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Herbert Dawid   Tim Hellmann

Bielefeld University
P.O. Box 10 01 31
33501 Bielefeld – Germany

www.wiwi.uni–bielefeld.de

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Herbert Dawid* Tim Hellmann †

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Abstract

We study the evolution of R&D networks in a Cournot model where firms may lower marginal costs due to bilateral R&D collaborations. Stochastically stable R&D networks exhibit the dominant group architecture, and, contrary to the existing literature, generically unique predictions about the size of the dominant group can be obtained. This size decreases monotonically with respect to the cost of link formation and there exists a lower bound on the size of the dominant group for non-empty networks. Stochastically stable networks are always inefficient and an increase in linking costs has a non-monotone effect on average industry profits.

JEL Classifications: C72, C73, L13, O30
Keywords: R&D Networks, Oligopoly, Stochastic Stability

1 Introduction

The formation of R&D networks, where firms cooperate with respect to their innovative activities, is an important feature of many industries (see e.g. Hagedoorn [12], Powell et al. [18], Roijakkers and Hagedoorn [19]). In many cases the firms cooperating on the R&D level are competitors in the market, which gives rise to intricate strategic considerations when selecting R&D cooperation partners. Given the empirical evidence of R&D collaborations it is important to gain a sound understanding of the factors determining the structure of R&D networks. From a theoretical perspective Goyal and Joshi [9] have studied the structure of pairwise Nash stable (PNS) R&D networks in a seminal contribution. They consider a setting where links reduce marginal production costs of firms, which compete a la Cournot. They

*Corresponding Author. Department of Business Administration and Economics and Institute of Mathematical Economics, Bielefeld University, P.O. Box 100131, 33501 Bielefeld, Germany. Phone: +49-521-1064843, Fax: +49-521-10689005, Email: hdawid@wiwi.uni-bielefeld.de.

†Institute of Mathematical Economics, Bielefeld University, Germany. Email: tim.hellmann@uni–bielefeld.de.

show that the pairwise Nash stable (PNS) networks exhibit the dominant group architecture (with one completely connected group and all other firms isolated). However, a wide range of these types of networks (with respect to the size of the dominant group) may be PNS. And although the sizes of the dominant group are sensitive to the cost of link formation, there is no unique prediction with respect to the networks which will be observed. Moreover surprisingly, the minimal size of the component in a non-empty network is increasing in the cost of link formation for a certain cost range. In a related setting of directed R&D networks and Cournot competition Billand and Bravard [2] obtain stable networks with a similar structure in a sense that a subset of nodes is heavily connected, whereas the other nodes do not form own links.

In a recent contribution Marinucci and Vergote [17] study the strategic formation of R&D networks in a patent-race setting where the number of links of a firm positively affects its expected valuation of the patent. The patent is awarded using an all-pay auction based on the R&D effort invested by all competitors. The authors study the stable networks in a two-stage setting, where links are chosen first and effort determined in the second stage. They show that similar to the findings of Goyal and Joshi [9] only networks with dominant group structure can be stable.

The models of R&D network formation in the papers discussed above are static[1]. Dynamic models of R&D network formation have recently been provided in different economic frameworks (see e.g. Baum et al. [1],König et al. [16]). The economic environment in these contributions differs substantially from the models of Goyal and Joshi [9], Billand and Bravard [2], and Marinucci and Vergote [17], and therefore the stable networks do not exhibit the dominant group structure. Hence, these dynamic studies do not provide an indication of how to select among the (typically numerous) stable networks with dominant group structure. In particular, a dynamic analysis of the standard Cournot setting considered in Goyal and Joshi [9] is so far missing.

In this paper, we fill this gap and focus on the dynamics of R&D networks in a Cournot oligopoly. We assume in this two stage game, where decisions about links are made in the first stage and quantities are chosen in the second, that interaction on the collaboration network is faster than the dynamics of the networks. This implies in our model that the unique Nash equilibrium of the second stage, i.e. the equilibrium choice of quantities, is immediately established.[2] For the evolution of collaboration links, we employ the dynamic model of network formation by Jackson and Watts

[1]It should be noted that the study of R&D network formation is closely related to the literature on R&D coalition formation (e.g. Bloch [3], Yi [22]). For a contribution in a setting related to our model see Greenlee [11].

[2]In a model with multiple second stage equilibria a slow-fast dynamic needs to be modeled explicitly as e.g. in Dawid and MacLeod [7].

[14]. In this framework, each link is considered one by one and the decision makers play a myopic best-reply to the current state with high probability and make mistakes with low probability. The resulting stochastically stable networks select among the pairwise stable networks, and are those which are observed most of the time in this dynamic model when the probability of mistakes vanishes.

In the main result of our paper, we characterize the set of stochastically stable networks. Trivially, they also exhibit the dominant group architecture. We find that the stochastically stable networks are typically unique (with respect to the size of the dominant group) and the size of the dominant group is monotonically decreasing in the cost of link formation, solving the puzzle of non-monotonicity in Goyal and Joshi [9]. Further, we show that there exists a threshold of the dominant group size, below which only the empty network can be stochastically stable. This result has interesting connections to analytical findings on efficient networks; e.g. in a similar two stage game, Westbrock [21] studies the efficient networks and also concludes that either the empty network is efficient or there exists a lower threshold on the size of the dominant group for efficient networks to have the dominant group structure. Our findings imply that for relatively large linking costs the structure of the stochastically stable networks differ from that of the efficient ones. For relatively small linking costs both the stochastically stable and the efficient networks have dominant group structure, where however numerical analysis suggests that both generically differ with respect to size such that stochastically stable networks are under-connected.[3] Since the concept of stochastic stability allows us to select generically unique R&D networks for all values of the linking costs, we are in a position to study the effects of changes in the linking costs on consumer surplus and industry profits under consideration of the resulting changes in structure of the R&D networks. It turns out that whereas consumer surplus moves in the intuitively anticipated direction, i.e. increasing linking costs imply decreasing consumer surplus, a non-monotone U-shaped relationship between linking costs and average industry profits emerges. In particular, for relatively high linking costs associated with small dominant group sizes, an increase in these costs induces an increase in the firms' profits.

Relative to the existing literature, one new contribution of this paper is thus to establish a monotone and explicit relationship between key parameters, like the costs of link formation or market size, and the size of the dominant group. Such a characterization is important since it allows to derive conclusions concerning the change in the shape of emerging networks as key parameters vary. Furthermore, our selection result allows to compare efficient and emergent networks within the set of dominant group networks.

[3]Under-connected networks are contained in welfare better networks, e.g. efficient networks, see Buechel and Hellmann [5].

This is different from the existing literature where statements about inefficiency of stable networks are based on *structural* differences between these networks and the efficient one (see Westbrock [21]). On the contrary, our results enable evaluation of the efficiency of an emergent network based on the *size* of the dominant group. Finally, we derive results concerning the effects of changes in the linking costs on key market indicators like consumer surplus and average industry profit. In the absence of a (generically) unique prediction about the shape of the R&D network for a given parameter setting, the existing literature did not provide any results in this respect.

The paper is organized as follows. In Section 2 we present the model and Section 3 is devoted to the characterization of the stochastically stable R&D networks for different levels of linking costs. In Section 4 we study the relationship between stochastically stable and efficient networks and explore the effect of changes in linking costs on consumer surplus and average industry profit. The paper ends with some conclusions in Section 5. All proofs are given in the Appendix.

2 The Model

A set of $N = \{1, ..., n\}$ ex ante identical firms participates in a two stage game. To exclude uninteresting cases we assume $n \geq 3$. Firms first form bilateral agreements of collaboration. We denote by $g^n := \{\{i, j\} | i, j \in N, i \neq j\}$ the set of all possible collaboration agreements, which we call the complete network. The set of all undirected networks is given by $G = \{g : g \subseteq g^n\}$. For notational convenience we denote by $ij = ji := \{i, j\} \in g$ a collaboration link between firm i and firm j in network g. Given a network $g \in G$, the neighbors of player i are represented by the set $N_i(g) := \{j \in N \mid ij \in g\}$. We denote by $\eta_i(g) := |N_i(g)|$ the degree of firm i and by $\eta_{-i} := \sum_{j \neq i} \eta_j$ the sum of all other firms' degree. For a network $g \in G$ and a set of links $l \subseteq g^n \setminus g$ (which is also a network) let $g + l := g \cup l$ be the network obtained by adding the links l to network g. Similarly, let $g - l := g \setminus l$ denote the network obtained by deleting the set of links $l \subseteq g$ from network $g \in G$. Collaboration links can be interpreted as R&D agreements lowering marginal costs of producing the homogeneous good. However, maintenance of links is costly, with constant cost f per formed link.

