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Abstract. We develop a basic convergence analysis for an adaptive $\mathsf{C^0IPG}$ method for the Biharmonic problem which provides convergence without rates for all practically relevant marking strategies and all penalty parameters assuring coercivity of the method. The analysis hinges on embedding properties of (broken) Sobolev and BV spaces, and the construction of a suitable limit space. In contrast to the convergence result of adaptive discontinuous Galerkin methods for elliptic PDEs, by Kreuzer and Georgoulis ([KG18]), here we have to deal with the fact that the Lagrange finite element spaces may possibly contain no proper C^1-conforming subspace. This prevents from a straight forward generalisation and requires the development of some new key technical tools.

1. Introduction

We develop here a basic convergence analysis for an adaptive C^0-interior penalty method ($\mathsf{AC^0IPGM}$) for fourth order boundary value problems. Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with Lipschitz boundary. For the ease of presentation we restrict ourselves to the Biharmonic problem

$$\Delta^2 u = f \quad \text{in } \Omega, \quad \text{and} \quad u = \frac{\partial u}{\partial \boldsymbol{n}_\Omega} = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $f \in L^2(\Omega)$ and $\boldsymbol{n}_\Omega$ denotes the outer normal on $\partial\Omega$. However, we emphasise that the presented techniques also apply to more general fourth order problems.

Conforming discretisations of fourth order problems require C^1-elements [AFS68, Cia74, DDPS79], which are typically very cumbersome to implement since they require polynomial degree $\geqslant 5$ in $2d$ or constructions via macrotriangulations. For this reason, mixed (see e.g. [BBF13, dB74, Joh73]) and non-conforming methods (e.g. [BCI65, Mor68]) gained attraction. In this work, we consider the non-conforming so-called C^0-interior penalty Galerkin discretisation ($\mathsf{C^0IPG}$) of (1.1). This method uses standard continuous Lagrange finite elements of order $\geqslant 2$. Consistency is ensured and jumps of the normal derivatives across element interfaces are penalised. For a thorough introduction to C^0-interior penalty methods see e.g. [BS05, EGH$^+$02, HL02]. A posteriori error estimators for the $\mathsf{C^0IPG}$ method were developed in [GHV11, BGS10] and can be used to design an $\mathsf{AC^0IPGM}$ based on the standard loop

$$\mathsf{SOLVE} \to \mathsf{ESTIMATE} \to \mathsf{MARK} \to \mathsf{REFINE}. \tag{1.2}$$

The convergence theory, however, turns out to be a particular challenging task for two reasons. First, the presence of the negative power of the mesh-size h in the discontinuity penalisation term. Second, the analysis of the $\mathsf{C^0IPG}$ method suffers additionally from the fact that, in general, no conforming subspace with proper approximation properties is available unless the polynomial degree exceeds e.g. 4 in $2d$; compare with [dBD83, GS02].

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The first issue also appears in adaptive discontinuous Galerkin methods for 2nd order problems. Here, resorting to Dörflers marking strategy, error reduction [KP07, HKW09] and even optimal convergence rates [BN10] of adaptive schemes are available. These results generalise the ideas for conforming methods in [D96, MNS00, CKNS08] based on the observation that the penalty is dominated by the 'conforming parts' of the estimator provided the penalisation parameter is chosen sufficiently large. This idea was taken up in [FHP15] in an attempt to prove convergence of the $\mathsf{AC^0IPGM}$ for the biharmonic problem (1.1), although the resulting argument is unclear to hold. However, there are generalisations for the Hellan-Hermann-Johnson element [HHX11] and a hybridisable C^0-discontinuous Galerkin method [SH18] where no negative power of the mesh-size is present; compare also with the discussion in [CNZ16].

Very recently in [KG18] the basic convergence results for conforming adaptive finite element methods [MSV08, Sie11] have been extended to adaptive discontinuous Galerkin methods for 2nd order problems. The result utilises a newly developed space limit of the discrete space sequence created by the adaptive loop (1.2). Replacing Cea's Lemma in [MSV08] by a version of the medius analysis of Gudi [Gud10] adapted to the limit space yields convergence of discrete approximations to the weak solution in the limit space. Coincidence with the exact solution follows thanks to properties of the marking strategy. The result is neither restricted to symmetric problems and discretisations nor to a particular marking strategy and holds for all values of the penalty parameter, for which the method is coercive. This has important consequences in practical computations: Since the condition number of the respective stiffness matrix grows as the penalty parameter grows, the magnitude of the penalisation affects the performance of iterative linear solvers. This fact becomes even more relevant for the here considered fourth order problem. We stress, however, that this technique does not provide linear or even optimal convergence rates.

In this work, we extend [KG18] to an $\mathsf{AC^0IPGM}$ for the Biharmonic problem (1.1). The main result states convergence of the adaptive loop (1.2) for most common marking strategies and all penalty parameters, for which the method is coercive. Unfortunately, [KG18] makes exhaustive use of conforming subspaces of the respective discrete spaces, which is prohibitive for the $\mathsf{AC^0IPGM}$ unless the polynomial degree of the Ansatz space is large enough. Therefore, the verification of certain properties of the limit space requires the development of essentially different techniques and also the convergence of discrete solutions cannot be concluded using the generalised medius analysis of Gudi [Gud10] from [KG18]. For the sake of presentation, in this paper, we restrict ourselves to quadratic C^0-elements. We emphasise, however, that the techniques apply to more general fourth order problems, arbitrary polynomial and even discontinuous Galerkin discretisations, however, the construction of suitable technical tools like interpolation operators and a posteriori error estimators is getting much more involved.

The rest of this paper is organised as follows. In Section 2, we introduce the $\mathsf{C^0IPG}$ discretisation and define the $\mathsf{AC^0IPGM}$ by a precise formulation of the adaptive loop (1.2). We conclude the section stating the main result, Theorem 7. For the sake of clarity, in Section 3, we first present the main ideas of its proof. The fact that the discrete C^0-spaces do in general not contain proper C^1-conforming subspaces mainly affects the proofs of the two key technical results, Lemma 10 and Theorem 12. They are presented in Section 4. Finally, in appendices A–C we elaborate on some auxiliary results in order to keep the presentation self consistent.

2. The adaptive C^0IPG finite element method and the main result

Let ω be a measurable set and $m \in \mathbb{N}$. We consider the usual Lebesgue spaces $L^p(\omega; \mathbb{R}^m)$, $1 \leqslant p \leqslant \infty$ over ω with values in $\mathbb{R}^m$. In the case $p = 2$, $L^2(\omega; \mathbb{R}^m)$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_\omega$ and associated norm $\|\cdot\|_\omega$. We also set $L^2(\omega) := L^2(\omega; \mathbb{R})$. The Sobolev space $H^k(\omega)$ is the space of all functions in $L^2(\omega)$ whose weak derivatives of up to order k are in $L^2(\omega)$. Thanks to the Poincaré-Friedrichs' inequality, the closure $H^2_0(\omega)$ of $C^\infty_0(\omega)$ in $H^2(\omega)$ is a Hilbert space with inner product $\langle D^2 \cdot, D^2 \cdot \rangle_\omega$ and norm $\|D^2 \cdot\|_\omega$, where $D^2 v$ denotes the Hessian of v. The dual space $H^{-2}(\omega)$ of $H^2_0(\omega)$ is equipped with the norm $\|v\|_{H^{-2}(\omega)} := \sup_{w \in H^2_0(\omega)} \frac{\langle v, w \rangle}{\|D^2 w\|_\omega}$, $v \in H^{-2}(\omega)$, with dual brackets defined by $\langle v, w \rangle := v(w)$, for $w \in H^2_0(\omega)$.

For $f \in L^2(\Omega)$, the weak formulation of (1.1) reads: find $u \in H^2_0(\Omega)$, such that

$$a(u, v) = \int_\Omega f v \, \mathrm{d}x \quad \forall v \in H^2_0(\Omega), \tag{2.1}$$

for the bilinear form

$$a(w, v) := \int_\Omega D^2 w : D^2 v \, \mathrm{d}x = \int_\Omega \sum_{i,j=1}^{2} \frac{\partial^2 w}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} \, \mathrm{d}x,$$

which is uniformly coercive and continuous on $H^2_0(\Omega)$. Consequently, Riesz' representation theorem provides a unique solution $u \in H^2_0(\Omega)$ of (2.1).

2.1. The C^0IPG finite element Method.

Let $\mathcal{T}$ be a conforming and shape regular subdivision of Ω into disjoint triangular elements $K \in \mathcal{T}$ such that $\overline{\Omega} = \bigcup_{K \in \mathcal{T}} K$. Let $\mathcal{F}_\mathcal{T} := \mathcal{F}(\mathcal{T})$ be the set of one-dimensional faces F, associated with the subdivision $\mathcal{T}$ (including $\partial\Omega$), and let $\mathring{\mathcal{F}}_\mathcal{T}$ be the subset of interior sides only. The corresponding *skeletons* are then defined by $\Gamma_\mathcal{T} = \Gamma(\mathcal{T}) := \bigcup\{F \in \mathcal{F}_\mathcal{T}\}$ and $\mathring{\Gamma}_\mathcal{T} := \bigcup\{F \in \mathring{\mathcal{F}}_\mathcal{T}\}$ respectively. We assume that $\mathcal{T}$ is derived by iterative or recursive bisection of an initial conforming mesh $\mathcal{T}_0$; compare with [Bae91, Kos94, Mau95]. We denote by $\mathbb{G}$ the family of shape-regular triangulations consisting of such refinements of $\mathcal{T}_0$. For $\mathcal{T}, \mathcal{T}_\star \in \mathbb{G}$, we write $\mathcal{T}_\star \geqslant \mathcal{T}$, whenever $\mathcal{T}_\star$ is a refinement of $\mathcal{T}$.

For $r \geqslant 2$, we define the *Lagrange finite-element space* by

$$\mathbb{V}(\mathcal{T}) := H^1_0(\Omega) \cap \mathbb{P}_r(\mathcal{T}) \quad \text{with} \quad \mathbb{P}_r(\mathcal{T}) := \{v \in L^1(\Omega) : v|_K \in \mathbb{P}_r(K) \ \forall K \in \mathcal{T}\}.$$

Obviously, we have $\mathbb{V}(\mathcal{T}) \subset H^1_0(\Omega)$ but $\mathbb{V}(\mathcal{T}) \not\subset H^2_0(\Omega)$ in general. Since each function $V \in \mathbb{V}(\mathcal{T})$ is locally a polynomial on each element $K \in \mathcal{T}$, we have, however, that

$$\mathbb{V}(\mathcal{T}) \subset H^2_0(\mathcal{T}) := H^2(\mathcal{T}) \cap H^1_0(\Omega),$$

where $H^2(\mathcal{T}) := \{v \in L^2(\Omega) : v|_K \in H^2(K), \ \forall K \in \mathcal{T}\}$.

The piecewise constant *mesh-size* function $h_\mathcal{T} : \Omega \to \mathbb{R}_{\geqslant 0}$ is defined by $h_\mathcal{T}(x) := h_K := |K|^{1/d}$ for $x \in K \backslash \partial K$ and $h_\mathcal{T}(x) := h_F := |F|^{1/(d-1)}$ for $x \in F \in \mathcal{F}$. Let $\mathcal{Z}_\mathcal{T}$ be the set of Lagrange nodes of $\mathbb{V}(\mathcal{T})$, which can be identified with its nodal degrees of freedom $\mathcal{N}_\mathcal{T}$. For $z \in \overline{\Omega}$, we denote its neighbourhood by $N_\mathcal{T}(z) := \{K' \in \mathcal{T} \mid z \in K'\}$, and the corresponding domain is defined by $\omega_\mathcal{T}(z) := \Omega(N_\mathcal{T}(z))$. Hereafter we use $\Omega(X) := \bigcup\{K \mid K \in X\}$ for a collection of elements X. With a little abuse of notation, for an element $K \in \mathcal{T}$ we define its j*th neighbourhood* recursively by $N^j_\mathcal{T}(K) := \left\{K' \in \mathcal{T} \mid K' \cap N^{j-1}_\mathcal{T}(K) \neq \varnothing\right\}$, where we set $N^0_\mathcal{T}(K) := K$, and the corresponding domain by $\omega^j_\mathcal{T}(K) := \Omega(N^j_\mathcal{T}(K))$. We shall skip the superindex if $j = 1$, e.g. we write $N_\mathcal{T}(K) = N^1_\mathcal{T}(K)$ and $\omega_\mathcal{T}(K) = \omega^1_\mathcal{T}(K)$ for simplicity. For a side $F \subset \mathcal{F}_\mathcal{T}$, we set $\omega_\mathcal{T}(F) := \bigcup\{K \in \mathcal{T} \mid F \subset K\}$. We extend the above definitions to subsets $\mathcal{M} \subset \mathcal{T}$ setting

$$N^j_\mathcal{T}(\mathcal{M}) := \{K \in \mathcal{T} \colon \exists K' \in \mathcal{M} \text{ such that } K \in N^j_\mathcal{T}(K')\}.$$

Note that the shape regularity and conformity of $\mathbb{G}$ implies local quasi-uniformity, i.e.

$$\sup_{\mathcal{T}\in\mathbb{G}} \max_{K'\in N_\mathcal{T}(K)} \frac{|K|}{|K'|} \lesssim 1 \qquad \text{and} \qquad \sup_{\mathcal{T}\in\mathbb{G}} \max_{K\in\mathcal{T}} \# N_\mathcal{T}(K) \lesssim 1.$$

In the sequel we use the notation $a \lesssim b$, when $a \leqslant Cb$ for a constant $C > 0$, which is independent of all essential quantities (e.g. the mesh-size of $\mathcal{T}$).

In order to formulate the discrete bilinear form, we first need to introduce the so-called jumps and averages of vector- respectively tensorfields on the skeleton $\Gamma_\mathcal{T}$. In fact, for $v \in \mathbb{V}(\mathcal{T})$, we define

$$[\![\partial_n v]\!]_F := [\![\nabla v \cdot \boldsymbol{n}]\!]\,|_F := \nabla v|_{K_1} \cdot \boldsymbol{n}_{K_1} + \nabla v|_{K_2} \cdot \boldsymbol{n}_{K_2}$$

for $F \in \mathring{\mathcal{F}}_\mathcal{T}$ and $F = K_1 \cap K_2$ with two disjoint elements $K_1, K_2 \in \mathcal{T}$. If $F \subset \partial K \cap \partial\Omega$, then $[\![\partial_n v]\!]_F := \nabla v|_K \cdot \boldsymbol{n}_K$. The average of the Hessian of $v \in \mathbb{V}(\mathcal{T})$ is defined by

$$\{\!\{\partial^2_n v\}\!\}_F := \{\!\{(D^2 v)\boldsymbol{n} \cdot \boldsymbol{n}\}\!\} := \frac{1}{2}\left(D^2 v|_{K_1} + D^2 v|_{K_2}\right) \boldsymbol{n}_{K_1} \cdot \boldsymbol{n}_{K_1}$$

whenever $F \in \mathring{\mathcal{F}}_\mathcal{T}$ with $F = K_1 \cap K_2$ and $\{\!\{\partial^2_n v\}\!\}_F := D^2 v|_K \boldsymbol{n}_K \cdot \boldsymbol{n}_K$ for sides $F \subset \partial K \cap \partial\Omega$. We stress that the above definitions do not depend on the choice of the ordering of the elements K_1 and K_2. This is not true for

$$[\![\partial^2_n v]\!]_F := [\![\partial_n(\nabla v \cdot \boldsymbol{n}_{K_1})]\!]\,|_F \quad \text{and} \quad \{\!\{\partial_n v\}\!\}_F := \frac{1}{2}\left(\nabla v|_{K_1} + \nabla v|_{K_2}\right) \cdot \boldsymbol{n}_{K_1} \tag{2.2}$$

for $F \in \mathring{\mathcal{F}}_\mathcal{T}$ with $F = K_1 \cap K_2$ for disjoint $K_1, K_2 \in \mathcal{T}$. However, the two expressions will only appear as products with each other, e.g. as $[\![\partial^2_n v]\!]_F \{\!\{\partial_n w\}\!\}_F$ or as $[\![\partial^2_n v]\!]^2_F$, which are then again unique.

For $v, w \in \mathbb{V}(\mathcal{T})$ we recall then the discrete bilinear form from [BS05, BGS10]

$$\begin{aligned}\mathfrak{B}_\mathcal{T}[v,\, w] :&= \int_\mathcal{T} D^2 v \colon D^2 w \,\mathrm{d}x - \int_{\mathcal{F}_\mathcal{T}} \{\!\{\partial^2_n v\}\!\} [\![\partial_n w]\!] + \{\!\{\partial^2_n w\}\!\} [\![\partial_n v]\!] \,\mathrm{d}s \\ &\quad + \int_{\mathcal{F}_\mathcal{T}} \frac{\sigma}{h_\mathcal{T}} [\![\partial_n v]\!] [\![\partial_n w]\!] \,\mathrm{d}s.\end{aligned}$$

Here, we used the following abbreviations

$$\int_\mathcal{T} \cdot \,\mathrm{d}x := \sum_{K\in\mathcal{T}} \int_K \cdot \,\mathrm{d}x \quad \text{and} \quad \int_{\mathcal{F}_\mathcal{T}} \cdot \,\mathrm{d}s := \sum_{F\in\mathcal{F}_\mathcal{T}} \int_F \cdot \,\mathrm{d}s,$$

where on each element $K \in \mathcal{T}$, the piecewise Hessian $(D^2_{\mathtt{pw}} v)|_K = D^2(v|_K) \in L^2(K)$ exists since $v \in H^2_0(\mathcal{T})$, i.e. we have $\int_\mathcal{T} D^2 v \colon D^2 w \,\mathrm{d}x = \int_\Omega D^2_{\mathtt{pw}} v \colon D^2_{\mathtt{pw}} w \,\mathrm{d}x$.

For sufficiently large σ, we have from [BS05] that $\mathfrak{B}_\mathcal{T}$ is continuous and coercive on $\mathbb{V}(\mathcal{T})$ with respect to the *energy norm*

$$|\!|\!| v |\!|\!|^2_\mathcal{T} := \int_\mathcal{T} D^2 v \colon D^2 v \,\mathrm{d}x + \int_{\mathcal{F}_\mathcal{T}} \frac{\sigma}{h_\mathcal{T}} \left|[\![\partial_n v]\!]\right|^2 \,\mathrm{d}s \quad \forall v \in H^2_0(\mathcal{T}).$$

In the following, instead of $\int_\Omega D^2_{\mathtt{pw}} v \colon D^2_{\mathtt{pw}} v \,\mathrm{d}x$, we will also write $\int_\Omega \left|D^2_{\mathtt{pw}} v\right|^2 \,\mathrm{d}x$ for brevity.

