Optimal R&D investment with learning-by-doing: Multiple steady-states and thresholds

Alfred Greiner   **Anton Bondarev**

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Abstract

In this paper we present an inter-temporal optimization problem of a representative R&D firm that simultaneously invests in horizontal and vertical innovations. We posit that learning-by-doing makes the process of quality improvements a positive function of the number of existing technologies with the function displaying a convex-concave form. We show that multiple steady-states can arise with two being saddle point stable and one unstable with complex conjugate eigenvalues. Thus, a threshold with respect to the variety of technologies exists that separates the two basins of attractions. From an economic point of view, this implies that a lock-in effect can occur such that it is optimal for the firm to produce only few technologies at a low quality when the initial number of technologies falls short of the threshold. Hence, history matters as concerns the state of development implying that past investments and innovations determine whether the firm produces a large or a small variety of high- or low-quality technologies, respectively.

Keywords: Optimal control, horizontal and vertical innovations, multiple steady-states, thresholds, lock-in

JEL classification: C61, D92, O32

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1 Introduction

The importance of innovations for the process of economic development has been acknowledged in the economics literature for quite a long time. Already (Schumpeter 1942) saw in innovations the major driving force for the evolution of economies giving rise to the so-called process of creative destruction. New methods of production or new products replace old ones, thus, generating economic growth. In the modern growth literature, both horizontal and vertical innovations play a major role in explaining ongoing growth. Thus, (Romer 1990) presents a model where horizontal innovations lead to a permanent increase of the number of technologies used to produce the final output good which leads to permanent growth. The process of creative destruction has been modelled by (Aghion and Howitt 1992) who assume that vertical innovations lead to a continuous improvement of the technologies in use replacing older ones. A combination of those two approaches has been presented in the contribution by (Bondarev and Greiner 2014a) that allows for simultaneous horizontal and vertical innovations in a model of economic growth, where both structural change and economic growth are endogenous phenomena.

Another important factor that determines economic development is learning-by-doing. One of the first studies that pointed to this sort of productivity raising activity was the one by (Wright 1936) who observed that the number of hours needed to produce one airframe is a decreasing function of the number of airframes already produced. (Arrow 1962) generalized this approach and argued that the acquisition of knowledge in general, i.e. learning, is strongly related to experience and he postulated that cumulated past gross investment is a good index for this type of learning.

In this paper we intend to analyze a microeconomic model of a representative R&D firm that invests in both horizontal and vertical innovations with learning-by-doing effects. Our goal is to study whether multiple equilibria may occur giving rise to lock-in effects. It is generally known that in a framework with co-existing vertical and horizontal innovations, firms tend to invest more into the development of existing products rather than into the creation of new products. This is a typical situation in industries where large firms are multi-product firms due to patent regimes, with examples being the pharmaceutical and packaging (Tetra Pak) industries. The problem of multi-product innovations has received

attention in the industrial organization literature starting with (Lambertini 2003). In that paper the optimal behaviour of a multi-product firm investing in R&D is analyzed. However, the question of multiple steady-states and lock-ins is beyond the scope of that paper. The description of multi-product innovations in our approach follows the ideas of (Lambertini and Orsini 2001; Lin 2004; Belyakov et al. 2011) and more closely the one by (Bondarev 2012), where the single-agent dynamic optimization problem with infinite life cycles of technologies has been treated and, in particular, the one presented in (Bondarev and Greiner 2014b).

In the paper by (Bondarev and Greiner 2014a) the existence of multiple steady-states has been shown. There, a maximum number of available technologies is exogenously given and when there is a restriction with respect to the amount of R&D spending, less technologies will be produced and multiple steady-states can occur. One way to overcome that situation is to raise the R&D budget such that the maximum number of technologies is produced and the steady-state becomes unique. But, in the latter study the focus is on the long-run outcome and transition dynamics and stability of the steady-states have not been not analyzed. In contrast to that, in this paper we give a complete characterization of the dynamics of the firm and we do not impose an exogenous constraint on R&D spending. In addition, we assume that learning-by-doing occurs such that the increase in productivity is a positive function of the number of technologies already produced, as in (Wright 1936) and in (Arrow 1962), and we allow for non-linear learning effects by assuming a convex-concave function. Thus, we take into account that learning is not a linear process but first accelerates as more technologies are invented but slows down later on and possibly converges to a constant.

