as given in Theorems 4.1 and 4.2 are compared to the exact ones obtained by employing the

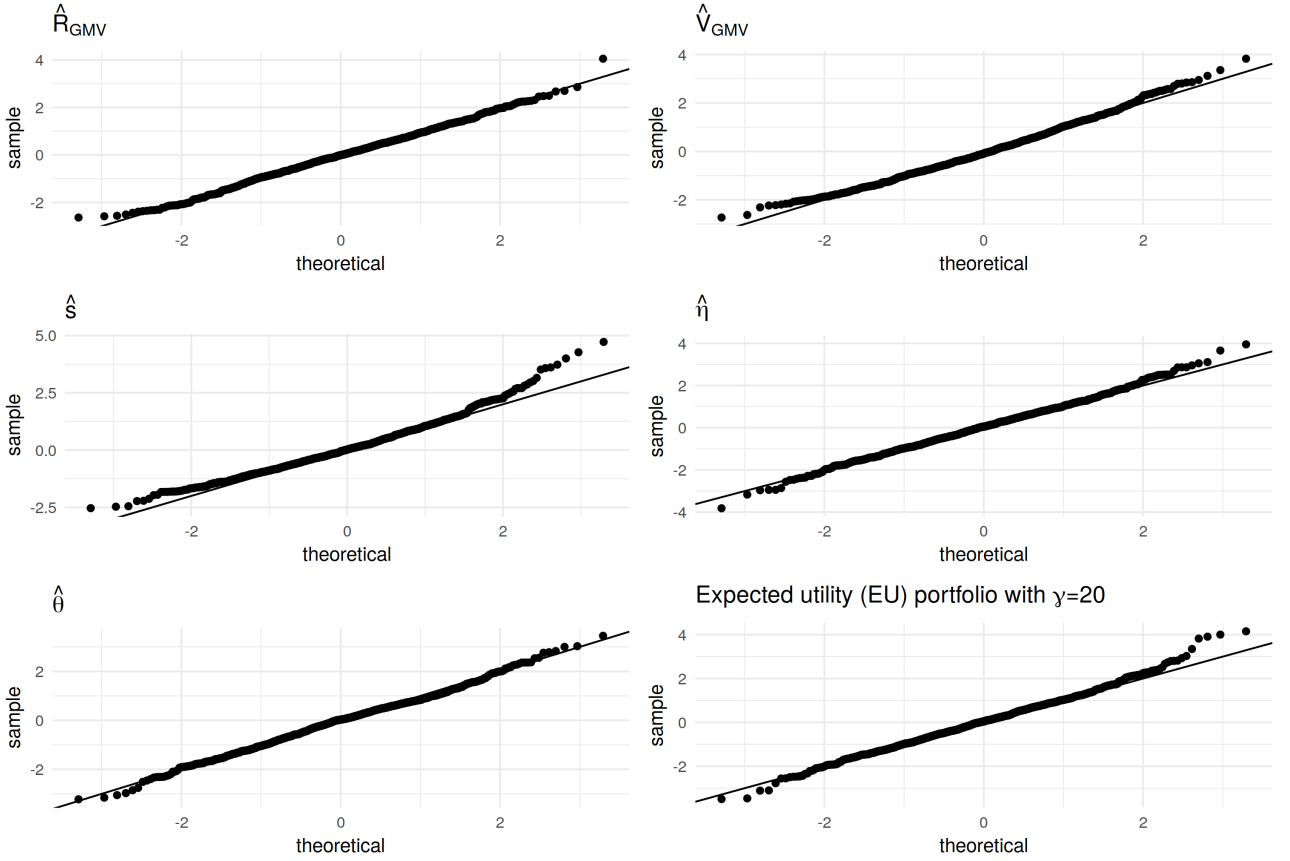


Figure 2: QQ-plots of the standardized quantities of $\hat{V}_{GMV}$, $\hat{\theta}_{GMV}$, $\hat{R}_{GMV}$, $\hat{s}$, $\hat{\eta}$, and $\mathbf{L}\hat{\mathbf{w}}_{EU}$ in comparison to their high-dimensional asymptotic distribution. Data generating from **Scenario 1** with $c = 0.9$.

stochastic representation of Theorems 2.1 and 3.1 from which the finite-sample distribution of each estimated quantity is approximated by using $B = 5000$ independent draws $\hat{V}_{GMV}^{(b)}$, $\hat{\theta}_{GMV}^{(b)}$, $\hat{R}_{GMV}^{(b)}$, $\hat{s}^{(b)}$, $\hat{\eta}^{(b)}$, and $\mathbf{L}\hat{\mathbf{w}}_{EU}^{(b)}$ for $b = 1, \dots, B$. To this end we note that the application of Theorems 2.1 and 3.1 provides an efficient way to generate the sample $\hat{V}_{GMV}^{(b)}$, $\hat{\theta}_{GMV}^{(b)}$, $\hat{R}_{GMV}^{(b)}$, $\hat{s}^{(b)}$, $\hat{\eta}^{(b)}$, and $\mathbf{L}\hat{\mathbf{w}}_{EU}^{(b)}$ which also avoids the computation of the inverse sample covariance matrix which might be an ill-defined object in large dimensions, especially when $c = 0.9$.

In Figures 1 and 2 we display the QQ-plots in the case of the multivariate normal distribution following **Scenario 1**. We observe in the figures that the high-dimensional asymptotic distributions provide a good approximation for the moderate value of the concentration ratio $c = 0.5$ and its large value $c = 0.9$. The approximation seems to be worst off in the context of approximating the distribution of $\hat{s}$ when $c = 0.9$ as the tails becomes much heavier than the approximation seem to be able to account for.

In Figure 3 and 4 we can how the high-dimensional asymptotic approximations of the sam-

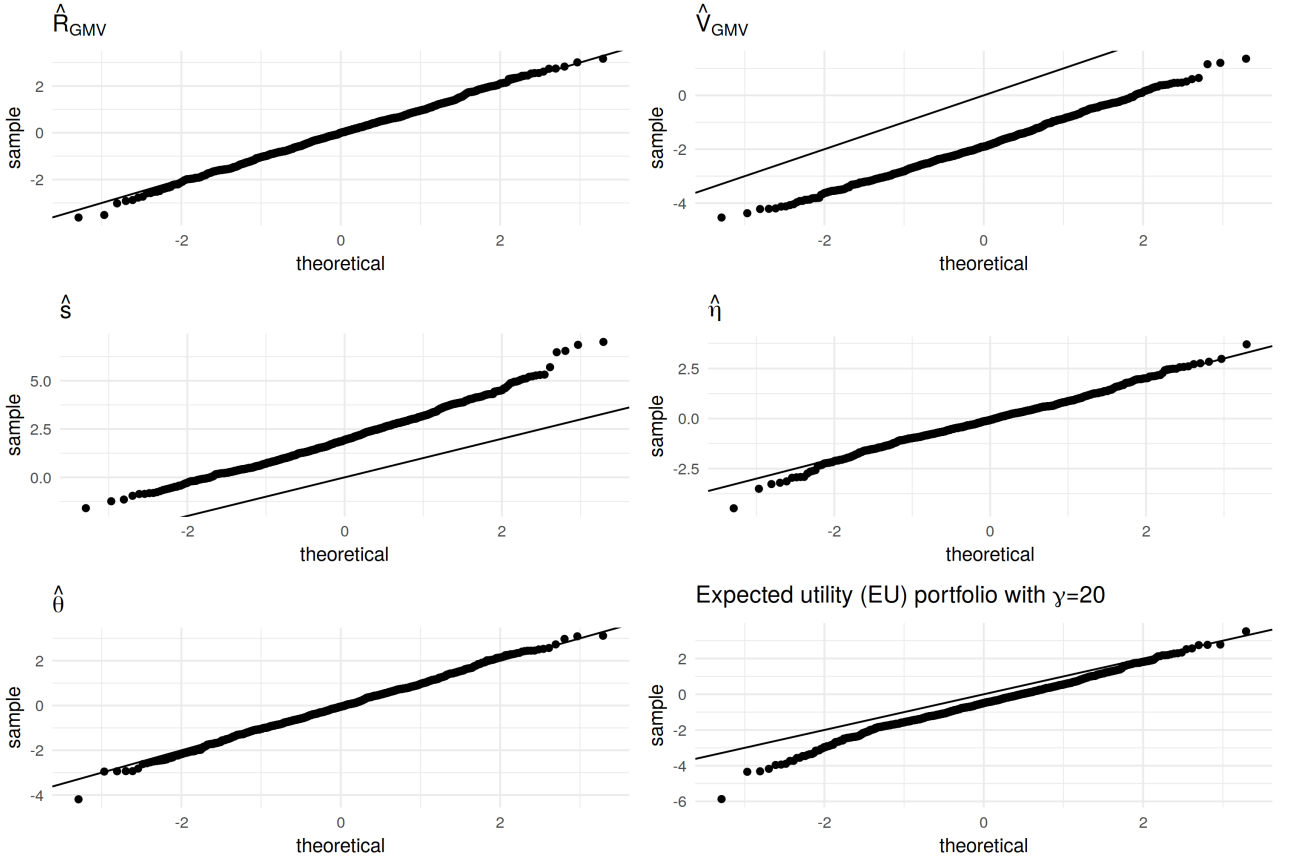


Figure 3: QQ-plots of the standardized quantities of $\hat{V}_{GMV}$, $\hat{\theta}_{GMV}$, $\hat{R}_{GMV}$, $\hat{s}$, $\hat{\eta}$, and $\mathbf{L}\hat{\mathbf{w}}_{EU}$ in comparison to their high-dimensional asymptotic distribution. Data generating from **Scenario 2** with $c = 0.5$.

pling distributions of $\hat{V}_{GMV}$, $\hat{\theta}_{GMV}$, $\hat{R}_{GMV}$, $\hat{s}$, $\hat{\eta}$, and $\mathbf{L}\hat{\mathbf{w}}_{EU}$ works well when the returns are assumed to be multivariate t -distributed. Small deviations from the asymptotic normality is observed only in the case of $\hat{s}$ and $\hat{V}_{GMV}$ when $c = 0.9$. Also, a small positive bias is present for these two quantities when $c = 0.5$ which is explained by the influence of heavy tails in the estimation of the inverse of the high-dimensional covariance matrix. On the other hand, the asymptotic variances seem to be well approximated by the results of Theorems 4.1 and 4.2. All other quantities show a good performance despite the violation of the distributional assumption. We also observe the same type of skewness as in **Scenario 1** in the case of $\hat{s}$ when the asset universe becomes large.

