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‚Change in space'-point estimation, Part I: Lower bound for rates of consistency

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'CHANGE IN SPACE'-POINT ESTIMATION, PART I: LOWER BOUND FOR RATES OF CONSISTENCY

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Given n discrete observations of a homogeneous diffusion process with a piecewise constant diffusion coefficient containing one point of discontinuity ρ_0, we study the semiparametric problem of estimating its 'change in space'-point ρ_0 in the high-frequency setting. We establish a lower bound for the minimax rate of convergence $n^{-3/4}$, which is slower than the n^{-1}-rate in traditional change-point problems.

1. Introduction.

Let

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t \tag{1.1}$$

be a homogeneous stochastic differential equation (SDE), with a Wiener process W, and a diffusion coefficient σ of the form

$$\sigma(x) = \sigma_{\alpha_0,\beta_0,\rho_0}(x) = \begin{cases} \alpha_0 & \text{if } x \le \rho_0; \\ \beta_0 & \text{if } x > \rho_0 \end{cases} \tag{1.2}$$

with $\alpha_0, \beta_0 > 0$, $\alpha_0 \neq \beta_0$, and the point of discontinuity $\rho_0 \in \mathbb{R}$. Subsequently, we refer to ρ_0 as 'change in space'-point in order to distinguish it from the classical change-point where the structural break occurs in time. As σ^{-2} is locally integrable, there exists for every initial distribution μ a weak solution of (1.1) which is unique in law Engelbert and Schmidt (1985).

We will denote the vector of high-frequency observations by

$$X^{(n)} = \left(X_0^{(n)}, X_1^{(n)}, \ldots, X_n^{(n)}\right) = \left(X_0, X_{T/n}, X_{2T/n}, \ldots, X_T\right).$$

In case $\sigma = \sigma_{\alpha,\beta,\rho}$, the transition density, that is the solution of the Kolmogorov backward or Fokker-Planck equation Liptser and Shiryayev (1977)

$$\frac{1}{2}\sigma^2(y)\frac{\partial^2}{\partial y^2}p_t(x,y) = \frac{\partial}{\partial t}p_t(x,y),$$

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is given by Harada (2011)

$$p_t(x,y) = \begin{cases} \frac{1}{\sqrt{2\pi t}\alpha}\left[\exp\left(-\frac{(y-x)^2}{2t\alpha^2}\right) + \frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(y-2\rho+x)^2}{2t\alpha^2}\right)\right] & \text{for } x,y \le \rho \\ \frac{1}{\sqrt{2\pi t}\beta}\left[\exp\left(-\frac{(y-x)^2}{2t\beta^2}\right) - \frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(y-2\rho+x)^2}{2t\beta^2}\right)\right] & \text{for } x,y > \rho \\ \frac{2}{\alpha+\beta}\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{1}{2t}\left(\frac{y-\rho}{\beta} - \frac{x-\rho}{\alpha}\right)^2\right) & \text{for } x \le \rho < y \\ \frac{2}{\alpha+\beta}\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{1}{2t}\left(\frac{y-\rho}{\alpha} - \frac{x-\rho}{\beta}\right)^2\right) & \text{for } y \le \rho < x. \end{cases}$$

Note that the transition density is highly non-differentiable in the parameter ρ of interest.

2. Main result.

THEOREM 1. *There exists some $\eta > 0$, such that*

$$\liminf_{n} \inf_{\hat{\rho}_n} \sup_{\rho_0 \in \mathbb{R}} \mathbb{P}_{\alpha,\beta,\rho_0}\left(n^{3/4}\left|\hat{\rho}_n - \rho_0\right| > \eta\right) > 0.$$

