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On a Useful Characterization of Nash Equilibria in Decision Trees

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Dedicated to Prof. Dr. em. Otto Moeschlin
on the occasion of his 80th birthday

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Abstract

The concept of subgame perfect equilibrium is broadly accepted in the theory of non-cooperative games in extensive form and offers a method of equilibrium calculation called backward induction. Nevertheless, there often exist many other equilibria, which might also be interesting, because they require some coordination by groups of players. The most frequently cited literature does not provide an effective mechanism to identify all equilibria of games in extensive form. The present paper gives an answer to this problem. First, a general characterization of equilibrium is derived. This result offers a way to specify an algorithm, which lists all paths to terminal points of the game arising from equilibrium strategies. Finally, an application of the results to the decision problem of the Cuban Missile Crisis will be discussed.

Keywords: *Game Theory, Decision Tree, Extensive Form Game, Subgame Perfectness, Backward Induction, Nash Equilibrium Strategies, Entangled Subgame, Equilibrium Path, Equilibrium Identification Algorithm*

1 Problem

Sequential Games have been a matter of discussions from various points of view. The original idea of E. Zermelo ([5]) to formulate sequential games as decision trees has gained in importance after J. F. Nash proposed his classical solution concept of equilibrium points for non-cooperative games ([1]). R. Selten (see e.g. [3]) found that not all Nash equilibria for extensive form games with complete information are of the same quality and accordingly introduced the concept of subgame perfect equilibrium in 1965. His valuable contributions to game theory have been honored by the Nobel price in 1994.

Game trees have the nice property that algorithms can easily step through and find Nash equilibria of the underlying game. The most frequently applied algorithm is backward induction, which starts from the leaves of the tree and solves all subgames successively until the root node is reached. This algorithm assigns strategies to all players which turn out to form a subgame perfect Nash equilibrium of the game. Moreover, this approach provides a method to proof the existence of at least one subgame perfect Nash equilibrium under relatively weak assumptions.

The concept of subgame perfect equilibrium is broadly accepted by game theorists as far as players have complete information about everybody's preferences and the stages of the game. In the case of strict preferences, there exists exactly one subgame perfect equilibrium. Nevertheless, even in this case there may be many other equilibria, which are not subgame perfect, but may also provide a plausible course of the game.

If we are interested in an algorithm, which identifies all equilibria of the game, we may check all strategy combinations of players and list all those which satisfy the equilibrium conditions. This approach is very time consuming and needs a lot of computing power even for trees of rather small size. Therefore, it seems to be reasonable to search for more efficient algorithms to solve this problem.

For this purpose, we first give a characterization of equilibrium paths in the tree. Considering the paths to equilibrium the number of checks for equilibrium conditions increases only with the number of terminal points of the tree and not with all combinations of decisions made by the players. On the one hand this characterization gives some insight how the other players can force a decision maker to follow a given path, on the other hand it offers a way to specify an algorithm which allows to identify paths to equilibrium. The algorithm draws on the Nash solution of certain zero-sum subgames starting from the decision points of the path, where the decision maker in each specific point on the path maximizes his own outcome in the subgames against the coalition of all other players.

2 Model

We assume to have a sequential game represented by a decision tree with finite horizon and complete information for all players. Such a game consists of a finite set $P := \{1, \ldots, n\}$ of players and a finite tree $G(V, E)$ of nodes V and edges E. The successors of nodes $v \in V$ will be denoted by $N(v)$ and the terminal nodes of the tree will be denoted by V^*. Accordingly, $V^0 := V \setminus V^*$ is the set of interior nodes of the tree. Each player i is assumed to have some transitive and complete preferences $\preceq_i$ on the set V^* of terminal nodes. At all interior points $v \in V^0$ of

V, a certain player $\pi(v)$ is invoked to make a decision, to which of the followers $N(v)$ of v in the tree he would like to move. The complete plan of all decisions of player i in the tree is called a strategy of player i. The set of all strategies of player i is denoted by Σ_i. As soon as all players have chosen a certain strategy, the corresponding moves starting from a node $v \in V^0$ in the game tree form a path $\omega(v, \sigma_1, \ldots, \sigma_n)$ for given strategies $\sigma_1 \in \Sigma_1, \ldots, \sigma_n \in \Sigma_n$, which ends up in a specific terminal node $\omega^*(v, \sigma_1, \ldots, \sigma_n)$ of the tree. If v is the root node of the decision tree, we simply write $\omega(\sigma_1, \ldots, \sigma_n)$ and $\omega^*(\sigma_1, \ldots, \sigma_n)$, respectively. With these notations we can formulate the equilibrium conditions for the game.