6 Summary

In this paper we derive the exact sampling distribution of the estimators for a large class of optimal portfolio weights and their estimated characteristics. The results are present in terms of

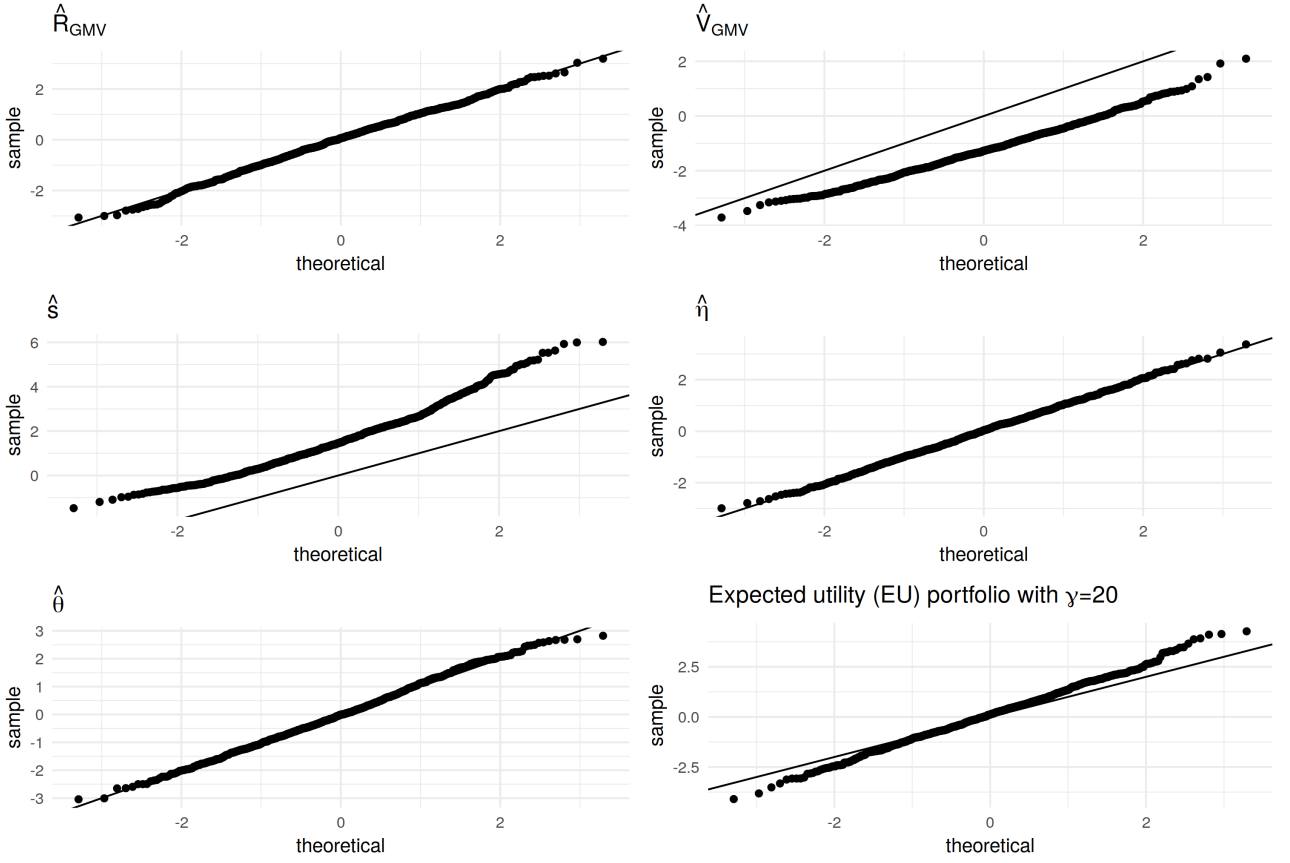


Figure 4: QQ-plots of the standardized quantities of $\hat{V}_{GMV}$, $\hat{\theta}_{GMV}$, $\hat{R}_{GMV}$, $\hat{s}$, $\hat{\eta}$, and $\mathbf{L}\hat{\mathbf{w}}_{EU}$ in comparison to their high-dimensional asymptotic distribution. Data generating from **Scenario 2** with $c = 0.9$.

stochastic representations which provide an easy way to assess the sampling distribution of the estimated optimal portfolio weights. Another important application of the derived stochastic representations is that it presents the way how samples from the corresponding (joint) sampling distribution can be generated in an efficient way that excludes the inversion of the sample covariance matrix in each simulation run. Furthermore, the derived stochastic simulation simplify considerably the study of the asymptotic properties of the estimated quantities under the high-dimensional asymptotic regime.

The finite sample performance of the obtained asymptotic approximations to the exact sampling distributions are investigated via an extensive simulation study where the departure from the model assumption is studied as well. While a very good performance is observed when the data sets are simulated from the normal distribution, some biases are present in the asymptotic means and the asymptotic variances when the assumption of normality is violated. Although, the normal approximations seem to provide a good fit also in the later case. Assessing

the biases in the asymptotic means and in the asymptotic (co)variances of the estimated optimal portfolio weights and their characteristic is an important challenge which will be treated in the consequent paper.

7 Appendix

In this section, the proofs of the theoretical results are given. In Lemma 7.1 we derive the conditional distribution of $(\hat{V}_{GMV}, \hat{\boldsymbol{\theta}}^\top, \hat{R}_{GMV}, \hat{s}, \hat{\boldsymbol{\eta}}^\top)^\top$ under the condition $\hat{\boldsymbol{\mu}} = \tilde{\boldsymbol{\mu}}$, i.e. the distribution of $(\hat{V}_{GMV}, \hat{\boldsymbol{\theta}}^\top, \hat{R}_{GMV}, \hat{s}, \hat{\boldsymbol{\eta}}^\top)^\top$.

Lemma 7.1. *Under the conditions of Theorem 2.1, the distribution of $(\hat{V}_{GMV}, \hat{\boldsymbol{\theta}}^\top, \hat{R}_{GMV}, \hat{s}, \hat{\boldsymbol{\eta}}^\top)^\top$ is determined by*

(i) $\hat{V}_{GMV}$ is independent of $(\hat{\boldsymbol{\theta}}^\top, \hat{R}_{GMV}, \hat{s}, \hat{\boldsymbol{\eta}}^\top)^\top$;

(ii) $(n-1)\frac{\hat{V}_{GMV}}{\hat{V}_{GMV}} \sim \chi_{n-p}^2$;

(iii) $\begin{pmatrix} \hat{\boldsymbol{\theta}} \\ \hat{R}_{GMV} \end{pmatrix} \sim t_{k+1} \left(n-p+1, \begin{pmatrix} \boldsymbol{\theta} \\ \check{R}_{GMV} \end{pmatrix}, \frac{V_{GMV}}{n-p+1} \check{\mathbf{G}} \right)$, with

$$\check{\mathbf{G}} = \begin{pmatrix} \mathbf{LQL}^\top & \mathbf{LQ}\tilde{\boldsymbol{\mu}} \\ \tilde{\boldsymbol{\mu}}^\top \mathbf{QL}^\top & \tilde{\boldsymbol{\mu}}^\top \mathbf{Q}\tilde{\boldsymbol{\mu}} \end{pmatrix} = \begin{pmatrix} \mathbf{LQL}^\top & \check{s}\check{\boldsymbol{\eta}} \\ \check{s}\check{\boldsymbol{\eta}}^\top & \check{s} \end{pmatrix};$$

(iv) $\check{s}$ and $\check{\boldsymbol{\eta}}$ are conditionally independent given $\hat{\boldsymbol{\theta}}$ and $\check{R}_{GMV}$

(v) $(n-1)\frac{\check{s}}{\check{s}} \left(1 + \frac{(\check{R}_{GMV} - \check{R}_{GMV})^2}{V_{GMV}\check{s}} \right) \sim \chi_{n-p+2}^2$;

