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How to test for debt sustainability? Some theoretical reflections on an empirical test

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How to test for debt sustainability? Some theoretical reflections on an empirical test

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Abstract

Testing the reaction of the primary surplus to variations in public debt, relative to GDP respectively, has been frequently resorted to in order to test for sustainability of a given debt policy. In this contribution, we analyze theoretically under which condition a positive reaction of the primary surplus to variations in debt implies a sustainable debt policy. We demonstrate that the evolution of the debt to GDP ratio plays a decisive role as concerns the validity of the test outcome. In addition, we demonstrate that a positive reaction coefficient does not guarantee sustainability unless it exceeds at least the difference between the interest rate on public debt and the GDP growth grate. Thus, this test allows judgements about sustainability of public debt policies only if the interest rate and the GDP growth rate are known.

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1 Introduction

Modern research on sustainability of debt policies that applies statistical tests has started with the contribution by Hamilton and Falvin (1986) who analyzed whether the series of public debt in the USA contains a bubble term. Since then, a great many papers have been written that try to answer the question of whether public debt policies can be considered as sustainable. That question is not only of academic interest but it has also a high practical relevance. Hence, if tests reach the conclusion that given debt policies cannot be considered as sustainable, governments should undertake corrective actions in order to prevent bankruptcy that goes along with hugh economic and social costs.

An important role in many of the studies on sustainability plays the interest rate, an aspect that was pointed out by Wilcox (1989) for example. Considering that the intertemporal budget constraint of the government requires that the present value of public debt asymptotically converges to zero, the role of the interest rate that is used in order to discount the stream of public debt becomes immediately clear. However, future interest rates are unknown. Therefore, tests have been conceived that reach results which are independent of the interest rate. One such test is to analyze whether public deficits inclusive of interest payments grow at most linearly, as suggested by Trehan and Walsh (1991). If that property is holds, a given series of public debt is sustainable because any time series that grows linearly converges to zero if it is discounted exponentially, provided the real interest rate is positive. Another test proposed by Trehan and Walsh (1991) is to analyze whether a quasi-difference of public debt is stationary and whether public debt and primary surpluses are co-integrated. If the government debt is quasi-difference stationary and public debt and primary surpluses are cointegrated, public debt is sustainable. Hence, these two tests present alternatives where the outcome is independent of the exact numerical value of the interest rate. A survey of analyses that have tested for sustainability of debt policies can be found in Afonso (2005) or in Neck and Sturm (2008).

Another test that has received great attention in the economics literature is the one proposed by Bohn (1995, 1998). There, it is suggested to test whether the primary surplus relative to GDP is a positive and at least linearly rising function of the debt to GDP ratio. If that property holds, the ratio of public debt relative to GDP should become a mean-reverting process so that the debt to GDP ratio remains bounded. This test is very plausible because it has a nice economic intuition: if governments run into debt today they have to take corrective actions in the future by increasing the primary surplus. Otherwise, public debt will not be sustainable. Testing real world public debt policies for that property, one can indeed find evidence that countries behave like that (see for example Bohn, 1998, for the USA and Ballabriga and Martinez-Mongay, 2005, Greiner et al., 2007, or Fincke and Greiner, 2008, for selected countries of the euro area).

In the rest of the paper we proceed as follows. In the next section we present the basic equations that govern the accumulation of public debt and the assumptions we make in analyzing which policies guarantee a sustainable debt policy. In section 3, we demonstrate how governments have to set primary surpluses such that public debt remains sustainable

and section 4, finally, concludes the paper.

2 Public debt, the debt to GDP ratio and the primary surplus

The evolution of the level of outstanding public debt in continuous time is governed by the following equation

$$\dot{B}(t) = r(t)B(t) - T(t) + G(t) = r(t)B(t) - S(t), \tag{1}$$

with $B(t)$ the level of public debt at time t, $r(t)$ the interest rate on outstanding debt, $T(t)$ and $G(t)$ give the tax revenue and public spending, respectively, and $S(t)$ is the primary surplus.[1]

The primary surplus can be controlled by the government by setting the taxes and by determining public spending. There do exist types of spending that are beyond the control of the government and, in addition, the government cannot set tax rates arbitrarily. Nevertheless, the government disposes of at least some discretionary scope to set tax rates and public spending.