2.1 Definition: *Given the described sequential game, we call each strategy combination $\sigma_1 \in \Sigma_1, \ldots, \sigma_n \in \Sigma_n$ which satisfies*

$$\omega^*(\sigma_1, \ldots, \sigma_{i-1}, \tau_i, \sigma_{i+1}, \ldots, \sigma_n) \preceq_i \omega^*(\sigma_1, \ldots, \sigma_n) \qquad \forall \tau_i \in \Sigma_i, \forall i \in P, \tag{1}$$

a ***Nash equilibrium*** *of the game.*

This is the standard definition of J. F. Nash for equilibrium. It means that each player choses a best response to the plans of all opposing players. The same definition applies to subgames with any root node $v \in V^0$ of the given game, if we confine strategies of players to the corresponding subtree $G^v(V, E)$ of the entire tree and denote it by Σ_i^v. This consideration gives us an opportunity to talk about subgame perfectness of Nash equilibria.

2.2 Definition: *For each player i and strategy $\sigma_i \in \Sigma_i$ we denote the restriction of σ_i to the subgame with root node $v \in V^0$ by σ_i^v. We call each strategy combination $\sigma_1 \in \Sigma_1, \ldots, \sigma_n \in \Sigma_n$ which satisfies*

$$\omega^*(v, \sigma_1^v, \ldots, \sigma_{i-1}^v, \tau_i, \sigma_{i+1}^v, \ldots, \sigma_n^v) \preceq_i \omega^*(v, \sigma_1^v, \ldots, \sigma_n^v) \qquad \forall \tau_i \in \Sigma_i^v, \forall i \in P, \forall v \in V^0, \tag{2}$$

a ***subgame perfect Nash equilibrium*** *of the game.*

Subgame perfectness is a widely accepted concept of equilibrium for decision trees and offers a method to calculate a subgame perfect strategy combination by backward induction starting with the terminal nodes and solving successively all subgames. Of course, backward induction is a bottom up approach, which considers the game upside down in terms of timing of decisions. Essentially, the common understanding of the game is top down decision making starting from the root node. For this process to end up successfully for the players, rational behavior of the players is necessary in each step. Nevertheless, simple examples exist, where this decision process may fail, if some player's preferences have indifference on terminal nodes. The assumption of strict preferences of players ensures that such defects cannot occur. Players have strict preferences, whenever

$$v \preceq_i w \text{ and } w \preceq_i v \Rightarrow v = w \tag{3}$$

holds for all $v, w \in V^*$ and $i \in P$. Moreover, for strict preferences a unique subgame perfect equililibrium point exists. This fact is a good justification for the concept of subgame perfectness.

But what may happen, if the decision process is performed in bottom up direction? There may arise an opportunity for players to cooperate and force other players to make decisions in advantage of the coalition. This can be illustrated by an example in the following remarks. Moreover, equilibrium points of this type turn out to be not necessarily subgame perfect. The criticism on such equilibria concerns incredible threats involved in the strategies of players. This objection is of course correct for top down decision processes. As we will show, for bottom up processes it does not apply in the same rigor. Therefore, there exists some justification to analyze the set of all equilibria regardless of their subgame perfectness.

3 Entangled Subgames

The following simple example with three players "chess bishop" , "chess king" and "chess knight" illustrates the problem of top down and bottom up decision processes. [1]

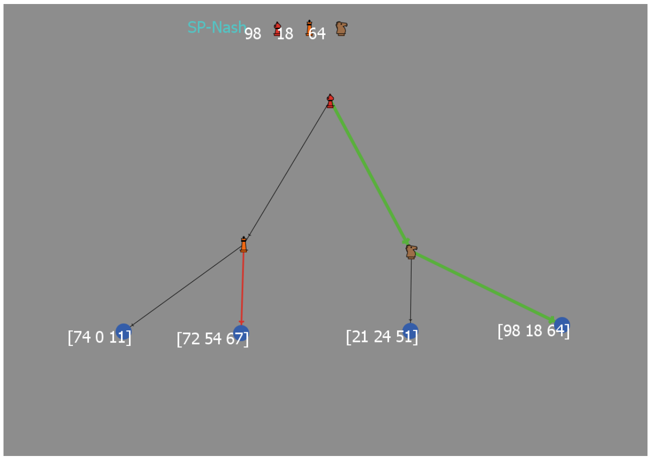

Figure 1: Equilibrium calculation top down

The unique subgame perfect equilibrium path is drawn in green color (Figure 1), the decision path in the left subgame is colored red. In this case, player

[1]All graphics has been produced by a model using the open source environment NetLogo ([4])

“chess bishop” will prefer to move along the subgame perfect equilibrium path. But there exists a second green colored equilibrium path (Figure 2). Players “chess knight” and “chess king” will prefer the terminal node with payoffs $[72, 54, 67]$ to the terminal node of the subgame perfect equilibrium path with payoffs $[98, 18, 64]$. Therefore, in the bottom up decision process player “chess knight” will play the terminal node with payoff $[21, 24, 51]$ and keep the “chess bishop” from handing the decision over to him. This kind of equilibrium requires some coordination between the “chess king” and the “chess knight”, but results in a better payoff for both. One may argue if player “chess bishop” starts with the decision path as in Figure 1 then there is no better way for the other players than to decide in the sense of the subgame perfect solution. But this argumentation draws on the top down decision sequence.