(vi)

$$\check{\boldsymbol{\eta}} | \hat{\boldsymbol{\theta}}^\top, \check{R}_{GMV} \sim t_k \left(n-p+3, \check{\boldsymbol{\eta}} + \mathbf{h}, \frac{(n-p+3)^{-1} \tilde{\mathbf{F}}}{\check{s} \left(1 + \frac{(\check{R}_{GMV} - \check{R}_{GMV})^2}{V_{GMV}\check{s}} \right)^2} \right),$$

where

$$\mathbf{h} = \left(1 + \frac{(\check{R}_{GMV} - \check{R}_{GMV})^2}{V_{GMV}\check{s}} \right)^{-1} \frac{(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta} - \check{\boldsymbol{\eta}}(\check{R}_{GMV} - \check{R}_{GMV}))(\check{R}_{GMV} - \check{R}_{GMV})}{V_{GMV}\check{s}}$$

and

$$\begin{aligned} \tilde{\mathbf{F}} &= (\mathbf{LQL}^\top - \check{s}\check{\boldsymbol{\eta}}\check{\boldsymbol{\eta}}^\top) \left(1 + \frac{(\check{R}_{GMV} - \check{R}_{GMV})^2}{V_{GMV}\check{s}} \right) \\ &+ \frac{\check{s}}{V_{GMV}} \left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta} - \check{\boldsymbol{\eta}}(\check{R}_{GMV} - \check{R}_{GMV}) \right) \left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta} - \check{\boldsymbol{\eta}}(\check{R}_{GMV} - \check{R}_{GMV}) \right)^\top. \end{aligned}$$

Proof of Lemma 7.1: Under the assumption of independent and normally distributed sample of the asset returns, we get that

- (a) $\hat{\boldsymbol{\mu}} \sim \mathcal{N}_p\left(\boldsymbol{\mu}, \frac{1}{n}\boldsymbol{\Sigma}\right)$;
- (b) $(n-1)\hat{\boldsymbol{\Sigma}} \sim \mathcal{W}_p(n-1, \boldsymbol{\Sigma})$ (p -dimensional Wishart distribution with $(n-1)$ degrees of freedom and covariance matrix $\boldsymbol{\Sigma}$);
- (c) $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$ are independent.

As a result, the conditional distribution of a random variable defined as a function of $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$ given $\hat{\boldsymbol{\mu}} = \tilde{\boldsymbol{\mu}}$ is equal to the distribution of a random variable defined by the same function where $\hat{\boldsymbol{\mu}}$ is replaced by $\tilde{\boldsymbol{\mu}}$.

Let $\tilde{\mathbf{M}} = (\mathbf{L}^\top, \tilde{\boldsymbol{\mu}}, \mathbf{1})^\top$ and define

$$\tilde{\mathbf{H}} = \tilde{\mathbf{M}}\hat{\boldsymbol{\Sigma}}^{-1}\tilde{\mathbf{M}}^\top = \begin{pmatrix} \tilde{\mathbf{H}}_{11} & \tilde{\mathbf{H}}_{12} \\ \tilde{\mathbf{H}}_{21} & \tilde{\mathbf{H}}_{22} \end{pmatrix}$$

with

$$\tilde{\mathbf{H}}_{11} = \begin{pmatrix} \mathbf{L}\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{L}^\top & \mathbf{L}\hat{\boldsymbol{\Sigma}}^{-1}\tilde{\boldsymbol{\mu}} \\ \tilde{\boldsymbol{\mu}}^\top\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{L}^\top & \tilde{\boldsymbol{\mu}}^\top\hat{\boldsymbol{\Sigma}}^{-1}\tilde{\boldsymbol{\mu}} \end{pmatrix}, \quad \tilde{\mathbf{H}}_{12} = \begin{pmatrix} \mathbf{L}\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1} \\ \tilde{\boldsymbol{\mu}}^\top\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1} \end{pmatrix}, \quad \tilde{\mathbf{H}}_{21} = \tilde{\mathbf{H}}_{12}^\top, \quad \tilde{\mathbf{H}}_{22} = \mathbf{1}^\top\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}$$

and

$$\mathbf{H} = \tilde{\mathbf{M}}\boldsymbol{\Sigma}^{-1}\tilde{\mathbf{M}}^\top = \begin{pmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{pmatrix}$$

with

$$\mathbf{H}_{11} = \begin{pmatrix} \mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{L}^\top & \mathbf{L}\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}} \\ \tilde{\boldsymbol{\mu}}^\top\boldsymbol{\Sigma}^{-1}\mathbf{L}^\top & \tilde{\boldsymbol{\mu}}^\top\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}} \end{pmatrix}, \quad \mathbf{H}_{12} = \begin{pmatrix} \mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{1} \\ \tilde{\boldsymbol{\mu}}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1} \end{pmatrix}, \quad \mathbf{H}_{21} = \mathbf{H}_{12}^\top, \quad \mathbf{H}_{22} = \mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}.$$

Also, let

$$\tilde{\mathbf{G}} = \tilde{\mathbf{H}}_{11} - \frac{\tilde{\mathbf{H}}_{12}\tilde{\mathbf{H}}_{21}}{\tilde{\mathbf{H}}_{22}} = \begin{pmatrix} \mathbf{L} \\ \tilde{\boldsymbol{\mu}}^\top \end{pmatrix} \hat{\mathbf{Q}} \begin{pmatrix} \mathbf{L}^\top & \tilde{\boldsymbol{\mu}} \end{pmatrix} = \begin{pmatrix} \mathbf{L}\hat{\mathbf{Q}}\mathbf{L}^\top & \mathbf{L}\hat{\mathbf{Q}}\tilde{\boldsymbol{\mu}} \\ \tilde{\boldsymbol{\mu}}^\top\hat{\mathbf{Q}}\mathbf{L}^\top & \tilde{\boldsymbol{\mu}}^\top\hat{\mathbf{Q}}\tilde{\boldsymbol{\mu}} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{G}}_{11} & \tilde{\mathbf{G}}_{12} \\ \tilde{\mathbf{G}}_{21} & \tilde{\mathbf{G}}_{22} \end{pmatrix} \quad (7.1)$$

and

$$\mathbf{G} = \mathbf{H}_{11} - \frac{\mathbf{H}_{12}\mathbf{H}_{21}}{\mathbf{H}_{22}} = \begin{pmatrix} \mathbf{L} \\ \tilde{\boldsymbol{\mu}}^\top \end{pmatrix} \mathbf{Q} \begin{pmatrix} \mathbf{L}^\top & \tilde{\boldsymbol{\mu}} \end{pmatrix} = \begin{pmatrix} \mathbf{L}\mathbf{Q}\mathbf{L}^\top & \mathbf{L}\mathbf{Q}\tilde{\boldsymbol{\mu}} \\ \tilde{\boldsymbol{\mu}}^\top\mathbf{Q}\mathbf{L}^\top & \tilde{\boldsymbol{\mu}}^\top\mathbf{Q}\tilde{\boldsymbol{\mu}} \end{pmatrix} = \begin{pmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{pmatrix}$$

(7.2)

with $\tilde{\mathbf{G}}_{22} = \tilde{\boldsymbol{\mu}}^\top \hat{\mathbf{Q}} \tilde{\boldsymbol{\mu}}$ and $\mathbf{G}_{22} = \tilde{\boldsymbol{\mu}}^\top \mathbf{Q} \tilde{\boldsymbol{\mu}}$.

In using the definitions of $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{G}}$, we get

$$\hat{V}_{GMV} = \frac{1}{\tilde{\mathbf{H}}_{22}}, \quad \begin{pmatrix} \hat{\boldsymbol{\theta}} \\ \tilde{R}_{GMV} \end{pmatrix} = \frac{\tilde{\mathbf{H}}_{12}}{\tilde{\mathbf{H}}_{22}}, \quad \tilde{s} = \tilde{\mathbf{G}}_{22}, \quad \tilde{\boldsymbol{\eta}} = \frac{\tilde{\mathbf{G}}_{12}}{\tilde{\mathbf{G}}_{22}}.$$

Moreover, from Muirhead [48, Theorem 3.2.11] we get $(n-1)\tilde{\mathbf{H}}^{-1} \sim \mathcal{W}_{k+2}(n-p+k+1, \mathbf{H}^{-1})$ and, consequently, (see, Gupta and Nagar [35, Theorem 3.4.1]) $(n-1)^{-1}\tilde{\mathbf{H}} \sim \mathcal{W}_{k+2}^{-1}(n-p+2k+4, \mathbf{H})$ ($(k+2)$ -dimensional inverse Wishart distribution with $n-p+2k+4$ degrees of freedom and parameter matrix $\mathbf{H}$). The application of Theorem 3 in Bodnar and Okhrin [15] leads to

(i) $\tilde{\mathbf{H}}_{22}$ is independent of $\tilde{\mathbf{H}}_{12}/\tilde{\mathbf{H}}_{22}$ and $\tilde{\mathbf{G}}$ and, consequently,

$$\hat{V}_{GMV} \text{ is independent of } (\hat{\boldsymbol{\theta}}^\top, \tilde{R}_{GMV}, \tilde{s}, \tilde{\boldsymbol{\eta}}^\top)^\top.$$

(ii) We get that $(n-1)^{-1}\tilde{\mathbf{H}}_{22} \sim \mathcal{W}_1^{-1}(n-p+2, \mathbf{H}_{22})$. Hence,

$$(n-1) \frac{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}}{\mathbf{1}^\top \hat{\boldsymbol{\Sigma}}^{-1} \mathbf{1}} = (n-1) \frac{\hat{V}_{GMV}}{V_{GMV}} \sim \chi_{n-p}^2; \quad (7.3)$$