The rest of the paper is organized as follows. The next section presents the structure of our microeconomic model of the firm. Section 3 derives the solution to the optimization problem and section 4 analyzes the question of whether multiple steady-states and a threshold can exist. Section 5, finally, concludes the paper and the appendix contains some proofs and derivations.

2 The inter-temporal optimization problem

We assume a representative R&D firm that wants to maximize the discounted stream of profits, given by the productivities of all technologies, i.e. intermediate capital goods, minus the cost from investments in horizontal and vertical innovations. The R&D firm produces intermediate capital goods it sells to producers in the final goods sector and it commits to permanently raise the productivity of the technologies it provides, even if the technologies are not produced any longer by the R&D firm.

The model follows the lines of (Belyakov et al. 2011), where the mathematical foundations for this type of models are discussed, but it is more closely to the stylized model in (Bondarev 2012) and the one in (Bondarev and Greiner 2014a). Investments into horizontal and vertical innovations are determined by an inter-temporal optimization problem subject to the laws of motion describing the productivity increase for each technology and subject to the process of variety expansion of the technologies. Denoting the discount rate of the firm by r, the objective functional to be maximized can be written as:

$$J = \max_{u(\cdot),g(\cdot)} \int_0^\infty \mathrm{e}^{-rt} \left(\int_0^{n(t)} \left[q(i,t) - \frac{1}{2}g(i,t)^2 \right] di - \frac{1}{2}u(t)^2 \right) dt. \tag{1}$$

with:

- $u(t)$: investments into variety expansion;

- $g(i,t)$: investments into the productivity growth of technology i at time t.

The firm continuously develops new technologies, $i \in \mathbf{I}$, from the potential spectrum of technologies that is assumed to be bounded but so large such that the upper bound does not constitute a binding constraint for the firm.[1] The process of acquiring new technologies follows a simple linear process. At the same time, the firm develops the productivity of all the technologies. The dynamics for the variety of technologies, $n(t)$,

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[1]This can always be done as long as the dynamic system has a maximal steady state. Then, it is sufficient to choose the boundary on $\mathbf{I}$ greater than the maximal steady state. The boundedness of the state space is necessary for the generic existence of a solution for the class of infinite dimensional control systems, see (Fattorini 1999).

and for the productivities of technologies, $q(t)$, are given by:

$$\dot{n}(t) = \xi u(t) - \delta n(t),$$

$$\dot{q}(i,t) = \psi(i)g(i,t) - \beta q(i,t), \ \forall i \in \mathbf{I} \subset \mathbb{R}_+, \tag{2}$$

with:

- $\xi > 0$: efficiency of investments in the expansion of the variety of technologies, with ξ set equal to one, $\xi = 1$, without loss of generality;

- $\delta > 0$: rate at which technologies disappear from the spectrum of the R&D firm and are not produced any longer;

- $\psi(i) > 0$: efficiency of investments in the productivity growth of technology i;

- $\beta > 0$: rate of decay of productivity of technology i, identical across technologies.

One also has a number of static constraints on controls and states:

$$q(i,t)|_{i=n(t)} = 0, \ 0 \leq q(i,t) < \infty, \ 0 \leq n(t) < \infty, \ u(t) \geq 0, \ g(i,t) \geq 0. \tag{3}$$

From (2), (3) it can be seen, that the productivity of each technology needs permanent maintenance at the rate β and a certain number of technologies becomes obsolete at the rate δ and they are not produced any longer. But note that the firm still has to improve the productivity of those outdated technologies since it has signed a contract guaranteeing the producers of the final goods, i.e. the buyers of the technology, to permanently improve the quality of the technology sold. Further, we differ between the invention of a technology and its innovation by postulating that each new technology has zero productivity at the time it is invented,

$$q(i, t_i(0)) = 0, \tag{4}$$

which makes sense from an economic point of view and where $t_i(0)$ denotes the time of invention of the technology i which is the inverse function of the process of variety expansion.