Proposition 1 (Continuity and coercivity)**.** *Let $\mathcal{T} \in \mathbb{G}$, then there exists $\sigma_\star > 0$, such that for all $\sigma > \sigma_\star$ there exist positive constants C_{cont}, C_{coer} such that*

$$\mathfrak{B}_\mathcal{T}[v,\, w] \leqslant C_{cont} |\!|\!| v |\!|\!|_\mathcal{T} \, |\!|\!| w |\!|\!|_\mathcal{T} \qquad \text{and} \qquad C_{coer} |\!|\!| v |\!|\!|^2_\mathcal{T} \leqslant \mathfrak{B}_\mathcal{T}[v,\, v].$$

for all $v, w \in \mathbb{V}(\mathcal{T})$. The constants $\sigma_\star$, C_{cont}, and C_{coer} solely depend on the shape regularity of $\mathcal{T}$ and the polynomial degree r.

Since $\mathbb{V}(\mathcal{T})$ is a Banach space with the energy-norm $|\!|\!|\cdot|\!|\!|_{\mathcal{T}}$, there exists a unique $u_{\mathcal{T}} \in \mathbb{V}(\mathcal{T})$ with

$$\mathfrak{B}_{\mathcal{T}}[u_{\mathcal{T}},\, v_{\mathcal{T}}] = \int_\Omega f v_{\mathcal{T}} \,\mathrm{d}x \qquad \forall v_{\mathcal{T}} \in \mathbb{V}(\mathcal{T}). \tag{2.3}$$

This is the C^0*-interior penalty Galerkin* approximation of (2.1), which depends continuously on f, i.e.

$$|\!|\!|u_{\mathcal{T}}|\!|\!|_{\mathcal{T}} \lesssim \|f\|_\Omega \tag{2.4}$$

thanks to the following broken Poincaré-Friedrichs inequalities; compare with [Bre03].

Proposition 2. *Let $v \in \mathbb{V}(\mathcal{T})$, then we have*

$$|v|^2_{H^1_0(\Omega)} \lesssim \sum_{K\in\mathcal{T}} |v|^2_{H^2(K)} + \sum_{F\in\mathcal{F}(\mathcal{T})} h_F^{-1} \int_F [\![\partial_n v]\!]^2 \, ds \lesssim |\!|\!|v|\!|\!|^2_{\mathcal{T}}\,.$$

Unfortunately, $\mathfrak{B}_{\mathcal{T}}$ cannot be applied to functions from $H^2_0(\mathcal{T})$ since no trace of second derivatives is available. For a side $F \in \mathcal{F}_{\mathcal{T}}$ we therefore define a local lifting operator $\mathcal{L}^F_{\mathcal{T}} \colon L^1(F) \to [\mathbb{P}_{r-2}(\mathcal{T})]^{2\times 2}$ by

$$\int_\Omega \mathcal{L}^F_{\mathcal{T}}(\varphi) \colon \boldsymbol{\tau} \,\mathrm{d}x = \int_F \{\!\!\{\boldsymbol{\tau}\boldsymbol{n}\cdot\boldsymbol{n}\}\!\!\}\,\varphi \,\mathrm{d}s \quad \forall \boldsymbol{\tau} \in [\mathbb{P}_{r-2}(\mathcal{T})]^{2\times 2}, \tag{2.5}$$

where the support of $\mathcal{L}^F_{\mathcal{T}}(\varphi)$ is given by $\omega_{\mathcal{T}}(F)$. Using a trace estimate, we have that

$$\left\|\mathcal{L}^F_{\mathcal{T}}(\varphi)\right\|_\Omega \lesssim \left\| h_{\mathcal{T}}^{-1/2} \varphi \right\|_F; \tag{2.6}$$

where the right-hand side is allowed to be infinity; compare also with e.g. [DPE12, Lemma 4.33]. We define the *global lifting operator* $\mathcal{L}_{\mathcal{T}} \colon L^1(\Gamma_{\mathcal{T}}) \to [\mathbb{P}_{r-2}(\mathcal{T})]^{2\times 2}$ by

$$\mathcal{L}_{\mathcal{T}}(\varphi) := \sum_{F\in\mathcal{F}_{\mathcal{T}}} \mathcal{L}^F_{\mathcal{T}}(\varphi) \quad \text{with} \quad \|\mathcal{L}_{\mathcal{T}}(\phi)\|_\Omega \lesssim \left\| h_{\mathcal{T}}^{-1/2}\phi \right\|_{\Gamma_{\mathcal{T}}}. \tag{2.7}$$

Noting that $\partial_n v \in L^2(\Gamma_{\mathcal{T}})$ for all $v \in H^2_0(\mathcal{T})$, we can extend the bilinear form $\mathfrak{B}_{\mathcal{T}}$ from $\mathbb{V}(\mathcal{T})$ to $H^2_0(\mathcal{T})$ by

$$\begin{aligned}\mathfrak{B}_{\mathcal{T}}[v,\, w] := \int_{\mathcal{T}} D^2 v \colon D^2 w \,\mathrm{d}x - \int_\Omega \mathcal{L}_{\mathcal{T}}([\![\partial_n w]\!]) \colon D^2_{\mathsf{pw}} v + \mathcal{L}_{\mathcal{T}}([\![\partial_n v]\!]) \colon D^2_{\mathsf{pw}} w \,\mathrm{d}x \\ + \int_{\mathcal{F}_{\mathcal{T}}} \frac{\sigma}{h_{\mathcal{T}}} [\![\partial_n v]\!]\,[\![\partial_n w]\!] \,\mathrm{d}s.\end{aligned}$$

Discontinuous Galerkin spaces can be embedded into the space of functions with bounded variation; compare e.g. with [BO09, Lemma 2]. In the context of $\mathsf{C}^0\mathsf{IPG}$ methods, this transfers to the following embedding; compare also with [LNSO04].

Proposition 3. *Let $v \in \mathbb{V}(\mathcal{T})$ and $|D(\nabla v)|(\Omega)$ the total variation of ∇v. Then we have*

$$|D(\nabla v)|(\Omega) \lesssim \int_\Omega \left|D^2_{pw} v\right| \,\mathrm{d}x + \int_{\mathcal{F}(\mathcal{T})} [\![\partial_n v]\!] \, ds \lesssim |\!|\!|v|\!|\!|_{\mathcal{T}}\,.$$

2.2. **A posteriori error bounds.** From here on, we restrict ourselves to quadratic C^0-elements, i.e., $r = 2$ and introduce the a posteriori error estimators from [BGS10]. For $v \in \mathbb{V}(\mathcal{T})$ and $K \in \mathcal{T}$ let

$$\eta(v,K) := \left(\int_K h_{\mathcal{T}}^4\,|f|^2\,\mathrm{d}x + \int_{\partial K\cap\Omega} h_{\mathcal{T}}\,[\![\partial_n^2 v]\!]^2 + \sigma^2\int_{\partial K} h_{\mathcal{T}}^{-1}\,[\![\partial_n v]\!]^2\right)^{1/2}. \tag{2.8}$$

When $v = u_{\mathcal{T}}$, we simply write $\eta_{\mathcal{T}}(K) := \eta(u_{\mathcal{T}},K)$. Moreover, for $\mathcal{M}\subset\mathcal{T}$, we set

$$\eta_{\mathcal{T}}(v,\mathcal{M}) := \left(\sum_{K\in\mathcal{M}}\eta(v,K)^2\right)^{1/2} \quad\text{and}\quad \eta_{\mathcal{T}}(\mathcal{M}) := \eta_{\mathcal{T}}(u_{\mathcal{T}},\mathcal{M}).$$

From [BGS10, Theorem 3.1], we have that (2.8) defines a reliable estimator.

Proposition 4. *Let $u \in H_0^2(\Omega)$ be the solution of* (2.1) *and $u_{\mathcal{T}}$ the discrete solution of* (2.3). *Then,*

$$|\!|\!| u-u_{\mathcal{T}}|\!|\!|_{\mathcal{T}} \lesssim \eta_{\mathcal{T}}(\mathcal{T}),$$

where the constants in $\lesssim$ depend only on the shape regularity of $\mathcal{T}$.

In [BGS10, Section 4] $\eta_{\mathcal{T}}$ is also proved to be efficient.

Proposition 5. *Let $u \in H_0^2(\Omega)$ be the solution of*(2.1) *and $\mathcal{T}\in\mathbb{G}$. Then, for all $v\in\mathbb{V}(\mathcal{T})$ and $K\in\mathcal{T}$, we have*

$$\int_K h_{\mathcal{T}}^4\,|f|^2\,\mathrm{d}x + \int_{\partial K\cap\Omega} h_{\mathcal{T}}\,[\![\partial_n^2 v]\!]^2 \lesssim \int_{\omega_{\mathcal{T}}(K)} \left|D_{pw}^2(u-v)\right|^2\mathrm{d}x + \operatorname{osc}(N_k(K),f)^2,$$

with data-oscillation defined by

$$\operatorname{osc}(\mathcal{M},f) := \left(\sum_{K\in\mathcal{M}}\operatorname{osc}(K,f)^2\right)^{1/2}, \quad \textit{where}\quad \operatorname{osc}(K,f)^2 := \int_K h_k^4\,|f-\Pi_0 f|^2\,\mathrm{d}x$$

for all $\mathcal{M}\subset\mathcal{T}$. Here, $\Pi_0 f$ denotes the $L^2(\Omega)$-orthogonal projection onto $\mathbb{P}_0(\mathcal{T})$,

$$\Pi_0 f|_K := \frac{1}{|K|}\int_K f\,\mathrm{d}x \quad \forall K\in\mathcal{T}.$$

2.3. **The adaptive C^0IPG method (AC^0IPGM).** Now, we are in the position to precisely formulate the adaptive algorithm (1.2) based on the modules SOLVE, ESTIMATE, MARK and REFINE, which are described in more detail below.

Algorithm 6 (AC^{0}IPGM)**.** Let $\mathcal{T}_0$ be an initial triangulation. The adaptive algorithm is an iteration of the following form:

(1) $u_k = \mathsf{SOLVE}(\mathbb{V}(\mathcal{T}_k))$;
(2) $\{\eta_k(K)\}_{K\in\mathcal{T}_k} = \mathsf{ESTIMATE}(u_k,\mathcal{T}_k)$;
(3) $\mathcal{M}_k = \mathsf{MARK}\left(\{\eta_k(K)\}_{K\in\mathcal{T}_k},\mathcal{T}_k\right)$;
(4) $\mathcal{T}_{k+1} = \mathsf{REFINE}(\mathcal{T}_k,\mathcal{M}_k)$; increment k and go to Step 1.

Here we have replaced the subscript triangulations $\{\mathcal{T}_k\}_{k\in\mathbb{N}_0}$ with the iteration counter k in $\eta_k(\mathcal{T}_k) = \eta_{\mathcal{T}_k}(\mathcal{T}_k)$ for brevity. Similar short hand notations will be frequently used below when no confusion can occur, e.g. we write also $N_k^j(K) = N_{\mathcal{T}_k}^j(K)$. Next we comment on the modules SOLVE, ESTIMATE, MARK and REFINE.

SOLVE. For a given mesh $\mathcal{T}$ we assume that

$$u_{\mathcal{T}} = \mathsf{SOLVE}(\mathbb{V}(\mathcal{T}))$$

is the exact C^{0}IPG solution of problem (2.3).

ESTIMATE. We suppose that

$$\{\eta_{\mathcal{T}}(K)\}_{K\in\mathcal{T}} := \mathsf{ESTIMATE}(u_{\mathcal{T}},K)$$

is the elementwise error defined in (2.8).

MARK. We assume that the output

$$\mathcal{M} := \mathsf{MARK}(\{\eta_{\mathcal{T}}(K)\}_{K\in\mathcal{T}}, \mathcal{T})$$

of marked elements satisfies

$$\eta_{\mathcal{T}}(K) \leqslant g(\eta_{\mathcal{T}}(\mathcal{M})), \qquad \text{for all } K \in \mathcal{T}\backslash\mathcal{M}. \tag{2.9}$$

Here $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a fixed function, which is continuous in 0, with $g(0) = 0$.

REFINE. We assume for $\mathcal{M} \subset \mathcal{T}$ that

$$\mathcal{T} \leqslant \widetilde{\mathcal{T}} := \mathsf{REFINE}(\mathcal{T}, \mathcal{M}) \in \mathbb{G},$$

such that

$$K \in \mathcal{M} \quad \Rightarrow \quad K \in \mathcal{T}\backslash\widetilde{\mathcal{T}}, \tag{2.10}$$

i.e., each marked element is at least refined once.

2.4. **The main result.** The main result of this work states that the sequence of $\mathsf{C}^0\mathsf{IPG}$ finite element approximations produced by the $\mathsf{AC}^0\mathsf{IPGM}$ (Algorithm 6) converges to the exact solution $u \in H^2_0(\Omega)$ of (2.1). From here on we will refer to $|\!|\!|\cdot|\!|\!|_{\mathcal{T}_k}$ as $|\!|\!|\cdot|\!|\!|_k$.

Theorem 7. *We have that*

$$\eta_k(\mathcal{T}_k) \to 0 \quad \textit{and} \qquad |\!|\!|u - u_k|\!|\!|_k \to 0 \quad \textit{as } k \to \infty.$$

3. Proof of the main result Theorem 7

The proof of convergence of the $\mathsf{AC}^0\mathsf{IPGM}$ is based on ideas of [MSV08, Sie11] for conforming elements and its generalisation [KG18] to adaptive discontinuous Galerkin methods for the Poisson problem. For the sake of clarity, in this section, we present the main ideas of the proof of Theorem 7 following the ideas of [KG18]. In contrast to the latter result here we are faced with the problem that $\mathbb{V}(\mathcal{T})$ contains no proper conforming subspace. This requires new techniques of proof for the two key auxiliary results, Theorem 12 and Lemma 10, which proofs are postponed to Section 4 below.

3.1. **Sequence of Partitions.** Following [MSV08, Sie11, KG18], we split the domain Ω into essentially two parts according to whether the mesh-size function $h_k := h_{\mathcal{T}_k}$ vanishes or not. In order to make this rigorous, we define the set of eventually never refined elements by

$$\mathcal{T}^+ := \bigcup_{k\geqslant 0} \bigcap_{l\geqslant k} \mathcal{T}_l \qquad \text{with corresponding domain} \qquad \Omega^+ := \Omega(\mathcal{T}^+). \tag{3.1}$$

Additionally, we denote the complementary domain $\Omega^- = \text{interior}(\Omega\backslash\Omega^+)$.

For $k \in \mathbb{N}_0$, we define $\mathcal{T}_k^- := \{K \in \mathcal{T}_k \colon K \subset \overline{\Omega^-}\}$ and $\mathcal{T}_k^+ := \mathcal{T}_k \cap \mathcal{T}^+$ as well as for $j \geqslant 1$

$$\begin{aligned}
\mathcal{T}_k^{j-} &:= \{K \in \mathcal{T}_k \colon N_k^j(K) \subset \mathcal{T}_k^-\} = \{K \in \mathcal{T}_k \colon N_k(K) \subset \mathcal{T}_k^{(j-1)-}\},\\
\mathcal{T}_k^{j+} &:= \{K \in \mathcal{T}_k \colon N_k^j(K) \subset \mathcal{T}_k^+\} = \{K \in \mathcal{T}_k \colon N_k(K) \subset \mathcal{T}_k^{(j-1)+}\},\\
\mathcal{T}_k^{j\star} &:= \mathcal{T}_k\backslash(\mathcal{T}_k^{j+} \cup \mathcal{T}_k^{j-}),
\end{aligned}$$

where we used $\mathcal{T}_k^{0+} := \mathcal{T}_k^+$ and $\mathcal{T}_k^{0-} := \mathcal{T}_k^-$ in the identities when $j = 0$. For the corresponding domains we denote $\Omega_k^{j-} := \Omega(\mathcal{T}_k^{j-})$, $\Omega_k^{j+} := \Omega(\mathcal{T}_k^{j+})$ and $\Omega_k^{j\star} := \Omega(\mathcal{T}_k^{j\star})$. Moreover, we adopt the above notations for the corresponding faces, e.g. $\mathcal{F}^{j-} := \mathcal{F}(\mathcal{T}_k^{j-})$, $\mathcal{F}^{j+} := \mathcal{F}(\mathcal{T}_k^{j+})$.

We remark that we need the above definitions of $\mathcal{T}_k^{j-}$ and $\mathcal{T}_k^{j+}$ for technical reasons. In fact, our analysis involves Clément type quasi-interpolations for which local stability estimates involve neighbourhoods. However, for different but fixed js the above sets behave asymptotically similar for $k \to \infty$. To see this, the next key result from [MSV08, Lemma 4.1] states that neighbours of never refined elements are eventually also never refined again.

Lemma 8. *For $K \in \mathcal{T}^+$ there exists a constant $L = L(K) \in \mathbb{N}_0$ such that*

$$N_k(K) = N_L(K)$$

for all $k \geqslant L$. In particular, we have $N_k(K) \subset \mathcal{T}^+$ for all $k \geqslant L$.

The next Lemma essentially goes back to [MSV08, (4.15) and Corollary 4.1] and was proved for $j = 2$ in [KG18, Lemma 11].

Lemma 9. *For fixed $j \in \mathbb{N}$, we have*

$$\lim_{k\to\infty} \left|\Omega_k^{j\star}\right| = 0 \qquad \text{and} \qquad \lim_{k\to\infty} \left\|h_k \chi_{\Omega_k^{j-}}\right\|_{L^\infty(\Omega)} = 0,$$

with $\chi_{\Omega_k^{j-}}$ denoting the characteristic function of Ω_k^{j-}. Moreover, we have $\mathcal{T}^+ = \bigcup_{k=0}^\infty \mathcal{T}_k^{j+}$.

Proof. We prove the first claim by induction over j. In fact, we have

$$|\Omega_k^{0\star}| \leqslant |\Omega_k^{1\star}| \to 0 \qquad \text{as } k \to \infty$$

from [KG18, Lemma 11]. In order to conclude $j \to j+1$, we observe that

$$|\Omega_k^{(j+1)\star}| = |\Omega^- \backslash \Omega_k^{(j+1)-}| + |\Omega^+ \backslash \Omega_k^{(j+1)+}| \tag{3.2}$$

and consider the two terms on the right-hand side separately. Since $\#\mathcal{T}_k^{(j+1)+} < \infty$, we have thanks to Lemma 8, that for all $k \in \mathbb{N}$ there exists $K = K(k) \geqslant k$, such that $\mathcal{T}_K^{(j+1)+} \supset \mathcal{T}_k^{j+}$, and consequently,

$$|\Omega^+ \backslash \Omega_{K(k)}^{(j+1)+}| \leqslant |\Omega^+ \backslash \Omega_k^{j+}| \leqslant |\Omega_k^{j\star}| \to 0 \qquad \text{as } k \to \infty.$$

Since $|\Omega^+ \backslash \Omega_k^{(j+1)+}|$ decreases monotonically, we conclude that $|\Omega^+ \backslash \Omega_k^{(j+1)+}| \to 0$ as $k \to \infty$.