(iii) Let

$$\Gamma_l\left(\frac{m}{2}\right) = \pi^{l(l-1)/4} \prod_{i=1}^l \Gamma\left(\frac{m-i+1}{2}\right).$$

be the multivariate gamma function. Then, the density of $\tilde{\mathbf{H}}_{12}/\tilde{\mathbf{H}}_{22} = \begin{pmatrix} \hat{\boldsymbol{\theta}}^\top & \tilde{R}_{GMV} \end{pmatrix}^\top$ is given by

$$\begin{aligned} f(\mathbf{y}) &= \frac{|\mathbf{G}|^{-\frac{1}{2}} |\mathbf{H}_{22}|^{\frac{(k+1)}{2}} \Gamma_{k+1}\left(\frac{n-p+k+2}{2}\right)}{\pi^{\frac{k+1}{2}} \Gamma_{k+1}\left(\frac{n-p+k+1}{2}\right)} \\ &\times |\mathbf{I} + \mathbf{G}^{-1}(\mathbf{y} - \mathbf{H}_{12}/\mathbf{H}_{22})\mathbf{H}_{22}(\mathbf{y} - \mathbf{H}_{12}/\mathbf{H}_{22})^\top|^{-\frac{n-p+k+2}{2}} \\ &= \frac{|\mathbf{G}/\mathbf{H}_{22}|^{-\frac{1}{2}} \Gamma_{k+1}\left(\frac{n-p+k+2}{2}\right)}{\pi^{\frac{k+1}{2}} \Gamma_{k+1}\left(\frac{n-p+k+1}{2}\right)} \\ &\times \left(1 + \mathbf{H}_{22}(\mathbf{y} - \mathbf{H}_{12}/\mathbf{H}_{22})^\top \mathbf{G}^{-1}(\mathbf{y} - \mathbf{H}_{12}/\mathbf{H}_{22})\right)^{-\frac{n-p+k+2}{2}} \end{aligned} \quad (7.4)$$

where the last equality is obtained by the use of the Sylvester determinant identity. The density presented in (7.4) corresponds to a $(k+1)$ -dimensional t distribution with $(n-p+1)$ degrees of freedom, location parameter $\mathbf{H}_{12}/\mathbf{H}_{22} = \begin{pmatrix} \boldsymbol{\theta}^\top & \check{R}_{GMV} \end{pmatrix}^\top$ and scale matrix $\frac{V_{GMV}}{n-p+1} \mathbf{G}$.

In the proof of parts (iv)-(vi) we use the following result (see Theorem 3.f of Bodnar and Okhrin [15])

$$(n-1)^{-1}\tilde{\mathbf{G}}|\hat{\boldsymbol{\theta}}^\top, \tilde{R}_{GMV} \sim \mathcal{W}_{k+1}^{-1}\left(n-p+2k+4, \tilde{\mathbf{B}}\right).$$

where

$$\tilde{\mathbf{B}} = \mathbf{G} + \frac{1}{V_{GMV}} \begin{pmatrix} \hat{\boldsymbol{\theta}} - \boldsymbol{\theta} \\ \tilde{R}_{GMV} - \check{R}_{GMV} \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{\theta}} - \boldsymbol{\theta} \\ \tilde{R}_{GMV} - \check{R}_{GMV} \end{pmatrix}^\top = \begin{pmatrix} \tilde{\mathbf{B}}_{11} & \tilde{\mathbf{B}}_{12} \\ \tilde{\mathbf{B}}_{21} & \tilde{\mathbf{B}}_{22} \end{pmatrix}$$

with $\tilde{\mathbf{B}}_{22} = \mathbf{G}_{22} + \frac{(\tilde{R}_{GMV} - \check{R}_{GMV})^2}{V_{GMV}}$

Hence,

(iv) $\tilde{s} = \tilde{\mathbf{G}}_{22}$ and $\tilde{\boldsymbol{\eta}} = \tilde{\mathbf{G}}_{12}/\tilde{\mathbf{G}}_{22}$ are conditionally independent given $\hat{\boldsymbol{\theta}}^\top$ and $\tilde{R}_{GMV}$.

(v) It holds that $(n-1)^{-1}\tilde{\mathbf{G}}_{22}|\hat{\boldsymbol{\theta}}^\top, \tilde{R}_{GMV} \sim \mathcal{W}_1^{-1}\left(n-p+4, \tilde{\mathbf{B}}_{22}\right)$. Hence,

$$(n-1) \frac{\tilde{s} + (\tilde{R}_{GMV} - \check{R}_{GMV})^2/V_{GMV}}{\tilde{s}} \sim \chi_{n-p+2}^2. \quad (7.5)$$

(vi) Finally, similarly to the proof of part (iii), we get

$$\tilde{\boldsymbol{\eta}}|\hat{\boldsymbol{\theta}}^\top, \tilde{R}_{GMV} \sim t_k \left(n-p+3, \frac{\tilde{\mathbf{B}}_{12}}{\tilde{\mathbf{B}}_{22}}, \frac{1}{n-p+3} \frac{\tilde{\mathbf{B}}_{11}\tilde{\mathbf{B}}_{22} - \tilde{\mathbf{B}}_{12}\tilde{\mathbf{B}}_{21}}{\tilde{\mathbf{B}}_{22}^2} \right),$$

where

$$\begin{aligned} \tilde{\mathbf{B}}_{11}\tilde{\mathbf{B}}_{22} - \tilde{\mathbf{B}}_{12}\tilde{\mathbf{B}}_{21} &= \left(\mathbf{G}_{11} + \frac{1}{V_{GMV}}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^\top \right) \left(\mathbf{G}_{22} + \frac{(\tilde{R}_{GMV} - \check{R}_{GMV})^2}{V_{GMV}} \right) \\ &- \left(\mathbf{G}_{12} + \frac{\tilde{R}_{GMV} - \check{R}_{GMV}}{V_{GMV}}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \right) \left(\mathbf{G}_{12} + \frac{\tilde{R}_{GMV} - \check{R}_{GMV}}{V_{GMV}}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \right)^\top \\ &= \mathbf{G}_{11}\mathbf{G}_{22} - \mathbf{G}_{12}\mathbf{G}_{21} + \frac{\mathbf{G}_{22}}{V_{GMV}}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^\top + \frac{(\tilde{R}_{GMV} - \check{R}_{GMV})^2}{V_{GMV}}\mathbf{G}_{11} \\ &- \frac{\tilde{R}_{GMV} - \check{R}_{GMV}}{V_{GMV}} \left((\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})\mathbf{G}_{12}^\top + \mathbf{G}_{12}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^\top \right) \\ &= \left(\mathbf{G}_{11} - \frac{\mathbf{G}_{12}\mathbf{G}_{21}}{\mathbf{G}_{22}} \right) \left(\mathbf{G}_{22} + \frac{(\tilde{R}_{GMV} - \check{R}_{GMV})^2}{V_{GMV}} \right) \\ &+ \frac{\mathbf{G}_{22}}{V_{GMV}} \left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta} - \frac{\mathbf{G}_{12}}{\mathbf{G}_{22}}(\tilde{R}_{GMV} - \check{R}_{GMV}) \right) \left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta} - \frac{\mathbf{G}_{12}}{\mathbf{G}_{22}}(\tilde{R}_{GMV} - \check{R}_{GMV}) \right)^\top \end{aligned}$$

□

Proof of Theorem 2.1: From Theorem 7.1.ii we get

$$\hat{V}_{GMV} \stackrel{d}{=} \frac{V_{GMV}}{n-1} \xi_1 \quad (7.6)$$

where $\xi_1 \sim \chi_{n-p}^2$. Moreover, Theorem 7.1.iii implies that $\hat{\boldsymbol{\theta}}$ and $\tilde{R}_{GMV}$ are jointly multivariate t -distributed and, hence, it holds that (see, e.g., Ding [28])

$$\tilde{R}_{GMV} \sim t \left(n-p+1, \check{R}_{GMV}, \frac{V_{GMV}\check{s}}{n-p+1} \right)$$

and

$$\begin{aligned} \hat{\boldsymbol{\theta}} | \tilde{R}_{GMV} &\sim t_k \left(n-p+2, \boldsymbol{\theta} + \check{\boldsymbol{\eta}}(\tilde{R}_{GMV} - \check{R}_{GMV}), \right. \\ &\quad \left. \frac{n-p+1 + (n-p+1)(\tilde{R}_{GMV} - \check{R}_{GMV})^2 / (V_{GMV}\check{s})}{n-p+2} \frac{V_{GMV}}{n-p+1} (\mathbf{L}\mathbf{Q}\mathbf{L}^\top - \check{s}\check{\boldsymbol{\eta}}\check{\boldsymbol{\eta}}^\top) \right) \\ &= t_k \left(n-p+2, \boldsymbol{\theta} + \check{\boldsymbol{\eta}}(\tilde{R}_{GMV} - \check{R}_{GMV}), \right. \\ &\quad \left. \frac{V_{GMV}}{n-p+2} \left(1 + \frac{(\tilde{R}_{GMV} - \check{R}_{GMV})^2}{V_{GMV}\check{s}} \right) (\mathbf{L}\mathbf{Q}\mathbf{L}^\top - \check{s}\check{\boldsymbol{\eta}}\check{\boldsymbol{\eta}}^\top) \right) \end{aligned}$$