It should be noted that the efficiency of R&D investments in quality improvements, $\psi(i)$, plays a crucial role in the determination of the dynamics of productivities. We make two important assumptions as concerns this function. First, we posit that the efficiency positively depends on the number of technologies already invented. The justification for that assumption are learning-by-doing effects stating that experience rises as more and more technologies are produced so that labor becomes more productive, as already described by (Wright 1936) and by (Arrow 1962). Second, we allow for non-linearities and assume that the efficiency function displays a convex-concave shape. The reason for that assumption is that at early stages of development, learning effects rise more than proportional since the stock of knowledge, or the experience, is still small such that it is relatively easy to acquire new knowledge. However, that cannot go on to infinity because the more knowledge has already been acquired, the more difficult it becomes to acquire additional knowledge. Therefore, sooner or later the function takes a concave form and converges to a finite value.

For the control problem to make sense we also require compactness of the state space and this requires the function $\psi(i)$ to be bounded. We choose a Gompertz function for this efficiency function that has the following specific form, allowing it to vary across technologies:

$$\psi(i) = a\,e^{-b\cdot e^{-d\cdot i}},\ a, b, d > 0. \tag{5}$$

This function disposes of the properties discussed above. In particular, due to learning-by-doing effects, the increase in productivity first rises more than proportional before it becomes slower and converges to a constant. The latter reflects the fact the learning effects are bounded such that the change in productivity is not an ever accelerating process for a given level of R&D investment. A different function that displays the same shape would be the logistic function. The qualitative analysis, however, would not change with that function just as with any other increasing convex-concave function.

3 Necessary optimality conditions

To solve the dynamic optimization problem given by (1) subject to (2), we construct the current-value Hamiltonian:

$$\mathcal{H} = \int_0^{n(t)} \Big[q(i,t) - \frac{1}{2}g(i,t)^2 \Big] di - \frac{1}{2}u(t)^2 + \lambda_n(t) \cdot (u(t) - \delta\, n(t)) +$$
$$+ \int_0^{n(t)} \lambda_q(i,t) \cdot \Big(\psi(i)g(i,t) - \beta q(i,t))di \Big), \tag{6}$$

where λ_n, λ_q are the shadow prices or co-state variables of the variety expansion and of the productivities of technologies, respectively.

The first order conditions for this problem are given by,

$$u(t) = \lambda_n(t); \tag{7}$$

$$g(i,t) = \psi(i)\lambda_q(i,t); \tag{8}$$

and the differential equation system for the co-state variables is:

$$\dot{\lambda}_n(t) = r\lambda_n(t) - \frac{\partial \mathcal{H}}{\partial n} =$$
$$= (r + \delta)\lambda_n(t) + \frac{1}{2}\, g(n(t),t)^2 - \lambda_q(n(t),t)\, \psi(n(t))\, g(n(t),t); \tag{9}$$

$$\forall i \leq n(t): \dot{\lambda}_q(i,t) = r\lambda_q(t) - \frac{\partial \mathcal{H}}{\partial q} = (r + \beta)\lambda_q(i,t) - 1. \tag{10}$$

In deriving that system, we made use of the following:

- in the first equation we use $q(i,t)|_{i=n(t)} = 0$,

- $g(n(t),t) = g(i,t)|_{i=n(t)}$ is the value of R&D investments into the productivity of the latest technology,

- $\lambda_q(n(t),t) = \lambda_q(i,t)|_{i=n(t)}$ is the shadow price of the productivity of the latest technology,

- $\psi(n(t)) = \psi(i)|_{i=n(t)}$ is the value of the efficiency function evaluated at the latest technology.

We can summarize our results as concerns the optimality conditions in the following lemma 1.