An important variable in determining sustainable debt policies is played by the debt to GDP ratio, $b := B/Y$, with Y denoting the GDP. The debt to GDP ratio evolves according to

$$\dot{b} = b(r - g - s/b), \tag{2}$$

where g denotes the growth rate of GDP and $s := S/Y$ is the primary surplus relative to GDP.

In our analysis of sustainable public debt policies we make the following two assumptions:[2]

- Assumption 1: $0 < \Delta_{rg} < \infty$, with $\Delta_{rg} := \bar{r} - \bar{g}$.
- Assumption 2:

  $$s(t) = f(b(t), Z(t)) = \beta(t)b(t) + \alpha(t) = \bar{\beta}\, b(t) + \bar{\alpha} \leq s_m,$$

  with $\bar{\beta} \in \mathbb{R}_{++}$, $\bar{\alpha} \in \mathbb{R}$ and $s_m < 1$.

Our Assumption 1 states that the difference between the interest rate on public debt and the GDP growth rate is strictly positive. It should be noted that this inequality does not have to hold in each period but only on average. That is, this difference may be negative for some time, however, on average it is positive over the whole time period. That assumption is made because sustainability would not pose a problem if the difference was negative. That holds since the government can grow out of debt when the GDP growth

[1] In the following, we delete the time argument t as long as no ambiguity arises.

[2] The bar over a coefficient ($^-$) denotes average values.

rate exceeds the interest rate on public debt, meaning that it does not have to run primary surpluses in order to pursue a sustainable debt policy.

Assumption 2 posits that the primary surplus relative to GDP is a function that positively depends on the debt to GDP ratio and on other variables in the economy, summarized by the vector $Z(t)$. Since any non-linear function can be written as a linear function with time-varying coefficients (cf. Granger, 2008) this function can be written in a linear form with $\beta(t)$ giving the reaction of the primary surplus to variations in public debt, relative to GDP respectively. Further, with a time-varying reaction coefficient, it is sufficient to consider the average value of that coefficient when analyzing sustainability (see Greiner, 2011).

Finally, it should be noted that the primary surplus relative to GDP is bounded from above, where s_m denotes the upper bound. It must also be pointed out that this is not an assumption but rather a consequence of national accounting. Since all expenditures of an economy are financed ouf of GDP, they cannot exceed the latter. Assume for the moment that public spending is equal to zero. Then, the primary surplus of the government is identical to its total tax revenue and it becomes immediately clear that taxes cannot exceed GDP so that the primary surplus is always smaller than GDP.

Given Assumption 2, the differential equations describing the evolution of public debt and of the debt to GDP ratio can be rewritten as

$$\begin{aligned} \dot{B} &= (\bar{r} - \bar{\beta})B - \bar{\alpha}\, Y \qquad (3) \\ \dot{b} &= b\left(\bar{r} - \bar{g} - \bar{\beta}\right) - \bar{\alpha}. \qquad (4) \end{aligned}$$

In the next section, we analyze under which conditions Assumption 2 guarantees a sustainable debt policy and how the debt to GDP ratio evolves in this case.

3 Sustainable public debt

In this section we want to analyze under which conditions a positive reaction coefficient $\bar{\beta}$ guarantees that public debt remains sustainable. But, before we define a sustainable debt policy in Definition 1

Definition 1 *A sustainable debt policy is defined as a policy such that*

$$\lim_{t\to\infty} e^{-rt}B(t) = 0 \leftrightarrow B(0) = \int_0^\infty e^{-rt}S(t)dt\ .$$

According to Definition 1 a sustainable debt policy is, as usual, a debt policy such that the present value of the public debt converges to zero asymptotically. The latter is equivalent to the requirement that the level of the outstanding public debt equals the sum of the present value of future primary surpluses.

Before we analyze under which conditions our Assumption 2 guarantees that public debt remains sustainable, we first state that sustainability can never be achieved if the ratio of public debt relative to GDP exceeds a certain threshold. This is the contents

of Proposition 1 that states that there exists an upper bound for the debt to GDP ratio beyond which a sustainable debt policy is excluded.