In the second stage, firms compete in the market by choosing quantities.[4] We assume that marginal cost of producing the homogeneous good is constant for each firm and for $i \in N$ given by $c_i(g) = \gamma_0 - \gamma\eta_i(g)$ with $\gamma < \frac{\gamma_0}{n-1}$. Let $q_i \in \mathbb{R}_+$ be the quantity chosen by firm i and let $q = (q_1, .., q_n) \in \mathbb{R}^n_+$ be the profile of quantities chosen. We assume that market demand is linear and given by $P(q) = \max[0, \alpha - \sum_{j \in N} q_j]$. As-

[4] A more detailed derivation of the second stage equilibria can be found in Goyal and Joshi [9].

suming positive prices, the profit of firm $i \in N$ in the second stage can be derived to be, $\tilde{\pi}_i(q,g) = (\alpha - \sum_{j \in N} q_j)q_i - q_i c_i(g)$. Taking the network g as given, firms try to maximize profits. The interior Cournot equilibrium can be calculated to be,

$$q_i^*(g) := \frac{(\alpha - \gamma_0) + n\gamma\eta_i(g) - \gamma \sum_{j \neq i} \eta_j(g)}{n+1},$$

which is strictly positive assuming $\alpha - \gamma_0 - \gamma(n-1)(n-2) > 0$. Thus, in equilibrium of the second stage, profits are $\tilde{\pi}_i(g) = (q_i^*(q))^2$. Adjusting for the cost of link formation and noting that the resulting payoff in the first stage only depends on the degree distribution as the only network statistic, we write, abusing notation:

$$\pi_i(\eta_i, \eta_{-i}) := \pi_i(g) := \frac{((\alpha - \gamma_0) + n\gamma\eta_i(g) - \gamma\eta_{-i}(g))^2}{(n+1)^2} - \eta_i(g)f. \qquad (1)$$

So far the static model is in line with Goyal and Joshi [9]. We now present a dynamic model of network formation. We assume that adjustment in the quantity choice stage is fast compared to the rate by which changes in the network occur. An interpretation of this is that adjustments in competitive decisions (in this case: quantities) happens on an everyday basis, while strategic choices on R&D partnerships are long-term decisions. Thus, we consider a dynamic model of network formation such that the (unique) equilibrium in the second stage is immediately adapted for each change in the network.[5] To model the network dynamics we employ the stochastic process introduced by Jackson and Watts [14]: time is discrete $t = 0, 1, \ldots$ and at $t = 0$ an arbitrary network is given (e.g. the empty network). We denote the network at time $t \in \mathbb{N}$ by g_t. At each point in time t, one link is selected by some probability distribution which is identical and independent over time with full support, i.e. $p(ij) > 0$ for all $ij \in g^n$. If the selected link is already contained in g_t, then both firms decide to keep or delete the link, and, if not, both firms decide whether to add or not to add the link. These decisions are myopic and based on marginal payoffs from the given link, $\Delta_i^+(\eta_i, \eta_{-i}) := \pi_i(\eta_i + 1, \eta_{-i} + 1) - \pi_i(\eta_i, \eta_{-i})$ and

[5] Although it is well known that the equilibrium in multi-firm Cournot oligopolies is unstable under a standard best response dynamics due to overshooting (see Theocharis [20]), assuming a certain degree of inertia in the dynamics makes the equilibrium stable (see Dawid [6]) and in our analysis it is implicitly assumed that the inertia in quantity adjustment is sufficiently large such that the unique Cournot equilibrium is reached for any given R&D network.

$\Delta_i^-(\eta_i, \eta_{-i}) := \pi_i(\eta_i, \eta_{-i}) - \pi_i(\eta_i - 1, \eta_{-i} - 1)$ which can be calculated to be:[6]

$$\begin{aligned}
\Delta_i^+(\eta_i, \eta_{-i}) &= \frac{\gamma(n-1)}{(n+1)^2}\Big[2(\alpha - \gamma_0) + \gamma(n-1) + 2\gamma n \eta_i - 2\gamma\eta_{-i}\Big] - f \\
\Delta_i^-(\eta_i, \eta_{-i}) &= \frac{\gamma(n-1)}{(n+1)^2}\Big[2(\alpha - \gamma_0) - \gamma(n-1) + 2\gamma n \eta_i - 2\gamma\eta_{-i}\Big] - f
\end{aligned}$$

The link $ij \notin g_t$ is then added if $\Delta_i^+(\eta_i, \eta_{-i}) > 0$ and $\Delta_j^+(\eta_j, \eta_{-j}) \geq 0$, i.e. if that link is beneficial for both involved firms (with one strict inequality) while it is not added else. Similarly a link $ij \in g_t$ is kept if $\Delta_k^-(\eta_k, \eta_{-k}) \geq 0$ for both $k \in \{i, j\}$, while it is deleted else. With high probability $1 - \epsilon$ the decision of the players is implemented while with low probability ϵ the decision is reversed (i.e. a mutation), which can be interpreted as firms making a mistake or a experimentation. The such defined stochastic process is an ergodic Markov process on the state space of G with unique limit distribution μ^ϵ depending on the probability of mistakes. The networks $g \in G$ such that $\lim_{\epsilon \to 0} \mu^\epsilon(g) > 0$ are called *stochastically stable* (see e.g. Young [23]). By construction, the absorbing states of the unperturbed process (for $\epsilon = 0$) are the pairwise stable networks (PS), i.e. the networks $g \in G$ such that for all $i \in N : \Delta_i^-(\eta_i(g), \eta_{-i}(g)) \geq 0$ and for all pairs $i, j \in N$ with $ij \notin g$: $\Delta_i^+(\eta_i(g), \eta_{-i}(g)) > 0 \Rightarrow \Delta_j^+(\eta_j(g), \eta_{-j}(g)) < 0$.[7]

The condition for pairwise stability is weaker than that for pairwise Nash stability, used in Goyal and Joshi [9], since PNS requires that in addition to the PS conditions that $\pi_i(\eta_i, \eta_{-i}) - \pi_i(0, \eta_{-i} - \eta_i) \geq \eta_i f$ holds for all $i \in N$. PNS also captures the opportunity to delete multiple links at a time. Thus, the dynamics of network formation introduced by Jackson and Watts [14] may converge to networks which are not pairwise Nash stable, i.e. where firms would be better off deleting all their links. The reason for this is that multiple link decisions are not considered in the dynamic model by Jackson and Watts [14]. A motivation for such a dynamics may be in our context that link revision opportunities only arrive at certain times due to long lasting contracts (for existing links) or occasional meetings between firms (to create new links). Therefore, a model where each link is considered one by one and firms behave myopically is reasonable.[8] Moreover, since the economic environment is rather complex, it is possible that firms

[6] For notational convenience we will drop the dependence of $\eta_i(g_t)$ on g_t whenever the reference is clear.

[7] The definition of Jackson and Wolinsky [15] is adapted here to our framework. The conditions simply mean that no firm wants to delete a single R&D collaboration and there do not exist two firms which both benefit (at least one strictly) from a mutual partnership.

[8] A dynamic process, where absorbing states are only the pairwise Nash stable networks, necessarily needs to include another decision stage where players (not links) are selected to revise (multiple) links. In the framework of directed network formation with best response this has been examined in Feri [8]. It is not straightforward to set up a similar process in the context of undirected networks considered here and the complexity of the dynamics would make the stochastic stability analysis infeasible.

make mistakes or experiment by not forming a myopically reasonable R&D partnership (with probability ϵ). However, as we consider the limit $\epsilon \to 0$, firms learn over time and decide myopically optimal. The networks such that $\lim_{\epsilon \to 0} \mu^{\epsilon}(g) > 0$, i.e. the stochastically stable networks, are those which for small ϵ are observed most of the time (as $t \to \infty$) in such a process.

3 The Evolution of Collaboration Networks

In order to study the set of stochastically stable networks, we first characterize the set of pairwise stable networks. Note that

$$\Delta_i^+(\eta_i + k, \eta_{-i} + k) - \Delta_i^+(\eta_i, \eta_{-i}) = \frac{2k\gamma^2(n-1)^2}{(n+1)^2} > 0 \tag{2}$$

which implies that $\pi_i(g)$ is convex in own links, i.e. externalities of additional own links on marginal payoff from a given link are positive. Moreover,

$$\Delta_i^+(\eta_i, \eta_{-i} + 2k) - \Delta_i^+(\eta_i, \eta_{-i}) = -\frac{4k\gamma^2(n-1)}{(n+1)^2} < 0 \tag{3}$$

which implies that $\pi_i(g)$ satisfies the strategic substitutes property, i.e. externalities of additional links of other firms on marginal payoff from a given link are negative.[9] The reason for the strategic substitutes property and convexity in own links is that additional R&D partnerships of other firms lower their marginal costs of production and hence decreases own output quantity and own benefit of a partnership, while additional own collaboration links lower own marginal costs and thus increases own output and the benefit of a given partnership.

From the convexity property (and ex ante identical firms), it follows directly that only networks with dominant group architecture such that there exists one completely connected group of firms of size k and all other firms isolated, denoted by g^k, can be pairwise stable.[10] In the following the pairwise stable networks g^k are characterized in terms of the size of the dominant group.

Proposition 1. *There exist numbers* $(0 <)F_0 < F_1 < F_2$ *with the following properties:*

1. *for* $f < F_0$ *the complete network* g^n *is the unique PS network,*
2. *for* $F_0 \leq f < F_1$*, there exists* $\underline{k}(f) \in \mathbb{N}$*,* $1 < \underline{k}(f) < n$ *such that* $PS = \{g^{\underline{k}(f)}, ..., g^n\}$*,*

[9] For formal definitions of the properties convexity and strategic substitutes see, among others, Goyal and Joshi [10] and Hellmann [13].