For the second term in (3.2), we have from the nestedness $\Omega_k^{(j+1)-} \subset \Omega_k^{j-}$ that

$$|\Omega^- \backslash \Omega_k^{(j+1)-}| = |\Omega^- \backslash \Omega_k^{j-}| + |\Omega_k^{j-} \backslash \Omega_k^{(j+1)-}|.$$

The first term vanishes by the induction assumption. For the second term, we have

$$K \in \mathcal{T}_k^{j-} \backslash \mathcal{T}_k^{(j+1)-} \quad \Rightarrow \quad N_k(K) \not\subset \mathcal{T}_k^{j-} \quad \text{but} \quad N_k(K) \subset \mathcal{T}_k^{(j-1)-}.$$

Therefore, there exists $K' \in \mathcal{T}_k$ with $K' \in N_k(K)$ or equivalently $K \in N_k(K')$, such that $K' \in \mathcal{T}_k^{(j-1)-} \backslash \mathcal{T}_k^{j-}$. We thus conclude that

$$|\Omega_k^{j-} \backslash \Omega_k^{(j+1)-}| = |\Omega(\mathcal{T}_k^{j-} \backslash \mathcal{T}_k^{(j+1)-})| \leqslant |\Omega(N_k(\mathcal{T}_k^{(j-1)-} \backslash \mathcal{T}_k^{j-}))| \lesssim |\Omega(\mathcal{T}_k^{(j-1)-} \backslash \mathcal{T}_k^{j-})|,$$

where the last inequality is a consequence of shape regularity. Finally, we have $|\Omega(\mathcal{T}_k^{(j-1)-} \backslash \mathcal{T}_k^{j-})| \leqslant |\Omega(\mathcal{T}_k^{-} \backslash \mathcal{T}_k^{j-})| \to 0$ as $k \to \infty$ by the induction assumption.

Since $\|h_k \chi_{\Omega_k^{(j+1)-}}\|_{L^\infty(\Omega)} \leqslant \|h_k \chi_{\Omega_k^{j-}}\|_{L^\infty(\Omega)}$, $j \geqslant 0$, the second claim follows from [MSV08, Corollary 4.1] noting that $\Omega_k^- \subset \Omega_k^0$ with Ω_k^0 as in [MSV08].

The last claim is a direct consequence of Lemma 8. □

3.2. **The limit space.** In this section we discuss the limit of the finite element spaces $\mathbb{V}_k$. Following the ideas in [KG18, Section 3.2], we define

$$\mathbb{V}_\infty := \Big\{v \in H^1_0(\Omega) \mid \nabla v \in BV(\Omega)^2,\ v|_{\Omega^-} \in H^2_{\partial\Omega\cap\partial\Omega^-}(\Omega^-),\ v|_K \in \mathbb{P}_r(K),\ \forall K \in \mathcal{T}^+,$$
$$\text{such that } \exists\{v_k\}_{k\in\mathbb{N}}, v_k \in \mathbb{V}_k \text{ with } \lim_{k\to\infty} \|v - v_k\|_k = 0$$
$$\text{and } \limsup_{k\to\infty} \|v_k\|_k < \infty\Big\}.$$

By $H^2_{\partial\Omega\cap\partial\Omega^-}(\Omega^-)$ we denote the space of functions from $H^2_0(\Omega)$ restricted to the domain Ω^-. The fact that $\nabla\mathbb{V}_\infty \subset BV(\Omega)^2$ is motivated by Proposition 3.

We will use the following bilinear form on $\mathbb{V}_\infty$: For $v, w \in \mathbb{V}_\infty$, we define

$$\langle v, w\rangle_\infty := \int_\Omega D^2_{\mathtt{pw}} v : D^2_{\mathtt{pw}} w \,\mathrm{d}x + \sigma \int_{\mathcal{F}^+} h_+^{-1} [\![\partial_n v]\!]\, [\![\partial_n w]\!] \,\mathrm{d}s,$$

where we set $h_+ := h_{\mathcal{T}^+}$ and $\mathcal{F}^+ := \mathcal{F}(\mathcal{T}^+)$. For a function $v \in \mathbb{V}_\infty$ the piecewise Hessian $D^2_{\mathtt{pw}} v$ is defined by

$$D^2_{\mathtt{pw}} v(x) := D^2 v(x) \quad \text{if } x \in \Omega^- \quad \text{and} \quad D^2_{\mathtt{pw}} v(x) := D^2 v(x) \quad \text{if } x \in K \in \mathcal{T}^+.$$

The induced norm is denoted by $\|v\|_\infty = \langle v, v\rangle_\infty^{1/2}$. Note that from the definition of $\mathbb{V}_\infty$, we have $\nabla v \in BV(\Omega)^2$. Consequently, we have from [AFP00, Theorem 3.88] that the L^1-trace of ∇v exists for all $F \in \mathcal{F}$ and for all $k \in \mathbb{N}$. Therefore, the jump terms are measurable with respect to the 1-dimensional Hausdorff measure on $\mathcal{F}$, and we are able to evaluate the k-norm $\|v\|_k$ for $v \in \mathbb{V}_\infty$.

The next Lemma is crucial for the existence of a generalised Galerkin solution in $\mathbb{V}_\infty$, its proof is postponed to Section 4.3.

Lemma 10. *The space $(\mathbb{V}_\infty, \langle\cdot,\cdot\rangle_\infty)$ is a Hilbert space.*

In order to extend the discrete problem (2.3) to the space $\mathbb{V}_\infty$, we have to extend the bilinear form $\mathfrak{B}_\mathcal{T}$ to the space $\mathbb{V}_\infty$. To this end, we define suitable liftings for the limit space. Thanks to Lemma 8, for each $F \in \mathcal{F}^+$, there exists $L = L(F)$ such that $F \in \mathcal{F}^{1+}_\ell$ for all $\ell \geqslant L$. We define the local lifting operators

$$\mathcal{L}^F_\infty := \mathcal{L}^F_L = \mathcal{L}^F_{\mathcal{T}_L}. \tag{3.3}$$

From the definition of the discrete local liftings (2.5), we see that $\mathcal{L}^F_\infty$ vanishes outside the two neighbouring element K', K, with $F = K \cap K'$. Consequently, we have $\mathcal{L}^F_\ell = \mathcal{L}^F_L$ for all $\ell \geqslant L$, and therefore this definition is unique. The global lifting operator is defined by

$$\mathcal{L}_\infty = \sum_{F\in\mathcal{F}^+} \mathcal{L}^F_\infty. \tag{3.4}$$

From estimate (2.7) we have that $\sum_{F\in\mathcal{F}^+} \mathcal{L}^F_\infty([\![\partial_n v]\!])$ is a Cauchy sequence in $L^2(\Omega)^{d\times d}$. Therefore, $\mathcal{L}_\infty([\![\partial_n v]\!]) \in L^2(\Omega^{d\times d})$ and the estimate

$$\|\mathcal{L}_\infty([\![\partial_n v]\!])\|_\Omega \lesssim \left\| h_+^{-1/2} [\![\partial_n v]\!] \right\|_{\Gamma^+} \tag{3.5}$$

holds. Here we used the notation $\Gamma^+ := \bigcup\{F \mid F \in \mathcal{F}^+\}$. Now we are in position to generalise the DG-bilinear form to $\mathbb{V}_\infty$ setting

$$\mathfrak{B}_\infty[v, w] : = \int_\Omega D^2_{\mathtt{pw}} v : D^2_{\mathtt{pw}} w \,\mathrm{d}x - \int_\Omega \mathcal{L}_\infty([\![\partial_n w]\!]) : D^2_{\mathtt{pw}} v + \mathcal{L}_\infty([\![\partial_n v]\!]) : D^2_{\mathtt{pw}} w \,\mathrm{d}x$$
$$+ \int_{\mathcal{F}^+} \frac{\sigma}{h_\mathcal{T}} [\![\partial_n v]\!]\, [\![\partial_n w]\!] \,\mathrm{d}s,$$

for all $v, w \in \mathbb{V}_\infty$.

Corollary 11. *There exists a unique $u_\infty \in \mathbb{V}_\infty$, such that*

$$\mathfrak{B}_\infty[u_\infty, v] = \int_\Omega fv \,\mathrm{d}x \quad \forall v \in \mathbb{V}_\infty. \tag{3.6}$$

Proof. From Lemma 10 we have that $\mathbb{V}_\infty$ is a Hilbert space. Moreover, stability of the lifting operators (3.5) and local scaled trace inequalities prove coercivity and continuity of $\mathfrak{B}_\infty[\cdot,\cdot]$ with respect to $|||\cdot|||_\infty$; compare also with Proposition 1. The assertion follows from the Riesz representation theorem. □

The following Theorem states that the solution of (3.6) is indeed the limit of the adaptive sequence produced by the $\mathsf{AC^0IPGM}$. Its proof is postponed to Section 4.

Theorem 12. *Let u_∞ the solution of* (3.6) *and let $\{u_k\}_{k\in\mathbb{N}_0}$ be the sequence of C^0IPG solutions produced by $\mathsf{AC^0IPGM}$. Then,*

$$|||u_\infty - u_k|||_k \to 0 \quad \text{as } k \to \infty.$$

3.3. **Convergence of the estimator.** In this section we shall conclude from Theorem 12, that the sequence of estimators $\{\eta_k(\mathcal{T}_k)\}_{k\in\mathbb{N}_0}$ produced by $\mathsf{AC^0IPGM}$ vanishes as $k \to \infty$. On $\mathcal{T}_k^{1-}$ this follows from the following local lower bound, which extends the result of Proposition 5 to our adaptively created limit space $\mathbb{V}_\infty$.

Proposition 13. *Let u_∞ be the solution of* (3.6)*. Then, for every $K \in \mathcal{T}_k^{1-}$ and $v \in \mathbb{V}_k$, $k \in \mathbb{N}$, we have*

$$\begin{aligned}\int_K h_k^4 |f|^2 \,\mathrm{d}x &+ \int_{\partial K\cap\Omega} h_k \left[\!\left[\partial_n^2 v\right]\!\right]^2 \mathrm{d}s \\ &\lesssim \left\| D_{pw}^2(u_\infty - v)\right\|_{\omega_k(K)}^2 + \left\| h_k^2(f - \Pi_0 f)\right\|_{\omega_k(K)}^2.\end{aligned}$$

In particular, we also have that

$$\begin{aligned}\sum_{K\in\mathcal{T}_k^{1-}} \int_K h_k^4 |f|^2 \,\mathrm{d}x &+ \int_{\partial K\cap\Omega} h_k \left[\!\left[\partial_n^2 v\right]\!\right]^2 \mathrm{d}s \\ &\lesssim |||u_\infty - v|||_k + \sum_{K\in\mathcal{T}_k^{1-}} \sum_{K'\in\omega_k(K)} \left\| h_k^2(f - \Pi_0 f)\right\|_{K'}^2.\end{aligned}$$

Proof. Verifying for suitable element bubble functions $b_K \in H_0^2(K) \subset \mathbb{V}_\infty$ for $K \in \mathcal{T}_k^{1-}$ and, correspondingly, for side bubble functions $b_F \in H_0^2(\omega_k(F)) \subset \mathbb{V}_\infty$ and $F = K_1 \cap K_2$ with $K_1, K_2 \in \mathcal{T}_k^-$, allows to use standard techniques in a posteriori analysis (see [Ver13, BGS10, GHV11]) resorting to (3.6) instead of (2.1). In order to keep the presentation self-contained, we present a precise proof in Appendix A. □

Now we are in a position to prove that the error estimator is vanishing.

Lemma 14. *We have for the sequence of error estimators produced by $\mathsf{AC^0IPGM}$*

$$\eta_k(\mathcal{T}_k^{1-}) \to 0 \quad \text{as } k \to \infty.$$

Proof. From Proposition 13 we deduce that

$$\begin{aligned}\sum_{K\in\mathcal{T}_k^{1-}} \int_K h_k^4 |f|^2 \,\mathrm{d}x &+ \int_{\partial K\cap\Omega} h_k \left[\!\left[\partial_n^2 u_k\right]\!\right]^2 \mathrm{d}s \\ &\lesssim |||u_\infty - u_k|||_k^2 + \sum_{K\in\mathcal{T}_k^{1-}} \operatorname{osc}(N_k(K), f)^2.\end{aligned}$$

The first term on the right-hand side vanishes, due to Theorem 12. For the second term, we have by the finite overlap of neighbourhoods that

$$\sum_{K\in\mathcal{T}_k^{1-}} \operatorname{osc}(N_k(K), f)^2 \lesssim \sum_{K\in\mathcal{T}_k^{-}} \operatorname{osc}(K, f)^2 \leqslant \left\| h_k \chi_{\Omega_k^-}\right\|_{L^\infty(\Omega)}^4 \|f\|_\Omega^2,$$

which vanishes thanks to Lemma 9.

It remains to prove that

$$\int_{\mathcal{F}_k^{1-}} h_k^{-1} [\![\partial_n u_k]\!]^2 \,\mathrm{d}s \to 0 \quad \text{as } k \to \infty.$$

To this end we observe that $u_\infty \in H^2_{\partial\Omega\cap\partial\Omega^-}(\Omega^-)$ and $\Omega_k^{1-} \subset \Omega^-$, and thus $[\![\partial_n u_\infty]\!]|_F = 0$ for all $F \in \mathcal{F}_k^{1-}$. From this we conclude from Theorem 12 that

$$\int_{\mathcal{F}_k^{1-}} h_F^{-1} [\![\partial_n u_k]\!]^2 \,\mathrm{d}s = \int_{\mathcal{F}_k^{1-}} h_F^{-1} [\![\partial_n (u_k - u_\infty)]\!]^2 \,\mathrm{d}s \leqslant |\!|\!| u_k - u_\infty |\!|\!|_k^2 \to 0$$

as $k \to \infty$. □

In view of Lemma 16 we have to analyse the limit behaviour of jumpterms stemming from functions located in $\mathbb{V}_\infty$. This is part of the following Proposition.

Proposition 15. *For $v \in \mathbb{V}_\infty$, we have*

$$|\!|\!| v |\!|\!|_k \nearrow |\!|\!| v |\!|\!|_\infty < \infty \quad \text{as } k \to \infty.$$

In particular, for fixed $\ell \in \mathbb{N}$, let $K \in \mathcal{T}_\ell$; then, we have

$$\int_{\{F\in\mathcal{F}_k:\, F\subset K\}} h_k^{-1} [\![\partial_n v]\!]^2 \,\mathrm{d}s \nearrow \int_{\{F\in\mathcal{F}^+:\, F\subset K\}} h_+^{-1} [\![\partial_n v]\!]^2 \,\mathrm{d}s \quad \text{as } k \to \infty.$$

Proof. The assertion follows along the same arguments used in [KG18, Proposition 12]. A detailed proof is provided in the Appendix B. □

Lemma 16. *We have that $\eta_k(\mathcal{T}_k^{2\star}) \to 0$ as $k \to \infty$.*

Proof. Thanks to Proposition 5, we have

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_k^{2\star}} \int_K h_{\mathcal{T}}^4 |f|^2 \,\mathrm{d}x + \int_{\partial K\cap\Omega} h_{\mathcal{T}} \left[\!\left[\partial_n^2 v\right]\!\right]^2 \,\mathrm{d}s \\
&\lesssim \sum_{K\in\mathcal{T}_k^{2\star}} \int_{\omega_k(K)} \left|D^2_{\mathtt{pw}}(u-u_k)\right|^2 \,\mathrm{d}x + \operatorname{osc}(N_k(K), f)^2, \\
&\lesssim \sum_{K\in\mathcal{T}_k^{2\star}} \left\{ \int_{\omega_k(K)} \left|D^2_{\mathtt{pw}} u\right|^2 + \left|D^2_{\mathtt{pw}}(u_k-u_\infty)\right|^2 + \left|D^2_{\mathtt{pw}} u_\infty\right|^2 \,\mathrm{d}x + \operatorname{osc}(N_k(K), f)^2 \right\}.
\end{aligned}$$

The right-hand side vanishes as $k \to \infty$ by Theorem 12, Lemma 9 and the uniform integrability of the terms involving u and u_∞. In this context we emphasise that $\left|\bigcup\left\{\omega_k(K)\colon K \in \mathcal{T}_k^{2\star}\right\}\right| \lesssim \left|\Omega_k^{2\star}\right|$, thanks to the finite overlap of neighbourhoods.

We are left to prove that

$$\int_{\mathcal{F}_k^{2\star}} h_k^{-1} [\![\partial_n u_k]\!]^2 \,\mathrm{d}s \to 0 \quad \text{as } k \to \infty.$$

To this end, we deduce for the solution $u_\infty \in \mathbb{V}_\infty$ of (3.6) that

$$\begin{aligned}
\int_{\mathcal{F}_k^{2\star}} h_k^{-1} [\![\partial_n u_k]\!]^2 \,\mathrm{d}s &\leqslant \int_{\mathcal{F}_k^{2\star}} h_k^{-1} [\![\partial_n (u_k-u_\infty)]\!]^2 \,\mathrm{d}s + \int_{\mathcal{F}_k^{2\star}} h_k^{-1} [\![\partial_n u_\infty]\!]^2 \,\mathrm{d}s \\
&\lesssim |\!|\!| u_k - u_\infty |\!|\!|_k + \int_{\mathcal{F}_k^{2\star}} h_k^{-1} [\![\partial_n u_\infty]\!]^2 \,\mathrm{d}s.
\end{aligned}$$

The first term vanishes thanks to Theorem 12. For the second term we have

$$\begin{aligned}\int_{\mathcal{F}_k^{2\star}} h_k^{-1} [\![\partial_n u_\infty]\!]^2 \,\mathrm{d}s &= \int_{\mathcal{F}_k\setminus\mathcal{F}_k^{1+}} h_k^{-1} [\![\partial_n u_\infty]\!]^2 \,\mathrm{d}s \\ &= \int_{\mathcal{F}_k} h_k^{-1} [\![\partial_n u_\infty]\!]^2 \,\mathrm{d}s - \int_{\mathcal{F}_k^{1+}} h_k^{-1} [\![\partial_n u_\infty]\!]^2 \,\mathrm{d}s \\ &\leqslant \int_{\mathcal{F}^+} h_+^{-1} [\![\partial_n u_\infty]\!]^2 \,\mathrm{d}s - \int_{\mathcal{F}_k^{1+}} h_+^{-1} [\![\partial_n u_\infty]\!]^2 \,\mathrm{d}s,\end{aligned}$$

where the last estimate follows from Proposition 15 and $h_k = h_+$ on $\mathcal{F}_k^{1+}$. Now the assertion follows from Lemma 9 and the fact that $\sum_{F\in\mathcal{F}^+}\int_F h_+^{-1} [\![\partial_n u_\infty]\!]^2 \,\mathrm{d}s \leqslant |\!|\!| u_\infty |\!|\!|_\infty^2 < \infty$. □

Lemma 17. *We have $\eta_k(\mathcal{T}_k^{1+}) \to 0$ as $k \to \infty$.*

Proof. Let $K \in \mathcal{T}_k^{1+}$, then $K \notin \mathcal{M}_k$ thanks to (2.10). Thus, by assumption (2.9) on the marking MARK, we conclude from Lemmas 14 and 16 for all $K \in \mathcal{T}_k^{1+}$ that

$$0 \leqslant \eta_k(K) \leqslant g(\eta_k(\mathcal{M}_k)) \leqslant g(\eta_k(\mathcal{T}_k^{2\star} \cup \mathcal{T}_k^{2-})) \to 0 \quad \text{as } k \to \infty.$$

This yields the element-wise convergence of $\eta_k(K)$ for all $K \in \mathcal{T}_k^{1+}$. The convergence $\eta_k(\mathcal{T}_k^{1+}) \to 0$ as $k \to \infty$ follows then from reformulating the element-wise convergence as pointwise convergence in an integral framework and a generalised Lebesgue dominated convergence theorem; for details see [MSV08, Proposition 4.3]. □

Proof of Theorem 7. Combining Lemmas 14, 16, and 17, we obtain

$$\eta_k(\mathcal{T}_k)^2 = \eta_k(\mathcal{T}_k^{2+})^2 + \eta_k(\mathcal{T}_k^{2-})^2 + \eta_k(\mathcal{T}_k^{2\star})^2 \to 0$$

as $k \to \infty$. Thanks to Proposition 4 this also implies convergence of the error. □

4. Proofs of Lemma 10 and Theorem 12

In order to close the proof of the main result, Theorem 7, we need to verify Lemma 10 and Theorem 12. The primer states that $\mathbb{V}_\infty$ is a Hilbert space with norm $|\!|\!|\cdot|\!|\!|_\infty$, and thus a unique solution $u_\infty \in \mathbb{V}_\infty$ of (3.6) exists; see Corollary 11. The latter proves that u_∞ is indeed the limit of the $\mathsf{C^0IPG}$ approximations $\{u_k\}_{k\in\mathbb{N}_0}$ produced by the $\mathsf{AC^0IPGM}$. We emphasise that in contrast to [KG18], the lack of proper H^2-conforming subspaces of $\mathsf{C^0IPG}$ spaces, does not allow for a straight forward generalisation: For example, in order to prove $|\!|\!| u_\infty - u_k |\!|\!|_k \to 0$, in [KG18] the best-approximation property for inf-sup stable conforming elements [MSV08, Sie11] is replaced by a variant of Gudi's medius analysis [Gud10]. However, this required a discrete smoothing operator into $\mathbb{V}_\infty$, whose construction is heavily based on the existence of a proper conforming subspace of $\mathbb{V}_k$.