Lemma 1 (Characterization of the optimality conditions)

For the inter-temporal optimization problem of the firm, given by the maximization of (1) subject to (2), the optimal solution is characterized as follows:

1. *The optimal controls for the problem are given by (7) and (8);*

2. *The dynamics of the shadow prices for the variety expansion and for the productivities of technologies are given by (9) and by (10), respectively.*

In addition, the limiting transversality conditions

$$\lim_{t\to\infty} e^{-rt}\lambda_n(t) = 0, \lim_{t\to\infty} e^{-rt}\lambda_q(i,t) = 0, \forall i \leq n(t), \tag{11}$$

must hold.

Given the necessary optimality conditions from lemma 1, we can derive the following proposition that states that the optimal policy of the firm is completely described by an autonomous system of differential equations.

Proposition 1 (Dynamics of the R&D firm)

The dynamics of the R&D firm are completely described by the following autonomous differential equation system,

$$\dot{u}(t) = (r+\delta)\,u(t) - \frac{\psi(i)^2}{2\,(r+\beta)^2}, \ \lim_{t\to\infty} e^{-rt}u(t) = 0; \tag{12}$$

$$\dot{n}(t) = u(t) - \delta\,n(t), \ n(0) = n_0; \tag{13}$$

$$\dot{q}(i,t) = \frac{\psi(i)^2}{r+\beta} - \beta\,q(i,t), \ q(i,0) = q_0(i). \tag{14}$$

The proof of that proposition follows from lemma 1 and is given in appendix A.

4   Multiple steady-states and thresholds

In this section we first present some general results and, then, report the outcome of a concrete numerical example in order to rigorously prove the results.

4.1 Analytical results

The closer analysis of the dynamic system describing the optimal R&D investment reveals that the presence of learning-by-doing with a convex-concave function leads to the possible occurence of multiple steady-states. Before we present the analysis in detail, we first define a steady-state for our model and, then, analyze under which conditions multiple steady-states can arise.

Definition 1 (Steady-state)
The steady-state of the model is characterized by the following conditions:

$$\dot{n}(t) = 0, \ \dot{\lambda}_n(t) = 0, \ \dot{q}(i,t) = 0, \ \dot{\lambda}_q(i,t) = 0, \ \forall i \in \boldsymbol{I}. \tag{15}$$

It should be noticed that, due to the form of the efficiency function $\psi(i)$, the levels of productivities of all existing technologies reach their respective steady-state values, too, as long as the system giving the evolution of the variety is in steady-state. That holds because $\dot{q}(i,t)$, equation (14), is a function of $n(t)$ and of $q(i,t)$, but $q(i,t)$ does not appear in the other two equations $\dot{u}(t)$ and $\dot{n}(t)$, equations (12) and (13), respectively. This implies that $\psi(i)$ is constant when n has reached its steady-state, i.e. for $\dot{n} = 0$. A constant function $\psi(i)$ gives $\dot{q} = \psi^2/(r+\beta) - \beta q$ and, since the first term is constant, q converges to its steady-state. Therefore, $\dot{u}(t)$ and $\dot{n}(t)$, (12) and (13), can be solved independently of $\dot{q}$ and these two equations determine the dynamics of the policy of the R&D firm.

Further, in steady-state the number of technologies produced has a positive effect on the productivity. This follows from $\dot{q} = 0$ which implies $\tilde{q}(\tilde{n}) = \psi(\tilde{n})^2/(\beta(r+\beta))$, where the ~ over a variable denotes its steady-state value. Since ψ is an increasing function of the variety of technologies, the quality of the latest technology, and also of the average technology, is the higher the more technologies are produced.

Analyzing equations (12) and (13) demonstrates that for certain parameter constellations multiple steady-states can arise. Proposition 2 gives the result.

Proposition 2 (Multiple steady-states)
For certain parameter constellations, the dynamic system (12)-(14) *is characterized by*

three distinct steady-states. The low steady-state and the high steady-state are saddle point stable with a small and large number of low- and high-quality technologies, respectively. The middle steady-state is an unstable focus.