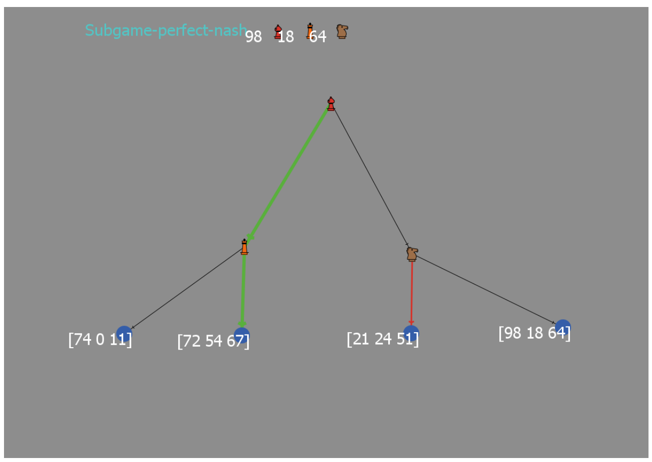

Figure 2: Equilibrium calculation bottom up

As a preparation for a more general analysis of equilibria we will first introduce a concept which will turn out to be helpful for a characterization of equilibrium paths.

3.1 Remark: Given some node $v \in V^0$, the preferences in the terminal nodes of the subgame with root node v will be defined by the original preference relation $\preceq_i$ for a given coalition $C \subset P$ and some key player $i \in C$. For all players in $P \setminus C$ the preferences will be inverse to those of i, i.e.

$$\begin{aligned} v \preceq_i w &\iff v \preceq_j w \;\; \forall j \in C, \\ v \preceq_i w &\iff w \preceq_j v \;\; \forall j \in P \setminus C. \end{aligned} \tag{4}$$

Since all players of C as well as all players of $P \setminus C$ have each the same preferences, the game can be considered as a two person game of C against $P \setminus C$. An equilibrium of this game with root node v is therefore a strategy combination $\sigma_1 \in \Sigma_1^v, \dots, \sigma_n \in \Sigma_n^v$ satisfying [2]

$$
\begin{aligned}
\omega^*(v, \tau_C, \sigma_{P\setminus C}) &\preceq_i \omega^*(v, \sigma_C, \sigma_{P\setminus C}) \quad \forall \tau_j \in \Sigma_j^v,\ \ j \in C \\
\omega^*(v, \sigma_C, \sigma_{P\setminus C}) &\preceq_i \omega^*(v, \sigma_C, \tau_{P\setminus C}) \quad \forall \tau_j \in \Sigma_j^v,\ \ j \in P \setminus C
\end{aligned}
\tag{5}
$$

Since these particularly modified subgames will become useful for an analysis of the set of all equilibria of the original game, we will address them by a separate notation.

3.2 Definition: *The resulting game will be called the* ***entangled subgame*** *of the original subgame with root node v, coalition C and key player $i \in C$. We will use the symbol $\Gamma(v, i, C)$ to address this game.*

Entangled subgames with root node v are games of a coalition C against the rest of the world, because players from C maximize their outcome with respect to the preferences of the key player against the strategies of the other players, who aim for minimizing this outcome. In case of numerical payoffs the equilibria of this (zero-sum) game are therefore maximin solutions. At least one (even subgame perfect) equilibrium exists, since backward induction also applies to this kind of game. Entangled subgames have some properties, which are immediate consequences of the equilibrium conditions.

3.3 Remarks:

1. For all equilibria of an entangled subgame with root node v, coalition C and key player $i \in C$ the corresponding paths in the subgame end up in equivalent terminal nodes. This is a well-known fact for zero-sum games and applies in the same way to games with complete and transitive preferences of two opposing players. Thus, the terminal nodes of equilibrium strategies specify a characteristic value of the entangled subgame $\Gamma(v, i, C)$. This value is given by $\phi(\Gamma(v, i, C)) := \omega^*(v, \sigma_1, \dots, \sigma_n)$, whenever the strategy combination $\sigma_1 \in \Sigma_1^v, \dots, \sigma_n \in \Sigma_n^v$ satisfies the equilibrium conditions of Remark 3.1 (5).

2. For two entangled subgames $\Gamma(v, i, C)$ and $\Gamma(v, i, D)$ with $i \in C \subset D$ the corresponding characteristic values satisfy

$$\phi(\Gamma(v, i, C)) \preceq_i \phi(\Gamma(v, i, D)) \tag{6}$$

   To verify this relation, let $\sigma := (\sigma_1, \dots, \sigma_n)$ be an equilibrium of $\Gamma(v, i, C)$ and let $\tau := (\tau_1, \dots, \tau_n)$ be an equilibrium of $\Gamma(v, i, D)$. Then, from the equilibrium conditions of Remark 3.1 (5) and $C \subset D$, we have

$$\phi(\Gamma(v, i, C)) = \omega^*(v, \sigma) \preceq_i \omega^*(v, \sigma_C, \tau_{P\setminus C}) \preceq_i \tag{7}$$
$$\preceq_i \omega^*(v, \tau_C, \tau_{P\setminus C}) = \omega^*(v, \tau) = \phi(\Gamma(v, i, D)). \tag{8}$$

[2] we use the notation $(\sigma_S, \tau_{P\setminus S})$ for the strategy combination with players $j \in S$ chosing their strategy $\sigma_j \in \Sigma_j$ and players $j \in P \setminus S$ chosing their strategy $\tau_j \in \Sigma_j$ for given $S \subset P$.