[10] See Goyal and Joshi [10], Lemma 4.1 for an analogous statement for pairwise Nash stable (PNS) networks. The proof trivially also holds for pairwise stable networks.

3. *for* $f = F_1$ *we have* $PS = \{g^1, ..., g^n\}$,
4. *for* $F_1 < f \leq F_2$, *there exists* $\underline{k}(f), \bar{k}(f) \in \mathbb{N} : 1 < \underline{k}(f) \leq \frac{n+2}{2} \leq \bar{k}(f) < n$ *and* $\underline{k}(f) + \bar{k}(f) = (n+2)$ *such that* $PS = \left\{g^1, g^{\underline{k}(f)}, ..., g^{\bar{k}(f)}\right\}$,
5. *for* $f > F_2$ *the empty network* g^1 *is the unique PS network.*

The pattern of pairwise stable networks exhibits similar structure as the pattern of pairwise Nash stable networks in Goyal and Joshi [9]. In fact the set of PS networks contains the set of PNS networks.[11] Two properties are notable when comparing the two sets. First, the cost threshold such that the complete network stops being PS coincides with the threshold such that the empty network starts becoming PS. Second, for the non-monotonicity part of $k(f)$, i.e. $F_1 < f < F_2$, the minimal k and the maximal k such that g^k, $k \neq 1$ is PS are symmetric around $\frac{n+2}{2}$. These two observations do not hold for the PNS networks in Goyal and Joshi [10]. For an illustration of the PS and PNS networks, see also Figure 1. In particular, it is worth noting that many networks can be PS, resp. PNS, and thus these static stability concepts do not provide precise predictions of which networks will emerge.

Proposition 1 completely characterizes the set of pairwise stable networks. With respect to the dynamics introduced above, the only other possible recurrent classes of the unperturbed process (s.t. $\epsilon = 0$) are closed cycles.[12] The following Lemma shows, that there do not exist closed cycles in our model.

Lemma 1. *In the model of collaboration networks where payoff satisfies* (1), *there does not exist a closed cycle.*

Thus the only recurrent classes are the singleton states of pairwise stable networks. We now employ the techniques by Jackson and Watts [14] to find the stochastically stable networks. Since the set of stochastically stable networks is the set of networks with minimal stochastic potential, this requires the computation of the stochastic potential of a network which is defined as the sum of all transition costs of the minimal cost (directed) tree connecting all networks, where the transition cost between two networks is given by the minimal number of mutations (i.e. mistakes) to move from one network to another. Since all other states are transient, we may restrict the construction of the minimal cost tree to the set of pairwise stable networks, i.e. we construct minimal resistance trees for each g^k. To denote the transition costs for $k \geq 2$, let $c^+(k)$ denote the minimal number of mutations necessary to move from g^k to g^{k+1} and let $c^-(k)$ denote the minimal number of mutations necessary to move from g^k to g^{k-1}. Moreover, denoting

$$\kappa(k) := \underset{\tilde{k} \in \{0...k\}}{\arg\min} \left(\Delta_i^+(\tilde{k}, k(k-1) + \tilde{k}) \geq 0\right), \tag{4}$$

[11] This holds trivially due to the definition of PS and PNS, see Bloch and Jackson [4].

[12] For a definition of improving paths and closed cycles, see Jackson and Watts [14].

we get $c^+(k) = \kappa(k)$ and $c^-(k+1) = k - \kappa(k)$, which is proved in Lemma 2.

Lemma 2. *Let $k \geq 2$ and let g^k and g^{k+1} be pairwise stable. Then the minimal number of mistakes to move from g^k to g^{k+1} is given by $c^+(k) = \kappa(k)$ and minimal number of mistakes to move from g^{k+1} to g^k is given by $c^-(k+1) = k - \kappa(k)$.*

Lemma 2 shows that the number of mistakes necessary to move between two dominant group networks, g^k, g^{k+1} is determined by $\kappa(k)$. For a PS network g^k, the number $\kappa(k)$ is the minimal number of links an isolated firm needs to be given in order to have an incentive to form a link, i.e. these are the number of myopically non-optimal links formed by an isolated firm in order to be willing to form links on their own. Note that a firm in the dominant group always has an incentive to form a link and thus will not decline a link.

From Proposition 1 we have for $f < F_1$ that if g^k and $g^{\tilde{k}}$ are PS for $k < \tilde{k}$ then also $g^{k'}$ is PS for all $k' \in \mathbb{N}$ such that $k < k' < \tilde{k}$. The only case of there being a gap (in terms of the size k) between two pairwise stable networks g^k is for g^1 and $g^{\underline{k}(f)}$ if $F_1 < f < F_2$. Thus, we get that the stochastic potential of a network g^k with $k \geq 2$ is given by

$$r(g^k) = c(g^1, g^{\underline{k}(f)}) + \sum_{l=\underline{k}(f)}^{k-1} c^+(l) + \sum_{l=k+1}^{\bar{k}(f)} c^-(l),$$

where $\underline{k}(f)$ and $\bar{k}(f)$ is the minimal respectively maximal number $k \in \{2, ..., n\}$ such that g^k is pairwise stable and $c(g^1, g^{\underline{k}(f)})$ is the minimal number of mistakes to move from the empty network to $g^{\underline{k}(f)}$, which is set to 0 if the empty network is not pairwise stable. Denoting by $\Delta^r(k)$ the difference in stochastic potentials between two networks, g^k and g^{k+1}, $k \geq 2$, we get:

$$\Delta^r(k) := r(g^{k+1}) - r(g^k) = 2\kappa(k) - k.$$

To characterize stochastically stable networks in Proposition 2, we show first that $\Delta^r(k)$ is weakly decreasing in $k \in \mathbb{N}$ up to $k = \frac{n-1}{4}$ and then weakly increasing. Thus, the network(s) g^k which satisfy the necessary condition, $\Delta^r(k) \geq 0$ and $\Delta^r(k-1) \leq 0$,[13] are the only candidates for stochastic stability besides the empty and complete network. In the following we characterize the stochastically stable networks.

Proposition 2. *There exist numbers $F_0^*, F_1^* \in \mathbb{R}$ such that $F_0 < F_0^* < F_1 < F_1^* < F_2^* < F_2$ such that:*

1. *for $f < F_0^*$ the complete network g^n is uniquely stochastically stable.*

[13] $\Delta^r(k) \geq 0$ and $\Delta^r(k-1) \leq 0$ are necessary conditions for a network g to be pairwise stable since $\Delta^r(k) \geq 0$ implies $r(g^{k+1}) \geq r(g^k)$ and $\Delta^r(k-1) \leq 0$ implies $r(g^{k-1}) \geq r(g^k)$, and the stochastically stable networks are those which minimize stochastic potential.

2. *for* $F_0^* < f < F_1^*$ *there exists a function* $k^*(f) : [F_0^*, F_1^*) \mapsto \{\frac{n-1}{4}, n-1\}$ *such that either the network* g^{k^*} *is uniquely stochastically stable or* g^{k^*} *and* g^{k^*+1} *are the only stochastically stable networks. Moreover,* $k^*(f)$ *is weakly decreasing in* f*.*

3. *for* $F_1^* \leq f \leq F_2^*$ *the empty network and the network* g^{k^*} *(respectively the networks* g^{k^*} *and* g^{k^*+1}*) are stochastically stable.*

4. *For* $f > F_2^*$ *the empty network* g^1 *is uniquely stochastically stable.*

The proof is presented in the appendix. It may be helpful to illustrate the result of Proposition 2 by Figure 1.

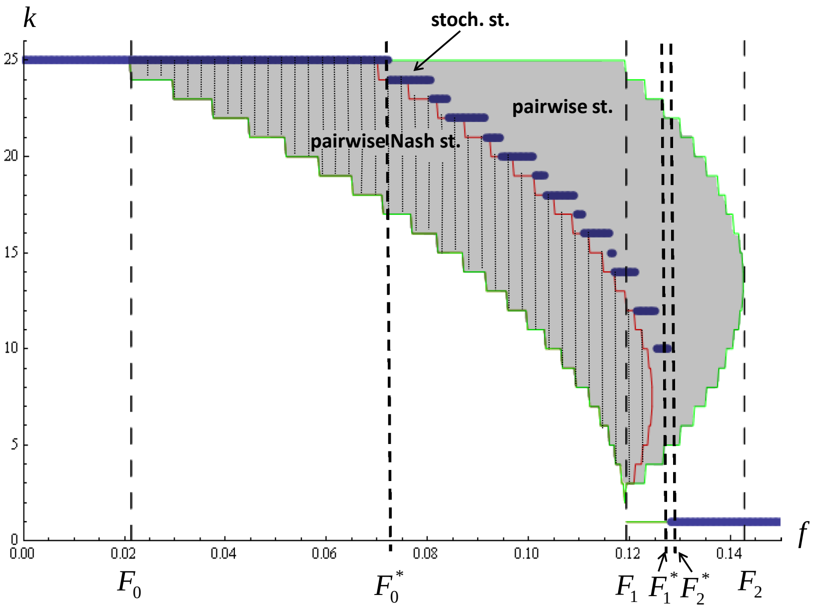

Figure 1: The set of pairwise stable (gray area), pairwise Nash stable (ruled area) and stochastically stable networks (blue).