After recalling auxiliary Poincaré- and Friedrichs-type inequalities, we shall introduce a smoothing operator, which maps $\mathbb{V}_k$ into $H_0^2(\Omega)$. This accounts for the fact that each $v \in \mathbb{V}_\infty$ on Ω^- is a restriction of an $H_0^2(\Omega)$ function. Moreover, we require an interpolation operator in order to deal with the piecewise discrete structure of $\mathbb{V}_\infty$ on Ω^+. Both operators need to satisfy some compatibility conditions. Finally, we conclude the section with the proofs of Lemma 10 and Theorem 12.

4.1. **Preliminary results.** The following Poincaré and Friedrichs estimates are subsequently used to prove stability of the smoothing and quasi-interpolation operators, defined below.

Lemma 18. *Let $\mathcal{T}, \mathcal{T}_*$ be some triangulations of Ω with $\mathcal{T} \leqslant \mathcal{T}_*$ and let $v \in \mathbb{V}(\mathcal{T}_*)$. Moreover, for $K \in \mathcal{T}$ let $D_K \subset \Omega$ be either $\omega_{\mathcal{T}}(K)$ or $\omega^2_{\mathcal{T}}(K)$. Then, there exists a linear polynomial Q, defined on D_K such that we have*

$$|v - Q|^2_{H^1(D_K)} \lesssim \int_{D_K} h^2_{\mathcal{T}} \left|D^2_{pw} v\right|^2 \,\mathrm{d}x + \sum_{\substack{F \in \mathcal{F}(\mathcal{T}_*)\\ F \subset D_K}} \int_F h^2_{\mathcal{T}} h^{-1}_{\mathcal{T}_*} [\![\partial_n v]\!]^2 \,\mathrm{d}s. \tag{4.1a}$$

If additionally $F \subset D_K \cap \partial\Omega$ for some $F \in \mathcal{F}_{\mathcal{T}}$, then

$$|v|^2_{H^1(D_K)} \lesssim \int_{D_K} h^2_{\mathcal{T}} \left|D^2_{pw} v\right|^2 \,\mathrm{d}x + \sum_{\substack{F \in \mathcal{F}(\mathcal{T}_*)\\ F \subset D_K}} \int_F h^2_{\mathcal{T}} h^{-1}_{\mathcal{T}_*} [\![\partial_n v]\!]^2 \,\mathrm{d}s. \tag{4.1b}$$

Proof. Let $Q \in \mathbb{P}_1(D_K)$ be the H^1-orthogonal projection of v into $\mathbb{P}_1(D_K)$, i.e.,

$$\langle \nabla(v - Q), \nabla P\rangle_{D_K} = 0 \quad \forall P \in \mathbb{P}_1(D_K) \qquad \text{and} \qquad \int_{D_K} Q \,\mathrm{d}x = \int_{D_K} v \,\mathrm{d}x.$$

Now the proof of (4.1a) is a direct consequence of [KG18, Proposition 1].

The second claim (4.1b) follows from [BO09, Corollary 4.3] together with [KG18, Proposition 1] and the definition of the jump terms on boundary sides. □

The following Lemma extends the previous result to the limit space $\mathbb{V}_\infty$.

Lemma 19 (Poincaré-Friedrichs $\mathbb{V}_\infty$)**.** *Let $v \in \mathbb{V}_\infty$ and let either $D_K = \omega_k(K)$ or $D_K = \omega^2_k(K)$ for some $K \in \mathcal{T}_k$ and $k \in \mathbb{N}_0$. Then, there exists $Q \in \mathbb{P}_1(D_K)$, such that*

$$|v - Q|^2_{H^1(D_K)} \lesssim \int_{D_K} h^2_k \left|D^2_{pw} v\right|^2 \,\mathrm{d}x + \sum_{\substack{F \in \mathcal{F}^+\\ F \subset D_K}} \int_F h^2_k h^{-1}_+ [\![\partial_n v]\!]^2 \,\mathrm{d}s.$$

If in addition $F \subset D_K \cap \partial\Omega$ for some $F \in \mathcal{F}_{\mathcal{T}}$, then

$$|v|^2_{H^1(D_K)} \lesssim \int_{D_K} h^2_k \left|D^2_{pw} v\right|^2 \,\mathrm{d}x + \sum_{\substack{F \in \mathcal{F}^+\\ F \subset D_K}} \int_F h^2_k h^{-1}_+ [\![\partial_n v]\!]^2 \,\mathrm{d}s.$$

Proof. We follow the ideas of [KG18, Lemma 13] and let $Q \in \mathbb{P}_1(D_K)$ be the H^1-orthogonal projection of v into $\mathbb{P}_1(D_K)$, defined by

$$\langle \nabla(v - Q), \nabla P\rangle_{D_K} = 0 \quad \forall P \in \mathbb{P}_1(D_K) \qquad \text{and} \qquad \int_{D_K} Q \,\mathrm{d}x = \int_{D_K} v \,\mathrm{d}x.$$

Since $v \in \mathbb{V}_\infty$, there exists a sequence $v_\ell \in \mathbb{V}_\ell$, $\ell \in \mathbb{N}$, with $\lim_{\ell\to\infty} |\!|\!| v - v_\ell |\!|\!|_\ell \to 0$ and $\limsup_{\ell\to\infty} |\!|\!| v_\ell |\!|\!|_\ell < \infty$. From Proposition 15 we have

$$\begin{aligned}\int_{D_K} |D^2_{\mathrm{pw}} v_\ell|^2 \,\mathrm{d}x &+ \sum_{\substack{F \in \mathcal{F}_\ell\\ F \subset D_K}} \int_F h^{-1}_\ell [\![\partial_n v_\ell]\!]^2 \,\mathrm{d}s \\ &\nearrow \int_{D_K} \left|D^2_{\mathrm{pw}} v\right|^2 \,\mathrm{d}x + \sum_{\substack{F \in \mathcal{F}^+\\ F \subset D_K}} \int_F h^{-1}_+ [\![\partial_n v]\!]^2 \,\mathrm{d}s\end{aligned}$$

as $\ell \to \infty$. Let $\ell \geqslant k$. Thanks to Lemma 18 there exists $Q_\ell \in \mathbb{P}_1(D_K)$ with

$$\begin{aligned}|v_\ell - Q_\ell|^2_{H^1(D_K)} &\lesssim \int_{D_K} h^2_k |D^2_{\mathrm{pw}} v_\ell|^2 \,\mathrm{d}x + \sum_{\substack{F \in \mathcal{F}_\ell\\ F \subset D_K}} \int_F h^2_k h^{-1}_\ell [\![\partial_n v_\ell]\!]^2 \,\mathrm{d}s \\ &\nearrow \int_{D_K} h^2_k |D^2_{\mathrm{pw}} v|^2 \,\mathrm{d}x + \sum_{\substack{F \in \mathcal{F}^+\\ F \subset D_K}} \int_F h^2_k h^{-1}_+ [\![\partial_n v]\!]^2 \,\mathrm{d}s,\end{aligned}$$

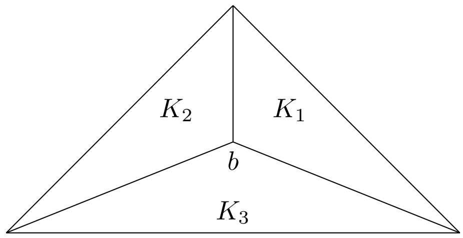

FIGURE 1. A macro triangle K subdivided into three small sub triangles which share a common point b.

as $\ell \to \infty$; compare also with Proposition 15. From the definition of Q and Q_ℓ, we have from Proposition 2 that

$$|Q_\ell - Q|^2_{H^1(D_K)} \lesssim |v_\ell - v|^2_{H^1(D_K)} \leqslant |v_\ell - v|^2_{H^1_0(\Omega)} \lesssim \|v_\ell - v\|^2_\ell \to 0$$

as $\ell \to \infty$. Therefore, Proposition 2 implies $|v_\ell - Q_\ell|^2_{H^1(D_K)} \to |v - Q|^2_{H^1(D_K)}$ as $\ell \to \infty$, which finishes the proof. □

4.2. **Smoothing and quasi-interpolation.** Before introducing the interpolation operator, we first discuss a smoothing operator $\mathcal{E}_\mathcal{T} : \mathbb{V}(\mathcal{T}) \to H^2_0(\Omega)$, $\mathcal{T} \in \mathbb{G}$. To this end, following the ideas of [BGS10, GHV11], we introduce the so-called Hsieh-Clough-Tocher (HCT) macro element constructed in [DDPS79].

Definition 20 (HCT element)**.** *Let $\mathcal{T} \in \mathbb{G}$ and $K \in \mathcal{T}$. Then the HCT nodal macro finite element $(K, \hat{\mathbb{P}}_4(K), \mathcal{N}_K^{HTC})$ is defined as follows.*

a) The local space is given by

$$\hat{\mathbb{P}}_4(K) = \left\{p \in C^1(K) : p|_{K_i} \in \mathbb{P}_4(K_i), i = 1, 2, 3\right\}.$$

Here the three triangles K_1, K_2 and K_3 denote subtriangulation of K obtained by connecting the vertices of K with its barycenter; compare with Figure 1.

b) The degrees of freedom $\mathcal{N}_K^{HTC}$ are given by (compare also with Figure 2)

- *the function value and the gradient at the vertices of K,*
- *the function value at one interior point of each side $F \in \mathcal{F}_\mathcal{T}$, $F \subset \partial K$.*
- *the normal derivative at two disctinct points in the interior of each side $F \in \mathcal{F}_\mathcal{T}$, $F \subset \partial K$.*
- *the function value and the gradient at the barrycenter of K.*

The corresponding global H^2-conforming finite element space is defined as

$$\widetilde{\mathbb{V}}(\mathcal{T}) := \{V \in C^1(\bar{\Omega}) : V|_K \in \hat{\mathbb{P}}_4(K) \text{ for all } K \in \mathcal{T}\}$$

and its global degrees of freedom are given by

$$\mathcal{N}_\mathcal{T}^{HTC} := \bigcup_{K \in \mathcal{T}} \mathcal{N}_K^{HTC},$$

which is well-posed thanks to conformity of $\widetilde{\mathbb{V}}(\mathcal{T}) \subset H^2(\Omega)$.

Since $\mathbb{P}_2(K) \subset \hat{\mathbb{P}}_4(K)$, we can apply $\mathcal{N}_K^{\mathtt{HTC}}$ to $\mathbb{P}_2(K)$. We therefore define the *smoothing operator* $\mathcal{E}_\mathcal{T} : \mathbb{V}(\mathcal{T}) \to \widetilde{\mathbb{V}}(\mathcal{T}) \subset H^2_0(\Omega)$, by setting for all degrees of freedom $N_z \in \mathcal{N}_\mathcal{T}^{\mathtt{HTC}}$:

$$N_z(\mathcal{E}_\mathcal{T}(v)) = \begin{cases} \frac{|K|}{|\omega_k(z)|} \sum\limits_{K \in \omega_k(z)} N_z^K(v|_K) & \text{if } z \in \mathcal{Z}_\mathcal{T}^{\mathtt{HTC}} \cap \Omega \\ 0 & \text{if } z \in \mathcal{Z}_\mathcal{T}^{\mathtt{HTC}} \cap \partial\Omega. \end{cases} \tag{4.2}$$

Here $\mathcal{Z}_\mathcal{T}^{\mathtt{HTC}}$ denotes the set of nodes z associated with some degree of freedom $N_z \in \mathcal{N}_\mathcal{T}^{\mathtt{HTC}}$ and corresponding local degree of freedom $N_z^K \in \mathcal{N}_K^{\mathtt{HTC}}$. Note that there may be different degrees of freedom associated with one node; compare with Figure 2.

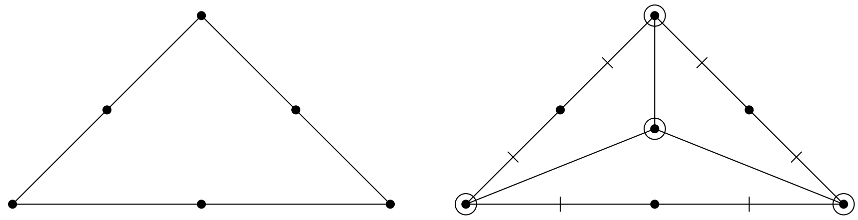

FIGURE 2. The Lagrange element of degree two and the corresponding macro element of degree four. Here point evaluations are denoted by small dots, (first) partial derivatives by circles and normal derivatives by lines.

Lemma 21 ($H^2_0(\Omega)$-smoothing)**.** *Let $\mathcal{T} \in \mathbb{G}$. The operator $\mathcal{E}_\mathcal{T} \colon \mathbb{V}(\mathcal{T}) \to H^2_0(\Omega)$ defined in* (4.2) *satisfies*

$$\sum_{K\in\mathcal{T}} \|D^\alpha(v - \mathcal{E}_\mathcal{T}(v))\|_K^2 \lesssim \int_{\mathcal{F}_\mathcal{T}} h_\mathcal{T}^{\frac{3}{2}-\alpha} \, |[\![\partial_n v]\!]|^2 \, \mathrm{d}s, \qquad \alpha = 0, 1, 2$$

where the hidden constant depends only on shape coefficient of $\mathcal{T}_0$.

Proof. See [GHV11, Lemma 3.1]. □

Denoting by $\mathcal{Z}_K^{\mathtt{HTC}}$ the set of points in K associated with the degrees of freedom $\mathcal{N}_\mathcal{T}^{\mathtt{HTC}}$, we have $\mathcal{Z}_K := \mathcal{Z}_\mathcal{T} \cap K \subset \mathcal{Z}_K^{\mathtt{HTC}}$. This enables us to define a Clément-type quasi-interpolation $\mathcal{I}_\mathcal{T} : L^1(\Omega) \to L^1(\Omega)$, which is locally a right inverse of the smoothing operator $\mathcal{E}_\mathcal{T}$ on $\mathbb{V}(\mathcal{T})$, i.e.,

$$\mathcal{I}_\mathcal{T} \circ \mathcal{E}_\mathcal{T}|_{\mathbb{V}(\mathcal{T})} = \mathrm{id}\,|_{\mathbb{V}(\mathcal{T})}.$$

To this end, we define the operator based on extensions of the local degrees of freedoms $\mathcal{N}_K^{\mathtt{HTC}}$ instead of $\mathcal{N}_K$.

To be more precise, for $K \in \mathcal{T}$, let $\{\phi_N^K \colon N \in \mathcal{N}_K^{\mathtt{HTC}}\}$ be the nodal basis of $\hat{\mathbb{P}}_4(K)$ and identify $\mathcal{N}_K^{\mathtt{HTC}}$ with the dual basis $\left\{\phi_N^{K,*} \colon N \in \mathcal{N}_K^{\mathtt{HTC}}\right\} \subset \hat{\mathbb{P}}_4(K)$, i.e.

$$\left\langle \phi_M^{K,*}, \phi_N^K \right\rangle_{L^2(K)} = M(\phi_N^K) = \delta_{NM} \quad N, M \in \mathcal{N}_K^{\mathtt{HTC}}.$$

Recalling Definition 20, we have that $\mathcal{N}_K^{\mathtt{HTC}}$ contains the point evaluation in the vertices and edge midpoints of $\mathcal{T}$ (the Lagrange nodes $\mathcal{Z}_K$ of $\mathbb{P}_2(K)$). For $z \in \mathcal{Z}_K$, we denote the corresponding dual basis functions by

$$\phi_z^{K,*} \in \left\{\phi_N^{K,*} \colon N \in \mathcal{N}_K^{\mathtt{HTC}}\right\} \quad \text{such that} \quad \langle \phi_z^{K,*}, v\rangle_{L^2(K)} = v(z) \text{ for all } v \in \hat{\mathbb{P}}_4(K).$$

Extending each local dual function by zero to an function in $L^2(\Omega)$ we define

$$\phi_z^* := \frac{1}{|\omega_\mathcal{T}(z)|} \sum_{K\in\omega_\mathcal{T}(z)} \phi_z^{K,*} \in \mathbb{V}(\mathcal{T})^*, \quad z \in \mathcal{Z}_\mathcal{T}.$$

Obviously, $\mathrm{supp}(\phi_z^*) \subset \omega_\mathcal{T}(z)$ and

$$\langle \phi_z^*, v\rangle_{L^2(\Omega)} = v(z) \qquad \text{for all } z \in \mathcal{Z}_\mathcal{T}, v \in \widetilde{\mathbb{V}}(\mathcal{T}).$$

We define a quasi-interpolation operator $\mathcal{I}_\mathcal{T} \colon L^1(\Omega) \to \mathbb{V}(\mathcal{T})$ by

$$(\mathcal{I}_\mathcal{T} v)(z) := \begin{cases} \langle \phi_z^*, v\rangle_{L^2(\Omega)}, & \text{if } z \in \mathcal{Z}_\mathcal{T} \cap \Omega \\ 0, & \text{if } z \in \mathcal{Z}_\mathcal{T} \cap \partial\Omega. \end{cases} \tag{4.3}$$

Since this definition differs from standard Clément interpolation in [Cle75] only by the choice of a different but nevertheless piecewise polynomial dual basis representation, we obtain the following results from standard arguments; see [Cle75].