This proposition demonstrates that multiple steady-states can be the result of the optimization problem of the R&D firm and it may be optimal to converge either to the high or to the low steady-state. Two properties of our model are necessary such that multiple steady-states can arise. First, the function $\psi(i)$ that determines the efficiency of quality improving R&D must depend on the number of technologies produced, which reflects the learning-by-doing effect in our framwork. Second, this learning-by-doing must be non-linear and display a convex-concave shape. Then, for certain parameter constellations, multiple steady-states can occur. Below, we will illustrate this rigorously with the help of a concrete numerical example. We will confine our presentation to one such example, but it is obvious that the multiplicity result is generic and there exist quite a many parameter constellations that generate this outcome.

The result of proposition 2 suggests that there exists a threshold[2] that separates the two basins of attraction so that it is optimal to converge to the low steady-state if the initial variety of technologies falls short of the critical value and to the high steady-state if the initial number of technologies exceeds the threshold. This outcome is stated as a corollary to proposition 2.

Corollary 1 (Threshold with respect to the number of technologies)
For certain parameter constellations, there exists a critical value of the initial technologies such that the dynamic system (12)-(14) converges to the low (high) steady-state when the initial number of technologies falls short of (exceeds) the threshold.

The proof of the corollary is given in appendix B. From an economic point of view, the existence of a threshold implies that, depending on the initial condition with respect to the variety of technologies, two different outcomes are feasible. If the firm starts with an intially high number of technologies that exceeds the threshold, it is optimal for the firm to choose an R&D policy such that it converges to the high steady-state with

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[2]Such a threshold is sometimes called Skiba point paying tribute to the seminal paper by (Skiba 1978).

relatively many technologies that are produced at a high quality. If, on the other hand, the initial variety is smaller than the critical value, but larger than the low steady-state number of technologies, it is optimal for the firm to invest only little in the creation of new technologies and in quality improvements, implying that the number of technologies further declines. Only if the initial number of technologies is smaller than the steady-state variety of technologies, the amount of R&D investment will be such that the number of technologies rises over time until it converges to the low steady-state. But, it should be underlined that the firm will not invest so much that it converges to the high stady-state with a large variety of technologies.

These considerations demonstrate that a so-called lock-in effect can result in our model, stating that the firm will find itself locked-in in a situation with only few technologies produced at a low quality. The latter occurs if the initial variety of technologies falls short of a certain critical value, showing that history matters. This means that it is of importance whether the firm had already invested successfully in innovations in the past such that it can start with a relativey large number of technologies that are produced. Then, any new project the management starts at a certain point in time, formally described by our control problem (1) subject to (2), will lead to an outcome where the firm will finally produce a large variety of technologies at a high quality.

4.2   A numerical example

Now, we present a numerical example in order to illustrate and to rigorously proof proposition 2 and corollary 1. To do so, we resort to the following parameter values given in table 1.

Table 1: Parameters values used in the numerical example.

Parameter	Value
r	0.035
δ, β	$0.22, 0.05$
a, b, d	$0.05, 3, 1.5$

With these parameter values, the steady-states are obtained as,

$$\tilde{u}_l = 0.002, \tilde{n}_l = 0.008, \tilde{g}_l = 0.03, \tilde{q}_l = 0.002; \tag{16}$$

$$\tilde{u}_m = 0.282, \tilde{n}_m = 1.281, \tilde{g}_m = 0.379, \tilde{q}_m = 0.244; \tag{17}$$

$$\tilde{u}_h = 0.623, \tilde{n}_h = 2.83, \tilde{g}_h = 0.563, \tilde{q}_h = 0.54; \tag{18}$$

where the index l, m, h denotes the values on the low, middle and high steady-state, respectively, and q gives the quality of the latest technology. The eigenvalues of the Jacobian matrix evaluated at the respective steady-state are as follows,

$$\mu_{1,l} = 0.25, \mu_{2,l} = -0.21; \quad \mu_{1,2,m} = 0.02 \pm 0.2\sqrt{-1}; \quad \mu_{1,h} = 0.21, \mu_{2,h} = -0.12. \tag{19}$$