Proposition 1 *A sustainable debt policy is excluded if the debt to GDP ratio exceeds the critical value b_{crit}, with $b_{crit} = s_m \int_0^\infty e^{-(\Delta_{rg})\cdot t} dt$.*

Proof: When the government sets the primary surplus to its maximum value, the evolution of the public debt is given by

$$\dot{B} = \bar{r}B - s_m Y.$$

Solving that equation and multiplying both sides by $e^{-\bar{r}t}$, the present value of public debt is obtained as

$$e^{-\bar{r}t}B(t) = B(0) - s_m Y(0) \int_0^t e^{-(\bar{r}-\bar{g})\nu} d\nu.$$

The intertemporal budget constraint is fulfilled for $\lim_{t\to\infty} e^{-\bar{r}t}B(t) = 0$ which implies $b(0) = s_m \int_0^\infty e^{-(\bar{r}-\bar{g})\nu} d\nu$. If the initial debt to GDP ratio, $b(0)$, exceeds $s_m \int_0^\infty e^{-(\bar{r}-\bar{g})\nu} d\nu$ sustainability of public debt is excluded. □

Proposition 1 is an immediate consequence of the fact that the primary surplus relative to GDP is bounded from above. Hence, once public debt relative to GDP becomes larger than the critical threshold, b_{crit}, governments will not be able to pay back public debt. The critical ratio of public debt relative to GDP is the larger, the higher the maximally achievable primary surplus and the smaller the difference between the interest rate and the GDP growth rate. It should be noted that we assume that neither the interest rate nor the growth rate of GDP depend on the level of public debt that is rather questionable for real world economies. Nevertheless, Proposition 1 would remain valid even if that assumption was given up, as long as the difference between the interest rate and the GDP growth rate is positive and finite.

An immediate consequence of Proposition 1 is that a permanently rising debt to GDP ratio is not compatible with a sustainable debt policy. This is stated in the following Corollary 1.

Corollary 1 *A permanently rising debt to GDP ratio is not compatible with a sustainable debt policy.*

Proof: Follows immediately from Proposition 1. □

The significance of Corollary 1 becomes clear when one considers that the public debt to GDP ratio may rise even if the reaction coefficient $\bar{\beta}$ is strictly positive. Then, the government reacts to higher public debt by increasing its primary surplus, but, the reaction is not sufficiently strong to stabilize the debt to GDP ratio. This leads to an ever increasing debt ratio, a policy that is clearly unsustainable. The following proposition states that in detail.

Proposition 2 *If the government pursues a debt policy such that $0 < \bar{\beta} < \Delta_{rg}$, public debt becomes unsustainable.*

Proof: Solving equation (4) gives

$$b(t) = \left(b(0) - \frac{\bar{\alpha}}{\Delta_{rg} - \bar{\beta}} \right) e^{(\Delta_{rg} - \bar{\beta})t} + \frac{\bar{\alpha}}{\Delta_{rg} - \bar{\beta}} \, .$$

For $0 < \bar{\beta} < \Delta_{rg}$ the debt to GDP ratio diverges to plus or minus infinity. □

Proposition 2 demonstrates that public debt is unsustainable even if the reaction coefficient is positive but smaller than the difference between the interest rate and the GDP growth rate. The reason is that such a policy implies a permanently rising debt to GDP ratio such that the critical debt to GDP ratio is reached sooner or later beyond which a sustainable public debt policy is excluded. A permanently rising debt to GDP ratio would require a permanently increasing primary surplus to GDP ratio for sustainability which, however, is excluded because the primary surplus relative to GDP is bounded from above. Hence, a positive reaction coefficient does not guarantee sustainability of public debt. But even a positive reaction coefficient that exceeds the difference between interest rate and the growth rate of GDP may not be sufficient to guarantee sustainability.

To see this, we first compute the limit of the debt to GDP ratio for the case when the reaction coefficient exceeds the difference between interest rate and the growth rate. The result is stated as Lemma 1.