The stochastically stable networks follow a clear pattern. First, the size of the connected component in stochastically stable networks is (weakly) decreasing with cost of link formation, although the sizes of PS and PNS networks exhibit a non-monotonicity property for a certain cost range. Second, there exists a lower bound of the component size of non-empty stochastically stable networks. Third, as Figure 1 indicates, the stochastically stable networks may lie outside the set of PNS networks characterized by Goyal and Joshi [9].[14] The observation that stochastically stable networks might not be pairwise Nash stable, shows that this concept can in general not be supported by a dynamic foundation, which has the usual properties of evolutionary dynamics, that changes in the state from one period to the next are local.

It should be noted that, although our discussion concentrates on the effects of changes of link formation costs f, it is straight forward to see that a qualitatively identical picture emerges if the market size parameter α is varied. In particular, we obtain that for a given level of link formation costs, a decrease of the market size might lead to the abrupt disappearance of an R&D network of strictly positive size.

4 Efficiency, Consumer- and Producer-Surplus

Westbrock [21] shows that efficient networks exhibit quite a similar structure to that found in the previous section for stochastically stable networks:[15] for large n there exists a cost threshold such that above that threshold no dominant group network other than the empty network can be efficient (Proposition 4 in Westbrock [21]). A natural question is then whether it is possible to compare stochastically stable networks with efficient ones. First, trivially there exists a cost threshold such that above that threshold, the empty network is both stochastically stable and efficient. The same is true for the complete network, if linking costs f are very low.

However, for intermediate cost levels, it follows straightforwardly from the proof of Proposition 2 and the findings in Westbrock [21] that for a certain range of linking costs stochastically stable networks are always inefficient. According to Westbrock [21] the network density, defined as $D(g) = \frac{\sum_{i=1}^{n} \eta_i}{n(n-1)}$ is an important factor in determining the efficiency of a network. In particular, no dominant group network g with density $0 < D(g) < 1/2$ can be efficient. Concerning the density of stochastically stable networks we obtain the following.

Corollary 1. *There exists* $\underline{F} \in [F_0^*, F_1]$ *such that for all* $f \in [\underline{F}, F_1^*]$ *all stochastically stable networks have density* $0 < D(g) < 1/2$.

We can thus immediately conclude that the stochastically stable network(s) for costs $f \in [\underline{F}, F_1^*]$ are inefficient and have the wrong structure. For $f < \underline{F}$ the structure of the stochastically stable networks appears to match that of the efficient networks, in the sense that both are dominant group networks and the size of the dominant group is decreasing in f. However, the sizes of the dominant groups do not necessarily coincide. In Figure

[14] The parameter constellation underlying this figure is $n = 25, \gamma_0 = 2, \gamma = 0.05, \alpha = 35$.

[15] A network is defined as efficient if it maximizes the sum of industry profits and consumer surplus among all networks.

2 we compare the size of the dominant group in the stochastically stable networks with that of the welfare maximizing dominant group network.[16] It can be clearly seen that the welfare maximizing network always has a larger dominant group than the stochastically stable one and this observation appears to be very robust with respect to parameter changes. In particular this implies that for values of f where the efficient network has dominant group structure the stochastically stable networks are under-connected[17].

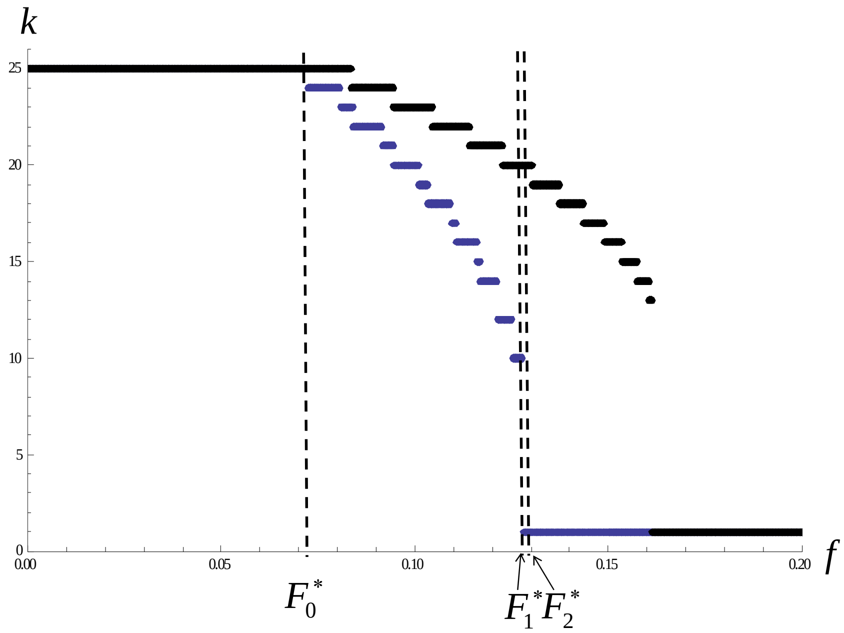

Figure 2: The stochastically stable networks (blue) and the networks maximizing welfare among all dominant group networks (black).

The observation that stochastically stable networks are under-connected relative to the efficient ones is very intuitive. An increase in the size of the dominant group induces a decrease in the market price, which has positive implications for consumer surplus. This positive welfare effect of link formation is not taken into account by firms when they decide whether to build respectively to delete a link. Based on this difference between social and

[16] Note that these welfare maximizing dominant networks are efficient for low linking costs. For high cost levels networks with a different structure are efficient, see Westbrock [21]. However, since stochastically stable networks always exhibit the dominant group structure, we compare these to those dominant group networks which are welfare maximizing within this class of networks.

[17] For a definition of over-connected or under-connected networks, see Buechel and Hellmann [5].

private returns of link formation it should be expected that the size of the dominant groups in stochastically stable networks tend to be smaller than those in efficient networks.

Having characterized the relationship between stochastically stable and efficient networks we will now evaluate how changes in linking costs affect consumer surplus and average industry profits. First intuition suggest that an increase in costs should decrease the surplus on both sides of the market, but we will demonstrate in this section that this intuition is not necessarily correct.

In what follows we assume that for all values of f a stochastically stable R&D network emerges and in case two stochastically stable networks co-exist the network with dominant group size $k^*(f)$ is selected. Consumer surplus and average industry profits can then be defined as functions of linking costs in the standard way. For the consumer surplus we have

$$CS^*(f) = \frac{(\alpha - p^*(f))^2}{2},$$

where $p^*(f) = P\big(Q^*(f)\big)$ denotes the equilibrium price and

$$Q^*(f) = \sum_{i=1}^{n} q_i^*(g^*) = \frac{n(\alpha - \gamma_0) + \gamma(k^*(f)^2 - k^*(f))}{n+1}$$

denotes the total equilibrium output under the stochastically stable R&D network g^* (of dominant group size $k^*(f)$). The average industry profit reads

$$\begin{aligned}\Pi^*(f) &= \frac{1}{n}\Big[k^*(f)\pi_i\big(k^*(f)-1, (k^*(f)-1)^2\big) \\ &\qquad + \big(n - k^*(f)\big)\pi_i\big(0, k^*(f)(k^*(f)-1)\big)\Big].\end{aligned}$$

Given these definitions it is a direct Corollary of Proposition 2 that consumer surplus goes down if linking costs increase.

Corollary 2. *The consumer surplus function $CS^*(f)$ is constant for $f < F_0^*$ and $f > F_2^*$, but weakly decreasing on the interval $[F_0^*, F_2^*]$.*

The intuition for this result is straightforward. The size of linking costs affects the market price only indirectly, because it determines the shape of the R&D network and thereby the size of the marginal production costs of the competitors. An increase of the linking costs induces a reduction in the number of links in the stochastically stable network. This results in an increase of the production costs of (some) producers and hence to an increase of the market price and a decrease in consumer surplus. The reduction of total linking costs that go along with a shrinking size of the dominant group does not influence the price and are therefore irrelevant for the size of the consumer surplus.

Considering average industry profits the implications of a change of linking costs are however much less obvious. Several countervailing effects arise. For a given R&D network the direct effect of an increase of f is clearly negative. However an increase in f might lead to a reduction of the size of the dominant group, which leads, on the one hand, to an increase of marginal costs of some producers, but, on the other hand, reduces the total linking costs in the market. The next Proposition shows that these countervailing effects indeed imply that the relationship between linking costs and industry profits is similar to a U-shape.

Proposition 3. *Assume that n is sufficiently large. For $f < F_0^*$ the average industry profit $\Pi^*(f)$ strictly decreases with respect to f. For $f > F_0^*$ the average industry profit exhibits an upward jump for all values of f where $k^*(f)$ is not continuous. In particular, $\Pi^*(f)$ exhibits an upward jump for $f = F_2^*$ and is constant for all $f > F_2^*$.*

Corollary 2 and Proposition 3 are illustrated in Figure 3, where consumer surplus and average industry profits in the Cournot equilibrium are shown under the stochastically stable R&D network[18].