PROOF. For simplicity of the proof we assume, that the diffusion starts in the 'change-in-space' point $X_0 = \rho$. We prove the lower bound by reduction on two hypotheses and bounding the corresponding Kullback-Leibler divergence, see the Kullback version of Theorem 2.2 in Tsybakov (2009). For this aim, we shall show

$$\sup_{r \in A_{n,\rho}} \mathbb{E}_{\rho,\alpha,\beta} \log \frac{\mathrm{d}\mathbb{P}_{\rho,\alpha,\beta}}{\mathrm{d}\mathbb{P}_{r,\alpha,\beta}}(X^{(n)}) < \infty, \tag{2.1}$$

with

$$A_{n,\rho} = \left\{r : n^{3/4}|\rho - r| \le \eta\right\}.$$

Due to the four different regimes of the transition density, the expression is of quite a complicated nature. By the Markov property,

$$\log \frac{\mathrm{d}\mathbb{P}_{\rho,\alpha,\beta}}{\mathrm{d}\mathbb{P}_{r,\alpha,\beta}}(X^{(n)}) = \sum_{k=1}^{n} \log\left(\frac{p_\Delta^\rho(X_{k-1},X_k)}{p_\Delta^r(X_{k-1},X_k)}\right),$$

which for $r < \rho$ is equal to

$$\sum_{k=1}^{n}\left\{\log\left[\frac{\beta}{\alpha}\frac{\exp\left(-\frac{(X_k-X_{k-1})^2}{2\Delta\alpha^2}\right) + \frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(X_k-2\rho+X_{k-1})^2}{2\Delta\alpha^2}\right)}{\exp\left(-\frac{(X_k-X_{k-1})^2}{2\Delta\beta^2}\right) - \frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(X_k-2r+X_{k-1})^2}{2\Delta\beta^2}\right)}\right]\mathbf{1}_{\{r<X_{k-1},X_k\le\rho\}}\right.$$

$$+\log\left[\frac{\alpha+\beta}{2\alpha}\frac{\exp\left(-\frac{(X_k-X_{k-1})^2}{2\Delta\alpha^2}\right)+\frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(X_k-2\rho+X_{k-1})^2}{2\Delta\alpha^2}\right)}{\exp\left(-\frac{1}{2\Delta}\left(\frac{X_k-r}{\beta}-\frac{X_{k-1}-r}{\alpha}\right)^2\right)}\right]\mathbf{1}_{\{X_{k-1}\le r<X_k\le\rho\}}$$

$$+\log\left[\frac{\alpha+\beta}{2\alpha}\frac{\exp\left(-\frac{(X_k-X_{k-1})^2}{2\Delta\alpha^2}\right)+\frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(X_k-2\rho+X_{k-1})^2}{2\Delta\alpha^2}\right)}{\exp\left(-\frac{1}{2\Delta}\left(\frac{X_k-r}{\alpha}-\frac{X_{k-1}-r}{\beta}\right)^2\right)}\right]\mathbf{1}_{\{X_k\le r<X_{k-1}\le\rho\}}$$

$$+\log\left[\frac{2\beta}{\alpha+\beta}\frac{\exp\left(-\frac{1}{2\Delta}\left(\frac{X_k-\rho}{\beta}-\frac{X_{k-1}-\rho}{\alpha}\right)^2\right)}{\exp\left(-\frac{(X_k-X_{k-1})^2}{2\Delta\beta^2}\right)-\frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(X_k-2r+X_{k-1})^2}{2\Delta\beta^2}\right)}\right]\mathbf{1}_{\{r<X_{k-1}\le\rho<X_k\}}$$

$$+\log\left[\frac{2\beta}{\alpha+\beta}\frac{\exp\left(-\frac{1}{2\Delta}\left(\frac{X_k-\rho}{\alpha}-\frac{X_{k-1}-\rho}{\beta}\right)^2\right)}{\exp\left(-\frac{(X_k-X_{k-1})^2}{2\Delta\beta^2}\right)-\frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(X_k-2r+X_{k-1})^2}{2\Delta\beta^2}\right)}\right]\mathbf{1}_{\{r<X_k\le\rho<X_{k-1}\}}$$