Lemma 22 (Quasi-interpolation onto $\mathbb{V}(\mathcal{T})$)**.** *For $\mathcal{T} \in \mathbb{G}$ let $\mathcal{I}_{\mathcal{T}}\colon L^1(\Omega) \to \mathbb{V}(\mathcal{T})$ be defined as in* (4.3)*. Then we have that*

a) $\mathcal{I}_{\mathcal{T}}\colon L^p(\Omega) \to L^p(\Omega)$ is a linear and bounded projection for all $1 \leqslant p \leqslant \infty$ and is stable in the following sense: If $v \in H_0^1(\Omega)$ and $\ell \in \mathbb{N}_0$, then

$$\int_{\omega_{\mathcal{T}}^{\ell}(K)} |\nabla \mathcal{I}_{\mathcal{T}} v|^2 \,\mathrm{d}x \lesssim \int_{\omega_{\mathcal{T}}^{\ell+1}(K)} |\nabla v|^2 \,\mathrm{d}x \qquad \textit{for all } K \in \mathcal{T}.$$

b) $\mathcal{I}_{\mathcal{T}} v \in \mathbb{V}(\mathcal{T})$ for all $v \in L^1(\Omega)$,
c) $\mathcal{I}_{\mathcal{T}} v|_K = v|_K$ on $K \in \mathcal{T}$ with $K \cap \partial\Omega = \emptyset$ if $v|_{\omega_{\mathcal{T}}(K)} \in \mathbb{P}_2(N_{\mathcal{T}}(K)) \cap C(\omega_{\mathcal{T}}(K))$,
d) $\mathcal{I}_{\mathcal{T}}(\mathcal{E}_{\mathcal{T}} v)|_K = v|_K$ on $K \in \mathcal{T}$ if $v|_{\omega_{\mathcal{T}}(K)} \in \mathbb{P}_2(N_{\mathcal{T}}(K)) \cap C(\omega_{\mathcal{T}}(K))$ and $K \cap \partial\Omega = \emptyset$ or $v|_{\partial\Omega \cap K} = 0$. Here $\mathcal{E}_{\mathcal{T}}\colon \mathbb{V}(\mathcal{T}) \to H_0^2(\Omega)$ is the enriching operator defined in (4.2)*.*

We remark that, in principle, one can also resort to a Scott-Zhang-type quasi interpolation [SZ90]. However, this complicates the construction of $\mathcal{I}_{\mathcal{T}}$, since a dual basis, bi-orthogonal to the nodal basis of traces of functions in $\hat{\mathbb{P}}_4(K)$, needs to be constructed on faces of boundary elements. The price we have to pay for the simpler construction is that the set of integration needs to be slightly increased in the right hand side of the following stability estimate. We are particularly interested in the interplay of different refinement levels related to the sequence $\{\mathcal{T}_k\}_{k\in\mathbb{N}_0}$ of meshes produced by the $\mathsf{AC^0IPGM}$. To simplify notation, we again replace subscripts $\mathcal{T}_k$ by k, e.g. we write $\mathcal{I}_k$ instead of $\mathcal{I}_{\mathcal{T}_k}$.

Lemma 23 (Stability of $\mathcal{I}_k$)**.** *Let $v \in \mathbb{V}_\ell$ for some $\ell \in \mathbb{N}_0 \cup \{\infty\}$. Then, for all $K \in \mathcal{T}_k$, $k \leqslant \ell$, we have*

$$\begin{aligned}\int_K \left|D^2 \mathcal{I}_k v\right|^2 \mathrm{d}x &+ \int_{\partial K} h_k^{-1} \left[\!\left[\partial_n \mathcal{I}_k v\right]\!\right]^2 \mathrm{d}s \\ &\lesssim \int_{\omega_k^3(K)} \left|D_{pw}^2 v\right|^2 \mathrm{d}x + \sum_{\substack{F \in \mathcal{F}_\ell \\ F \subset \omega_k^3(K)}} \int_F h_\ell^{-1} \left[\!\left[\partial_n v\right]\!\right]^2 \mathrm{d}s,\end{aligned}$$

where $\mathcal{F}_\ell := \mathcal{F}^+$ and $h_\ell := h_+$, when $\ell = \infty$. In particular, we have $|\!|\!| \mathcal{I}_k v |\!|\!|_k \lesssim |\!|\!| v |\!|\!|_\ell$.

Proof. Let $\ell < \infty$ and assume that $K \in \mathcal{T}_k$ such that $\omega_k^2(K) \cap \partial\Omega = \emptyset$. Let Q be the linear polynomial from Lemma 18 with $\mathcal{T} = \mathcal{T}_k$, $\mathcal{T}^\star = \mathcal{T}_\ell$, and $D_K = \omega_k(K)$. Then Lemma 22a) and c) yields

$$\begin{aligned}\int_K \left|D^2 \mathcal{I}_k v\right|^2 \mathrm{d}x &= \int_K \left|D^2 \mathcal{I}_k (v - Q)\right|^2 \mathrm{d}x \lesssim \int_K h_k^{-2} \left|\nabla \mathcal{I}_k (v - Q)\right|^2 \mathrm{d}x \\ &\lesssim \int_{\omega_k(K)} h_k^{-2} \left|\nabla (v - Q)\right|^2 \mathrm{d}x \\ &\lesssim \int_{\omega_k(K)} \left|D_{\mathrm{pw}}^2 v\right|^2 \mathrm{d}x + \sum_{\substack{F \in \mathcal{F}_\ell \\ F \subset \omega_k(K)}} \int_F h_\ell^{-1} \left[\!\left[\partial_n v\right]\!\right]^2 \mathrm{d}s.\end{aligned}$$

In order to bound the jump terms, let Q be the linear polynomial from Lemma 18 with $\mathcal{T} = \mathcal{T}_k$, $\mathbb{V}(\mathcal{T}^\star) = \mathbb{V}_\ell$, and $D_K = \omega_k^2(K)$. We observe that $\nabla Q \equiv$ *const* and hence does not jump across interelement boundaries. Consequently, using Lemma

22a) and c), together with a scaled trace theorem and inverse estimates, we obtain

$$\begin{aligned}\int_{\partial K} h_k^{-1} [\![\partial_n \mathcal{I}_k v]\!]^2 \, \mathrm{d}s &= \int_{\partial K} h_k^{-1} [\![\partial_n \mathcal{I}_k (v-Q)]\!]^2 \, \mathrm{d}s \\ &\lesssim \int_{\omega_k(K)} h_k^{-2} \left|\nabla \mathcal{I}_k (v-Q)\right|^2 \, \mathrm{d}x \lesssim h_K^{-2} \int_{\omega_k^2(K)} \left|\nabla (v-Q)\right|^2 \, \mathrm{d}x \\ &\lesssim \int_{\omega_k^2(K)} \left|D^2_{\mathrm{pw}} v\right|^2 \, \mathrm{d}x + \sum_{\substack{F\in\mathcal{F}_\ell \\ F\subset \omega_k^2(K)}} \int_F h_\ell^{-1} [\![\partial_n v]\!]^2 \, \mathrm{d}s,\end{aligned}$$

where we also used $\bigcup\{\omega_k(F)\colon F\subset \partial K\}\subset \omega_k(K)$ and Lemma 18(4.1a).

If $\omega_k^2(K)\cap\partial\Omega \neq \varnothing$, then there exists a side $F\in\mathcal{F}_k$ with $F\subset\omega_k^3(K)\cap\partial\Omega$. Now applying (4.1b) instead of (4.1a) the desired assertion follows similar as above.

For $\ell=\infty$ we replace Lemma 18 by Lemma 19 and proceed as before. □

In view of the proof of Lemma 10 below, we need a stability estimate comparable to Lemma 23 for $w\in H_0^2(\Omega)$. This estimate follows by analogous arguments as in the proof above but replacing Lemma 18 by the classical Poincaré-Friedrichs inequality for functions in $H_0^2(\Omega)$ together with scaling arguments. In particular for $w\in H_0^2(\Omega)$ we have $|\!|\!|\mathcal{I}_k w|\!|\!|_k \lesssim \left\|D^2 w\right\|_\Omega$.

The following Corollary is an immediate consequence of Lemma 23.

Corollary 24 (Interpolation estimate)**.** *Let $v\in\mathbb{V}_\ell$, $\ell\in\mathbb{N}\cup\{\infty\}$ and $K\in\mathcal{T}_k$ for some $k\leqslant\ell$. Then*

$$\begin{aligned}\int_K \left|D^2\mathcal{I}_k v - D^2_{pw} v\right|^2 \, \mathrm{d}x &+ \int_{\partial K} h_k^{-1} [\![\partial_n(\mathcal{I}_k v - v)]\!]^2 \, \mathrm{d}s \\ &\lesssim \int_{\omega_k^3(K)} \left|D^2_{pw} v\right|^2 \, \mathrm{d}x + \sum_{\substack{F\in\mathcal{F}_\ell \\ F\subset\omega_k^3(K)}} \int_F h_\ell^{-1} [\![\partial_n v]\!]^2 \, \mathrm{d}s,\end{aligned}$$

where we write $\mathcal{F}_\ell := \mathcal{F}^+$ and $h_\ell := h_+$ if $\ell=\infty$ as in Lemma 23.

The next Lemma states the convergence of the interpolation operator

Lemma 25. *Let $v\in\mathbb{V}_\infty$, then $|\!|\!|\mathcal{I}_k v - v|\!|\!|_k \to 0$ as $k\to\infty$.*

Proof. Fix some arbitrary $\epsilon>0$. For $k\in\mathbb{N}$, we split

$$|\!|\!|\mathcal{I}_k v - v|\!|\!|_k^2 = \sum_{K\in\mathcal{T}_k}\left[\int_K \left|D^2\mathcal{I}_k v - D^2_{\mathrm{pw}} v\right|^2 \, \mathrm{d}x + \int_{\partial K} h_k^{-1} [\![\partial_n(\mathcal{I}_k v - v)]\!]^2 \, \mathrm{d}s\right]$$

according to $\mathcal{T}_k = \left(\mathcal{T}_k\setminus\left(\mathcal{T}_k^{2+}\cup\mathcal{T}_k^{2-}\right)\right)\cup\mathcal{T}_k^{2+}\cup\mathcal{T}_k^{2-}$.

[1] We first consider the terms of $|\!|\!|\mathcal{I}_k v - v|\!|\!|_k^2$ according to $\mathcal{T}_k\setminus\left(\mathcal{T}_k^{2+}\cup\mathcal{T}_k^{2-}\right)$. Using Corollary 24, Proposition 15 and the finite overlap of the patches $\omega_k^3(K)$, $K\in\mathcal{T}_k$, we have

$$\begin{aligned}&\sum_{K\in\mathcal{T}_k\setminus(\mathcal{T}_k^{2+}\cup\mathcal{T}_k^{2-})}\left[\int_K \left|D^2\mathcal{I}_k v - D^2_{\mathrm{pw}} v\right|^2 \, \mathrm{d}x + \int_{\partial K} h_k^{-1} [\![\partial_n(\mathcal{I}_k v - v)]\!]^2 \, \mathrm{d}s\right] \\ &\lesssim \sum_{K\in\mathcal{T}_k\setminus(\mathcal{T}_k^{2+}\cup\mathcal{T}_k^{2-})}\left[\int_{\omega_k^3(K)} \left|D^2_{\mathrm{pw}} v\right|^2 \, \mathrm{d}x + \sum_{\substack{F\in\mathcal{F}^+ \\ F\subset\omega_k^3(K)}} \int_F h_+^{-1} [\![\partial_n v]\!]^2 \, \mathrm{d}s\right] \qquad (4.4) \\ &\lesssim \sum_{K\in\mathcal{T}_k\setminus(\mathcal{T}_k^{2+}\cup\mathcal{T}_k^{2-})} \int_{\omega_k^3(K)} \left|D^2_{\mathrm{pw}} v\right|^2 \, \mathrm{d}x + \sum_{F\in\mathcal{F}^+\setminus\mathcal{F}_k^{5+}} \int_F h_+^{-1} [\![\partial_n v]\!]^2 \, \mathrm{d}s.\end{aligned}$$

Thanks to $\Omega(N_k^3(\mathcal{T}_k\backslash(\mathcal{T}_k^{2+}\cup\mathcal{T}_k^{2-})))\subset\Omega\backslash(\Omega_k^{5+}\cup\Omega_k^{5-})$, Lemma 9, and the finite overlap of neighbourhoods, we can employ the uniform integrability of $D^2_{\mathtt{pw}}v$ and conclude for the first term on the right-hand side of (4.4), that

$$\sum_{K\in\mathcal{T}_k\backslash(\mathcal{T}_k^{2+}\cup\mathcal{T}_k^{2-})}\int_{\omega_k^3(K)}\left|D^2_{\mathtt{pw}}v\right|^2\,\mathrm{d}x\lesssim\left\|D^2_{\mathtt{pw}}v\right\|^2_{\Omega\backslash(\Omega_k^{5+}\cup\Omega_k^{5-})}\leqslant\epsilon^2 \tag{4.5}$$

for all $k\geqslant K_1=K_1(\epsilon)$.

In order to estimate the second term on the right-hand side of (4.4), we observe from $\mathcal{F}^+=\bigcup_{k\in\mathbb{N}_0}\mathcal{F}_k^{5+}$ (see Lemma 9) and the fact that $|\!|\!|v|\!|\!|_\infty<\infty$, that

$$\sum_{F\in\mathcal{F}_k^+\backslash\mathcal{F}_k^{5+}}\int_F h_+^{-1}\left[\!\left[\partial_n v\right]\!\right]^2\,\mathrm{d}s\leqslant\sum_{F\in\mathcal{F}^+\backslash\mathcal{F}_k^{5+}}\int_F h_+^{-1}\left[\!\left[\partial_n v\right]\!\right]^2\,\mathrm{d}s<\epsilon^2$$

for all $k\geqslant K_2=K_2(\epsilon)$. Thus, for all $k\geqslant\max\{K_1,K_2\}$, we have

$$\sum_{K\in\mathcal{T}_k\backslash(\mathcal{T}_k^{2+}\cup\mathcal{T}_k^{2-})}\int_K\left|D^2\mathcal{I}_kv-D^2_{\mathtt{pw}}v\right|^2\,\mathrm{d}x+\int_{\partial K}h_k^{-1}\left[\!\left[\partial_n(\mathcal{I}_kv-v)\right]\!\right]^2\,\mathrm{d}s\lesssim\epsilon^2.$$

$\boxed{2}$ We next bound the terms of $|\!|\!|\mathcal{I}_kv-v|\!|\!|_k^2$ according to $\mathcal{T}_k^{2-}$. Recall, that $H^2_{\partial\Omega\cap\partial\Omega^-}(\Omega^-)$ is defined to be the space of restrictions of $H^2_0(\Omega)$-functions to Ω^-. From the density of $H^3_0(\Omega)$ in $H^2_0(\Omega)$ we have that there exists $v_\epsilon\in H^3_0(\Omega)$ with $\|v-v_{\epsilon'}\|^2_{H^2(\Omega^-)}<\epsilon^2/2$. Therefore, stability of $\mathcal{I}_k$ (Lemma 23) and the fact, that $\Omega_k^-\subseteq\overline{\Omega^-}$ imply

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_k^{2-}}\left[\int_K\left|D^2\mathcal{I}_kv-D^2_{\mathtt{pw}}v\right|^2\,\mathrm{d}x+\int_{\partial K}h_k^{-1}\left[\!\left[\partial_n(\mathcal{I}_kv-v)\right]\!\right]^2\,\mathrm{d}s\right]\\
&\quad\leqslant\sum_{K\in\mathcal{T}_k^{2-}}\left[\int_K\left|D^2\mathcal{I}_k(v-v_\epsilon)\right|^2+\left|D^2\mathcal{I}_kv_\epsilon-D^2_{\mathtt{pw}}v_\epsilon\right|^2+\left|D^2_{\mathtt{pw}}v_\epsilon-D^2v\right|^2\,\mathrm{d}x\right]\\
&\qquad+\sum_{K\in\mathcal{T}_k^{2-}}\left[\int_{\partial K}h_k^{-1}\left[\!\left[\partial_n\mathcal{I}_k(v-v_\epsilon)\right]\!\right]^2+h_k^{-1}\left[\!\left[\partial_n\mathcal{I}_kv_\epsilon\right]\!\right]^2\,\mathrm{d}s\right]\\
&\quad\lesssim\int_{\Omega_k^-}\left|D^2_{\mathtt{pw}}v-D^2v_\epsilon\right|^2\,\mathrm{d}x\\
&\qquad+\sum_{K\in\mathcal{T}_k^{2-}}\int_K\left|D^2\mathcal{I}_kv_\epsilon-D^2_{\mathtt{pw}}v_\epsilon\right|^2+\int_{\partial K}h_k^{-1}\left[\!\left[\partial_n\mathcal{I}_kv_\epsilon\right]\!\right]^2\,\mathrm{d}s\\
&\quad\leqslant\epsilon^2+\sum_{K\in\mathcal{T}_k^{2-}}\int_K\left|D^2\mathcal{I}_kv_\epsilon-D^2_{\mathtt{pw}}v_\epsilon\right|^2+\int_{\partial K}h_k^{-1}\left[\!\left[\partial_n\mathcal{I}_kv_\epsilon\right]\!\right]^2\,\mathrm{d}s.
\end{aligned}$$

Employing the trace Theorem and Lemma 22a) and c), we can further bound the last two terms on the right-hand side by

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_k^{2-}}\int_K\left|D^2\mathcal{I}_kv_\epsilon-D^2v_\epsilon\right|^2+\int_{\partial K}h_k^{-1}\left[\!\left[\partial_n\mathcal{I}_kv_\epsilon\right]\!\right]^2\,\mathrm{d}s\\
&\quad\leqslant\sum_{K\in\mathcal{T}_k^{2-}}\int_K\left|D^2\mathcal{I}_k(v_\epsilon-Q_K)\right|^2+\left|D^2(v_\epsilon-Q_K)\right|^2\,\mathrm{d}x\\
&\qquad+\sum_{K\in\mathcal{T}_k^{2-}}\int_{\partial K}h_k^{-1}\left[\!\left[\partial_n\mathcal{I}_k(v_\epsilon-Q_K)\right]\!\right]^2\,\mathrm{d}s\\
&\quad\lesssim\sum_{K\in\mathcal{T}_k^{2-}}\int_{\omega_k^3(K)}h_k^{-2}\left|\nabla(v_\epsilon-Q_K)\right|^2+\left|D^2_{\mathtt{pw}}(v_\epsilon-Q_K)\right|^2\,\mathrm{d}x,
\end{aligned}$$

for some arbitrary $Q_K \in \mathbb{P}_2(\omega_k^3(K))$, $K \in \mathcal{T}_k$. Using the Bramble-Hilbert Lemma [DS80] and the finite overlap of neighbourhoods, we obtain

$$\sum_{K \in \mathcal{T}_k^{2-}} \int_K \left|D^2 \mathcal{I}_k v_\epsilon - D^2 v_\epsilon\right|^2 + \int_{\partial K} h_k^{-1} [\![\partial_n \mathcal{I}_k v_\epsilon]\!]^2 \, \mathrm{d}s$$
$$\lesssim \epsilon^2 + \int_{\Omega_k^-} h_k^2 \sum_{|\alpha|=3} |D^\alpha v_\epsilon|^2 \, \mathrm{d}x \leqslant \epsilon^2 + \left\|h_k \chi_{\Omega_k^-}\right\|_{L^\infty(\Omega)}^2 \int_\Omega \sum_{|\alpha|=3} |D^\alpha v_\epsilon|^2 \, \mathrm{d}x.$$