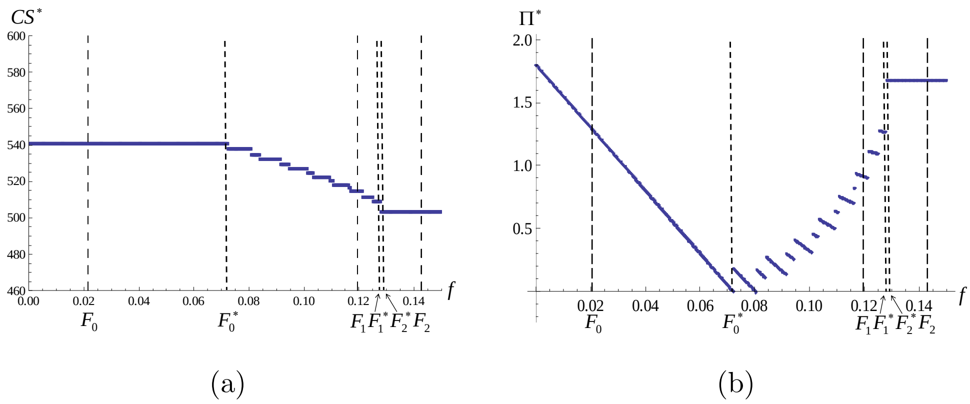

Figure 3: Consumer surplus (a) and average industry profit (b) in the Cournot equilibrium under the stochastically stable R&D networks.

Consumer surplus is constant with respect to linking costs on all intervals of f where the shape of the stochastically stable network does not change. Whenever the size of the dominant group in the stochastically stable network decreases the consumer surplus goes down. Average industry

[18]The parameter constellation is the same as that for Figure 1

profits decrease in response to increasing linking costs on all intervals where the R&D networks does not change. However, it can be clearly seen that average industry profit exhibits an upward jump at all values of f where the structure of the stochastically stable network changes and these jumps overcompensate the negative impact of an increase of f on the intervals where $\Pi^*(f)$ is continuous. Overall, this generates a U-shaped relationship where average industry profits are highest for very small and large values of the linking costs and lowest if f is in an intermediate range. The observation that average profits increase in response to an increase in f (which triggers a decrease in the size of the dominant group) is due to the fact that the deletion of links of one firm implies a positive externality on all other firms in addition to the two direct effects inducing the firms to delete their links. The two direct effects (one negative, one positive) of link deletion by one firm on average profits are that marginal costs (of the formerly connected) firms increase and linking costs decrease. The positive externality on the other firms results from the price increase triggered by the changes in marginal costs. As shown in Proposition 3 and illustrated in Figure 3 the sum of the positive externality and the positive implications for total linking costs that result from the shrinking dominant group size induced by increasing linking costs dominates the negative effect.

An implication of this insight is that for R&D networks, where the number of links is substantially lower than in the fully connected network, a reduction in linking costs is not in the interest of the average firm in the industry. This is particularly important since real world R&D networks typically are far from being fully connected and hence it could be concluded that the linking costs are in the range where average industry profits increase with f. Considering welfare, the observation in the first part of this section, that the size of the dominant component in the stochastically stable networks is smaller than that in the efficient networks, suggests that a decrease in the dominant component size due to an increase in f should lead to a welfare loss. Numerical calculations confirm this and show that welfare under stochastically stable networks is strictly decreasing in f with downward jumps at all levels of the links costs where the dominant component shrinks. This means that the negative effect of an increase of f on consumer surplus always dominates the positive implications for industry profits.

5 Conclusion

Considering a stochastic evolutionary process of network formation for collaboration networks between firms which later compete in a Cournot oligopoly, we find that the long–run equilibria, i.e. the stochastically stable networks, exhibit interesting properties. First, we get a generically unique selection of the pairwise stable networks. Second, the size of the dominant group is monotonically decreasing in the cost of link formation. For a certain cost range, static stability notions, like pairwise stable and pairwise Nash stable networks, do not exhibit such a monotonicity property. Third, there exists a lower threshold on the size of the dominant group such that below that threshold only the empty network is pairwise stable. This may be interpreted in a way such that there needs to be a number of firms to join a certain project in order for the project to succeed in the long–run. Interestingly, our fourth observation of the long–run equilibria is that these stochastically stable networks are usually not contained in the set of pairwise Nash stable networks. Thus, even though firms may be better off leaving the dominant group, in the long–run the large networks survive. Comparing the stochastically stable networks with efficient networks, we find that for some cost range the stochastically stable networks do not have the correct network structure, while simulation shows that for all values of linking costs stochastically stable networks appear to be under-connected.

An important implication of our findings is that a decrease in link formation costs induces an increase of the number of connections for a subset of (well-connected) firms and the accession of some firms to the well connected 'core', whereas the remaining firms stay isolated. The empirical evidence of an increasing number of R&D connections (e.g. Roijakkers and Hagedoorn [19]) suggests that costs of R&D links, relative to the market returns, are decreasing over time. Our results suggest that the distribution across firms of newly formed links should have heavy tails with a few firms adding a large number of links, few firms (the well connected ones) adding a few and many firms adding none. Also our findings concerning the relationship between linking costs and average industry profits provide an empirically testable hypothesis. An empirical evaluation of these qualitative implications of our analysis is left for future research.

Acknowledgments

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APPENDIX

Proof of Proposition 1

Since only the networks of type g^k (with one completely connected component of size k and all other firms isolated) can be pairwise stable,[19] we only have to consider incentives to add a link for an isolated player or incentives to delete a link for a connected player. In a network g^k, an isolated firm has no incentive to add a link if $\Delta_i^+(0, k(k-1)) < 0$ and a connected firm has no incentive to delete a link if $\Delta_i^-(k-1, (k-1)^2) \geq 0$. Note that we have

$$\begin{aligned} &\Delta_i^+(0, k(k-1)) < 0 \\ \Leftrightarrow \quad &\frac{\gamma(n-1)}{(n+1)^2}\Big[2(\alpha-\gamma_0)+\gamma(n-1)-2\gamma k(k-1)\Big] < f \end{aligned} \tag{5}$$

and

$$\begin{aligned} &\Delta_i^-(k-1, (k-1)^2) > 0 \\ \Leftrightarrow \quad &\frac{\gamma(n-1)}{(n+1)^2}\Big[2(\alpha-\gamma_0)-\gamma(n-1)+2\gamma(k-1)(n+1-k)\Big] > f \end{aligned} \tag{6}$$

Thus, the complete network is stable as long as $\Delta_i^-(n-1, (n-1)^2) \geq 0$ which implies that $f \leq F_1 := \frac{\gamma(n-1)}{(n+1)^2}\Big[2(\alpha-\gamma_0)+\gamma(n-1)\Big]$ and the empty network is stable as long as $\Delta_i^+(0,0) \leq 0$ which implies that $f \geq \frac{\gamma(n-1)}{(n+1)^2}\Big[2(\alpha-\gamma_0)+\gamma(n-1)\Big] = F_1$. Moreover, note that $\Delta_i^+(0, k(k-1))$ is strictly decreasing for $k \geq 1$ and $\Delta_i^-(k-1, (k-1)^2)$ is strictly increasing for $k < \frac{n+2}{2}$ and strictly decreasing for $k > \frac{n+2}{2}$. In particular we then get that for $f = F_1$ all networks g^k are pairwise stable. Since $\Delta_i^+(0, k(k-1))$ is strictly decreasing for $k \geq 1$, the complete network is uniquely pairwise stable if $\Delta_i^+\big(0, (n-1)(n-2)\big) > 0$ which implies that $f < F_0 := \frac{\gamma(n-1)}{(n+1)^2}\Big[2(\alpha-\gamma_0)+\gamma(n-1)(5-2n)\Big]$. Moreover, since $\Delta_i^-(k-1, (k-1)^2)$ attains its maximum at and is symmetric around $k = \frac{n+2}{2}$ and since k can only adopt natural numbers between 1 and $n-1$, we get that the empty network is uniquely pairwise stable if $\Delta_i^-\left(\lceil\frac{n+2}{2}\rceil - 1, \left(\lceil\frac{n+2}{2}\rceil - 1\right)^2\right) < 0$ which implies that $f > F_2 := \frac{\gamma(n-1)}{(n+1)^2}\Big[2(\alpha-\gamma_0)-\gamma(n-1)+2\gamma(\lceil\frac{n}{2}\rceil)(n-\lceil\frac{n+2}{2}\rceil)\Big]$. The remainder of the statement follows straightforwardly from the slope of $\Delta_i^+(0, k(k-1))$ and $\Delta_i^-(k-1, (k-1)^2)$.[20] □

Proof of Lemma 1 We show that from any network $g \in G$ there exists an improving path to a pairwise stable network.[21] Without loss of gener-

[19] Goyal and Joshi [9], Lemma 4.1 trivially also holds for pairwise stability.

[20] See also Figure 1 for an illustration. The proof here is rather kept concise. Goyal and Joshi [9] provide a more elaborate proof for the result on PNS networks, see Goyal and Joshi [9], Proposition 4.1.