$$+\log\left[\frac{\exp\left(-\frac{(X_k-X_{k-1})^2}{2\Delta\alpha^2}\right)+\frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(X_k-2\rho+X_{k-1})^2}{2\Delta\alpha^2}\right)}{\exp\left(-\frac{(X_k-X_{k-1})^2}{2\Delta\alpha^2}\right)+\frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(X_k-2r+X_{k-1})^2}{2\Delta\alpha^2}\right)}\right]\mathbf{1}_{\{X_{k-1},X_k\le r\}}$$

$$+\log\left[\frac{\exp\left(-\frac{(X_k-X_{k-1})^2}{2\Delta\beta^2}\right)-\frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(X_k-2\rho+X_{k-1})^2}{2\Delta\beta^2}\right)}{\exp\left(-\frac{(X_k-X_{k-1})^2}{2\Delta\beta^2}\right)-\frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(X_k-2r+X_{k-1})^2}{2\Delta\beta^2}\right)}\right]\mathbf{1}_{\{X_{k-1},X_k>\rho\}}$$

$$+\log\left[\frac{\exp\left(-\frac{1}{2\Delta}\left(\frac{X_k-\rho}{\alpha}-\frac{X_{k-1}-\rho}{\beta}\right)^2\right)}{\exp\left(-\frac{1}{2\Delta}\left(\frac{X_k-r}{\alpha}-\frac{X_{k-1}-r}{\beta}\right)^2\right)}\right]\mathbf{1}_{\{X_k\le r<\rho<X_{k-1}\}}$$

$$+\log\left[\frac{\exp\left(-\frac{1}{2\Delta}\left(\frac{X_k-\rho}{\beta}-\frac{X_{k-1}-\rho}{\alpha}\right)^2\right)}{\exp\left(-\frac{1}{2\Delta}\left(\frac{X_k-r}{\beta}-\frac{X_{k-1}-r}{\alpha}\right)^2\right)}\right]\mathbf{1}_{\{X_{k-1}\le r<\rho<X_k\}}\Bigg\}$$

$$=\sum_{k=1}^{n}\left\{L_k^{(1)}+L_k^{(2)}+L_k^{(3)}+L_k^{(4)}+L_k^{(5)}+L_k^{(6)}+L_k^{(7)}+L_k^{(8)}+L_k^{(9)}\right\}.$$

While it is sufficient for our purpose to bound the expectations of the first five summands

$$L_k^{(1)},L_k^{(2)},L_k^{(3)},L_k^{(4)},L_k^{(5)}$$

separately, a subtle combination of parts of the last four terms seems necessary to achieve the final goal. Without loss of generality, we may assume

$\alpha > \beta$. The analysis is structured as follows. In Subsections 2.1 – 2.4, bounds on the first five summands are deduced separately. In Subsections 2.5 and 2.6 bounds are achieved by suitable combinations of the remaining summands.

2.1. *Proof of* $\sum_k \mathbb{E}L_k^{(1)} < \infty$. First note that in case $r < X_k, X_{k-1} \le \rho$, we have

$$\begin{aligned} &0 \le (\rho - X_k)(\rho - X_{k-1}) \\ \iff &\exp\left(-\frac{(X_k - 2\rho + X_{k-1})^2}{2\Delta\alpha^2}\right) \le \exp\left(-\frac{(X_k - X_{k-1})^2}{2\Delta\alpha^2}\right) \end{aligned}$$

and analogical

$$\begin{aligned} &0 \le (r - X_k)(r - X_{k-1}) \\ \iff &\exp\left(-\frac{(X_k - 2r + X_{k-1})^2}{2\Delta\beta^2}\right) \le \exp\left(-\frac{(X_k - X_{k-1})^2}{2\Delta\beta^2}\right). \end{aligned}$$

