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Selection of Cluster Heads within Communication Networks by Voting Schemes

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Dedicated to Prof. Dr. em. Otto Moeschlin
on the occasion of his 80th birthday

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Dedication

As several recent political and economic developments show, decision making is a time consuming and complex task for good governance. Although often criticized as too scientific and impracticable the concepts of Game Theory have proven to provide useful insights within a variety of application fields. Even in cases, where the Theory of Games predicts results which disagree with observations from real life, these inconsistencies have initiated valuable discussions about the modeling of situations and the reasons of conflicting outcomes.

One of our first deeper contacts with Game Theory took place in the year 1978, when Otto Moeschlin and Diethard Pallaschke organized the symposium on Game Theory and Related Topics at the Fernuniversität of Hagen and the Gesellschaft für Mathematik und Datenverarbeitung (GMD) near Bonn. Many important researchers from international institutions participated in the editorial board of this symposium. The topics of the contributions represented accordingly widespread areas of research. The proceedings of the symposium are still an inspiring resource for further research activities.

Several doctoral thesis as well as a series of diploma thesis striking the area of Game Theory, Mathematical Statistics and Probability Theory have been coached under the guidance of Otto Moeschlin, partially arising from topics concerning the above mentioned symposium. In addition, a variety of publications emerged from the periphery of the addressed fields.

Abstract

We consider totally connected networks of nodes forming a cluster within a broader community of agents exchanging messages. These networks are also addressed by mesh networks. We examine a scenario where the community is partitioned into multiple clusters, in each cluster one node acting as cluster head. The function of the cluster head is to send and receive messages from remote clusters while the other members within the cluster will be informed by the cluster head and can therefore keep silent in order to save resources. The way how this cluster head will be selected is an internal voting scheme based on a majority rule and preferences for all voters. The problem arises from the fact that even complete and transitive preferences of all voters on the agent set do not induce transitive collective preferences, and hence do not ensure the existence of an undominated agent, who would be a suitable candidate for a cluster head. Therefore, the selection process must become more complicated. We propose a selection procedure based on sequential voting and an assessment of nodes in the style of a Shapley Value approach. Since the weakness of the Shapley Value approach in practical applications is based on its numerical complexity, we will investigate the properties of the voting process in more detail. Moreover, we will analyze the different structures of preference schemes of the voters and establish the relationship to the Shapley value approach. This offers a numerically more tractable method for the selection of the cluster head than the calculation of the Shapley value in a straight forward way.

Keywords: *Game Theory, Nash Equilibrium Strategies, Voting Scheme, Cluster Head Selection, Shapley Value, Biform Games, Preference Scheme*

1 Problem

Recently, networks of sensors have been a matter of research in the context of energy saving and reduction of data exchange in wireless networks, Car2X Communication and cooperation between moving vehicles (cf. for instance [3], [6], [1]). We take up some underlying problems in connection with such networks and continue the approach discussed in the paper [7]. We propose a voting mechanism for selection of a point of contact based on the ideas developed in [4]. Given a network with n agents communicating with each other, it makes sense for different reasons like energy saving or filtering and processing of messages that one node acts as point of contact for incoming and outgoing messages from remote clusters and dispatches all information to the members of his own cluster. The crucial question is the procedure how this cluster head can be selected in a fair manner. We assume that all members of the cluster have preferences on the set of all agents. It is not self-evident that they will put themselves in the top position of their preference scheme, but they may do it this way. According to the voting procedure described in [4], we put all agents of the concerning cluster in a sequence $(a_1, \ldots, a_n)$ and start with the proposal a_1. Next, the agent a_2 is posed as alternative to a_1 and will be the new proposal, if a majority of agents prefers a_2 to a_1. Otherwise, a_1 is asserted against a_2 and will be set as hypothetical candidate in the next step against a_3. The selection process is continued until a final agent persists as candidate for the cluster head. We allow strategic voting in this process. Therefore, all agents will act in a goal-oriented way, in accordance with their own preferences and will use best responses to the voting strategies of the opponents. Consequently, the resulting strategies will form a Nash Equilibrium as introduced by J. F. Nash ([5]) in his pathbreaking work.