[21] For the definitions of improving paths and cycles, see Jackson and Watts [14]

ality the players are ordered to size of η_i, such that $\eta_1 \geq \eta_2 \geq ... \geq \eta_n$ (otherwise reorder according to a permutation). By convexity and strategic substitutes we have that also $\Delta_i^+(\eta_i(g), \eta_{-i}(g)) \geq \Delta_j^+(\eta_j(g), \eta_{-j}(g))$ and $\Delta_i^-(\eta_i(g), \eta_{-i}(g)) \geq \Delta_j^-(\eta_j(g), \eta_{-j}(g))$ for all $i > j$. We now employ the following algorithm which preserves the order: If there exist players $j \in N$ who want to delete a link, i.e. such that $\Delta_j^-(\eta_j(g), \eta_{-j}(g)) < 0$, then start with the player with the largest number k for whom $\eta_k(g) > 0$, i.e. start with the player $k \in N$ with smallest positive η_k which implies $\Delta_k^-(\eta_k(g), \eta_{-k}(g)) < 0$ since Δ_k^- is lowest among all players with positive $\eta_k(g)$. By convexity, k has an incentive to then delete all of her links. Continue with players deleting all of their links according to this order (starting from the last player such that $\Delta_k^-(\eta_k(g), \eta_{-k}(g)) < 0$ and $\eta_k(g) > 0$) until this is no longer possible. Then this network is pairwise stable or there exist players who want to add links. Due to the predefined order and the fact that $\Delta_i^+(\eta_i, \eta_{-i})$ and $\Delta_i^-(\eta_i, \eta_{-i})$ are increasing in η_i and decreasing in η_{-i} this can only be the players with connections. Now start with the players first in order and add links for any player until one recipient declines a connection. Either that network is pairwise stable or we apply the procedure again by deleting links from the last player in order such that $\eta_k > 0$. Since the number of players is finite, the algorithm finally terminates at a pairwise stable network g^k. $\square$

Proof of Lemma 2 Note that in any pairwise stable network g^k, $k \neq n-1$ any connected player wants to form a link with an isolated player $i \in N$ by Proposition 1. However, since g^k is assumed to be pairwise stable, we have that $\Delta_i^+(0, k(k-1)) < 0$, i.e. the isolated players decline the connection. Note that there are two ways to increase player i's incentive to form a link, by deleting links between two connected players because of strategic substitutes or build links between i and connected players. However, the effect of the former is dominated by the latter since (2) and (3) holds. Thus, player i wants to form links by herself as soon as she has formed $\arg\min_{\tilde{k} \in \{0...k\}} \left(\Delta_i^+(\tilde{k}, k(k-1) + \tilde{k}) \geq 0 \right)$ links. Note that because of convexity in own links and strategic substitutes the connected players still want a link with player i, implying that there exists a zero resistance path to the network g^{k+1}. The other direction is analogous. $\square$

Proof of Proposition 2 If there exists a unique pairwise stable network, then it follows directly that it is stochastically stable. Hence, Proposition 1 directly implies that the fully connected network is stochastically stable for $f < F_0$ and the empty network for $f > F_2$. Hence we restrict attention to $F_0 \leq f \leq F_2$. In this range there are several PS networks and the subset of the PS networks with minimal stochastic potential gives the set of stochastically stable networks (see e.g. Young [23]).

Let us first compare the stochastic potential of networks g^k with $k \geq \underline{k}$. The stochastic potential of a PS network g^k, $k \in [\underline{k}(f), \bar{k}(f)]$ is given by $r(g^k) = c(g^1, g^{\underline{k}}) + \sum_{l=\underline{k}}^{k-1} c^+(l) + \sum_{l=k+1}^{\bar{k}} c^-(l)$, where $c(g^1, g^{\underline{k}(f)})$ is the minimal number of mistakes necessary to move from the empty network g^1 to $g^{\underline{k}(f)}$, which is zero if g^1 is not pairwise stable and where $\bar{k}(f) = n$, if the complete network is pairwise stable. By Lemma 2 we have for the difference in stochastic potential between two adjacent PS networks $\Delta^r(k) = r(g^{k+1}) - r(g^k) = 2\kappa(k) - k$, where $\kappa(k)$ is given by (4). In order to characterize the discrete $\kappa(k)$ let $h(k)$ be the implicit function $h(k) := \{h \in \mathbb{R} : \Delta_i^+(h, k(k-1)+h) = 0\}$. Since $\Delta_i^+(h, k(k-1)+h)$ is strictly increasing in h the solution is unique for every $k \in \{1, ..., n-1\}$. Solving for h we get

$$h(k) = \frac{k^2 - k}{n-1} + \frac{(n+1)^2 f}{2\gamma^2(n-1)^2} - \frac{2(\alpha - \gamma_0) + \gamma(n-1)}{2\gamma(n-1)}, \tag{7}$$

and hence $\kappa(k) = \lceil h(k) \rceil$, if $0 \leq h(k) \leq k$. Otherwise if $h(k) < 0$ then $\kappa(k) = 0$ and if $h(k) > k$ then $\kappa(k) = k$. Taking the continuous approximation of Δ^r we get $\tilde{\Delta}^r(k) = 2h(k) - k$ which yields $\frac{\partial \tilde{\Delta}^r(k)}{\partial k} = 2h'(k) - 1 = \frac{(4k-2)-n-1}{n-1}$. Thus, $\tilde{\Delta}^r(k)$ is strictly decreasing/increasing for $k < / > (n+1)/4$, has a global minimum at $k = (n+1)/4$ and $\tilde{\Delta}^r(k) < 0$ for $k = (n+1)/4$. Considering the continuous approximation $\tilde{\Delta}^r(k)$ the main intuition of the proof can be seen straightforwardly. For small enough f, we have $\tilde{\Delta}^r(k) < 0$ for all k implying that the complete network is stochastically stable. Otherwise there exists a unique $k^*(f)$ with $\tilde{\Delta}^r(k^*(f)) = 0$ and $\tilde{\Delta}^r(k) > 0$ for all $k > k^*(f)$. This means that $k^*(f)$ is a local minimizer of the stochastic potential $r(g^k)$. Moreover, $k^*(f) \geq \frac{n+1}{4}$. The only other candidate for a global minimizer is $k = 1$, i.e. the empty network. Since k is the size of the dominant group, k can only be an integer. Moreover, the number of mistakes $\kappa(k)$ can only take on integer values. In the following we therefore prove the statement by considering $\Delta^r(k) = 2\kappa(k) - k = 2\lceil h(k) \rceil - k$. Note that a necessary condition for a minimizer $k^*(f)$ of $r(g^k)$ is that $\Delta^r(k^*(f)) \geq 0$ (since this implies that $r(g^{k^*(f)+1}) \geq r(g^{k^*(f)})$) and $\Delta^r(k^*(f) - 1) \leq 0$ (since this implies that $r(g^{k^*(f)}) \leq r(g^{k^*(f)-1})$).

We first show the following auxiliary Lemmas which are helpful in restricting the set of possible minimizers of the stochastic potential. We then show the statement.

Lemma 3. *There does not exists a stochastically stable network g^k such that $2 \leq k \leq \frac{n-1}{4}$.*

Proof. Note that $h(k+1) - h(k) = \frac{2k}{n-1} \leq \frac{1}{2}$ if and only if $k \leq \frac{n-1}{4}$ which yields $\lceil h(k+1) \rceil - \lceil h(k) \rceil \leq 1$ for all $k \leq \frac{n-1}{4}$ and from $\lceil h(k) \rceil - \lceil h(k-1) \rceil = 1$ it follows that $\lceil h(k+1) \rceil - \lceil h(k) \rceil = 0$ for all $k \leq \frac{n-1}{4}$ and from $\lceil h(k+1) \rceil - \lceil h(k) \rceil = 1$ it follows that $\lceil h(k) \rceil - \lceil h(k-1) \rceil = 0$ for all $k \leq \frac{n-1}{4}$.

Suppose now that there exists a $2 \leq k \leq \frac{n-1}{4}$ such that g^k is stochastically stable. Necessary for stochastic stability of g^k is that $\Delta^r(k) \geq 0$ and $\Delta^r(k-1) \leq 0$.

First, let k be odd. Then $\Delta^r(k) \geq 0 \;\Leftrightarrow\; \lceil h(k) \rceil \geq \frac{k}{2}$ implies that both inequalities must be strict, since $\frac{k}{2} \notin \mathbb{Z}$. Then because of $k \leq \frac{n-1}{4}$ either $\lceil h(k-1) \rceil = \lceil h(k) \rceil$ which trivially implies that $\lceil h(k-1) \rceil > \frac{k-1}{2}$ or $\lceil h(k-1) \rceil = \lceil h(k) \rceil - 1$ which implies that $\lceil h(k-1) \rceil > \frac{k}{2} - 1$, and, hence $\lceil h(k-1) \rceil \geq \frac{k}{2} - \frac{1}{2}$ since $\lceil \frac{k}{2} - 1 \rceil = \frac{k}{2} - \frac{1}{2}$. Thus, $\Delta^r(k) \geq 0$ for k odd implies $\Delta^r(k-1) \geq 0$. Moreover if $\Delta^r(k-1) = 0$ then $\lceil h(k-1) \rceil = \lceil h(k) \rceil - 1$ and thus we have $\lceil h(k-1) \rceil = \lceil h(k-2) \rceil$ implying that $\Delta^r(k-2) > 0$, and hence, $r(g^{k-2}) < r(g^k)$, contradicting stochastic stability of g^k.