Since players in the coalition C and D can be considered as groups of supporters of player i in the entangled subgame, the above relation shows that more supporters improve the outcome for player i in this game.

3. The previous remark shows that the value $\phi(\Gamma(v,i,C))$ of an entangled subgame $\Gamma(v,i,C)$ always dominates $\phi(\Gamma(v,i,\{i\})$. Therefore, $C := \{i\}$ is the worst case for player i, which occurs, whenever nobody supports him in the entangled subgame.

4 A Characterization of Equilibrium Paths

Starting from a given terminal node of the decision tree, graph theory guarantees a unique path back to the root of the tree. We will investigate those paths more accurately, which belong to equilibrium strategies of the community of players. Since the decisions of players on the path are fixed, we are interested in appropriate decisions of all players at nodes contiguous to the path, which do not tempt the decision makers on the path to deviate from the path. In this sense, the question is, whether the partially defined strategies on the path can be extended in such a way that they form a Nash equilibrium. All paths of this type will be called **equilibrium paths**. First, we make a note of the following fact about equilibrium paths.

4.1 Lemma: *Let be given a path $\rho = (v_0, \ldots, v_k)$ from the root node v_0 to a terminal node v_k of the tree together with strategies $\sigma_1, \ldots, \sigma_n$ of all players satisfying*

$$\sigma_{\pi(v_j)}(v_j) = v_{j+1} \quad (j = 0, \ldots, k-1) \tag{9}$$

as well as

$$\omega^*(w, \tau_{\{i\}}, \sigma_{P\setminus\{i\}}) \preceq_i v_k \tag{10}$$
$$(w \in N(v_j) \setminus \{v_{j+1}\}, i := \pi(v_j), \tau_i \in \Sigma_i^w, j = 0, \ldots, k-1),$$

then ρ is an equilibrium path and $\sigma_1, \ldots, \sigma_n$ is a Nash equilibrium.

Proof. Suppose, there exists a player i and a strategy $\tau_i \in \Sigma_i$ with the property

$$v_k \prec_i \omega^*(\tau_{\{i\}}, \sigma_{P\setminus\{i\}}). \tag{11}$$

Then, since all players from $P \setminus \{i\}$ retain their strategies, there must exist a node v_j on the path ρ, where player i deviates from the path ρ, while all previous moves remain on the path. Therefore, there exists $w \in N(v_j) \setminus \{v_{j+1}\}$ with

$$\omega^*(\tau_{\{i\}}, \sigma_{P\setminus\{i\}}) = \omega^*(w, \tau_{\{i\}}, \sigma_{P\setminus\{i\}}). \tag{12}$$

But (11) and (12) contradict assumption (10). Together with (9) we have proven that there cannot exist any player i and any strategy $\tau_i \in \Sigma_i$ with

$$\omega^*(\sigma_1, \ldots, \sigma_n) = v_k \prec_i \omega^*(\tau_{\{i\}}, \sigma_{P\setminus\{i\}}). \tag{13}$$

Hence $\sigma_1, \ldots, \sigma_n$ is a Nash equilibrium and ρ is an equilibrium path. □

The following result provides a complete characterization of equilibrium paths.

4.2 Theorem: *A given path $\rho = (v_0, \ldots, v_k)$ from the root node v_0 to a terminal node v_k of the tree is an equilibrium path if and only if for all nodes v_j on the path and all entangled subgames with root node $w \in N(v_j) \setminus \{v_{j+1}\}$ and key player $i := \pi(v_j)$ the entanglement condition*

$$\phi(\Gamma(w, i, \{i\})) \preceq_i v_k \tag{14}$$

is satisfied.

Proof. Let $\rho = (v_0, \ldots, v_k)$ be any path from the root node v_0 to a terminal node v_k of the tree. If ρ is an equilibrium path of the game, there exist equilibrium strategies $\sigma_1, \ldots, \sigma_n$, for which at each node v_j the decision of player $\pi(v_j)$ invoked at v_j is v_{j+1}, and hence

$$v_k = \omega^*(\sigma_1, \ldots, \sigma_n). \tag{15}$$

Now, let be given a node v_j on the path ρ and consider an equilibrium $\xi_1, \ldots, \xi_n$ of the entangled subgame $\Gamma(w, \pi(v_j), \{\pi(v_j)\})$ for a given root node $w \in N(v_j) \setminus \{v_{j+1}\}$. Since all players $j \neq i := \pi(v_j)$ tend to downgrade the outcome of player i in the entangled subgame, we get

$$\phi(\Gamma(w, i, \{i\})) = \omega^*(w, \xi_1, \ldots, \xi_n) \preceq_i \omega^*(w, \tau_1, \ldots, \tau_{i-1}, \xi_i, \tau_{i+1}, \ldots, \tau_n) \quad \forall \tau_j \in \Sigma_j^w, \;\; j \in P \setminus \{i\}, \tag{16}$$

and therefore particularly

$$\phi(\Gamma(w, i, \{i\})) \preceq_i \omega^*(w, \sigma_1, \ldots, \sigma_{i-1}, \xi_i, \sigma_{i+1}, \ldots, \sigma_n). \tag{17}$$