Thanks to Lemma 9, there exists $K_3 = K_3(\epsilon)$, sucht that for all $k \geqslant K_3$, we have

$$\sum_{K \in \mathcal{T}_k^{2-}} \int_K \left|D^2 \mathcal{I}_k v - D^2_{\mathtt{pw}} v\right|^2 \, \mathrm{d}x + \int_{\partial K} h_k^{-1} [\![\partial_n (\mathcal{I}_k v - v)]\!]^2 \, \mathrm{d}s \lesssim \epsilon^2.$$

$\boxed{3}$ By the definition of $\mathbb{V}_\infty$, we have that $v|_{\omega_k(K)} \in \mathbb{P}_2(N_k(K)) \cap C(\omega_k(K))$ for all $K \in \mathcal{T}_k^{2+}$. Therefore, Lemma 22c) implies $\mathcal{I}_k v = v$ on Ω_k^{2+} and thus

$$\sum_{K \in \mathcal{T}_k^{2+}} \int_K \left|D^2 \mathcal{I}_k v - D^2_{\mathtt{pw}} v\right|^2 \, \mathrm{d}x + \int_{\partial K} h_k^{-1} [\![\partial_n (\mathcal{I}_k v - v)]\!]^2 \, \mathrm{d}s = 0.$$

Concluding, we have $\|\!|\mathcal{I}_k v - v\|\!|_k^2 \lesssim \epsilon$ whenever $k \geqslant \max\{K_1, K_2, K_3\}$. This is the desired result. □

4.3. **Proof of Lemma 10.** Note that for all $v \in \overline{\mathbb{V}}_\infty^{\|\!|\cdot\|\!|_\infty}$ the bounds

$$|v|_{H_0^1(\Omega)} \lesssim \|\!|v\|\!|_\infty \quad \text{and} \quad |D(\nabla v)|(\Omega) \lesssim \|\!|v\|\!|_\infty,$$

are inherited from Propositions 2 and 3. In particular, the trace of $\nabla v \in BV(\Omega)^2$ is measurable on sides $F \in \mathcal{F}_k$, $k \in \mathbb{N}$, (c.f. [AFP00, Theorem 3.88]) and thus we conclude also for $v \in \overline{\mathbb{V}}_\infty^{\|\!|\cdot\|\!|_\infty}$ that $\|\!|v\|\!|_k \nearrow \|\!|v\|\!|_\infty$ as $k \to \infty$ from Proposition 15 and the density of $\mathbb{V}_\infty$ in $\overline{\mathbb{V}}_\infty^{\|\!|\cdot\|\!|_\infty}$.

$\boxed{1}$ Let $0 \neq v \in \overline{\mathbb{V}}_\infty^{\|\!|\cdot\|\!|_\infty}$ arbitrary, then there exist $\{v^\ell\}_{\ell \in \mathbb{N}} \subset \mathbb{V}_\infty$ such that $\|\!|v - v^\ell\|\!|_\infty \to 0$ as $\ell \to \infty$. Using norm equivalence on finite dimensional spaces, we readily conclude that $v|_K \in \mathbb{P}_2(K)$ for all $K \in \mathcal{T}^+$.

$\boxed{2}$ In order to prove $v|_{\Omega^-} \in H^2_{\partial\Omega \cap \partial\Omega^-}(\Omega^-)$ we need to show that v is a restriction of a $H_0^2(\Omega)$-function. To this end, let $\{m_\ell\}_{\ell \in \mathbb{N}_0} \subset \mathbb{N}$ such that $\|\!|v^\ell - v^\ell_{m_\ell}\|\!|_{m_\ell} \leqslant \frac{1}{\ell}$ for $v_{m_\ell} := \mathcal{I}_{m_\ell} v^\ell \in \mathbb{V}_{m_\ell}$; see Lemma 25. Then $\|\!|v^\ell_{m_\ell}\|\!|_{m_\ell}$ is uniformly bounded, since

$$\|\!|v^\ell_{m_\ell}\|\!|_{m_\ell} \leqslant \|\!|v^\ell_{m_\ell} - v^\ell\|\!|_{m_\ell} + \|\!|v^\ell - v\|\!|_{m_\ell} + \|\!|v\|\!|_{m_\ell} \to \|\!|v\|\!|_\infty \qquad \text{as } \ell \to \infty.$$

We can now apply the smoothing operator defined in (4.2) to $v^\ell_{m_\ell} \in \mathbb{V}_{m_\ell}$ together with Lemma 21 ($\alpha = 2$) and obtain

$$\left\|D^2 \mathcal{E}_{m_\ell}(v^\ell_{m_\ell})\right\|_\Omega \lesssim \left\|D^2_{\mathtt{pw}}(\mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) - v^\ell_{m_\ell})\right\|_\Omega + \left\|D^2_{\mathtt{pw}} v^\ell_{m_\ell}\right\|_\Omega \lesssim \|\!|v^\ell_{m_\ell}\|\!|_{m_\ell}.$$

Hence, there exists $w \in H_0^2(\Omega)$ such that, for a not relabelled subsequence

$$\mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) \rightharpoonup w \quad \text{weakly in } H_0^2(\Omega), \qquad \text{as } \ell \to \infty. \tag{4.6}$$

Again from Lemma 21 (for $\alpha = 0$) and the trace theorem with scaling we have that

$$\left\|\mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) - v^\ell_{m_\ell}\right\|^2_{\Omega^-_{m_\ell}} \lesssim \int_{\mathcal{F}^-_{m_\ell}} h^3_{m_\ell} [\![\partial_n v^\ell_{m_\ell}]\!]^2 \, \mathrm{d}s \lesssim \left\|h_{m_\ell} \chi_{\Omega^-_{m_\ell}}\right\|^4_{L^\infty(\Omega)} \|\!|v^\ell_{m_\ell}\|\!|^2_{m_\ell},$$

where we used $\|h_{m_\ell}\|_{L^\infty(\mathcal{F}^-_{m_\ell})} \lesssim \|h_{m_\ell}\chi_{\Omega^-_{m_\ell}}\|_{L^\infty(\Omega)}$. Applying Lemma 9 yields

$$\begin{aligned}\|\mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) - v^\ell_{m_\ell}\|_{\Omega^-} &= \|\mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) - v^\ell_{m_\ell}\|_{\Omega^-_{m_\ell}} + \|\mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) - v^\ell_{m_\ell}\|_{\Omega^-\setminus\Omega^-_{m_\ell}} \\ &\lesssim \left\|h_{m_\ell}\chi_{\Omega^-_{m_\ell}}\right\|^2_{L^\infty(\Omega)} \|v^\ell_{m_\ell}\|_{m_\ell} + \|\mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) - w\|_\Omega + \|v^\ell_{m_\ell} - v\|_\Omega \\ &\qquad + \|v\|_{\Omega^-\setminus\Omega^-_{m_\ell}} + \|w\|_{\Omega^-\setminus\Omega^-_{m_\ell}} \\ &\to 0 \qquad \text{as } \ell \to \infty.\end{aligned}$$

Here we also used uniform integrability of v, w and (4.6) together with the fact that $H^2_0(\Omega)$ is compactly embedded into $L^2(\Omega)$. As a consequence, we have that $v|_{\Omega^-} = w|_{\Omega^-}$ and thus $v|_{\Omega^-} \in H^2_{\partial\Omega\cap\partial\Omega^-}(\Omega^-)$.

[3] We conclude by showing that for $v_k := \mathcal{I}_k w \in \mathbb{V}_k$, $k \in \mathbb{N}$, we have $\|v - v_k\|_k \to 0$ as $k \to \infty$, and $\limsup_{k\to\infty} \|v_k\|_k < \infty$; here $w \in H^2_0(\Omega)$ is the function defined in (4.6). The uniform boundedness follows since from Lemma 23, we have

$$\|v_k\|_k \lesssim \sum_{K\in\mathcal{T}_k} \int_{\omega^3_k(K)} |D^2 w|^2 \,\mathrm{d}x \lesssim \|w\|_{H^2_0(\Omega)} < \infty.$$

Fix now $\epsilon > 0$. Similarly as in the proof of Lemma 25, we split $\|v - v_k\|^2_k$ according to $\mathcal{T}_k = \left(\mathcal{T}_k \setminus \left(\mathcal{T}^{2+}_k \cup \mathcal{T}^{2-}_k\right)\right) \cup \mathcal{T}^{2+}_k \cup \mathcal{T}^{2-}_k$ and consider the corresponding terms separately. Thanks to Lemma 23, we have

$$\begin{aligned}&\sum_{K\in\mathcal{T}_k\setminus(\mathcal{T}^{2+}_k\cup\mathcal{T}^{2-}_k)} \left[\int_K |D^2\mathcal{I}_k w - D^2_{\mathtt{pw}} v|^2 \,\mathrm{d}x + \int_{\partial K} h_k^{-1} [\![\partial_n(\mathcal{I}_k w - v)]\!]^2 \,\mathrm{d}s\right] \\ &\lesssim \sum_{K\in\mathcal{T}_k\setminus(\mathcal{T}^{2+}_k\cup\mathcal{T}^{2-}_k)} \left[\int_{\omega^3_k(K)} |D^2 w|^2 + |D^2_{\mathtt{pw}} v|^2 \,\mathrm{d}x + \sum_{\substack{F\in\mathcal{F}^+\\ F\subset K}} \int_F h_+^{-1} [\![\partial_n v]\!]^2 \,\mathrm{d}s\right] \\ &\lesssim \sum_{K\in\mathcal{T}_k\setminus(\mathcal{T}^{2+}_k\cup\mathcal{T}^{2-}_k)} \int_{\omega^3_k(K)} |D^2 w|^2 + |D^2_{\mathtt{pw}} v|^2 \,\mathrm{d}x + \sum_{F\in\mathcal{F}^+\setminus\mathcal{F}^{2+}_k} \int_F h_+^{-1} [\![\partial_n v]\!]^2 \,\mathrm{d}s.\end{aligned}$$

Arguing as in (4.5), we can employ the uniform integrability of $D^2_{\mathtt{pw}} v$ and $D^2 w$ to obtain

$$\sum_{K\in\mathcal{T}_k\setminus(\mathcal{T}^{2+}_k\cup\mathcal{T}^{2-}_k)} \int_{\omega^3_k(K)} |D^2_{\mathtt{pw}} v|^2 + |D^2 w|^2 \,\mathrm{d}x < \epsilon,$$

for all $k \geqslant K_1 = K_1(\epsilon)$. According to $\mathcal{F}^+ = \bigcup_{k\in\mathbb{N}_0} \mathcal{F}^{2+}_k$ and $\|v\|_\infty < \infty$, we have

$$\sum_{F\in\mathcal{F}^+\setminus\mathcal{F}^{2+}_k} \int_F h_+^{-1} [\![\partial_n v]\!]^2 \,\mathrm{d}s < \epsilon,$$

for all $k \geqslant K_2 = K_2(\epsilon)$ and consequently we conclude for all $k \geqslant \max\{K_1, K_2\}$ that

(4.7)
$$\sum_{K\in\mathcal{T}_k\setminus(\mathcal{T}^{2+}_k\cup\mathcal{T}^{2-}_k)} \left[\int_K |D^2\mathcal{I}_k w - D^2_{\mathtt{pw}} v|^2 \,\mathrm{d}x + \int_{\partial K} h_k^{-1} [\![\partial_n(\mathcal{I}_k w - v)]\!]^2 \,\mathrm{d}s\right] \lesssim \epsilon.$$

Considering the terms of $\|v - v_k\|^2_k$ according to $\mathcal{T}^{2-}_k$, we recall $v|_{\Omega^-} = w|_{\Omega^-}$ and it suffices to prove

$$\sum_{K\in\mathcal{T}^{2-}_k} \int_K |D^2\mathcal{I}_k v - D^2_{\mathtt{pw}} v|^2 \,\mathrm{d}x + \int_{\partial K} h_k^{-1} [\![\partial_n(\mathcal{I}_k v - v)]\!]^2 \,\mathrm{d}s \lesssim \epsilon \tag{4.8}$$

for all $k \geqslant K_3 = K_3(\epsilon)$. This follows exactly as in step [2] of Lemma 25.

Let now $K \in \mathcal{T}_k^{2+}$. Then we have for all $m_\ell \geqslant k$ that $\mathcal{T}_k^+ \subset \mathcal{T}_{m_\ell}^+$ and thus $v_{m_\ell}^\ell|_{\omega_k(K)} \in \mathbb{P}_2(N_k(K)) \cap C(\omega_k(K))$ (see step $\boxed{2}$ for the definition of $v_{m_\ell}^\ell$ and m_ℓ). Therefore, Lemma 22d) implies

$$v_k = \mathcal{I}_k w \leftarrow \mathcal{I}_k \mathcal{E}_{m_\ell} v_{m_\ell}^\ell = \mathcal{I}_k \mathcal{E}_k v_{m_\ell}^\ell = v_{m_\ell}^\ell \to v \quad \text{in } \mathbb{P}_2(K)$$

as $\ell \to \infty$. Consequently, for all $k \in \mathbb{N}$, we have

$$\sum_{K \in \mathcal{T}_k^{2+}} \int_K \left|D^2 v_k - D^2 v\right|^2 \,\mathrm{d}x + \int_{\partial K} h_k^{-1} \left[\!\left[\partial_n (v_k - v)\right]\!\right]^2 \,\mathrm{d}s = 0.$$

Combining this with (4.7) and (4.8), we have $\|\mathcal{I}_k w - v\|_k^2 \lesssim \epsilon$ whenever $k \geqslant \max\{K_1, K_2, K_3\}$.

Overall, we have thus showed that $v \in \mathbb{V}_\infty$, which concludes the proof. □

4.4. **Proof of Theorem 12.** To identify a candidate for the limit of the sequence $\{u_k\}_{k\in\mathbb{N}_0}$ of discrete approximations computed by the AC^{0}IPGM, we employ Proposition 2 and (2.4), and conclude that

$$u_{k_j} \rightharpoonup \overline{u}_\infty \qquad \text{weakly in } H_0^1(\Omega) \quad \text{as } j \to \infty \tag{4.9}$$

for some subsequence $\{k_j\}_{j\in\mathbb{N}_0} \subset \{k\}_{k\in\mathbb{N}_0}$ and $\overline{u}_\infty \in H_0^1(\Omega)$. In the following, we shall see that in fact $u_\infty = \overline{u}_\infty \in \mathbb{V}_\infty$. Thus $\{u_k\}_{k\in\mathbb{N}_0}$ has only one weak accumulation point and the whole sequence converges. Finally we will conclude the section with proving the strong convergence $\lim_{k\to\infty} \|u_k - u_\infty\|_k = 0$ claimed in Theorem 12.

Lemma 26. *We have $\overline{u}_\infty \in \mathbb{V}_\infty$.*

Proof. $\boxed{1}$ For each $K \in \mathcal{T}^+$, the weak convergence (4.9) implies strong convergence of the restrictions $u_{k_j}|_K$ in the finite dimensional $\mathbb{P}_2(K)$ and thus $\overline{u}_\infty|_K \in \mathbb{P}_2(K)$.

Thanks to the uniform boundedness (2.4) of $\|u_{k_j}\|_{k_j}$, we conclude with Propositions 2 and 3 that

$$\nabla u_{k_j} \rightharpoonup^* \nabla \overline{u}_\infty \qquad \text{weakly* in } BV(\Omega)^2 \quad \text{as } j \to \infty; \tag{4.10}$$

compare also with [AFP00, Theorem 3.23]. Moreover, Lemma 21 ($\alpha = 2$) yields for the smoothing operator from (4.2) that

$$\left\|D^2 \mathcal{E}_{k_j}(u_{k_j})\right\|_\Omega \leqslant \left\|D^2_{\mathtt{pw}}(\mathcal{E}_{k_j}(u_{k_j}) - u_{k_j})\right\|_\Omega + \left\|D^2_{\mathtt{pw}} u_{k_j}\right\|_\Omega \lesssim \|u_{k_j}\|_{k_j}.$$

We thus have

$$\mathcal{E}_{k_j}(u_{k_j}) \rightharpoonup w \quad \text{weakly in } H_0^2(\Omega) \tag{4.11}$$

for a not relabelled subsequence. Arguing as in step $\boxed{2}$ of Lemma 10, we obtain thanks to compact embeddings, that $\left\|\mathcal{E}_{k_j}(u_{k_j}) - u_{k_j}\right\|_{\Omega_{k_j}^-} \to 0$ as $j \to \infty$ and thus

$$\overline{u}_\infty|_{\Omega^-} = w|_{\Omega^-} \in H^2_{\partial\Omega \cap \partial\Omega^-}(\Omega^-).$$

$\boxed{2}$ For w from (4.11), defining

$$v_k := \mathcal{I}_k w \in \mathbb{V}_k,$$

we have by Lemma 23 that $\|v_k\|_k \lesssim \left\|D^2 w\right\|_\Omega < \infty$. Therefore, in order to conclude the proof, it remains to show that $\|v_k - \overline{u}_\infty\|_k \to 0$ as $k \to \infty$. To see this, we first observe that, thanks to Lemma 22d), we have $\mathcal{I}_k w = \overline{u}_\infty$ on all $K \in \mathcal{T}_k^{1+}$ and thus

$$\begin{aligned}\sum_{F \in \mathcal{F}^+} \int_F h_+^{-1} \left[\!\left[\partial_n \overline{u}_\infty\right]\!\right]^2 \,\mathrm{d}s &= \lim_{k\to\infty} \sum_{F \in \mathcal{F}_k^{1+}} \int_F h_k^{-1} \left[\!\left[\partial_n \overline{u}_\infty\right]\!\right]^2 \,\mathrm{d}s \\ &= \lim_{k\to\infty} \sum_{F \in \mathcal{F}_k^{1+}} \int_F h_k^{-1} \left[\!\left[\partial_n v_k\right]\!\right]^2 \,\mathrm{d}s \leqslant \sup_k \|v_k\|_k^2 < \infty.\end{aligned}$$

In the same vein, we have that $D^2_{\mathtt{pw}} v_k = D^2_{\mathtt{pw}} \overline{u}_\infty$ for all $K \in \mathcal{T}_k^{1+}$, which implies $D^2_{\mathtt{pw}} v_k \to D^2_{\mathtt{pw}} \overline{u}_\infty$ a.e. in Ω^+ as $k \to \infty$ and thus $D^2_{\mathtt{pw}} \overline{u}_\infty \in L^2(\Omega^+)$. Together with $D^2 \overline{u}_\infty = D^2 w$ in Ω^-, this yields $D^2_{\mathtt{pw}} \overline{u}_\infty \in L^2(\Omega)$ and we conclude $\|\overline{u}_\infty\|_\infty < \infty$.