Since the outcome of this voting procedure, in particular the Nash solution, depends crucially on the sequence the agents are introduced as alternatives, we have to improve this method of voting in a more fair manner. This problem leads us to one of the classical concepts of Cooperative Game Theory, the Shapley Value (cf. [8]), which can be considered as a fair assessment of the agents in the cluster concerning their qualification as cluster head. To apply this concept to the underlying voting model, we examine all possible sequences of agents and count the appearance of all agents as Nash solution of the voting process. One of the most frequently elected agents will finally be chosen as cluster head.

The underlying game can be subsumed under the category of so called **Biform Games**. This is a hybrid form between non-cooperative and cooperative games. Biform Games are for instance investigated in the paper [2]. The way how non-cooperative and cooperative concepts can be integrated is usually based on a somehow defined Shapley Value for all players in combination with a decision process, which offers the players to adjust their Shapley Value as best response to the decisions of the other players. In our situation, the setting is reverse. We start with a decision process and determine the Shapley Values of all players as its outcome.

2 Formal model and notations

As already mentioned, we will investigate voting games, which can be described by the following setting. Let A be the community of a finite number n of agents. The voting process is then given by a sequence $(a_1, \ldots, a_n)$, which describes the order in which the candidates are introduced in the process. Each agent $a \in A$ has some preference relation $\preceq_a$ on the set A of agents. [1] So, we consider a

[1]We use a very general concept of preferences: The relation $\preceq$ on the set A is called *preference relation*, if $s \preceq s$ holds for all $s \in A$.

special case of a voting game as described in [4]. The set of alternatives coincides with the set of players. Without going into details, the strategies of the players will be their decisions in each step of the voting process. Whenever all players have decided how to vote in each step, they will finally select a certain candidate for the cluster head. Although the decision process follows the majority rule in each step, the nature of the game is non-cooperative. Each player will try to optimize the outcome with respect to his preferences against the same behavior of the other players. This means that the concept of Nash Equilibrium makes sense in this context and serves as guideline for the behavior of players in the voting process. As extensively discussed in [4], we have to refine the concept of Nash Equilibrium in the given game, because not all Nash Equilibria make sense. Some of them even ignore the preferences of players completely. To overcome these undesired effects, we introduced the concept of **consistent strategies**. This concept leads not only to a more plausible behavior of the players, but offers also a rather efficient algorithm to find the outcome of Nash Equilibria in consistent strategies. Before we go into the details of this algorithm, we will first define collective preferences arising from the individual preferences. We introduce the preference relation $\preceq$ on A by

$$a \preceq b :\Leftrightarrow |\{i \in S : a \preceq_i b\}| \geq \frac{1}{2}|A|. \tag{1}$$

These collective preferences represent the majority rule, i.e. $a \preceq b$ is satisfied if and only if a majority of all players votes for b, whenever a is the alternative. As usual, we denote the situation, where $a \preceq b$ but not $b \preceq a$ holds, by $a \prec b$. Each alternative $a \in A$ is said to be **undominated** in the set A of alternatives, if no alternative $b \in A$ exists with $a \prec b$. Now we are in a position to describe an algorithm to identify Nash Equilibria in consistent strategies.

2.1 Algorithm: Given the voting process $(a_1, \ldots, a_n)$, we define

$$\nu(S) := \max\{i | a_i \in S\} \qquad \forall S \subset A, S \neq \emptyset. \tag{2}$$

On the set of all subsets of all alternatives A we define the mapping ϕ by

$$\phi(S) := \{a \in S | a_{\nu(S)} \prec a\} \qquad \forall S \subset A, S \neq \emptyset. \tag{3}$$

As a consequence of $a_{\nu(S)} \notin \phi(S)$, we have

$$\phi(S) \subset S \;\text{ und }\; \phi(S) \neq S \qquad \forall S \subset A, S \neq \emptyset. \tag{4}$$

We make use of the abbreviation $\mu(j) := \nu(\phi^j(A))$ for $j = 0, \ldots, k$. Hence, we have $a_{\mu(k-1)} \prec a \;\; \forall a \in \phi^k(A)$, whenever $k > 0$, and $a \prec a_{\mu(k)} \;\; \forall a \in \phi^k(A)$, whenever $\phi^{k+1}(A) = \emptyset$.