Lemma 1 *For $\bar{\beta} > \Delta_{rg} > 0$ the limit of the debt to GDP ratio is given by the expression $b^\star := \lim_{t \to \infty} b(t) = \bar{\alpha}/(\Delta_{rg} - \bar{\beta})$.*

Proof: Follows immediately from the proof of Proposition 2. □

Lemma 1 gives the limit of the debt to GDP ratio when the government chooses a debt policy with a positive reaction coefficient that exceeds the difference between the interest rate and the GDP growth rate. However, that policy is only feasible if that limit is smaller than the critical debt ratio given in Proposition 1. If the limit, $b^\star$, exceeds the critical debt to GDP ratio, b_{crit}, public debt becomes unsustainable once the actual debt to GDP ratio becomes larger than the critical debt ratio. This holds because in this case the upper bound of the debt to GDP ratio becomes binding before convergence has been achieved.

Since both the critical debt to GDP ratio, b_{crit}, as well as the limit of the debt to GDP ratio, $b^\star$, are determined endogenously, it is possible to formulate a condition with respect to the parameters so that the public debt policy of the government is sustainable. This is the contents of Proposition 3.

Proposition 3 *For $\bar{\alpha} < 0$, sustainability is given if and only if $\bar{\beta} > \Delta_{rg}(1 - \bar{\alpha}/s_m)$ holds. For $\bar{\alpha} > 0$, sustainability is given if and only if $\bar{\beta} > \Delta_{rg}$ holds.*

Proof: For $\bar{\alpha} < 0$, sustainability is given if and only if $b^\star < b_{crit}$ which is equivalent to $\alpha \Delta_{rg} > s_m(\Delta_{rg} - \bar{\beta})$ which gives $\bar{\beta} > \Delta_{rg}(1 - \bar{\alpha}/s_m)$. For $\bar{\alpha} > 0$, the inequality $b^\star < b_{crit}$ always holds so that it must be only assured that the debt to GDP ratio converges to a finite value, which is given for $\bar{\beta} > \Delta_{rg}$. □

Proposition 3 shows that the reaction coefficient that guarantees sustainability of public debt depends on the value of $\bar{\alpha}$ that reflects the discretionary scope of the government. For $\bar{\alpha} < 0$, that is when the government reduces the level of the primary surplus as GDP rises, the reaction coefficient must exceed the difference between the interest rate and the growth rate multiplied by a factor larger than one. That makes sense from an economic point of view because a negative value of $\bar{\alpha}$ implies that governments do not care much about public debt and tend to raise debt with a higher GDP so that sustainability can only be achieved for a larger value of the reaction coefficient $\bar{\beta}$. If the government raises its primary surplus as GDP grows, i.e. for $\bar{\alpha} > 0$, the reaction coefficient must only be larger than the difference between the interest rate and the GDP growth rate because, otherwise, the debt to GDP would become unbounded, thus, excluding a sustainable debt policy, as shown in Proposition 2.

4 Conclusion

Testing how the primary surplus relative to GDP reacts, as the debt to GDP ratio rises, is a powerful test that yields valuable insight concerning the fiscal behaviour of governments. However, care must be taken in drawing conclusions with respect to the sustainability of a given time series of public debt. In particular, a positive reaction coefficient alone does not guarantee that the debt policy of a government is sustainable.

If the reaction coefficient is positive but smaller than the difference between the interest rate and the GDP growth rate, the debt to GDP ratio becomes unbounded asymptotically, thus, violating the inter-temporal budget constraint of the government. That holds beause an unbounded debt to GDP ratio is not compatible with a sustainable debt policy since it would require an unbounded primary surplus to GDP ratio that, however, is excluded because the primary surplus cannot exceed the GDP of a country.

In case the reaction coefficient exceeds the difference between the interest rate and the GDP growth rate, the debt to GDP ratio converges to a finite value. Nevertheless, the limiting value, to which the debt ratio converges, may be larger than the critical debt to GDP ratio, beyond which sustainability is excluded. This implies that even such a debt policy is not sustainable. The major shortcoming of this test is that it does not take into account that the ability of governments to achieve primary surpluses is limited so that the primary surplus to GDP ratio is bounded from above.

Hence, testing the reaction of the primary surplus to variations in public debt gives important information about the debt policy of governments and can, under certain conditions, also be used to make statements about sustainability of public debt. Nevertheless, that test should be accompanied by additional tests, such as testing for the cointegration of public spending and revenues, for example. This holds, in particular, when the debt to GDP ratio is characterized by a rising trend over time as in most European countries.

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