As a result, we get

$$\hat{R}_{GMV} \stackrel{d}{=} \frac{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \hat{\boldsymbol{\mu}}}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}} + \frac{\sqrt{V_{GMV}} \sqrt{\hat{\boldsymbol{\mu}}^\top \mathbf{Q} \hat{\boldsymbol{\mu}}}}{\sqrt{n-p+1}} t_1 \quad (7.7)$$

and

$$\begin{aligned} \hat{\boldsymbol{\theta}} &\stackrel{d}{=} \boldsymbol{\theta} + \sqrt{V_{GMV}} \frac{t_1}{\sqrt{n-p+1}} \frac{\mathbf{L}\mathbf{Q}\hat{\boldsymbol{\mu}}}{\sqrt{\hat{\boldsymbol{\mu}}^\top \mathbf{Q} \hat{\boldsymbol{\mu}}}} \\ &\quad + \sqrt{1 + \frac{t_1^2}{n-p+1} \frac{\sqrt{V_{GMV}}}{\sqrt{n-p+2}}} \left(\mathbf{L}\mathbf{Q}\mathbf{L}^\top - \frac{\mathbf{L}\mathbf{Q}\hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}^\top \mathbf{Q}\mathbf{L}^\top}{\hat{\boldsymbol{\mu}}^\top \mathbf{Q} \hat{\boldsymbol{\mu}}} \right)^{1/2} \mathbf{t}_2 \\ &= \boldsymbol{\theta} + \sqrt{V_{GMV}} \left(\frac{\mathbf{L}\mathbf{Q}\hat{\boldsymbol{\mu}}}{\sqrt{\hat{\boldsymbol{\mu}}^\top \mathbf{Q} \hat{\boldsymbol{\mu}}}} \frac{t_1}{\sqrt{n-p+1}} \right. \\ &\quad \left. + \left(\mathbf{L}\mathbf{Q}\mathbf{L}^\top - \frac{\mathbf{L}\mathbf{Q}\hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}^\top \mathbf{Q}\mathbf{L}^\top}{\hat{\boldsymbol{\mu}}^\top \mathbf{Q} \hat{\boldsymbol{\mu}}} \right)^{1/2} \sqrt{1 + \frac{t_1^2}{n-p+1} \frac{\mathbf{t}_2}{\sqrt{n-p+2}}} \right) \end{aligned} \quad (7.8)$$

where $t_1 \sim t(n-p+1)$, $\mathbf{t}_2 \sim t_k(n-p+2)$ are independent and also they are independent of ξ_1 .

Similarly, the application of Theorem 7.1.v leads to

$$\hat{s} \stackrel{d}{=} (n-1) \left(1 + \frac{t_1^2}{n-p+1} \right) \frac{\hat{\boldsymbol{\mu}}^\top \mathbf{Q} \hat{\boldsymbol{\mu}}}{\xi_2} \quad (7.9)$$

where $\xi_2 \sim \chi_{n-p+2}^2$ and is independent of t_1 , $\mathbf{t}_2$, and ξ_1 .

Finally, the application of Theorem 7.1.vi leads to

$$\begin{aligned}
\hat{\eta} &\stackrel{d}{=} \frac{\mathbf{LQ}\hat{\boldsymbol{\mu}}}{\hat{\boldsymbol{\mu}}^\top \mathbf{Q}\hat{\boldsymbol{\mu}}} + \frac{\sqrt{1 + \frac{t_1^2}{n-p+1}} \left(\mathbf{LQ}\mathbf{L}^\top - \frac{\mathbf{LQ}\hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}^\top \mathbf{Q}\mathbf{L}^\top}{\hat{\boldsymbol{\mu}}^\top \mathbf{Q}\hat{\boldsymbol{\mu}}} \right)^{1/2} \frac{\mathbf{t}_2}{\sqrt{n-p+2}} \frac{1}{\sqrt{\hat{\boldsymbol{\mu}}^\top \mathbf{Q}\hat{\boldsymbol{\mu}}}} \frac{t_1}{\sqrt{n-p+1}}}{1 + \frac{t_1^2}{n-p+1}} \\
&+ \frac{1}{\sqrt{\hat{\boldsymbol{\mu}}^\top \mathbf{Q}\hat{\boldsymbol{\mu}}}} \frac{1}{1 + \frac{t_1^2}{n-p+1}} \left(\left(\mathbf{LQ}\mathbf{L}^\top - \frac{\mathbf{LQ}\hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}^\top \mathbf{Q}\mathbf{L}^\top}{\hat{\boldsymbol{\mu}}^\top \mathbf{Q}\hat{\boldsymbol{\mu}}} \right) \left(1 + \frac{t_1^2}{n-p+1} \right) \right. \\
&+ \frac{\hat{\boldsymbol{\mu}}^\top \mathbf{Q}\hat{\boldsymbol{\mu}}}{V_{GMV}} \left(1 + \frac{t_1^2}{n-p+1} \right) \frac{V_{GMV}}{n-p+2} \left(\mathbf{LQ}\mathbf{L}^\top - \frac{\mathbf{LQ}\hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}^\top \mathbf{Q}\mathbf{L}^\top}{\hat{\boldsymbol{\mu}}^\top \mathbf{Q}\hat{\boldsymbol{\mu}}} \right)^{1/2} \\
&\times \mathbf{t}_2 \mathbf{t}_2^\top \left(\left(\mathbf{LQ}\mathbf{L}^\top - \frac{\mathbf{LQ}\hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}^\top \mathbf{Q}\mathbf{L}^\top}{\hat{\boldsymbol{\mu}}^\top \mathbf{Q}\hat{\boldsymbol{\mu}}} \right)^{1/2} \right)^\top \left. \right)^{1/2} \frac{\mathbf{t}_3}{\sqrt{n-p+3}} \\
&= \frac{\mathbf{LQ}\hat{\boldsymbol{\mu}}}{\hat{\boldsymbol{\mu}}^\top \mathbf{Q}\hat{\boldsymbol{\mu}}} + \frac{1}{\sqrt{\hat{\boldsymbol{\mu}}^\top \mathbf{Q}\hat{\boldsymbol{\mu}}}} \left(\mathbf{LQ}\mathbf{L}^\top - \frac{\mathbf{LQ}\hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}^\top \mathbf{Q}\mathbf{L}^\top}{\hat{\boldsymbol{\mu}}^\top \mathbf{Q}\hat{\boldsymbol{\mu}}} \right)^{1/2} \\
&\times \left(\frac{1}{\sqrt{1 + \frac{t_1^2}{n-p+1}}} \frac{\mathbf{t}_2}{\sqrt{n-p+2}} \frac{t_1}{\sqrt{n-p+1}} + \left(\mathbf{I}_k + \hat{\boldsymbol{\mu}}^\top \mathbf{Q}\hat{\boldsymbol{\mu}} \frac{\mathbf{t}_2 \mathbf{t}_2^\top}{n-p+2} \right)^{1/2} \frac{\mathbf{t}_3}{\sqrt{n-p+3}} \right)
\end{aligned} \tag{7.10}$$

where $\mathbf{t}_3 \sim t_k(n-p+3)$ and is independent of t_1 and $\mathbf{t}_2$. Moreover, due to Theorem 7.1.i and 7.1.iv we get that $\xi_1, \xi_2, t_1, \mathbf{t}_2$, and $\mathbf{t}_3$ are mutually independent.

Next, we derive stochastic representations for the linear and quadratic forms in $\hat{\boldsymbol{\mu}}$, namely of $\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \hat{\boldsymbol{\mu}}$, $\mathbf{LQ}\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\mu}}^\top \mathbf{Q}\hat{\boldsymbol{\mu}}$ which are present in the derived above stochastic representations. Let $\mathbf{P} = \mathbf{Q}\mathbf{L}^\top (\mathbf{LQ}\mathbf{L}^\top)^{-1/2}$ and $\mathbf{A} = \mathbf{Q} - \mathbf{P}\mathbf{P}^\top = \mathbf{Q} - \mathbf{Q}\mathbf{L}^\top (\mathbf{LQ}\mathbf{L}^\top)^{-1} \mathbf{LQ}$. Then

$$\hat{\boldsymbol{\mu}}^\top \mathbf{Q}\hat{\boldsymbol{\mu}} = \hat{\boldsymbol{\mu}}^\top \mathbf{A}\hat{\boldsymbol{\mu}} + (\mathbf{P}^\top \hat{\boldsymbol{\mu}})^\top (\mathbf{P}^\top \hat{\boldsymbol{\mu}}). \tag{7.11}$$

Moreover, the equality $\mathbf{1}^\top \mathbf{Q} = \mathbf{0}^\top$ implies

$$\begin{pmatrix} \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \\ \mathbf{P}^\top \end{pmatrix} \boldsymbol{\Sigma} \mathbf{A} = \begin{pmatrix} \mathbf{1}^\top \mathbf{A} \\ \mathbf{P}^\top \boldsymbol{\Sigma} \mathbf{A} \end{pmatrix} = \begin{pmatrix} \mathbf{0}^\top \\ \mathbf{P}^\top - \mathbf{P}^\top \end{pmatrix} = \mathbf{0}$$

and, consequently, we get from Theorem 5.5.1 in Mathai and Provost [46] that $\hat{\boldsymbol{\mu}}^\top \mathbf{A}\hat{\boldsymbol{\mu}}$ is independent of $\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \hat{\boldsymbol{\mu}}$ and $\mathbf{P}^\top \hat{\boldsymbol{\mu}}$, while Corollary 5.1.3a in Mathai and Provost [46] implies that

$$n \hat{\boldsymbol{\mu}}^\top \mathbf{A}\hat{\boldsymbol{\mu}} \stackrel{d}{=} \xi_3 \tag{7.12}$$

where $\xi_3 \sim \chi_{p-k-1; n \hat{\boldsymbol{\mu}}^\top \mathbf{A}\hat{\boldsymbol{\mu}}}^2$.