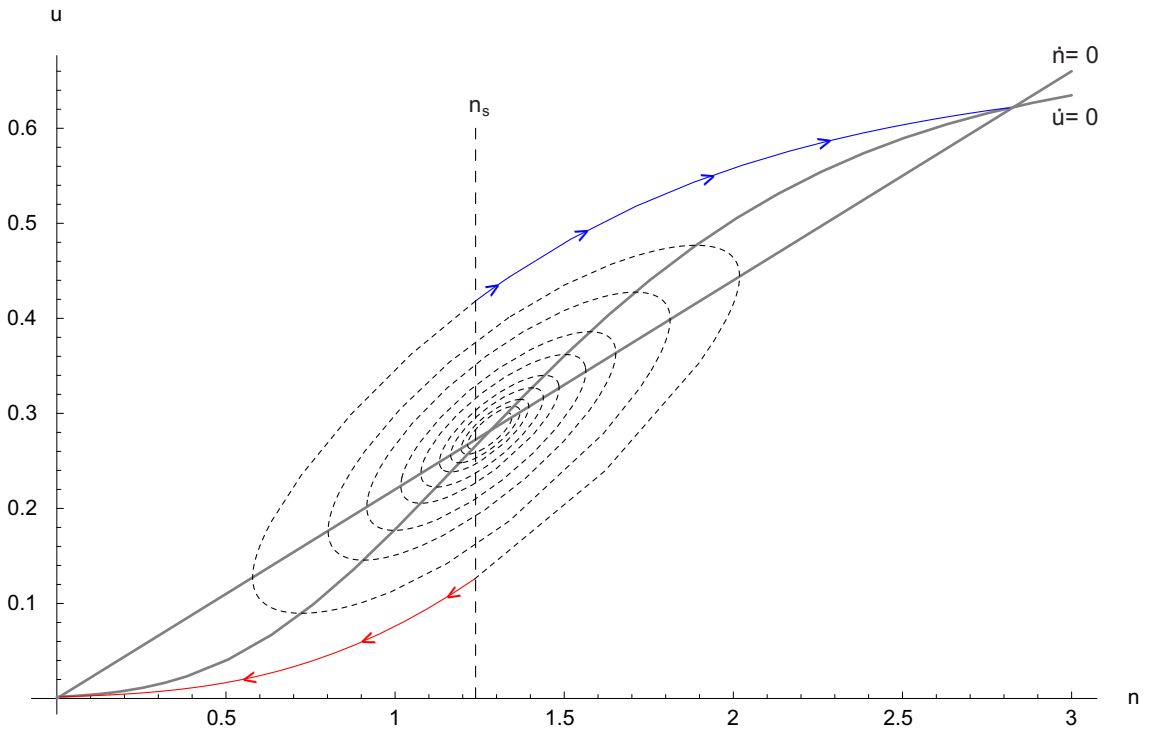The phase diagram for this parameter constellation is illustrated in figure 1.

Figure 1: Two saddle point stable steady-states and the threshold.

The dotted vertical line gives the threshold which takes the value $n_s = 1.238$ and the arrows indicate the stable manifolds of the two saddle points and they show to which

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steady-state u and n converge. If the firm starts to the left of the critical value n_s, convergence to the low steady-state, $\tilde{u}_l = 0.002, \tilde{n}_l = 0.008, \tilde{g}_l = 0.03, \tilde{q}_l = 0.002$, is optimal, while convergence to the high steady-state, $\tilde{u}_h = 0.623, \tilde{n}_h = 2.83, \tilde{g}_h = 0.563, \tilde{q}_h = 0.54$, is optimal when the initial variety of technologies exceeds the threshold.

5   Conclusion

In this paper we have analyzed the inter-temporal optimization problem of a representative firm that invests in R&D to generate both horizontal and vertical innovations. Assuming that non-linear learning-by-doing effects are present, we show that multiple steady-states can arise. In that case, the initial condition with respect to the variety of technologies is decisive as concerns the long-run outcome.