We start the algorithm with $S := A$ and apply successively the mapping ϕ. In each step of the algorithm, the initial set is reduced by at least one element. The algorithm generates a sequence $\phi^0(A), \ldots, \phi^k(A)$ of non-empty sets and ends after k steps, as soon as $\phi^{k+1}(A) = \emptyset$.

The algorithm 2.1 provides an efficient method to determine the Shapley Values of the agents in the voting process. For each agent we count the number of occurrences, where the agent is elected as candidate for the cluster head and take the mean over all permutations of the sequence of decision steps. The Shapley Value determined in this way will be denoted by $SHV(a) \ \forall a \in A$.

The following example shows how the algorithm works.

2.2 Example: We consider the case where the preference scheme of seven agents is given by the matrix

$$\begin{array}{l} \text{Ranking} \\ [0 \quad 1 \quad 2 \quad 5 \quad 4 \quad 3 \quad 6] \\ [2 \quad 4 \quad 1 \quad 6 \quad 5 \quad 0 \quad 3] \\ [1 \quad 4 \quad 2 \quad 3 \quad 6 \quad 0 \quad 5] \\ [3 \quad 6 \quad 5 \quad 2 \quad 0 \quad 1 \quad 4] \\ [2 \quad 0 \quad 6 \quad 5 \quad 3 \quad 4 \quad 1] \\ [6 \quad 4 \quad 2 \quad 3 \quad 5 \quad 0 \quad 1] \\ [4 \quad 5 \quad 0 \quad 1 \quad 2 \quad 6 \quad 3] \end{array} \tag{5}$$

and make use of an implemented version of the algorithm in the NetLogo ([9]) environment. The algorithm starts with the last alternative and successively

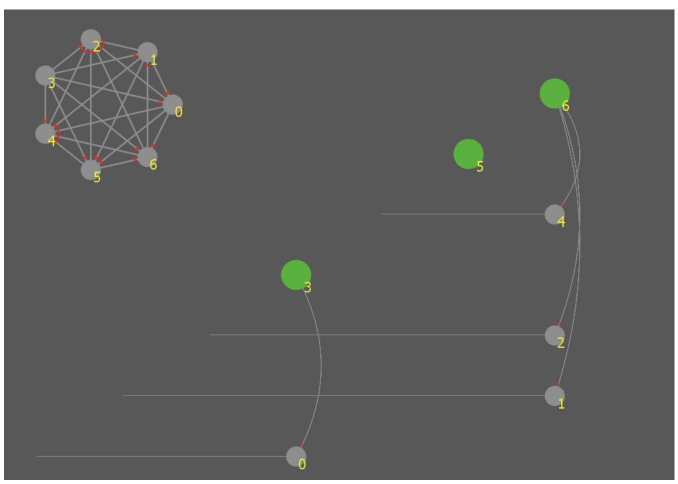

Figure 1: Sequence of algorithmic steps

excludes all alternatives dominated by the actually examined alternative (green point with arrows for dominated other alternatives). Finally, the algorithm stops at alternative 3, which dominates the only remaining alternative 0. Hence, alternative 3 is the solution of the decision process based on the given decision sequence. In the left upper corner, the collective preferences are shown by arrows with red arrowheads building a network.

3 Electing the Cluster Head

We draw on the assumptions and results of [4]. In the sequel, we continue therefore with the following premises.

3.1 Assumptions: For simplicity we assume:

1. The preference relations $\preceq_i$ of all agents i are complete.
2. The preference relations $\preceq_i$ of all agents i are transitive.
3. The preference relations $\preceq_i$ of all agents i are strict.
4. The number of agents is odd.

As a consequence of these assumptions the collective preference relation $\preceq$ is also complete and strict, but not necessarily transitive. Nevertheless, in many cases an undominated agent exists, and due to the strictness of the collective preference relation this undominated agent is unique. The following result is a justification for the chosen approach via voting processes.