Finally, the identity $\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\Sigma} \mathbf{P} = \mathbf{0}$ ensures that $\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \hat{\boldsymbol{\mu}}$ and $\mathbf{P}^\top \hat{\boldsymbol{\mu}}$ are independent (c.f., Rencher [51, Chapter 2.2]) with

$$\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \hat{\boldsymbol{\mu}} \stackrel{d}{=} \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} + \sqrt{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}} \frac{z_1}{\sqrt{n}} = \frac{R_{GMV}}{V_{GMV}} + \frac{1}{\sqrt{V_{GMV}}} \frac{z_1}{\sqrt{n}} \tag{7.13}$$

and

$$\mathbf{P}^\top \hat{\boldsymbol{\mu}} \stackrel{d}{=} \mathbf{P}^\top \boldsymbol{\mu} + (\mathbf{P}^\top \boldsymbol{\Sigma} \mathbf{P})^{1/2} \frac{\tilde{\mathbf{z}}_2}{\sqrt{n}} = (\mathbf{L} \mathbf{Q} \mathbf{L}^\top)^{-1/2} s \boldsymbol{\eta} + \frac{\mathbf{z}_2}{\sqrt{n}} \quad (7.14)$$

where $z_1 \sim \mathcal{N}(0, 1)$ and $\tilde{\mathbf{z}}_2 \sim \mathcal{N}_k(\mathbf{0}, \mathbf{I}_k)$ are independent. Inserting (7.11) – (7.14) in (7.6) – (7.10) and performing some algebra, we get the statement of the theorem. $\square$

Proof of Theorem 3.1: The statement of the theorem follows directly from the results of Theorem 2.1. $\square$

Proof of Theorem 3.2: The mutual independence of ξ , ψ , and z follows from Theorem 2.1, while Theorem 2.1.i provides the stochastic representation for $\hat{V}_{GMV}$.

Next, we derive the joint stochastic representation for $\hat{R}_{GMV}$ and $\hat{s}$. Let $\xi_2 = \xi_2^{-1}$. Then, the distribution of $(\hat{R}_{GMV}, \hat{s}, t_1, f)$ is obtained as a transformation of $(z_1, \tilde{\xi}_2, t_1, f)$ where $\tilde{\xi}_2 = 1/\xi_2$ with the Jacobian matrix given by

$$\mathbf{J} = \begin{pmatrix} \frac{\sqrt{V_{GMV}}}{\sqrt{n}} & 0 & \frac{\sqrt{f}\sqrt{V_{GMV}}}{\sqrt{n-p+1}} & \frac{1}{2} \frac{\sqrt{V_{GMV}} t_1}{\sqrt{n-p+1}\sqrt{f}} \\ 0 & (n-1) \left(1 + \frac{t_1^2}{n-p+1}\right) f & \frac{2(n-1)}{n-p+1} f t_1 \tilde{\xi}_2 & (n-1) \left(1 + \frac{t_1^2}{n-p+1}\right) \tilde{\xi}_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which implies that $|\mathbf{J}| = \frac{(n-1)}{\sqrt{n}} \sqrt{V_{GMV}} \left(1 + \frac{t_1^2}{n-p+1}\right) f$.

Let $d_f(\cdot)$ denote the marginal density of the distribution of f . Ignoring the normalizing constants, we get the joint density of $(\hat{R}_{GMV}, \hat{s}, t_1, f)$ expressed as

$$\begin{aligned} d(\hat{R}_{GMV}, \hat{s}, t_1, f) &\propto \exp \left\{ -\frac{n}{2} \frac{\left(\hat{R}_{GMV} - R_{GMV} - \sqrt{f} \frac{t_1 \sqrt{V_{GMV}}}{\sqrt{n-p+1}}\right)^2}{V_{GMV}} \right\} \\ &\times \left(\frac{(n-1)f}{\hat{s}} \left(1 + \frac{t_1^2}{n-p+1}\right) \right)^{\frac{n-p+2}{2}+1} \exp \left\{ -\frac{(n-1)f}{2\hat{s}} \left(1 + \frac{t_1^2}{n-p+1}\right) \right\} \\ &\times \left(1 + \frac{t_1^2}{n-p+1}\right)^{-\frac{n-p+2}{2}} \left(f \left(1 + \frac{t_1^2}{n-p+1}\right) \right)^{-1} d_f(f) \\ &\propto \left(\frac{f}{\hat{s}} \right)^{\frac{n-p+2}{2}+1} \frac{1}{f} \exp \left\{ -\frac{n}{2} \frac{\left(\hat{R}_{GMV} - R_{GMV}\right)^2}{V_{GMV}} + \frac{n \left(\hat{R}_{GMV} - R_{GMV}\right) \sqrt{f} \frac{t_1}{\sqrt{n-p+1}}}{\sqrt{V_{GMV}}} \right. \\ &\quad \left. - \frac{(n-1)f}{2\hat{s}} - \frac{1}{2} \left(n + \frac{n-1}{\hat{s}} \right) \frac{f t_1^2}{n-p+1} \right\} d_f(f). \end{aligned}$$

We now notice that

$$\begin{aligned}
& \exp \left\{ \frac{n \left(\hat{R}_{GMV} - R_{GMV} \right) \sqrt{f} \frac{t_1}{\sqrt{n-p+1}} - \frac{(n\hat{s} + (n-1))f}{2\hat{s}(n-p+1)} t_1^2}{\sqrt{V_{GMV}}} \right\} \\
&= \exp \left\{ -\frac{(n\hat{s} + (n-1))f}{2\hat{s}(n-p+1)} \left(t_1 - \frac{n^2\hat{s}\sqrt{n-p+1} \left(\hat{R}_{GMV} - R_{GMV} \right)}{\sqrt{V_{GMV}}f(n\hat{s} + (n-1))} \right)^2 \right\} \\
&\times \exp \left\{ \frac{n^2\hat{s} \left(\hat{R}_{GMV} - R_{GMV} \right)^2}{2V_{GMV}(n\hat{s} + (n-1))} \right\},
\end{aligned}$$

where the first factor is the kernel of a normal distribution. Hence,

$$\begin{aligned}
d(\hat{R}_{GMV}, \hat{s}) &= \int_{\mathbb{R}_+} \int_{\mathbb{R}} d(\hat{R}_{GMV}, \hat{s}, t_1, f) dt_1 df \\
&\propto \exp \left\{ -\frac{n \left(\hat{R}_{GMV} - R_{GMV} \right)^2}{2V_{GMV}} \right\} \exp \left\{ \frac{n^2\hat{s} \left(\hat{R}_{GMV} - R_{GMV} \right)^2}{2V_{GMV}(n\hat{s} + (n-1))} \right\} \\
&\times \int_{\mathbb{R}_+} \left(\frac{f}{\hat{s}} \right)^{\frac{n-p+2}{2}+1} \frac{e^{-\frac{f}{2\hat{s}}}}{f} d_f(f) \int_{\mathbb{R}} e^{-\frac{((n-1)+n\hat{s})f}{2\hat{s}(n-p+1)} \left(t_1 - \frac{\hat{s}\sqrt{n-p+1}(\hat{R}_{GMV}-R_{GMV})}{\sqrt{V_{GMV}}f(\hat{s}-1+1/n)} \right)^2} dt_1 df \\
&\propto \left(1 + \frac{n}{n-1} \hat{s} \right)^{-1/2} \exp \left\{ -\frac{n \left(\hat{R}_{GMV} - R_{GMV} \right)^2}{2 \left(1 + \frac{n}{n-1} \hat{s} \right) V_{GMV}} \right\} \tag{7.15}
\end{aligned}$$

$$\int_{\mathbb{R}_+} \left(\frac{f}{\hat{s}} \right)^{\frac{n-p+1}{2}+1} \frac{e^{-\frac{(n-1)f}{2\hat{s}}}}{f} d_f(f) df. \tag{7.16}$$

where (7.15) determines the conditional distribution of $\hat{R}_{GMV}$ given $\hat{s}$ which is a normal distribution with mean R_{GMV} and variance $\left(1 + \frac{n}{n-1} \hat{s} \right) \frac{V_{GMV}}{n}$. The expression in (7.16) specifies the marginal distribution of $\hat{s}$ which appears to be the integral representation of the density of the ratio of two independent variables f and ζ with $(n-1)\zeta \sim \chi_{n-p+1}^2$ and $nf \sim \chi_{p-1}^2(ns)$ (c.f., Mathai and Provost [46, Theorem 5.1.3]). Hence, $n(n-p+1)/((n-1)(p-1))\hat{s}$ has a noncentral F -distribution with $(p-1)$ and $(n-p+1)$ degrees of freedom and noncentrality parameter ns . $\square$

Proof of Theorem 4.1. If $\xi \sim \chi_{m,\delta}^2$, then it holds that (see, e.g., Bodnar and Reiß [18, Lemma 3])

$$\left(\frac{\xi}{m} - 1 - \frac{\delta}{m} \right) \xrightarrow{a.s.} 0 \quad \text{and} \quad \sqrt{m} \left(2 \left(1 + 2\frac{\delta}{m} \right) \right)^{-1/2} \left(\frac{\xi}{m} - 1 - \frac{\delta}{m} \right) \xrightarrow{d} N(0, 1) \tag{7.17}$$

for $m \rightarrow \infty$.