Thus, there exists a threshold with respect to the initial variety of technologies. This implies that the firm chooses its R&D policy in a way such that it converges to the low steady-state, with a small number of low-quality technologies, if the initial range of technologies falls short of the critical threshold and to the high steady-state, with a large number of high-quality technologies, if the initial variety of technologies exceeds the threshold. Hence, we can state that history matters, in the sense that it is of significance whether the firm had been successful in the past as concerns the outcome of a new project started by the management, where the project is formally described by the solution to an inter-temporal optimization problem subject to constraints as in our framework.

This result may have policy implications with respect to the R&D policy of the government. It is often of interest to identify those technologies which have to be first supported by subsidies and which can then be further developed by pure market forces. Our analysis shows that in those industries where learning-by-doing effects are substantial, firms may fall into the technology lock-in if the initial spectrum of technologies is not very high (low initial range). In this case, the set of potentially available technologies will not be developed by the firm, unless it receives some supporting subsidies from the government that switch the regime beyond the indifference point of the system leading to a more efficient level of innovations.

A   Proof of proposition 1

In (11) we posit that the standard transversality conditions for the finite time horizon T also hold for the infinite time horizon, with $\lim\limits_{t\to\infty}$ replacing $\lim\limits_{t\to T}$. For λ_q this implies:

$$\forall i \le n(t) : \lim_{t\to\infty} e^{-rt}\lambda_q(i,t) = 0. \tag{A.1}$$

Then, the co-state for each technology productivity can be obtained from (10) as:

$$\forall i \le n(t) : \lambda_q(i,t) = \frac{1 - e^{(r+\beta)(t-T)}}{(r+\beta)}, \tag{A.2}$$

which yields a constant shadow price in time for each technology for the infinite horizon case, i.e. for $T \to \infty$,

$$\forall i \le n(t) : \lambda_q(i,t)|_{T\to\infty} = \frac{1}{(r+\beta)}. \tag{A.3}$$

Using $\lambda_q(i,t) = 1/(r+\beta)$ gives $g(i,t) = \psi(i)/(r+\beta)$ and, thus, (14). The first two relations together with $\lambda_n(t) = u(t)$ yield equation (12).

B   Derivation of figure 1 and of the threshold

To draw the phase diagram in figure 1 we solved (12) and (13) by backward integration, where we started in an ϵ-environment around the two saddle point stable steady-states, with the initial values determined by the stable manifold of the linearized system. We stopped at $t = t_f$ and then solved (12) and (13) by forward integration, where we now took $u(t_f)$ and $n(t_f)$ as the starting values. Thus, we obtained the stable manifold of the two saddle point stable steady-states. To obtain the trajectories of the unstable foci, we took $u(t_f)$ and $n(t_f)$ as the starting values and solved (12) and (13) by backward integration giving the trajectories converging to the unstable steady-state.

To determine the threshold n_s we proceeded as follows. We know that the maximum of (1) is given by $H^0(u(0), n(0), q(i,0))/r$, with H^0 the maximized Hamiltonian. Further, the maximized Hamiltonian is convex in the control u. Therefore, for any $n(0) = n_0$ in between the low and the high steady-state, the optimal initial control, $u(0)$, lies on either

the lowest or highest branch of the spirals converging to the low or high steady-state. Then, we computed the difference $\Delta H^0 = H^0(u_1(0), n(0), \cdot)/r - H^0(u_2(0), n(0), \cdot)/r$, with $u_1(0)$ the initial value of u such that the system converges to the low steady-state, for a given $n(0)$, and $u_2(0)$ the initial value of u such that the system converges to the high steady-state, for the same $n(0)$. Doing so shows that $\Delta H^0 = 0$ for $n(0) = n_s = 1.238$. For $n(0) < (>) n_s$ we get $H^0(u_1(0), n(0), \cdot)/r > (<) H^0(u_2(0), n(0), \cdot)/r$ so that convergence to the low (high) steady-state is optimal.

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