Since $\sigma_1, \ldots, \sigma_n$ is an equilibrium of the entire game, the deviation of player i at node v_j from the path ρ to node w will not improve his outcome as long as the other players keep their strategies. Therefore,

$$\omega^*(w, \sigma_1, \ldots, \sigma_{i-1}, \xi_i, \sigma_{i+1}, \ldots, \sigma_n) \preceq_i \omega^*(\sigma_1, \ldots, \sigma_n) \tag{18}$$

is satisfied. Relations (15), (17) and (18) together with the transitivity of preferences, complete the $'' \Longrightarrow ''$ part of the theorem.

Let now the entanglement condition be satisfied. For each of the entangled subgames $\Gamma(w, \pi(v_j), \{\pi(v_j)\})$ with root node $w \in N(v_j) \setminus \{v_{j+1}\}$ for some node v_j on the path ρ, we can find equilibrium strategies $\sigma_1 \in \Sigma_1^w, \ldots, \sigma_n \in \Sigma_n^w$ such that

$$\omega^*(w, \sigma_1, \ldots, \sigma_n) = \phi(\Gamma(w, \pi(v_j), \{\pi(v_j)\})). \tag{19}$$

Since $i := \pi(v_j)$ is the maximizing players in the entangled subgame $\Gamma(w, i, \{i\})$, we have

$$\omega^*(w, \tau_{\{i\}}, \sigma_{P \setminus \{i\}}) \preceq_i \omega^*(w, \sigma_1, \ldots, \sigma_n). \tag{20}$$

for all strategies $\tau_i \in \Sigma_i^w$. By the transitivity of preferences together with (19), we have thus proven that

$$\omega^*(w, \tau_{\{i\}}, \sigma_{P\setminus\{i\}}) \preceq_i v_k \tag{21}$$

is satisfied for all strategies $\tau_i \in \Sigma_i^w$ of player $i = \pi(v_j)$. Therefore, there exists no better option for player i than to move to v_{j+1} at node v_j.

Since all the entangled subgames with root nodes $w \in N(v_j) \setminus \{v_{j+1}\}$ for some node v_j on the path ρ have pairwise disjoint subtrees and comprise all nodes outside the path ρ, we can define the strategies of players for the whole decision tree by choosing their equilibrium strategies in the entangled subgames $\Gamma(w, \pi(v_j), \{\pi(v_j)\})$ and setting

$$\sigma_{\pi(v_j)}(v_j) := v_{j+1} \ \ (j = 0, \ldots, k-1). \tag{22}$$

Together with (21), Lemma 4.1 completes the $'' \Longleftarrow''$ part of the theorem. $\square$

Of course, there exist many equilibrium paths, which do not represent a plausible course of action. Therefore, the set of equilibria should be examined concerning additional qualities.

4.3 Remark: As Theorem 4.2 states, the set of equilibrium paths $\rho = (v_0, \ldots, v_k)$ is characterized by the entanglement conditions (14) for all entangled subgames $\Gamma(w, \pi(v_j), \{\pi(v_j)\})$ with root node $w \in N(v_j) \setminus \{v_{j+1}\}$. Since, by Remark 3.3 (2)

$$\phi(\Gamma(v, \pi(v_j), \{\pi(v_j)\})) \preceq_{v_j} \phi(\Gamma(v, \pi(v_j), C \cup \{\pi(v_j)\})) \tag{23}$$

holds for $C \subset P$, it would be a stronger entanglement condition to claim

$$\phi(\Gamma(w, \pi(v_j), C \cup \{\pi(v_j)\})) \preceq_{\pi(v_j)} v_k \tag{24}$$

for some player coalition $C \subset P$, leading to a smaller set of equilibrium paths satisfying this condition. Equilibrium paths of this type can be considered to be robust against the coalition C, or conversely, the coalition $P \setminus C$ is able to force players to follow the path.

The given characterization offers an approach to identify equilibrium paths. The next section focuses on the corresponding algorithm. Clearly, each subgame perfect equilibrium satisfies the condition in Theorem 4.2. So, they will be part of the resulting set of paths.