Throughout the proof of the theorem the asymptotic results are derived under the high-dimensional asymptotic regime, that is under $p/n \rightarrow c \in (0, 1)$ as $n \rightarrow \infty$. The applications of Slutsky's lemma (c.f., DasGupta [26, Theorem 1.5]) and Theorem 2.1, and the fact that a t-distribution with infinite degrees of freedom tends to the standard normal distribution yield the following results:

- (i) The application of Theorem 2.1.i and (7.17) with $m = n - p$ leads to

$$\sqrt{n-p} \left(\hat{V}_{GMV} - \frac{1-p/n}{1-1/n} V_{GMV} \right) \stackrel{d}{=} \frac{1-p/n}{1-1/n} V_{GMV} \sqrt{n-p} \left(\frac{\xi_2}{n-p} - 1 \right) \xrightarrow{d} \sqrt{2}(1-c) V_{GMV} u_1,$$

where $u_1 \sim N(0, 1)$.

- (ii) Using (7.17) with $m = p - k - 1$ and $\delta = n\boldsymbol{\mu}^\top \mathbf{A}\boldsymbol{\mu}$, we get

$$\begin{aligned} f &\stackrel{d}{=} \frac{\xi_3}{n} + \left(s\boldsymbol{\eta} + \frac{\mathbf{z}_2}{\sqrt{n}} \right)^\top (\mathbf{L}\mathbf{Q}\mathbf{L}^\top)^{-1} \left(s\boldsymbol{\eta} + \frac{\mathbf{z}_2}{\sqrt{n}} \right) \\ &= \frac{(p-k-1)}{n} \left(\frac{\xi_3}{p-k-1} - 1 - \frac{n\boldsymbol{\mu}^\top \mathbf{A}\boldsymbol{\mu}}{p-k-1} \right) + \frac{(p-k-1)}{n} + \boldsymbol{\mu}^\top \mathbf{Q}\boldsymbol{\mu} \\ &\quad + \frac{1}{\sqrt{n}} \left(2s\boldsymbol{\eta}(\mathbf{L}\mathbf{Q}\mathbf{L}^\top)^{-1}\mathbf{z}_2 + \frac{1}{\sqrt{n}}\mathbf{z}_2^\top (\mathbf{L}\mathbf{Q}\mathbf{L}^\top)^{-1}\mathbf{z}_2 \right) \xrightarrow{a.s.} s + c \end{aligned} \quad (7.18)$$

and, hence,

$$\sqrt{n-p} (f - (s + p/n)) \xrightarrow{d} \sqrt{2(1-c)} (c + 2\boldsymbol{\mu}^\top \mathbf{A}\boldsymbol{\mu}) u_2 + 2s\sqrt{(1-c)} \boldsymbol{\eta}^\top (\mathbf{L}\mathbf{Q}\mathbf{L}^\top)^{-1/2} \mathbf{u}_3,$$

where $u_2 \sim N(0, 1)$ and $\mathbf{u}_3 \sim N_k(\mathbf{0}, \mathbf{I}_k)$ which are independent of u_1 following Theorem 2.1. Furthermore, the application of (7.18) yields

$$\begin{aligned} \sqrt{n-p} (\hat{R}_{GMV} - R_{GMV}) &\stackrel{d}{=} \sqrt{V_{GMV}} \left(\sqrt{1-p/n} z_1 + \left(\frac{1-p/n}{1-p/n+1/n} \right)^{1/2} \sqrt{f} t_1 \right) \\ &\xrightarrow{d} \sqrt{V_{GMV}} (\sqrt{1-c} u_4 + \sqrt{s+c} u_5) \end{aligned} \quad (7.19)$$

where $u_4, u_5 \sim N(0, 1)$ and $u_1, u_2, \mathbf{u}_3, u_4, u_5$ independent.

- (iii) Furthermore, by the stochastic representation of $\hat{\boldsymbol{\theta}}$ as given in Theorem 2.1.iii we have that

$$\begin{aligned} \sqrt{n-p} (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) &\stackrel{d}{=} \sqrt{V_{GMV}} \left(\frac{s\boldsymbol{\eta} + \mathbf{z}_2/\sqrt{n}}{\sqrt{f}} \sqrt{\frac{1-p/n}{1-p/n+1/n}} t_1 \right. \\ &\quad \left. + \left(\mathbf{L}\mathbf{Q}\mathbf{L}^\top - \frac{(s\boldsymbol{\eta} + \mathbf{z}_2/\sqrt{n})(s\boldsymbol{\eta} + \mathbf{z}_2/\sqrt{n})^\top}{f} \right)^{1/2} \sqrt{1 + \frac{t_1^2}{n-p+1}} \frac{\sqrt{n-p}}{\sqrt{n-p+2}} \mathbf{t}_2 \right) \\ &\xrightarrow{d} \sqrt{V_{GMV}} \left(\frac{s\boldsymbol{\eta}}{\sqrt{s+c}} u_5 + \left(\mathbf{L}\mathbf{Q}\mathbf{L}^\top - \frac{s^2}{s+c} \boldsymbol{\eta}\boldsymbol{\eta}^\top \right)^{1/2} \mathbf{u}_6 \right) \end{aligned} \quad (7.20)$$

where $\mathbf{u}_6 \sim N_k(\mathbf{0}, \mathbf{I}_k)$ and is independent of $u_1, u_2, \mathbf{u}_3, u_4$, and u_5 .

(iv) The application of Theorem 2.1.iv and (7.17) leads to

$$\begin{aligned} & \sqrt{n-p} \left(\hat{s} - \frac{(s+p/n)(1-1/n)}{1-p/n+2/n} \right) \\ \stackrel{d}{=} & \frac{1-1/n}{1-p/n+2/n} \left(\left(1 + \frac{t_1^2}{n-p+1} \right) \frac{\sqrt{n-p}(f-(s+p/n))}{\xi_2/(n-p+2)} \right. \\ & + (s+p/n) \left(\frac{\frac{t_1^2}{n-p+1} - \left(\frac{\xi_2}{n-p+2} - 1 \right)}{\xi_2/(n-p+2)} \right) \Bigg) \\ \stackrel{d}{\rightarrow} & \frac{1}{1-c} \left(\sqrt{2(1-c)(c+2\boldsymbol{\mu}^\top \mathbf{A} \boldsymbol{\mu})} u_2 + 2s\sqrt{(1-c)} \boldsymbol{\eta}^\top (\mathbf{L} \mathbf{Q} \mathbf{L}^\top)^{-1/2} \mathbf{u}_3 + \sqrt{2}(s+c)u_7 \right), \end{aligned}$$

where $u_7 \sim N(0, 1)$ and is independent of $u_1, u_2, \mathbf{u}_3, u_4, u_5$, and $\mathbf{u}_6$.

(v) Similarly, from Theorem 2.1.v we get

$$\begin{aligned} & \sqrt{n-p} \left(\hat{\boldsymbol{\eta}} - \frac{s}{s+p/n} \boldsymbol{\eta} \right) \stackrel{d}{=} \frac{1}{f} \left(\frac{-s}{s+p/n} \sqrt{n-p} (f-(s+p/n)) \boldsymbol{\eta} + \sqrt{1-p/n} \mathbf{z}_2 \right) \\ & + \frac{1}{\sqrt{f}} \left(\mathbf{L} \mathbf{Q} \mathbf{L}^\top - \frac{(s\boldsymbol{\eta} + \mathbf{z}_2/\sqrt{n})(s\boldsymbol{\eta} + \mathbf{z}_2/\sqrt{n})^\top}{f} \right)^{1/2} \\ & \times \left(\frac{1}{\sqrt{1+\frac{t_1^2}{n-p+1}}} \frac{\mathbf{t}_2}{\sqrt{n-p+2}} \left(\frac{n-p}{n-p+1} \right)^{1/2} t_1 \right. \\ & + \left(\mathbf{I}_k + f \frac{\mathbf{t}_2 \mathbf{t}_2^\top}{n-p+2} \right)^{1/2} \left(\frac{n-p}{n-p+3} \right)^{1/2} \mathbf{t}_3 \Bigg) \\ & \stackrel{d}{\rightarrow} \frac{1}{\sqrt{s+c}} \left(\mathbf{L} \mathbf{Q} \mathbf{L}^\top - \frac{s^2 \boldsymbol{\eta} \boldsymbol{\eta}^\top}{s+c} \right)^{1/2} \bar{\mathbf{u}}_8 \\ & + \frac{\sqrt{1-c}}{(s+c)} \left(\mathbf{L} \mathbf{Q} \mathbf{L}^\top - 2 \frac{s^2 \boldsymbol{\eta} \boldsymbol{\eta}^\top}{s+c} \right) (\mathbf{L} \mathbf{Q} \mathbf{L}^\top)^{-1/2} \mathbf{u}_3 - \frac{s\sqrt{2(1-c)(c+2\boldsymbol{\mu}^\top \mathbf{A} \boldsymbol{\mu})} u_2}{(s+c)^2} \boldsymbol{\eta}, \end{aligned}$$

where $\mathbf{u}_8 \sim N_k(\mathbf{0}, \mathbf{I}_k)$ and $u_1, u_2, \mathbf{u}_3, u_4, u_5, \mathbf{u}_6, u_7, u_8$ are mutually independent distributed.