The assertion follows now as in step $\boxed{3}$ of the proof of Lemma 10 by splitting $\|v_k - \overline{u}_\infty\|_k^2$ according to $\mathcal{T}_k = \big(\mathcal{T}_k \backslash \big(\mathcal{T}_k^{2+} \cup \mathcal{T}_k^{2-}\big)\big) \cup \mathcal{T}_k^{2+} \cup \mathcal{T}_k^{2-}$ and investigating the resulting terms separately. □

Lemma 27. *We have that $\overline{u}_\infty \in \mathbb{V}_\infty$ solves* (3.6) *and thus $\overline{u}_\infty = u_\infty$. In particular, the limit in* (4.9) *is unique and the full sequence $\{u_k\}_{k \in \mathbb{N}_0}$ converges to u_∞ weakly in $H^1_0(\Omega)$.*

Proof. Let $v \in \mathbb{V}_\infty$ and $\{v_k\}_{k\in\mathbb{N}}$, $v_k \in \mathbb{V}_k$ such that $\|v_k - v\|_k \to 0$ as $k \to \infty$. Consequently, for the subsequence (4.9) of discrete solutions $\{u_{k_j}\}_{j\in\mathbb{N}}$, we have

$$\mathfrak{B}_{k_j}[u_{k_j}, v_{k_j}] = \langle f, v_{k_j} \rangle_\Omega \to \langle f, v \rangle_{L^2(\Omega)} \quad \text{as } j \to \infty. \tag{4.12}$$

Using $\|v_k - v\|_k \to 0$ as $k \to \infty$ again, it suffices to prove $\mathfrak{B}_{k_j}[u_{k_j}, v] \to \mathfrak{B}_\infty[\overline{u}_\infty, v]$ as $j \to \infty$. To see this, we split the bilinear form according to

$$\begin{aligned}\mathfrak{B}_{k_j}[u_{k_j}, v] &= \int_\Omega (D^2_{\mathtt{pw}} u_{k_j} - \mathcal{L}_{k_j}(\llbracket \partial_n u_{k_j} \rrbracket)) : D^2_{\mathtt{pw}} v \,\mathrm{d}x \\ &\quad - \int_\Omega \mathcal{L}_{k_j}(\llbracket \partial_n v \rrbracket) : D^2_{\mathtt{pw}} u_{k_j} \,\mathrm{d}x + \int_{\mathcal{F}_{k_j}} \frac{\sigma}{h_{k_j}} \llbracket \partial_n u_{k_j} \rrbracket \llbracket \partial_n v \rrbracket \,\mathrm{d}s \\ &=: I_j - II_j + III_j.\end{aligned}$$

and consider the limit of each term separately.

$\boxed{1}$ In order to analyse the limit of I_j, we split the domain Ω according to

$$\Omega = \Omega \backslash (\Omega_\ell^{1+} \cup \Omega_\ell^{1-}) \cup \Omega_\ell^{1-} \cup \Omega_\ell^{1+}$$

for some $\ell \leqslant k_j$. We recall from (2.7) and (2.4) that

$$\|D^2_{\mathtt{pw}} u_{k_j}\|_\Omega \lesssim \|u_{k_j}\|_{k_j} \lesssim \|f\|_\Omega \quad \text{and} \quad \|\mathcal{L}_{k_j}(\llbracket \partial_n u_{k_j} \rrbracket)\|_\Omega \lesssim \|u_{k_j}\|_{k_j} \lesssim \|f\|_\Omega .$$

Hence, thanks to Lemma 9 and the stability of liftings, for $\epsilon > 0$ there exists $K(\epsilon)$ such that

$$\begin{aligned}\Big| \int_{\Omega \backslash (\Omega_\ell^{1+} \cup \Omega_\ell^{1-})} (D^2_{\mathtt{pw}} u_{k_j} - \mathcal{L}_{k_j}(\llbracket \partial_n u_{k_j} \rrbracket) - D^2_{\mathtt{pw}} \overline{u}_\infty + \mathcal{L}_\infty(\llbracket \partial_n \overline{u}_\infty \rrbracket)) : D^2_{\mathtt{pw}} v \,\mathrm{d}x \Big| \\ \lesssim \Big(\|u_{k_j}\|_{k_j} + \|\overline{u}_\infty\|_\infty \Big) \|D^2_{\mathtt{pw}} v\|_{\Omega \backslash (\Omega_\ell^{1+} \cup \Omega_\ell^{1-})} \leqslant \epsilon\end{aligned}$$

for each fixed $\ell \geqslant K(\epsilon)$. Moreover, on Ω_ℓ^{1-} we have, similar to [BO09, Theorem 5.2] (for details, compare Lemma 28, Appendix C), that

$$\int_{\Omega_\ell^{1-}} (D^2_{\mathtt{pw}} u_{k_j} - \mathcal{L}_{k_j}(\llbracket \partial_n u_{k_j} \rrbracket)) : D^2 v \,\mathrm{d}x \to \int_{\Omega_\ell^{1-}} D^2 \overline{u}_\infty : D^2 v \,\mathrm{d}x \quad \text{as } j \to \infty.$$

For the terms according to Ω_ℓ^{1+}, we observe from (4.9) that $D^2_{\mathtt{pw}} u_{k_j}|_{\Omega_\ell^{1+}} \to D^2_{\mathtt{pw}} \overline{u}_\infty|_{\Omega_\ell^{1+}}$ strongly in $L^2(\Omega_\ell^{1+})$ as $j \to \infty$ since $\mathbb{P}_0(\mathcal{T}_\ell^{1+})^{2\times 2}$ is finite dimensional for fixed ℓ. Therefore, we have

$$\int_{\Omega_\ell^{1+}} D^2_{\mathtt{pw}} u_{k_j} : D^2_{\mathtt{pw}} v \,\mathrm{d}x \to \int_{\Omega_\ell^{1+}} D^2_{\mathtt{pw}} \overline{u}_\infty : D^2_{\mathtt{pw}} v \,\mathrm{d}x \quad \text{as } j \to \infty.$$

Similar arguments prove $\llbracket \partial_n u_{k_j} \rrbracket |_{\mathcal{F}_\ell^{1+}} \to \llbracket \partial_n \overline{u}_\infty \rrbracket |_{\mathcal{F}_\ell^{1+}}$ strongly in $L^2(\mathcal{F}_\ell^{1+})$ as $j \to \infty$ and, thanks to the fact that the local definition (2.5) of the liftings eventually does not change on $\mathcal{T}_\ell^{1+}$, we have

$$\int_{\Omega_\ell^{1+}} \mathcal{L}_{k_j}([\![\partial_n u_{k_j}]\!])\colon D^2_{\mathtt{pw}} v \,\mathrm{d}x = \int_{\Omega_\ell^{1+}} \mathcal{L}_{\infty}([\![\partial_n u_{k_j}]\!])\colon D^2_{\mathtt{pw}} v \,\mathrm{d}x$$
$$\to \int_{\Omega_\ell^{1+}} \mathcal{L}_{\infty}([\![\partial_n \overline{u}_\infty]\!])\colon D^2_{\mathtt{pw}} v \qquad \text{as } j \to \infty.$$

Since $\epsilon > 0$ was arbitrary, combining the above results with the fact that $|\Omega\backslash(\Omega_\ell^{1+} \cup \Omega_\ell^{1-})| \to 0$ as $\ell \to \infty$, we have proved that

$$\lim_{j\to\infty} \Big| \int_\Omega (D^2_{\mathtt{pw}} u_{k_j} - \mathcal{L}_{k_j}([\![\partial_n u_{k_j}]\!]))\colon D^2_{\mathtt{pw}} v \,\mathrm{d}x - \int_\Omega (D^2_{\mathtt{pw}} \overline{u}_\infty - \mathcal{L}_\infty([\![\partial_n \overline{u}_\infty]\!]))\colon D^2_{\mathtt{pw}} v \,\mathrm{d}x \Big|$$
$$\leqslant \epsilon + \Big| \int_{\Omega\backslash(\Omega_\ell^{1+}\cup\Omega_\ell^{1-})} (D^2_{\mathtt{pw}} \overline{u}_\infty - \mathcal{L}_\infty([\![\partial_n \overline{u}_\infty]\!]))\colon D^2_{\mathtt{pw}} v \,\mathrm{d}x \Big| \to \epsilon$$

as $\ell \to \infty$. Since $\epsilon > 0$ was chosen arbitrary, for $j \to \infty$ we conclude

(4.13)
$$\int_\Omega (D^2_{\mathtt{pw}} u_{k_j} - \mathcal{L}_{k_j}([\![\partial_n u_{k_j}]\!]))\colon D^2_{\mathtt{pw}} v \,\mathrm{d}x \to \int_\Omega (D^2_{\mathtt{pw}} \overline{u}_\infty - \mathcal{L}_\infty([\![\partial_n \overline{u}_\infty]\!]))\colon D^2_{\mathtt{pw}} v \,\mathrm{d}x.$$

[2] In order to identify the limit of II_j, we split the domain Ω according to

$$\Omega = \Omega\backslash\Omega_\ell^{1+} \cup \Omega_\ell^{1+}$$

for some $\ell \leqslant k_j$. Thanks to uniform boundedness $|\!|\!| u_k |\!|\!|_k \lesssim \|f\|_\Omega$, for $\epsilon > 0$, we have

$$\Big| \int_{\Omega\backslash\Omega_\ell^{1+}} \mathcal{L}_{k_j}([\![\partial_n v]\!])\colon D^2_{\mathtt{pw}} u_{k_j} \,\mathrm{d}x \Big| \lesssim \|\mathcal{L}_{k_j}([\![\partial_n v]\!])\|_{\Omega\backslash\Omega_\ell^{1+}} \|f\|_\Omega < \epsilon \tag{4.14}$$

for all $k_j \geqslant \ell \geqslant K(\epsilon)$. Indeed, the stability of the lifting operator (2.6) together with Proposition 15 yields

$$\|\mathcal{L}_{k_j}([\![\partial_n v]\!])\|_{\Omega\backslash\Omega_\ell^{1+}} \lesssim \left(\int_{\mathcal{F}_{k_j}\backslash\mathcal{F}_\ell^{2+}} h_{k_j}^{-1} [\![\partial_n v]\!]^2 \,\mathrm{d}s \right)^{1/2} \to 0 \qquad \text{as } k_j \geqslant \ell \to \infty.$$

As in [1], on Ω_ℓ^{1+} we employ the strong convergence $D^2_{\mathtt{pw}} u_{k_j}|_{\Omega_\ell^{1+}} \to D^2_{\mathtt{pw}} \overline{u}_\infty|_{\Omega_\ell^{1+}} \in \mathbb{P}_0(\mathcal{T}_\ell^{1+})^{2\times 2}$ in $L^2(\Omega_\ell^{1+})$ as $j \to \infty$, in order to obtain from the local definitions of the liftings (2.5) and (3.3) that

$$\int_{\Omega_\ell^{1+}} \mathcal{L}_{k_j}([\![\partial_n v]\!])\colon D^2_{\mathtt{pw}} u_{k_j} \,\mathrm{d}x = \int_{\Omega_\ell^{1+}} \mathcal{L}_{\infty}([\![\partial_n v]\!])\colon D^2_{\mathtt{pw}} u_{k_j} \,\mathrm{d}x$$
$$\to \int_{\Omega_\ell^{1+}} \mathcal{L}_{\infty}([\![\partial_n v]\!])\colon D^2_{\mathtt{pw}} \overline{u}_\infty \,\mathrm{d}x \qquad \text{as } j \to \infty.$$

Combining this with (4.14) and $|\Omega\backslash(\Omega_\ell^{1+} \cup \Omega_\ell^{1-})| \to 0$ as $\ell \to \infty$, we thus obtain

$$\Big| \int_\Omega \mathcal{L}_{k_j}([\![\partial_n v]\!])\colon D^2_{\mathtt{pw}} u_{k_j} - \mathcal{L}_\infty([\![\partial_n v]\!])\colon D^2_{\mathtt{pw}} \overline{u}_\infty \,\mathrm{d}x \Big|$$
$$\leqslant \epsilon + \Big| \int_{\Omega^+\backslash\Omega_\ell^{1+}} \mathcal{L}_\infty([\![\partial_n v]\!])\colon D^2_{\mathtt{pw}} \overline{u}_\infty \,\mathrm{d}x \Big| \to \epsilon \qquad \text{as } \ell \to \infty.$$

Since $\epsilon > 0$ was arbitrary, this yields

$$\int_\Omega \mathcal{L}_{k_j}([\![\partial_n v]\!])\colon D^2_{\mathtt{pw}} u_{k_j} \,\mathrm{d}x \to \int_\Omega \mathcal{L}_\infty([\![\partial_n v]\!])\colon D^2_{\mathtt{pw}} \overline{u}_\infty \,\mathrm{d}x \quad \text{as } k \to \infty. \tag{4.15}$$

$\boxed{3}$ For the last term III_j, we observe from $\mathcal{F}_\ell^+ \subset \mathcal{F}_{k_j}^+$, $\ell \leqslant k_j$, that

$$\int_{\mathcal{F}_{k_j}} \frac{\sigma}{h_{k_j}} [\![\partial_n u_{k_j}]\!] [\![\partial_n v]\!] \,\mathrm{d}s = \int_{\mathcal{F}_\ell^+} \frac{\sigma}{h_{k_j}} [\![\partial_n u_{k_j}]\!] [\![\partial_n v]\!] \,\mathrm{d}s + \int_{\mathcal{F}_{k_j}\setminus\mathcal{F}_\ell^+} \frac{\sigma}{h_{k_j}} [\![\partial_n u_{k_j}]\!] [\![\partial_n v]\!] \,\mathrm{d}s.$$

For the second term on the right-hand side, we conclude from Proposition 15 that for arbitrary fixed ϵ there exists $K(\epsilon) > 0$ such that

$$\begin{aligned}\int_{\mathcal{F}_{k_j}\setminus\mathcal{F}_\ell^+} &\frac{\sigma}{h_{k_j}} [\![\partial_n u_{k_j}]\!] [\![\partial_n v]\!] \,\mathrm{d}s \\ &\leqslant \left(\int_{\mathcal{F}_{k_j}\setminus\mathcal{F}_\ell^+} \frac{\sigma}{h_{k_j}} [\![\partial_n u_{k_j}]\!]^2 \,\mathrm{d}s\right)^{1/2} \left(\int_{\mathcal{F}_{k_j}\setminus\mathcal{F}_\ell^+} \frac{\sigma}{h_{k_j}} [\![\partial_n v]\!]^2 \,\mathrm{d}s\right)^{1/2} \\ &\lesssim |\!|\!| u_{k_j} |\!|\!|_{k_j} \left(\int_{\mathcal{F}_{k_j}\setminus\mathcal{F}_\ell^+} \frac{\sigma}{h_{k_j}} [\![\partial_n v]\!]^2 \,\mathrm{d}s\right)^{1/2} \\ &\lesssim \|f\|_{L^2(\Omega)} \left(\int_{\mathcal{F}^+\setminus\mathcal{F}_\ell^+} \frac{\sigma}{h_+} [\![\partial_n v]\!]^2 \,\mathrm{d}s\right)^{1/2} \leqslant \epsilon\end{aligned}$$

whenever $k_j \geqslant \ell \geqslant K(\epsilon)$. As in $\boxed{1}$, we use for fixed ℓ that $[\![\partial_n u_{k_j}]\!]|_{\mathcal{F}_\ell^{1+}} \to [\![\partial_n \overline{u}_\infty]\!]|_{\mathcal{F}_\ell^{1+}}$ as $j \to \infty$ strongly in $L^2(\mathcal{F}_\ell^{1+})$ and consequently

$$\int_{\mathcal{F}_\ell^+} \frac{\sigma}{h_{k_j}} [\![\partial_n u_{k_j}]\!] [\![\partial_n v]\!] \,\mathrm{d}s \to \int_{\mathcal{F}_\ell^+} \frac{\sigma}{h_+} [\![\partial_n \overline{u}_\infty]\!] [\![\partial_n v]\!] \,\mathrm{d}s \qquad \text{as } j \to \infty.$$

The desired convergence

$$\int_{\mathcal{F}_{k_j}} \frac{\sigma}{h_{k_j}} [\![\partial_n u_{k_j}]\!] [\![\partial_n v]\!] \,\mathrm{d}s \to \int_{\mathcal{F}_+} \frac{\sigma}{h_+} [\![\partial_n \overline{u}_\infty]\!] [\![\partial_n v]\!] \,\mathrm{d}s \qquad \text{as } j \to \infty \tag{4.16}$$

follows from $\int_{\mathcal{F}_+\setminus\mathcal{F}_\ell} \frac{\sigma}{h_+} [\![\partial_n \overline{u}_\infty]\!] [\![\partial_n v]\!] \,\mathrm{d}s \to 0$ as $\ell \to \infty$.

Now combining (4.13), (4.15) and (4.16), we have proved

$$\begin{aligned}\mathfrak{B}_{k_j}[u_{k_j}, v] \to &\int_{\Omega^-} D^2\overline{u}_\infty : D^2 v \,\mathrm{d}x + \int_{\Omega^+} (D^2_{\mathtt{pw}}\overline{u}_\infty - \mathcal{L}_\infty([\![\partial_n \overline{u}_\infty]\!]) : D^2_{\mathtt{pw}} v \,\mathrm{d}x \\ &+ \int_{\Omega^+} \mathcal{L}_\infty([\![\partial_n v]\!]) : D^2_{\mathtt{pw}}\overline{u}_\infty \,\mathrm{d}x + \int_{\mathcal{F}^+} \frac{\sigma}{h_+} [\![\partial_n \overline{u}_\infty]\!] [\![\partial_n v]\!] \,\mathrm{d}s \\ &= \mathfrak{B}_\infty[\overline{u}_\infty, v] \qquad \text{as } j \to \infty.\end{aligned}$$

Hence, by (4.12) we conclude $\overline{u}_\infty = u_\infty$, thanks to $\overline{u}_\infty \in \mathbb{V}_\infty$ and the uniqueness of the generalised Galerkin solution of (3.6). ☐

We conclude the section by finally proving Theorem 12.