5 Identification of Equilibrium Paths

A recapitulation of the characterization result 4.2 gives rise to a method how equilibrium paths can be identified. It suffices to step through each path starting from a terminal node of the decision tree and leading back to the root node. At each node of the path the entangled subgames for deviation points from

```
For each terminal node of the decision tree
   Calculate path to root node
      For each node on the path
         Check all entangled subgames
          to satisfy entanglement condition (14)
         if any negative answer:  stop
   if all checks positiv:  add path to set of equilibria
```

Table 1: Identification algorithm

the path have to be checked for their equilibrium to satisfy the entanglement condition (14). Formally the algorithm works as described in Table 1.

As a consequence of the numerical complexity of the backward induction algorithm, the identification algorithm runs in polynomial time depending on the number of nodes of the tree. The subsequent examples illustrate the results of the algorithm (Table 1).

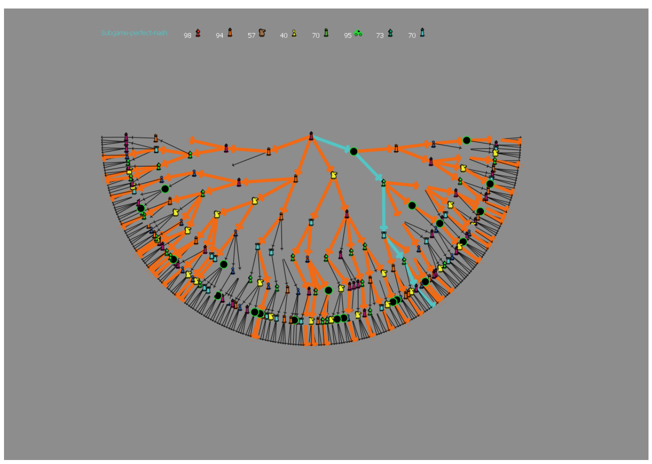

Figure 3: Identification of all equilibria of a decision tree

Equilibrium paths are colored orange, in addition, the subgame perfect equilibrium is colored blue. The next figure shows how entangled subgames are solved and contains the subgame paths satisfying the entanglement condition in red color.

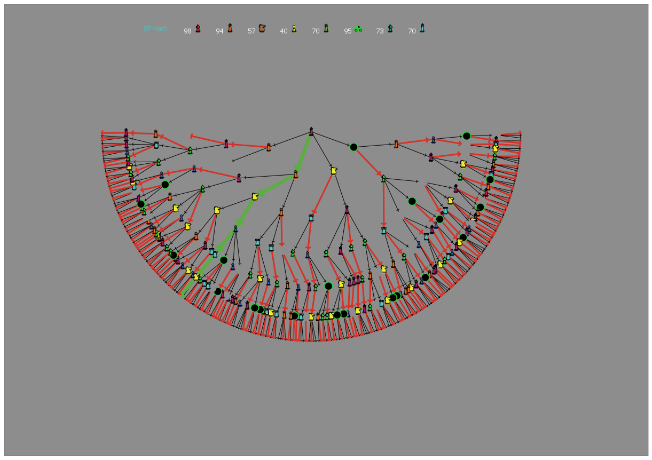

Figure 4: Solving entangled subgames

6 An Application in the Political-Military Environment

Using the previous results, we will analyze the situation of the Cuba Crisis in 1962, which was perhaps the most dangerous conflict during the cold war period between the US and the Soviet government. There is a comprehensive documentation and analysis of the backgrounds of this conflict (see for instance L. Scott and R.G. Hughes [2]). In brief, the balance of military capabilities was perceived by the Soviet side to be in favor of the US side. Therefore the Soviets tried to install nuclear war heads and launchers on the island of Cuba, which threatened at least a part of the US area. Troops, equipment and launchers were currently shipped to Cuba by the Soviets. This was a strong violation of the US integrity demand and required an adequate reaction. The available options of a nuclear strike against the Soviets, an air attack against the installations on Cuba, a sea blockade of the island and further negotiations with the counter part turned out to be a matter of decision.

The corresponding outcomes are the expected military capabilities of both sides after a decision for one of the four options and possible reactions (see Figure 5). As we know, for the identification of equilibria the numerical values are less important than the preferences behind them. Therefore there exists some scope for adjustment of these numbers and the given numbers should be considered as of exemplary nature. The equilibrium analysis shows that there is a subgame perfect solution with an air attack, following negotiations and an agreement. Two other entangled equilibria exist, one with direct unfruitful negotiations (see Figure 6) and another with a sea blockade and subsequent negotiations with concessions on both sides (see Figure 7). As history shows, this was the preferred option for the US side leading to a deescalation of the conflict. The