Now let k be even and suppose $\Delta^r(k) \geq 0$. First, consider $\Delta^r(k) > 0 \Leftrightarrow \lceil h(k) \rceil > \frac{k}{2}$. Thus, $\lceil h(k) \rceil \geq \frac{k}{2} + 1$ since $\frac{k}{2} \in \mathbb{Z}$. As above let $\lceil h(k-1) \rceil = \lceil h(k) \rceil - 1$ (the other case $\lceil h(k-1) \rceil = \lceil h(k) \rceil$ trivially implies $\Delta^r(k-1) > 0$.) Then $\lceil h(k-1) \rceil = \lceil h(k) \rceil - 1 \geq \frac{k}{2} > \frac{k-1}{2}$, implying $\Delta^r(k-1) > 0$. Finally suppose that $\Delta^r(k) = 0$. First, if $\lceil h(k-1) \rceil = \lceil h(k) \rceil$ then $\Delta^r(k-1) > 0$ and we are in the case above, where k is odd. Second if $\lceil h(k-1) \rceil = \lceil h(k) \rceil - 1$ then we must have $\lceil h(k+1) \rceil = \lceil h(k) \rceil$, implying that $\Delta(g^{k+1}) < 0$ which implies that $r(g^{k+2}) < r(g^{k+1}) = r(g^k)$, contradicting stochastic stability of g^k. $\square$

Lemma 4. *Assume that* $\min_{k \in \{1,\ldots n-1\}} \Delta^r(k) < 0$ *and* $\Delta^r(n-1) \geq 0$. *Then, there either exists a unique* $k^*(f) \in \left\{\frac{n-1}{4}, \ldots, n-1\right\}$ *with* $\Delta^r(k^*-1) < 0, \Delta^r(k^*) > 0, \Delta^r(k) \leq 0,\ \forall \frac{n-1}{4} \leq k < k^* - 1$ *and* $\Delta^r(k) \geq 0,\ \forall k^* < k \leq n-1$ *or a unique* $k^*(f) \in \left\{\frac{n-1}{4}, \ldots, n-1\right\}$ *with* $\Delta^r(k^*) = 0, \Delta^r(k^*-1) < 0, \Delta^r(k^*+1) > 0, \Delta^r(k) \leq 0,\ \forall \frac{n-1}{4} \leq k < k - 1^*$ *and* $\Delta^r(k) \geq 0,\ \forall k^* + 1 < k \leq n-1$. *Furthermore,* $k^*(f)$ *is weakly decreasing with respect to* f.

Proof. We show first that $\Delta^r(k) > 0$ implies $\Delta^r(l) \geq 0$ for all $l > k$. Suppose that there is a $k > \frac{n-1}{4}$ such that $\Delta^r(k) > 0 \Leftrightarrow \lceil h(k) \rceil > \frac{k}{2}$. If k is even then $\frac{k}{2} \in \mathbb{Z}$ and hence $\lceil h(k) \rceil > \frac{k+1}{2}$ implying $\lceil h(k+1) \rceil \geq \lceil h(k) \rceil > \frac{k+1}{2}$, and thus $\Delta^r(k+1) > 0$. If k is odd then $\frac{k}{2} \notin \mathbb{Z}$ and hence $\lceil h(k) \rceil \geq \frac{k+1}{2}$ implying $\lceil h(k+1) \rceil \geq \lceil h(k) \rceil \geq \frac{k+1}{2}$, and thus $\Delta^r(k+1) \geq 0$. Note however that if $\Delta^r(k+1) = 0$ then it must be that $\lceil h(k+1) \rceil = \lceil h(k) \rceil$ implying that $\lceil h(k+2) \rceil \geq \lceil h(k+1) \rceil + 1$,[22] and, hence, $\Delta^r(k+2) > 0$. Thus if $\Delta^r(k) > 0$ then $\Delta^r(l) \geq 0$ for all $l > k$.

Second, we note that $\min_{k \in \left\{\frac{n-1}{4},\ldots n-1\right\}} \Delta^r(k) < 0$. Assume to the contrary that $\Delta^r(k) \geq 0$ for all $k \in \left\{\frac{n-1}{4}, \ldots, n-1\right\}$. From Lemma 3 it then follows that $\Delta^r(k) \geq 0$ for all $k \in \{1, ..., \frac{n-1}{4}\}$, since $\Delta^r(k) \geq 0$ for $k = \frac{n-1}{4}$ contradicting the the assumption $\min_{k \in \{1,\ldots n-1\}} \Delta^r(k) < 0$.

[22] Since for $k \geq \frac{n-1}{4}$, $h(k+1) - h(k) = \frac{2k}{n-1} \geq \frac{1}{2}$ and thus $\lceil h(k+1) \rceil - \lceil h(k) \rceil = 0$ implies $\lceil h(k+2) \rceil - \lceil h(k+1) \rceil = 1$.

Given that $\min_{k\in\left\{\frac{n-1}{4},\ldots n-1\right\}} \Delta^r(k) < 0$ define $k^*(f)$ by $k^*(f) := 1 + max\left[k \in \left\{\frac{n-1}{4}, \ldots, n-1\right\} | \Delta^r(k) < 0\right]$.[23] If $\Delta^r(k^*) > 0$ the statements of the first of the two cases given in the text of the Lemma follow directly from our arguments above. If $\Delta^r(k^*) = 0$, it follows, due to the definition of $\Delta^r(k)$, from $\Delta^r(k^*) = 0$ that $\Delta^r(k^*+1) \neq 0$, and due to the definition of k^* we must have $\Delta^r(k^*+1) > 0$. Similarly, we must have $\Delta^r(k^*-1) < 0$. Hence, we obtain the statements concerning the second case given in the Lemma. Finally, the claim that k^* is weakly decreasing with respect to f follows from the observation that $h(k)$ is increasing in f, which implies that $\Delta^r(k)$ is weakly increasing in f. Accordingly, k^* decreases (weakly) as f is increased. $\square$

In order to prove the claims of the Proposition, we first observe that for sufficiently small values of f, where $\Delta^r(n-1) < 0$, we have that $\Delta^r(k) < 0$ for all $k \in \{1, \ldots n-1\}$ implying that the stochastic potential is minimized for the complete network and the complete network is the unique stochastically stable network.

In what follows we therefore focus on values of f where $\Delta^r(n-1) \geq 0$. We consider first the case $F_0 \leq f < F_1$. As shown in Proposition 1 the set of candidates for stochastically stable networks is given by $\{g^{\underline{k}(f)}, \ldots, g^n\}$, where $\underline{k}(f)$ is the smallest k such that $\Delta_i^+(0, k(k-1)) < 0$. This property implies that for $k^*(f) < \underline{k}(f)$ we must have $c^+(k^*(f)) = 0$. Given that we have $c^-(k) > 0$ for all $k \in \{2, \ldots, n\}$ and $f < F_1$, this implies $\Delta^r(k^*(f)) = c^+(k^*(f)) - c^-(k^*(f)+1) < 0$, which contradicts Lemma 4. Hence, we must have $k^*(f) \geq \underline{k}(f)$ and therefore $\min_{\{k\in\{1,\ldots n-1\}} \Delta^r(k) < 0$. Direct application of Lemma 4 now establishes that among the PS networks the minimal stochastic potential is attained for g^{k^*}, if $\Delta^r(k^*) > 0$, or for each of the networks g^{k^*} and g^{k^*+1}, if $\Delta^r(k^*) = 0$.

Considering $F_1 \leq f < F_2$ we observe first that $r(g^1) - r(g^{\underline{k}(f)}) = c(g^{\underline{k}(f)}, g^1) - c(g^1, g^{\underline{k}(f)})$ is (weakly) decreasing in f and negative for sufficiently large f. On the one hand, we have that $c(g^1, g^{\underline{k}(f)})$ is (weakly) increasing in f, which follows because if $\Delta_i^+(\eta_i, \eta_{-i}) < 0$ for some f, then $\Delta_i^+(\eta_i, \eta_{-i}) < 0$ for all $f' > f$. Moreover, $\underline{k}(f)$ is increasing in f. The same argument implies that $c(g^{\underline{k}(f)}, g^1) = c^-(\underline{k}(f))$ is (weakly) decreasing in f. Obviously, we have $r(g^1) = 0$ for sufficiently large f, which implies that $r(g^1) - r(g^{\underline{k}(f)}) < 0$ for sufficiently large f. From the arguments above it follows that there exists an interval $[\tilde{f}_l, \tilde{f}_h]$ such that

$$r(g^1) - r(g^{\underline{k}(f)}) \begin{cases} > 0 & f < \tilde{f}_l \\ = 0 & f \in [\tilde{f}_l, \tilde{f}_h] \\ < 0 & f > \tilde{f}_h. \end{cases}$$

[23]For convenience we will drop the dependence on f.