Proof of Theorem 12. Using the coercivity of the bilinear form, Lemmas 25 and 27, and the interpolation operator $\mathcal{I}_k u_\infty \in \mathbb{V}_k$, we observe

$$\begin{aligned}C_{coer} |\!|\!| \mathcal{I}_k u_\infty - u_k |\!|\!|_k^2 &\leqslant \mathfrak{B}_k[\mathcal{I}_k u_\infty - u_k, \mathcal{I}_k u_\infty - u_k] \\ &= \mathfrak{B}_k[\mathcal{I}_k u_\infty, \mathcal{I}_k u_\infty] - 2\mathfrak{B}_k[\mathcal{I}_k u_\infty, u_k] + \mathfrak{B}_k[u_k, u_k] \\ &= \mathfrak{B}_k[\mathcal{I}_k u_\infty, \mathcal{I}_k u_\infty] - 2\langle f, \mathcal{I}_k u_\infty\rangle_{L^2(\Omega)} + \langle f, u_k\rangle_{L^2(\Omega)} \\ &\to \mathfrak{B}_\infty[u_\infty, u_\infty] - \langle f, u_\infty\rangle_{L^2(\Omega)} = 0 \qquad \text{as } k \to \infty.\end{aligned}$$

Hence, again with Lemma 25, we conclude

$$|\!|\!| u_\infty - u_k |\!|\!|_k^2 \leqslant |\!|\!| \mathcal{I}_k u_\infty - u_\infty |\!|\!|_k^2 + |\!|\!| \mathcal{I}_k u_\infty - u_k |\!|\!|_k^2 \to 0$$

as $k \to \infty$. □

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Appendix A. Proof of Proposition 13

For $K \in \mathcal{T}_k^{1-}$, we define the element bubble function

$$b_K := \beta_K \prod_{z \in \mathcal{Z}_{\mathcal{T}_k} \cap K} \lambda_{K,z}^2,$$

where $\lambda_{K,z}$ denotes the barycentric coordinate of K with respect to the node $z \in \mathcal{Z}_{\mathcal{T}_k} \cap K$. The scaling factor $\beta_K > 0$ is such that $\max_{x \in K} b_K(x) = 1$; compare with [Ver13]. Note that $b_K \in H_0^2(K)$ and since $K \subset \Omega^-$, extending b_K by zero to Ω, we have that $b_K \in \mathbb{V}_\infty$. Setting $\phi = b_K \Pi_0 f \in H_0^2(K)$ we have

$$\|\phi\|_{L^2(K)} = \|b_K \Pi_0 f\|_{L^2(K)} \leqslant \|\Pi_0 f\|_{L^2(K)}. \tag{A.1}$$

Recalling $\Delta^2 v|_K = 0$ for $v \in \mathbb{V}(\mathcal{T})$, we conclude from (2.1) and integration by parts

$$\begin{aligned}\int_K f\phi \,\mathrm{d}x &= \mathfrak{B}_\infty[u_\infty, \phi] = \int_K D^2 u_\infty : D^2\phi \,\mathrm{d}x \\ &= \int_K D^2 u_\infty : D^2\phi - (\Delta^2 v)\phi \,\mathrm{d}x = \int_K D^2(u_\infty - v) : D^2\phi \,\mathrm{d}x.\end{aligned}$$

Consequently, thanks to equivalence of norms on finite dimensional spaces, standard inverse estimates, and (A.1), we obtain

$$\begin{aligned}\|\Pi_0 f\|^2_{L^2(K)} &\eqsim \int_K \Pi_0 f\phi \,\mathrm{d}x = \int_K (\Pi_0 f - f)\phi \,\mathrm{d}x + \int_K f\phi \,\mathrm{d}x \\ &= \int_K (\Pi_0 f - f)\phi \,\mathrm{d}x + \int_K D^2(u_\infty - v) : D^2\phi \,\mathrm{d}x \\ &\lesssim \Big(\|\Pi_0 f - f\|_{L^2(K)} + h_k^{-2}|u_\infty - v|_{H^2(K)}\Big)\|\Pi_0 f\|_{L^2(K)}.\end{aligned} \tag{A.2}$$

This proves the assertion of Proposition 13 for the element residual.

In order to bound the jump residual let $K_1, K_2 \in \mathcal{T}_k^{1-}$ with $F = K_1 \cap K_2 \in \mathcal{F}_k^{1-}$ and set $\gamma_F := [\![\partial_n^2 v]\!]\,|_F$. We define $p_F \in \mathbb{P}_1$ by

$$p|_F = 0 \quad \text{and} \quad \frac{\partial p}{\partial \boldsymbol{n}_F} = \gamma_F, \tag{A.3}$$

where $\boldsymbol{n}_F = \boldsymbol{n}_{K_1}|_F$ assuming the convention in (2.2). Since $\gamma_F \in \mathbb{R}$, we have

$$|p|_{H^1(\omega_k(F))} = |\omega_k(F)|^{1/2}|\gamma_F| \eqsim h_F|\gamma_F| = \left(\int_F h_F [\![\partial_n^2 v]\!]^2 \,\mathrm{d}s\right)^{1/2},$$

recalling that $\omega_k(F) = K_1 \cup K_2$. Moreover, we have

$$\|p\|_{L^\infty(\omega_k(F))} \eqsim h_F|\gamma_F| = \left(\int_F h_F [\![\partial_n^2 v]\!]^2 \,\mathrm{d}s\right)^{1/2}.$$

Next, we define the side bubble function

$$b_F|_{K_i} := \beta_F \prod_{z \in \mathcal{Z}_k \cap F} \lambda^2_{K_i,z}|_{K_i}, \qquad i = 1, 2.$$

Here $\beta_F > 0$ is such that $\max_{x\in F} b_F(x) = \max_{x \in K_i} b_F(x) = 1$, $i = 1, 2$. Consequently, we have $b_F \in H_0^2(\omega_k(F))$ and extending b_F to Ω by zero yields $b_F \in H_0^2(\Omega)$. Standard scaling arguments prove

$$\int_F b_F \,\mathrm{d}s \eqsim |F| = h_F \eqsim \|b_F\|_{L^2(\omega_k(F))}.$$

Therefore, combining this with (A.3), (3.6), and integration by parts, from $p|_F = 0$, that

$$\begin{aligned}\int_F [\![\partial_n^2 v]\!]^2 \,\mathrm{d}s &\lesssim \int_F [\![\partial_n^2 v]\!]^2 b_F \,\mathrm{d}s = \int_F [\![\partial_n^2 v]\!] \frac{\partial p}{\partial n_F} b_F + \frac{\partial b_F}{\partial n_F} p \,\mathrm{d}s \\ &= \int_F [\![\partial_n^2 v]\!] \frac{\partial (p b_F)}{\partial n_F} \,\mathrm{d}s = \sum_{K \in N_k(F)} \int_K D^2 v : D^2(p b_F)\,\mathrm{d}x \\ &= \int_{\omega_k(F)} D^2(v - u_\infty) : D^2(p b_F)\,\mathrm{d}x + \mathfrak{B}_\infty[u_\infty, p b_F] \\ &= \int_{\omega_k(F)} D^2(v - u_\infty) : D^2(p b_F)\,\mathrm{d}x + \int_{\omega_k(F)} f(p b_F)\,\mathrm{d}x \\ &\leqslant \Big(|v - u_\infty|_{H^2(\omega_k(F))} + \big|h_k^2 f\big|_{L^2(\omega_k(F))}\Big) h_F^{-2}\|p b_F\|_{L^2(\omega_k(F))}.\end{aligned}$$

Here, we have used equivalence of norms on finite dimensional spaces in the first as well as Cauchy-Schwartz and an inverse inequalities in the last estimate. Combining this with

$$\|pb_F\|_{L^2(\omega_k(F))} \leqslant \|p\|_{L^\infty(\omega_k(F))} |b_F|_{L^2(\omega_k(F))} \lesssim h_F^2 |\gamma_F| = h_F \left(\int_F h_F \left[\!\left[\partial_n^2 v \right]\!\right]^2 \, \mathrm{d}s \right)^{1/2}$$

proves

$$\left(\int_F h_F \left[\!\left[\partial_n^2 v \right]\!\right]^2 \, \mathrm{d}s \right)^{1/2} \lesssim |v - u_\infty|_{H^2(\omega_k(F))} + \left\| h_k^2 f \right\|_{L^2(\omega_k(F))}.$$

The assertion of Proposition 13 for the jump residual follows then from applying (A.2) for the last term on the right-hand side. □

Appendix B. Proof of Proposition 15

For $v \in \mathbb{V}_\infty$ there exists a sequence $v_k \in \mathbb{V}_k$, $k \in \mathbb{N}$, such that $\|v - v_k\|_k \to 0$ as $k \to \infty$ and $\limsup_{k\to\infty} \|v_k\|_k < \infty$. Therefore, $\{\|v\|_k\}_{k\in\mathbb{N}}$ is bounded, since $\|v\|_k \leqslant \|v - v_k\|_k + \|v_k\|_k < \infty$ uniformly in k. For $m \geqslant k$ we have, by inclusion $\bigcup_{F\in\mathcal{F}_k} F \subset \bigcup_{F\in\mathcal{F}_m} F$ and mesh-size reduction $h_k^{-1} \leqslant h_m^{-1}$, that

$$\int_{\mathcal{F}_k} h_k^{-1} \left[\!\left[\partial_n v \right]\!\right]^2 \, \mathrm{d}s \leqslant \int_{\mathcal{F}_k} h_m^{-1} \left[\!\left[\partial_n v \right]\!\right]^2 \, \mathrm{d}s \leqslant \int_{\mathcal{F}_m} h_m^{-1} \left[\!\left[\partial_n v \right]\!\right]^2 \, \mathrm{d}s.$$

Consequently, we have $\|v\|_k \leqslant \|v\|_m$ and $\{\|v\|_k\}_{k\in\mathbb{N}}$ converges. In particular, for $\epsilon > 0$, the exists $L = L(\epsilon) \in \mathbb{N}$ such that for all $k \geqslant L$ and some sufficiently large $m > k$, we have

$$\begin{aligned} \epsilon > \left| \|v\|_m^2 - \|v\|_k^2 \right| &= \sigma \int_{\mathcal{F}_m \setminus (\mathcal{F}_k \cap \mathcal{F}_m)} h_m^{-1} \left[\!\left[\partial_n v \right]\!\right]^2 \, \mathrm{d}s - \sigma \int_{\mathcal{F}_k \setminus (\mathcal{F}_k \cap \mathcal{F}_m)} h_k^{-1} \left[\!\left[\partial_n v \right]\!\right]^2 \, \mathrm{d}s \\ &\geqslant \sigma \int_{\mathcal{F}_k \setminus \mathcal{F}_k^+} h_k^{-1} \left[\!\left[\partial_n v \right]\!\right]^2 \, \mathrm{d}s. \end{aligned}$$

This follows from the fact that $h_m|_F \leqslant 2^{-1} h_k|_F$ for all $F \in \mathcal{F}_m \setminus (\mathcal{F}_k \cap \mathcal{F}_m)$, and $\mathcal{F}_k^+ = \mathcal{F}_m \cap \mathcal{F}_k$ for sufficiently large $m > k$. Therefore, $\int_{\mathcal{F}_m \setminus \mathcal{F}_m^+} h_m^{-1} \left[\!\left[\partial_n v \right]\!\right]^2 \, \mathrm{d}s \to 0$ as $m \to \infty$ and thus

$$\begin{aligned} \|v\|_k^2 &= \int_\Omega \left| D_{\mathtt{pw}}^2 v \right|^2 \, \mathrm{d}x + \sigma \int_{\mathcal{F}_k^+} h_k^{-1} \left[\!\left[\partial_n v \right]\!\right]^2 \, \mathrm{d}s + \sigma \int_{\mathcal{F}_k \setminus \mathcal{F}_k^+} h_k^{-1} \left[\!\left[\partial_n v \right]\!\right]^2 \, \mathrm{d}s \\ &\to \|v\|_\infty + 0 \qquad \text{as } k \to \infty. \end{aligned}$$

The second claim is a localised version and follows by analogous arguments. □

Appendix C. Auxiliary results

Lemma 28. *Let $\{u_k\}_{k\in\mathbb{N}}$ be the sequence of discrete solutions generated by the AC⁰IPGM and $\overline{u}_\infty \in \mathbb{V}_\infty$ as in (4.9). Then, for arbitrary fixed $\ell \in \mathbb{N}$, we have*

$$\left[D_{pw}^2 u_k - \mathcal{L}_k(\left[\!\left[\partial_n u_k \right]\!\right]) \right]|_{\Omega_\ell^{1-}} \rightharpoonup D^2 \overline{u}_\infty|_{\Omega_\ell^{1-}} \quad \textit{weakly in } L^2(\Omega_\ell^{1-})^{2\times 2} \quad \textit{as } k \to \infty.$$

Proof. For $\ell \leqslant k$ we have $\Omega_\ell^{1-} \subset \Omega_k^{1-}$ and thus $\|h_k \chi_{\Omega_\ell^{1-}}\|_{L^\infty(\Omega)} \leqslant \|h_k \chi_{\Omega_k^{1-}}\|_{L^\infty(\Omega)} \to 0$ as $k \to \infty$, thanks to Lemma 9. We proceed similar as in [BO09, Theorem 5.2].

By the stability of the lifting operator (2.7) and the definition of the energy-norm we have

$$\left\| D_{\mathtt{pw}}^2 u_k \right\|_\Omega \lesssim \|u_k\|_k \quad \text{and} \quad \|\mathcal{L}_k(\left[\!\left[\partial_n u_k \right]\!\right])\|_\Omega \lesssim \|u_k\|_k.$$

Consequently, thanks to (2.4), both terms are bounded uniformly and thus, for a not relabelled subsequence, we obtain

$$D_{\mathtt{pw}}^2 u_k \rightharpoonup \boldsymbol{T_a} \quad \text{and} \quad \mathcal{L}_k(\left[\!\left[\partial_n u_k \right]\!\right]) \rightharpoonup \boldsymbol{T_j} \tag{C.1}$$

weakly in $\in L^2(\Omega)^{2\times 2}$ as $k \to \infty$. It therefore remains to prove that

$$\boldsymbol{T_a} - \boldsymbol{T_j} = D^2\overline{u}_\infty \qquad \text{in } L^2(\Omega_\ell^{1-})^{2\times 2}.$$

Proposition 3 and (2.4) imply that $\{\nabla u_k\}_{k\in\mathbb{N}_0}$ is uniformly bounded in $BV(\Omega_\ell^{1-})^2$. Therefore, as in (4.10), we have that $\nabla u_k \rightharpoonup^* \nabla\overline{u}_\infty$ in $BV(\Omega_\ell^{1-})^2$, i.e., for $\boldsymbol{\varphi} \in C_c^1(\Omega_\ell^{1-})^{2\times 2}_{sym}$, we have

$$\int_{\Omega_\ell^{1-}} \boldsymbol{\varphi}\colon \, \mathrm{d}D(\nabla u_k) \to \int_{\Omega_\ell^{1-}} \boldsymbol{\varphi}\colon \, \mathrm{d}D(\nabla\overline{u}_\infty) = \int_{\Omega_\ell^{1-}} \boldsymbol{\varphi}\colon D^2\overline{u}_\infty \,\mathrm{d}x,$$

as $k \to \infty$. Here $\mathrm{d}D(\nabla u_k)$ is a finite Radon-measure on Ω (compare [AFP00, Chapter 3.1]) and the last identity follows since $\overline{u}_\infty|_{\Omega^-} \in H^2_{\partial\Omega\cap\partial\Omega^-}(\Omega^-)$ and $\Omega_\ell^{1-} \subset \Omega^-$. By element-wise integration by parts formula ([Com89]) we have

$$\int_\Omega \boldsymbol{\varphi}\colon \, \mathrm{d}D(\nabla u_k) = \int_{\Omega_\ell^{1-}} D^2_{\mathtt{pw}} u_k : \boldsymbol{\varphi}\,\mathrm{d}x - \sum_{\substack{F\in\mathcal{F}_k\\ F\subset\Omega_\ell^{1-}}} \int_F \boldsymbol{\varphi}\boldsymbol{n}\cdot\boldsymbol{n}\,[\![\partial_n u_k]\!]\,\mathrm{d}s,$$

for all $\boldsymbol{\varphi} \in C_c^1(\Omega_\ell^{-1})^{2\times 2}_{sym}$. The assertion thus follows if

$$\sum_{\substack{F\in\mathcal{F}_k\\ F\subset\Omega_\ell^{1-}}} \int_F \boldsymbol{\varphi}\boldsymbol{n}\cdot\boldsymbol{n}\,[\![\partial_n u_k]\!]\,\mathrm{d}s \to \int_{\Omega_\ell^{1-}} \boldsymbol{T_j} : \boldsymbol{\varphi}\,\mathrm{d}x \quad \text{as } k\to\infty. \tag{C.2}$$

In order to verify this, let $\boldsymbol{\pi}_k = \boldsymbol{\pi}_k(\boldsymbol{\varphi})$ be the $L^2(\Omega_\ell^{1-})^{d\times d}$-orthogonal projection of $\boldsymbol{\phi}$ onto $\mathbb{P}_0(\{K\in\mathcal{T}_k : K\subset\overline{\Omega_\ell^{1-}}\})^{d\times d}$. Then, we have

$$\begin{aligned}
&\sum_{\substack{F\in\mathcal{F}_k^-\\ F\subset\Omega_\ell^{1-}}} \int_F \boldsymbol{\varphi}\boldsymbol{n}\cdot\boldsymbol{n}\,[\![\partial_n u_k]\!]\,\mathrm{d}s\\
&= \sum_{\substack{F\in\mathcal{F}_k^-\\ F\subset\Omega_\ell^{1-}}} \int_F (\boldsymbol{\varphi}-\boldsymbol{\pi}_k)\boldsymbol{n}\cdot\boldsymbol{n}\,[\![\partial_n u_k]\!]\,\mathrm{d}s + \sum_{\substack{F\in\mathcal{F}_k^-\\ F\subset\Omega_\ell^{1-}}} \int_F \boldsymbol{\pi}_k\boldsymbol{n}\cdot\boldsymbol{n}\,[\![\partial_n u_k]\!]\,\mathrm{d}s\\
&= \sum_{\substack{F\in\mathcal{F}_k^-\\ F\subset\Omega_\ell^{1-}}} \int_F (\boldsymbol{\varphi}-\boldsymbol{\pi}_k)\boldsymbol{n}\cdot\boldsymbol{n}\,[\![\partial_n u_k]\!]\,\mathrm{d}s + \int_{\Omega_\ell^{1-}} \mathcal{L}_k([\![\partial_n u_k]\!])\colon \boldsymbol{\pi}_k\,\mathrm{d}x\\
&= \sum_{\substack{F\in\mathcal{F}_k^-\\ F\subset\Omega_\ell^{1-}}} \int_F (\boldsymbol{\varphi}-\boldsymbol{\pi}_k)\boldsymbol{n}\cdot\boldsymbol{n}\,[\![\partial_n u_k]\!]\,\mathrm{d}s + \int_{\Omega_\ell^{1-}} \mathcal{L}_k([\![\partial_n u_k]\!])\colon (\boldsymbol{\pi}_k-\boldsymbol{\varphi})\,\mathrm{d}x\\
&\quad \int_{\Omega_\ell^{1-}} \mathcal{L}_k([\![\partial_n u_k]\!])\colon \boldsymbol{\varphi}\,\mathrm{d}x.
\end{aligned}$$

Thanks to $\|h_k\chi_{\Omega_\ell^{1-}}\|_{L^\infty(\Omega)} \to 0$ as $k\to\infty$, we have that the first two terms on the right hand side vanish as $k\to\infty$ since $\boldsymbol{\varphi} \in C_c^1(\Omega_\ell^{-1})^{2\times 2}_{sym}$. This concludes the proof since (C.2) follows then from (C.1). □

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