□

Proof of Theorem 4.2: The application of Theorem 4.1 and of the continuous mapping theorem (c.f., DasGupta [26, Theorem 1.14]) leads to

$$\mathbf{L} \hat{\mathbf{w}}_g \xrightarrow{a.s.} \boldsymbol{\theta} + \frac{sg(R_{GMV}, (1-c)V_{GMV}, (s+c)/(1-c))}{s+c} \boldsymbol{\eta}$$

for $p/n \rightarrow c$ as $n \rightarrow \infty$.

Let $\hat{\lambda}$ and λ be defined as in (4.5). Then, the first order Taylor series expansion yields

$$\begin{aligned}
& \sqrt{n-p} \left(\mathbf{L}\hat{\mathbf{w}}_g - \left(\boldsymbol{\theta} + \frac{sg(\lambda)}{s+p/n} \boldsymbol{\eta} \right) \right) \\
&= \sqrt{n-p} (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) + \sqrt{n-p} \left(\hat{\boldsymbol{\eta}} - \frac{s}{s+p/n} \boldsymbol{\eta} \right) g(\hat{\lambda}) + \sqrt{n-p} \left(g(\hat{\lambda}) - g(\lambda) \right) \frac{s\boldsymbol{\eta}}{s+p/n} \\
&= \sqrt{n-p} (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) + \sqrt{n-p} \left(\hat{\boldsymbol{\eta}} - \frac{s}{s+p/n} \boldsymbol{\eta} \right) g(\hat{\lambda}) \\
&+ \sqrt{n-p} \begin{pmatrix} \hat{R}_{GMV} - R_{GMV} \\ \hat{V}_{GMV} - (1-p/n)V_{GMV} \\ \hat{s} - \frac{s+p/n}{1-p/n} \end{pmatrix}^\top \begin{pmatrix} g_1(\lambda) \\ g_2(\lambda) \\ g_3(\lambda) \end{pmatrix} \frac{s}{s+p/n} \boldsymbol{\eta} + o_P(1) \tag{7.21}
\end{aligned}$$

Hence, from Theorem 4.1 we get

$$\begin{aligned}
& \sqrt{n-p} \left(\mathbf{L}\hat{\mathbf{w}}_g - \left(\boldsymbol{\theta} + \frac{sg(\lambda)}{s+p/n} \boldsymbol{\eta} \right) \right) \\
&\xrightarrow{d} \sqrt{V_{GMV}} \left(\frac{su_5}{\sqrt{s+c}} \boldsymbol{\eta} + \left(\mathbf{LQL}^\top - \frac{s^2}{s+c} \boldsymbol{\eta}\boldsymbol{\eta}^\top \right)^{1/2} \mathbf{u}_6 \right) \\
&+ \left(\frac{1}{\sqrt{s+c}} \left(\mathbf{LQL}^\top - \frac{s^2}{s+c} \boldsymbol{\eta}\boldsymbol{\eta}^\top \right)^{1/2} \mathbf{u}_8 + \frac{\sqrt{1-c}}{(s+c)} \left(\mathbf{LQL}^\top - 2\frac{s^2}{s+c} \boldsymbol{\eta}\boldsymbol{\eta}^\top \right) (\mathbf{LQL}^\top)^{-1/2} \mathbf{u}_3 \right. \\
&\left. - \frac{s\sqrt{2(1-c)}(c+2\boldsymbol{\mu}^\top \mathbf{A}\boldsymbol{\mu})u_2}{(s+c)^2} \boldsymbol{\eta} \right) g(\lambda) + g_1(\lambda) \left(\sqrt{V_{GMV}} (\sqrt{1-c}u_4 + \sqrt{s+cu_5}) \right) \frac{s}{s+c} \boldsymbol{\eta} \\
&+ g_2(\lambda) \left(\sqrt{2(1-c)}V_{GMV}u_1 \right) \frac{s}{s+c} \boldsymbol{\eta} + g_3(\lambda) \left(\frac{1}{1-c} \left(\sqrt{2(1-c)}(c+2\boldsymbol{\mu}^\top \mathbf{A}\boldsymbol{\mu})u_2 \right. \right. \\
&\left. \left. + 2s\sqrt{(1-c)}\boldsymbol{\eta}^\top (\mathbf{LQL}^\top)^{-1/2} \mathbf{u}_3 + \sqrt{2}(s+c)u_7 \right) \right) \frac{s}{s+c} \boldsymbol{\eta} \\
&= g_2(\lambda) \frac{\sqrt{2(1-c)}V_{GMV}s}{s+c} \boldsymbol{\eta}u_1 + \left(\frac{g_3(\lambda)}{1-c} - \frac{g(\lambda)}{(s+c)} \right) \frac{\sqrt{2(1-c)}(c+2\boldsymbol{\mu}^\top \mathbf{A}\boldsymbol{\mu})s}{s+c} \boldsymbol{\eta}u_2 \\
&+ \frac{\sqrt{1-c}}{s+c} \left(g(\lambda) \mathbf{LQL}^\top + 2s^2 \left(\frac{g_3(\lambda)}{1-c} - \frac{g(\lambda)}{(s+c)} \right) \boldsymbol{\eta}\boldsymbol{\eta}^\top \right) (\mathbf{LQL})^{-1/2} \mathbf{u}_3 \\
&+ \left(\frac{s\sqrt{V_{GMV}}\sqrt{1-c}}{s+c} g_1(\lambda) \right) \boldsymbol{\eta}u_4 + (g_1(\lambda) + 1) \frac{s\sqrt{V_{GMV}}}{\sqrt{s+c}} \boldsymbol{\eta}u_5 \\
&+ \sqrt{V_{GMV}} \left(\mathbf{LQL}^\top - \frac{s^2}{s+c} \boldsymbol{\eta}\boldsymbol{\eta}^\top \right)^{1/2} \mathbf{u}_6 + \left(\sqrt{2} \frac{s}{1-c} g_3(\lambda) \right) \boldsymbol{\eta}u_7 \\
&+ \frac{g(\lambda)}{\sqrt{s+c}} \left(\mathbf{LQL}^\top - \frac{s^2}{s+c} \boldsymbol{\eta}\boldsymbol{\eta}^\top \right)^{1/2} \mathbf{u}_8.
\end{aligned}$$

Using that $u_1, u_2, \mathbf{u}_3, u_4, u_5, \mathbf{u}_6, u_7, u_8$ are mutually independent and standard (multivariate) normally distributed, the expression of the asymptotic covariance matrix of $\mathbf{L}\hat{\mathbf{w}}_g$ is obtained. $\square$

Proof of Theorem 4.4: Using (4.10)-(4.14) together with a first order Taylor expansion we get

that

$$\begin{aligned}
& \sqrt{n-p} (\mathbf{L}\hat{\mathbf{w}}_{g;c} - \mathbf{L}\mathbf{w}_g) \\
\stackrel{d}{=} & \sqrt{n-p} \left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta} \right) - \sqrt{n-p} (\hat{s}_c - s) \frac{p/n}{\hat{s}_c(p/n+s)} g \left(\hat{R}_{GMV;c}, \hat{V}_{GMV;c}, \hat{s}_c \right) \boldsymbol{\eta} \\
& + \sqrt{n-p} \left(\hat{\boldsymbol{\eta}} - \frac{s}{s+p/n} \boldsymbol{\eta} \right) \frac{\hat{s}_c + p/n}{\hat{s}_c} g \left(\hat{R}_{GMV;c}, \hat{V}_{GMV;c}, \hat{s}_c \right) \\
& + \sqrt{n-p} \begin{pmatrix} \hat{R}_{GMV;c} - R_{GMV} \\ \hat{V}_{GMV;c} - V_{GMV} \\ \hat{s}_c - s \end{pmatrix}^\top \begin{pmatrix} g_1(R_{GMV}, V_{GMV}, s) \\ g_2(R_{GMV}, V_{GMV}, s) \\ g_3(R_{GMV}, V_{GMV}, s) \end{pmatrix} \boldsymbol{\eta} + o_P(1) \\
= & \sqrt{n-p} \left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta} \right) + \sqrt{n-p} \left(\hat{\boldsymbol{\eta}} - \frac{s}{s+p/n} \boldsymbol{\eta} \right) \frac{\hat{s}_c + p/n}{\hat{s}_c} g \left(\hat{R}_{GMV;c}, \hat{V}_{GMV;c}, \hat{s}_c \right) \\
& + \sqrt{n-p} \begin{pmatrix} \hat{R}_{GMV} - R_{GMV} \\ \hat{V}_{GMV} - (1-p/n)V_{GMV} \\ \hat{s} - \frac{s+p/n}{1-p/n} \end{pmatrix}^\top \\
& \times \begin{pmatrix} g_1(R_{GMV}, V_{GMV}, s) \\ (1-p/n)^{-1} g_2(R_{GMV}, V_{GMV}, s) \\ (1-p/n) \left(g_3(R_{GMV}, V_{GMV}, s) - \frac{p/n}{\hat{s}_c(p/n+s)} g \left(\hat{R}_{GMV;c}, \hat{V}_{GMV;c}, \hat{s}_c \right) \right) \end{pmatrix} \boldsymbol{\eta} + o_P(1)
\end{aligned}$$

The rest of the proof of part (a) follows from the proof of Theorem 4.2. Similarly, the statement of part (b) is obtained. $\square$

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