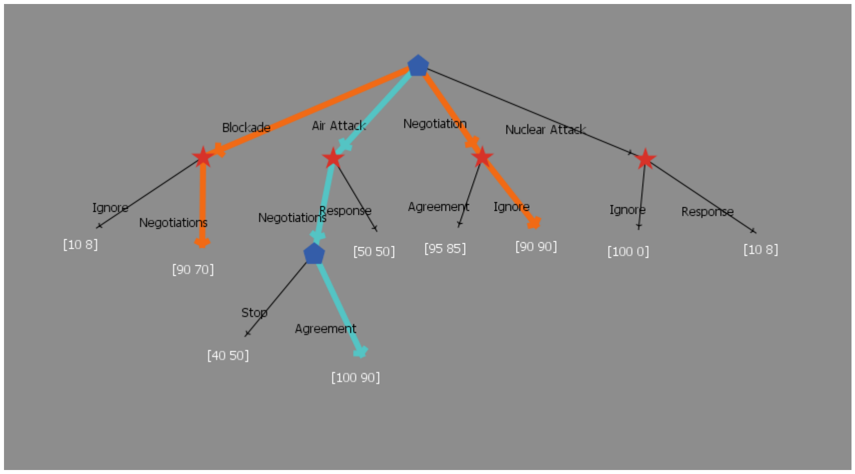

Figure 5: Cuba Crisis decision tree

concessions of the US side were to uninstall missiles already deployed in Turkey as well as to give some grantees not to attack Cuba any more. In return the Soviets withdrew their missiles from Cuba.

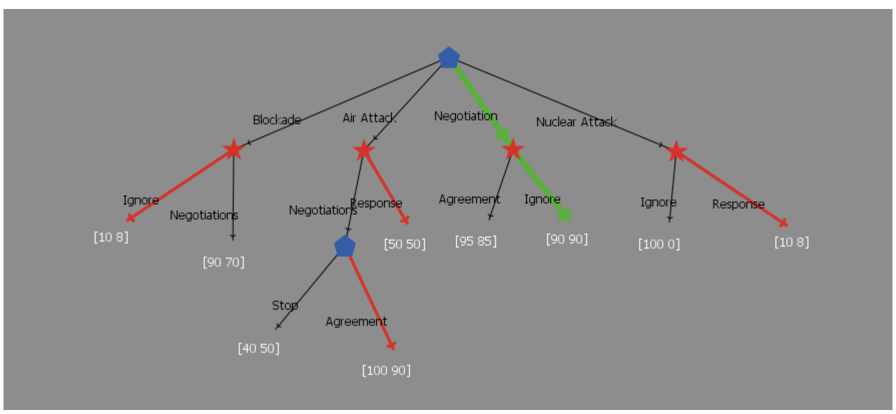

Figure 6: Cuba Crisis negotiation option

During the decision process a massive nuclear attack with disastrous consequences turned out to be not an appropriate option. Immediate negotiations were at risk to fail and give the Soviets the chance to complete their installations on Cuba and thus to achieve a better position for further activities. The entangled subgames show that the assumptions for this option would have been the Soviets to disregard the sea blockade and to respond to an air attack by a massive counter strike. In addition, the success of an air attack seemed to be questionable.

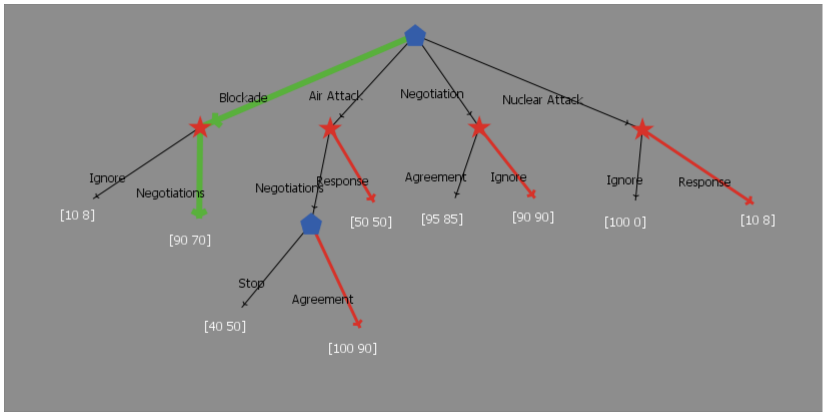

Figure 7: Cuba Crisis sea blockade option

The sea blockade was considered to be the best way to establish at least a chance for fruitful negotiations.

7 Management by Coalitions

As Figure 3 shows, there may exist many equilibria besides the subgame perfect solutions. Some of them may be less credible than others. Therefore, the question arises, which additional qualifier should be used to sort out equilibria of lower attraction. We propose a criterion which concerns a deeper analysis of entangled subgames. Theorem 4.2 states that an equilibrium path is characterized by the feature that in each point the decision maker can be kept on the path by all other players making the alternatives less favorable to him. In some situations this may result in paradox behavior of players who are not very interested in the terminal outcome of the decision path. To avoid this effect the situation could be modified in such a way that only a group of players with credible intentions manages the entangled subgames in their desired direction. This idea was already addressed by Remark 4.3. Starting from a subgame perfect solution for all equilibrium paths a coalition of players can be identified whose members all prefer the according terminal node of some alternative path. The problem is, whether they can manage the corresponding entangled subgames on behalf of their preferences.

This approach gives rise to find an algorithm which allows to identify equilibria which can be managed by coalitions in the above sense. A slight modification of the identification algorithm in Table 1 can do this job and gives an answer, whether equilibria of the desired type exist.