3.2 Theorem: *Suppose, there exists an undominated agent a^*. Then, a^* is the unique result of all voting processes, independent of the order of the agents in the process.*

Proof. We make use of the results of [4]. The algorithm 2.1 stops at a certain stage k with $\phi^k(A) \neq \emptyset$ and $\phi^{k+1}(A) = \emptyset$, where $a_{\nu(\phi^k(A))}$ is the solution of the voting process. Now, if a^* is undominated,

$$a^* \in \phi^j(A) \ \text{ for } \ j = 0, \ldots, k. \tag{6}$$

In the case $a_{\nu(\phi^k(A))} \neq a^*$, because of $a_{\nu(\phi^k(A))} \preceq a^*$ and the strictness of $\preceq$, step k would not be the last step of the algorithm. Therefore, we have shown $a_{\nu(\phi^k(A))} = a^*$. The argumentation of the proof does not make use of any order of the agents within the voting process. □

The previous theorem shows that in case of the existence of an undominated agent with respect to strict collective preferences, the voting process will always end up in a unique result. Therefore, considering all permutations of voting sequences, there exists a unique candidate for the role of the cluster head. The Shapley Value of this candidate is equal to 1, while all other agents will be dummies in the voting game.

In addition, we will derive a result, which can be considered as the worst case concerning the Shapley Values of the voting process, because all agents are assessed equally. To this end we first introduce a concept of cycling individual preferences.

3.3 Definition: *Let be given a voting game with alternatives $A = \{a_1, \ldots, a_n\}$ and individual preferences $\preceq_a$ for all players $a \in A$. We say that the preference* *relations* $\preceq_a$ $(a \in A)$ *are* ***floating preferences****, if for the shift operation* τ *on* A*, i.e.*

$$\tau(a_i) = \begin{cases} a_{i+1} & \textit{for } \; i < n, \\ a_1 & \textit{for } \; i = n \end{cases}$$

the individual preference relation of player a_i *is given by*

$$\tau^i(a_1) \prec_{a_i} \ldots \prec_{a_i} \tau^i(a_n). \tag{7}$$

For floating preferences we can realize the following properties.

3.4 Remarks: 1. For each voting game $A = \{a_1, \ldots, a_n\}$ with floating preferences, the only situation, where alternative $b \in A$ is preferred to $\tau(b)$ by a player $a_i \in A$ occurs in the case $\tau^{-i}(b) = a_n$. Consequently,

$$|\{b \in A | a \prec_b \tau(a)\}| = n - 1 \;\; \forall a \in A, \tag{8}$$

holds true and we have

$$a \prec \tau(a) \;\; \forall a \in A. \tag{9}$$

Continuing with this argumentation, we realize

$$|\{b \in A | a \prec_b \tau^2(a)\}| = n - 2 \;\; \forall a \in A, \tag{10}$$

and get by induction

$$|\{b \in A | a \prec_b \tau^k(a)\}| = n - k \;\; \forall a \in A. \tag{11}$$

Therefore, $a \prec \tau^k(a)$ holds true as long as $k \leq \frac{n}{2}$. Setting $\lfloor c \rfloor := \max\{i \in \mathbf{N} | i \leq c\}$ for all $c \in \mathbf{R}_+$, these findings lead us to the result

$$\begin{aligned} a \prec \tau^k(a) \quad & \text{for } \; 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor, \\ \tau^k(a) \prec a \quad & \text{for } \; \left\lfloor \frac{n}{2} \right\rfloor < k \leq n - 1 \\ & \forall a \in A. \end{aligned} \tag{12}$$

2. First, the previous considerations (cf. (9)) show that the alternatives can be arranged as a cycle

$$a_1 = \tau^1(a_n) \prec \ldots \prec \tau^n(a_n) = a_n \prec a_1 \tag{13}$$

with respect to the collective preferences.

3. We consider again the algorithm 2.1. Now, by the definition of the algorithm, we get $a_n \notin \phi(A)$, and from (12), we see that

$$\phi(A) = \{\tau^j(a_n) | j = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor, a_n \prec \tau^j(a_n)\}. \tag{14}$$

From this fact, we derive by induction

$$\phi^k(A) = \{\tau^j(a_n) | j = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor, a_{\mu(k-1)} \prec \tau^j(a_n)\} \text{ for } k > 0. \qquad (15)$$

Therefore, because the $\tau^j(a_n)$ with the highest j always dominates all other members of $\phi^k(A)$, we conclude by induction that

$$\tau^{\lfloor \frac{n}{2} \rfloor}(a_n) \in \phi^k(A) \qquad (16)$$

for $k \leq \lfloor \frac{n}{2} \rfloor$. The algorithm stops as soon as $\mu(k) = \lfloor \frac{n}{2} \rfloor$. Hence, $\tau^{\lfloor \frac{n}{2} \rfloor}(a_n)$ is the identified solution of the algorithm.