As the next step of the proof we establish that $g^{\underline{k}(f)}$ is never stochastically stable. To this end, we show that $k^*(f) \geq \underline{k}(f)$ for all $f \leq \tilde{f}_h$. Given the (weak) monotonicity of $k^*(f)$ and $\underline{k}(f)$ it suffices to show this claim for $f = \tilde{f}_h$. Assume that $k^*(\tilde{f}_h) < \underline{k}(\tilde{f}_h)$. Then, $k^*(f)$ is not pairwise stable and thus we have $c(g^{k^*}, g^1) = 0$. Furthermore, due to the definition $k^*(\tilde{f}_h)$ it follows from Lemma 4 that $r(g^{\underline{k}(\tilde{f}_h)}) > r(g^{k^*(\tilde{f}_h)})$. This implies

$$r(g^1) \leq r(g^{k^*(\tilde{f}_h)}) + c(g^{k^*}, g^1) = r(g^{k^*(\tilde{f}_h)}) < r(g^{\underline{k}(\tilde{f}_h)})$$

and we obtain a contradiction to $r(g^1) = r(g^{\underline{k}(\tilde{f}_h)})$. Hence $k^*(f) \geq \underline{k}(f)$ for all $f \leq \tilde{f}_h$. Since, by definition g^{k^*} always has a lower stochastic potential than $g^{\underline{k}}$, this shows that the only candidates for stochastically stable networks are g^1 and g^{k^*} (sometimes together with g^{k^*+1}). Considering the difference in stochastic potential between these two networks we have

$$r(g^1) - r(g^{k^*(f)}) = c(g^{\underline{k}(f)}, g^1) - c(g^1, g^{\underline{k}(f)}) + \sum_{k=\underline{k}}^{k^*-1} (-\Delta^r(k))$$

We know already that the first term is (weakly) decreasing in f. For the sum, we know that for each k the term $(-\Delta^r(k))$ is decreasing in f. Furthermore, the number of summands (weakly) decreases for increasing f and each summand is non-negative, because of $k < k^*$. Altogether, we obtain that $r(g^1) - r(g^{k^*(f)})$ is weakly decreasing with respect to f. Arguments analogous to above establish that the difference is negative for sufficiently large f. The claims of the Proposition follow now directly by setting $F_0^* = \min[f|k^*(f) = n-1], F_1^* = \min[f|r(g^1) - r(g^{k^*(f)}) = 0], F_2^* = \max[f|r(g^1) - r(g^{k^*(f)}) = 0]$. □

Proof of Corollary 1 We only need to show that for some $f \in [F_0^*, F_1]$ there exist stochastically stable networks g such that $0 < D(g) < 1/2$. The remainder follows from Proposition 2. Moreover, from the proof of Proposition 2 we have that the approximation $\tilde{\Delta}^r(k)$ is symmetric around its minimum $\frac{n+1}{4}$. For $f = F_1$ we have $\tilde{\Delta}^r(1) = 0$ and the empty network cannot be stochastically stable, implying by Proposition 2 that $k^*(f)$ (and possibly $k^*(f) + 1$) is stochastically stable. Moreover, for $f = F_1$, we have $\tilde{\Delta}^r(\frac{n-1}{2}) = 0$ by symmetry of $\tilde{\Delta}^r$ around $\frac{n+1}{4}$. Hence, for the size of the stochastically network(s) g^{k^*} (and possibly g^{k^*+1}) we obtain $k^* < \frac{n}{2}$. Therefore, for n large enough, we have $k^* < \frac{n-1}{\sqrt{2}} - 1$ and hence $0 < D(g^{k^*}) < D(g^{k^*+1}) = \frac{k^*(k^*+1)}{n(n-1)} < \frac{1}{2}$. □

Proof of Corollary 2 The statement of the corollary is equivalent to the statement that $p^*(f)$ is constant with respect to f for $f < F_0^*$ and $f > F_2^*$, but weakly increasing on the interval $[F_0^*, F_2^*]$. Since the equilibrium price in a Cournot oligopoly with linear demand and constant marginal costs is given by the arithmetic mean of the reservation price and the marginal costs of all producers, we get in our model,

$$p^*(f) = \frac{1}{n+1}\Big(\alpha + k^*(f)\big(\gamma_0 - \gamma(k^*(f)-1)\big) + \big(n - k^*(f)\big)\gamma_0\Big).$$

It is easy to see that this expression is decreasing with respect to $k^*(f)$ and the claim of the Corollary follows directly from the monotonicity of $k^*(f)$ with respect to f as shown in Proposition 2. □

Proof of Proposition 3 The first part of the Proposition is straightforward. According to Proposition 2, $k^*(f) = n$ for all $f < F_0^*$ and hence $\Pi^*(f)$ is a decreasing function of f. To show the second part we define

$$\begin{aligned}\bar{\Pi}(k) = \;& k\Big(\alpha - (n-k+1)\big(\gamma_0 - \gamma(k-1)\big) + (n-k)\gamma_0\Big)^2 + (n-k)\Big(\alpha \\ & -(k+1)\gamma_0 + k\big(\gamma_0 - \gamma(k-1)\big)\Big)^2 - (n+1)^2 k(k-1)f.\end{aligned}$$

It is easy to check that $\Pi^*(f) = \frac{1}{n(n+1)^2}\bar{\Pi}(k^*(f))$. Proposition 2 implies that for all values $\tilde{f}$ where $k^*(.)$ is not continuous we have $\lim_{f\downarrow\tilde{f}} k^*(f) = \lim_{f\uparrow\tilde{f}} k^*(f) + l$ for some positive integer l. Therefore, in order to show that $\Pi^*(f)$ exhibits an upward-jump at $\tilde{f}$ it is sufficient to show that $\bar{\Pi}(k)$ is decreasing with respect to k on the interval $[\lim_{f\downarrow\tilde{f}+} k^*(f), \lim_{f\uparrow\tilde{f}-} k^*(f)]$. In the remainder of the proof we show that there exists a threshold $\bar{F} < F_0^*$ such that $\bar{\Pi}(k)$ is a decreasing function for all $k \in [1,n]$ and $f > \bar{F}$.

Differentiating $\bar{\Pi}$ and collecting terms yields

$$\begin{aligned}\bar{\Pi}'(k) \;=\;& \gamma\Big[\alpha - (k+1)\gamma_0 + k\big(\gamma_0 - \gamma(k-1)\big)\Big]\Big(-(k-1)(n-3)+2\Big) \\ & +(n+1)\Big[\gamma(k-1)\Big(\alpha - \gamma_0 + \gamma\big(k(3(n+1)-5k) - (n+1-3k)\big)\Big)\Big] \\ & -(2k-1)(n+1)^2 f.\end{aligned}$$

If $k > 1$ the expression in the first line is negative, whereas for $k = 1$ this expression is independent of n. Hence, we have $\bar{\Pi}'(k) < 0$ for sufficiently large n if the the expression of the second and third line is negative. Thus, the condition that

$$f > \frac{\gamma(k-1)}{(2k-1)(n+1)}(\alpha - \gamma_0 + \gamma(k(3(n+1)-5k) - (n+1-3k)))$$

for all $k \in [1, n]$ is a sufficient condition for $\bar{\Pi}(k)$ to be decreasing. Concerning the right hand side of this inequality we obtain

$$\begin{aligned}
&\frac{\gamma(k-1)}{(2k-1)(n+1)}(\alpha - \gamma_0 + \gamma(k(3(n+1) - 5k) - (n+1-3k))) \\
&< \frac{\gamma}{2(n+1)}(\alpha - \gamma_0 + \gamma(k(3(n+1) - 5k) + 3n)) \\
&\leq \frac{\gamma}{2(n+1)}(\alpha - \gamma_0 + \gamma(0.45(n+1)^2 + 3n)) =: \bar{F},
\end{aligned}$$

where we have used that $\max_{k \in [1,n]} [k(3(n+1) - 5k] = \frac{9}{20}(n+1)^2$. Therefore, $\bar{\Pi}(k)$ is a decreasing function on $[1, n]$ if $f > \bar{F}$ and n sufficiently large.

In order to compare $\bar{F}$ with F_0^* we use that at $f = F_0^*$ we must have $\Delta_r(n) \geq 0$. Using the notation of the proof of Proposition 2 it is easy to see that $2(h(n)+1) - n \geq 0$ is a necessary condition for $\Delta_r(n) \geq 0$. Hence, we must have $F_0^* > \underline{F}_0^*$, where $\underline{F}_0^*$ is such that $2(h(n)+1) - n = 0$ for $f = \underline{F}_0^*$. Using (7) we obtain

$$\underline{F}_0^* = \frac{\gamma(n-1)}{(n+1)^2}\left(2(\alpha - \gamma_0) - \gamma(n-1)(n+1)\right).$$

Comparing this expression with $\bar{F}$ yields

$$\begin{aligned}
& \bar{F} < \underline{F}_0^* \\
\Leftrightarrow\ & (n+1)(\alpha - \gamma_0 + \gamma(0.45(n+1)^2 + 3n)) \\
& < 2\gamma(n-1)\left(2(\alpha - \gamma_0) - \gamma(n-1)(n+1)\right) \\
\Leftrightarrow\ & \gamma(0.45(n+1)^3 + 3n(n+1) + 2(n-1)^2(n+1)) \\
& < (3n-5)(\alpha - \gamma_0) \qquad (8)
\end{aligned}$$

Due to our assumption that $(\alpha - \gamma_0) > \gamma(n-1)(n-2)$ (to ensure strictly positive quantities) inequality (8) must hold if the following holds:

$$(3n-5)(n-1)(n-2) > (0.45(n+1)^3 + 3n(n+1) + 2(n-1)^2(n+1)).$$

It is easy to see that this inequality holds for sufficiently large n because the coefficient of n^3 on the left hand side is larger than on the right hand side. This shows that $\bar{F} < \underline{F}_0^* \leq F_0^*$ and completes the proof. □