The only difference to the already mentioned algorithm in Table 1 is the method how entangled subgames are treated. They are rather games of a group of supporters of the actual decision maker against their complementary group than a single decision maker against the rest of the world. The complementary group manages the entangled subgames in such a way that they keep the decision process on the given path. Their intention is to reach the better outcome in comparison to the subgame perfect result. In detail the algorithm works as decribed in Table 2.

```
Calculate a subgame perfect equilibrium
For each terminal node of the decision tree:
   Identify player group C which prefers the outcome to the subgame perfect solution
   Calculate path to root node
      For each node on the path
         Check all entangled subgames to satisfy entanglement condition
          for player group C and the corresponding key player (Remark 3.1)
         if any negative answer:  stop
   if all checks positiv:  add path to set of equilibria
```

Table 2: Identification algorithm for player groups

Unfortunately, examples of game trees with no such equilibria exist (see Figure 8). In this case, the algorithm only identifies the subgame perfect equilibrium (blue color) together with some less plausible equilibrium paths (orange color). Therefore, the concept of subgame perfectness becomes corroborated to be the only reasonable solution in this situation.

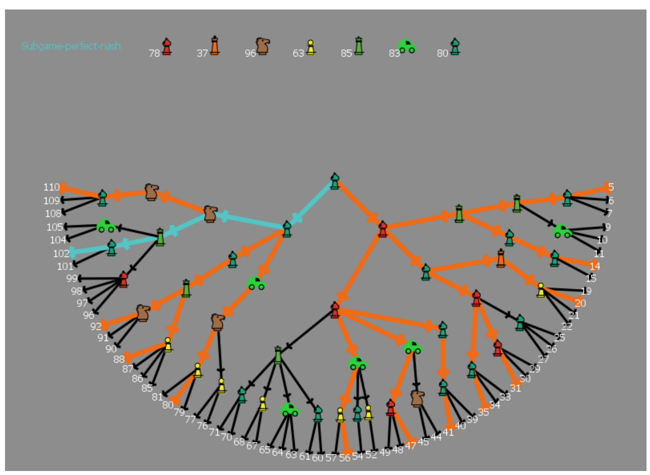

Figure 8: No better equilibria than the subgame perfect outcome

Nevertheless, in many cases with other assignments of utilities for the same decision tree, equilibria of the discussed type can be found (see Figure 9). Equilibrium paths are colored orange, in addition, the subgame perfect equilibrium is colored blue, while the paths managed by a coalition are painted in red color.

Clearly, solving the entangled subgames with two opposing coalitions results in stronger requirements for the path to become an equilibrium path. Therefore, less equilibria exist satisfying the conditions of Remark 4.3 than in the case of Theorem 4.2.

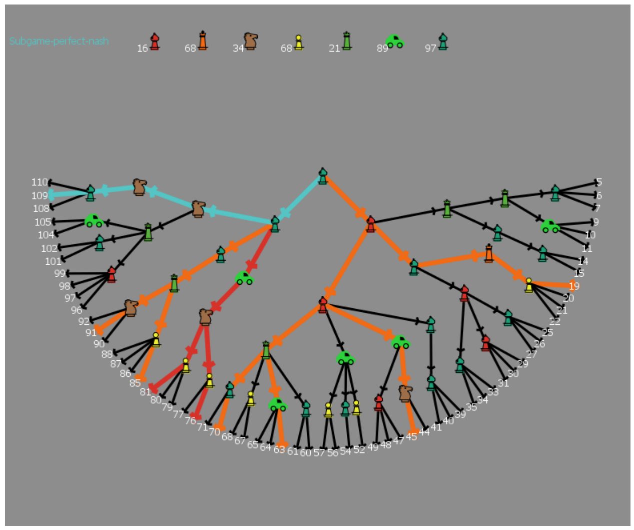

Figure 9: Equilibria managed by a coalition

The following listing (Table 3) contains the data corresponding to the above example in Figure 9.

Subgame perfect equilibrium (terminal vertex 109)

```
Outcome [16 68 34 68 21 89 97]
```

Prefered terminal nodes of other equilibria

```
[(terminal vertex 76) preferred by players [1 3 5]]
Outcome [78 0 97 30 30 66 59]
[(terminal vertex 81) preferred by players [1 2 3 4 5]]
Outcome [83 78 36 76 95 47 24]
```

Table 3: Management by coalitions

The table shows that there exists a strong incentive for some players to deviate from the subgame perfect strategy.

8 Summary

We have shown that for many game trees there exist alternative equilibria besides the subgame perfect solution, which may also be attractive for some coalitions of players and which can be realized by cooperation of these groups. If no such alternatives exist, there is no stimulation of any player group to deviate from the subgame perfect solution. Moreover, we can find sufficient conditions for decision paths in the game tree to form an equilibrium path managed by certain groups of players. This approach provides the opportunity to formulate an algorithm which identifies appropriate equilibrium paths. The algorithm is efficient and takes the same time as for backward induction times the number of terminal nodes in the tree.

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