4. The previous remark provides a method to identify the solution of the algorithm without going through all steps. Starting the algorithm with a_n will always result in $\tau^{\lfloor \frac{n}{2} \rfloor}(a_n)$. Thus, having regard to (13), we obtain a one-to-one mapping of the starting point to the solution of the algorithm.

3.5 Theorem: *If the individual preference relations of the voting game with alternatives $A = \{a_1, \ldots, a_n\}$ are floating preferences, then the Shapley Value SHV of the given game satisfies*

$$SHV(a) = SHV(b) \text{ for all } a, b \in A. \qquad (17)$$

Proof. All the proof of the theorem is already provided by the previous remarks 3.4 . Specifically, we draw on remark 3.4 (4.) and conclude that for different last alternatives a_n and b_n the cluster head identification algorithm finds different solutions $\tau^{\lfloor \frac{n}{2} \rfloor}(a_n)$ and $\tau^{\lfloor \frac{n}{2} \rfloor}(b_n)$. Now the probability for a permutation to assign the last position in the sequence of alternatives to a_n is the same as for b_n, namely $\frac{1}{n}$. Since the mapping $a \to \tau^{\lfloor \frac{n}{2} \rfloor}(a)$ is one-to-one, the proof is completed. □

Both Theorems 3.2 as well as 3.5 cover two extreme cases of preference schemes. Nonetheless, in many other cases the Shapley Value approach can also provide a method to identify a cluster head in an accurate way.

3.6 Example: We consider the case where the preference scheme of seven agents is given by the matrix

$$\begin{array}{l} \text{Ranking} \\ [1 \;\; 4 \;\; 0 \;\; 2 \;\; 3 \;\; 6 \;\; 5] \\ [1 \;\; 4 \;\; 3 \;\; 2 \;\; 6 \;\; 5 \;\; 0] \\ [6 \;\; 5 \;\; 2 \;\; 4 \;\; 0 \;\; 1 \;\; 3] \\ [5 \;\; 6 \;\; 2 \;\; 1 \;\; 4 \;\; 3 \;\; 0] \\ [6 \;\; 1 \;\; 5 \;\; 0 \;\; 4 \;\; 2 \;\; 3] \\ [0 \;\; 3 \;\; 5 \;\; 6 \;\; 4 \;\; 1 \;\; 2] \\ [4 \;\; 0 \;\; 6 \;\; 3 \;\; 1 \;\; 2 \;\; 5] \end{array} \qquad (18)$$

and calculate the corresponding Shapley Values using a NetLogo ([9]) environment. The network of collective preferences shows that there exists no undominated alternative. All collective preferences are represented by arrows with red arrowheads. The Shapley Values for the given alternatives are unfolded as

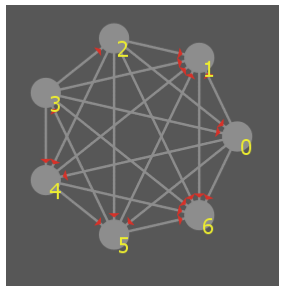

Figure 2: Network of collective preferences

the vector $[0 \quad 0 \quad 1008 \quad 3024 \quad 0 \quad 1008 \quad 0]$ without normative factor. It turns out that alternative 3 is significantly better than all other alternatives and is therefore qualified as cluster head.

The previous considerations lead us to the question how often an undominated agent will appear, when the preferences of all players are the result of a random process. More precisely, we will be interested in the most preferred agents of the individual preference schemes of all players. Clearly, by Theorem 3.2, an agent a^* is the favorite for the majority all agents, if a^* is undominated in the set A of all agents with respect to collective preferences. As an example, we will analyze the situation of a network with three agents.

3.7 Example: The individual preferences of the three players can be represented by a matrix

$$\begin{aligned} &a_{11} \prec_1 a_{12} \prec_1 a_{13} \\ &a_{21} \prec_2 a_{22} \prec_2 a_{23} \\ &a_{31} \prec_3 a_{32} \prec_3 a_{33} \end{aligned} \tag{19}$$

where $a_{ij} \in \{1, 2, 3\}$ and line i defines the ranking of player i concerning the alternatives. The overall number of such matrices is given by $(3!)^3 = 216$. Now, we can list all the cases, where an undominated alternative exists. We start with a fixed alternative a. It dominates all other alternatives in the following cases:

1: a has three times rank 3
2: a has twice rank 3, once rank 1
3: a has twice rank 3, once rank 2
4: a has once rank 3, twice rank 2 with different alternatives on rank 1

And these are the only cases, where a given alternative is undominated. Counting all these cases, we get

1: $2^3 = 8$ cases
2: $3 \times 2^3 = 24$ cases
3: $3 \times 2^3 = 24$ cases
4: $3 \times 2^2 = 12$ cases

Totally, this makes 68 cases. We can set up the same table for both other alternatives and arrive at a number of $3 \times 68 = 204$ cases, where an undominated alternative exists. This is somehow surprising and shows that for 3 agents we have a probility of $\frac{17}{18}$ to find a cluster head, who dominates all other agents. Of course, this result depends on the way how collective preferences are composed by individual preferences in consideration of the majority rule.

Of course, further analysis on the asymptotic behavior of these probabilities would be desirable. Monte-Carlo-Simulations show that the probabilities decrease only slowly for small networks. For instance, we got an estimated probability of $78,4\%$ for networks with 5 members and 1000 runs as well as 64% for networks with 7 members and 100 runs. A theoretical analysis, whether the probabilities approach zero or have a positive lower limit is actually not available.

4 Concluding Remarks

Some of the results have been inspired rather by experimental exploration using computer models than by strong top down theoretical derivation, in particular the algorithm 2.1 and some probabilistic analysis. This way of research is specifically helpful to underpin hypothesis or to reject them. Since Artificial Intelligence (AI) is a fashion topic in the area of computer science, it seems to be adequate to pose the question how this discipline could contribute to the development of theories and in particular to the organization of communication networks. Of course, there exists some progress in the field of logical argumentation and proof of theorems. But the most successful application of AI seems to be the evaluation of experiments for the formulation of hypothesis and testing. The theoretical derivation is then a matter of traditional methods.

References

[1] Arena F. and Pau G., 2019, *An Overview of Vehicular Communications*, future internet MDPI, Basel, Switzerland

[2] Brandenburger, Adam; Harborne, Stuart, 1993, *Biform Games*, www.stern.nyu.edu/ abranden, www.columbia.edu/ hws7

[3] Brisha, M., 2010, *Comparison between Methods of Selecting Cluster Head*, International Journal of Online and Biomedical Engineering, Vol 6, No 4, 0doi:10.3991/ijoe.v6i4.1325

[4] Meister, Helmut, 2016, *Abstimmungsprozesse aus spieltheoretischer Sicht*, Seminarberichte aus der Fakultät für Mathematik und Informatik, Band 88, S. 99 - 119

[5] Nash, John Forbes, 1950, *Non-cooperative games*, Dissertation, Princeton University.

[6] Ramesh K. and Somasundaram K., 2011, *A Comparative Study of Clusterhead Selection Algorithms in Wireless Sensor Networks*, International Journal of Computer Science & Engineering Survey (IJCSES) Vol.2, No.4, DOI:10.5121/ijcses.2011.2411 153

[7] Schulz, Katharina; Meister, Gisela, 2009, *Bayesian approach for reputation modelling in wireless sensor networks*, Seminarberichte aus der Fakultät für Mathematik und Informatik, Band 82, S. 125 - 132

[8] Shapley, Lloyd S., 1953, *A Value for n-person Games*, in Kuhn, H. W.; Tucker, A. W. Contributions to the Theory of Games. Annals of Mathematical Studies. 28. Princeton University Press. pp. 307–317, doi:10.1515/9781400881970-018. ISBN 9781400881970

[9] Wilensky, U., 1999, NetLogo, http://ccl.northwestern.edu/netlogo/, Center for Connected Learning and Computer-Based Modeling, Northwestern University, Evanston, IL.