Therefore for $k \ge 2$,

$$\begin{aligned} &\mathbb{E}_{\rho,\alpha,\beta}\left[L_k^{(1)}\right] \\ &\le \mathbb{E}_{\rho,\alpha,\beta}\left[\frac{(X_k - X_{k-1})^2}{2\Delta}\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)\mathbf{1}_{\{r < X_{k-1}, X_k \le \rho\}}\right] \\ &\le \frac{(\rho - r)^2}{2\Delta}\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)\mathbb{P}_{\rho,\alpha,\beta}\left(r < X_{k-1}, X_k \le \rho\right) \\ &= \frac{(\rho - r)^2}{2\Delta}\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)\int_r^\rho\int_r^\rho p_\Delta^\rho(x,y)p_{(k-1)\Delta}^\rho(\rho,x)dydx \\ &\le \frac{(\rho - r)^2}{2\Delta}\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)\int_r^\rho \frac{\rho - r}{\sqrt{2\pi\Delta}\alpha}\frac{2\alpha}{\alpha + \beta}p_{(k-1)\Delta}^\rho(\rho,x)dx \\ &= \frac{(\rho - r)^3}{2\Delta\sqrt{2\pi\Delta}}\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)\frac{2}{\alpha + \beta}\int_r^\rho \frac{1}{\sqrt{2\pi(k-1)\Delta}}\frac{2}{\alpha + \beta}\exp\left(-\frac{(x - \rho)^2}{2(k-1)\Delta\alpha^2}\right)dx \\ &\le \frac{(\rho - r)^3}{2\Delta\sqrt{2\pi\Delta}}\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)\frac{2}{\alpha + \beta}\frac{\rho - r}{\sqrt{2\pi(k-1)\Delta}}\frac{2}{\alpha + \beta} \\ &= \frac{(\rho - r)^4}{4\Delta^2\pi}\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)\frac{4}{(\alpha + \beta)^2}\frac{1}{\sqrt{k - 1}}. \end{aligned}$$

And

$$\mathbb{E}_{\rho,\alpha,\beta}\left[L_1^{(1)}\right]$$

$$
\begin{aligned}
&\le \mathbb{E}_{\rho,\alpha,\beta}\left[\frac{(X_k-\rho)^2}{2\Delta}\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)\mathbf{1}_{\{r<X_1\le\rho\}}\right]\\
&\le \frac{(\rho-r)^2}{2\Delta}\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)\mathbb{P}_{\rho,\alpha,\beta}\left(r<X_1\le\rho\right)\\
&= \frac{(\rho-r)^2}{2\Delta}\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)\int_r^\rho p_\Delta^\rho(\rho,x)\mathrm{d}x\\
&= \frac{(\rho-r)^2}{2\Delta}\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)\frac{2}{\alpha+\beta}\int_r^\rho \frac{1}{\sqrt{2\pi\Delta}}\exp\left(-\frac{(x-\rho)^2}{2\Delta\alpha^2}\right)\mathrm{d}x\\
&\le \frac{(\rho-r)^3}{2\sqrt{2\pi}\Delta^{3/2}}\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)\frac{2}{\alpha+\beta}.
\end{aligned}
$$

For $\rho-r\le 2n^{-3/4}=2\Delta^{3/4}$ and summing over k,

$$
\begin{aligned}
&\sum_{k=1}^n \mathbb{E}_{\rho,\alpha,\beta}\left[L_k^{(1)}\right]\\
&\quad\le \frac{(\rho-r)^3}{2\sqrt{2\pi}\Delta^{3/2}}\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)\frac{2}{\alpha+\beta}+\sum_{k=2}^n\frac{(\rho-r)^4}{4\Delta^2\pi}\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)\frac{4}{(\alpha+\beta)^2}\frac{1}{\sqrt{k-1}}\\
&\quad=\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)\left[\frac{2^3\Delta^{9/4}}{2\sqrt{2\pi}\Delta^{3/2}}\frac{2}{\alpha+\beta}+\sum_{k=2}^n\frac{2^4\Delta^3}{4\Delta^2\pi}\frac{4}{(\alpha+\beta)^2}\frac{1}{\sqrt{k-1}}\right]\\
&\quad=\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)\left[\frac{4\Delta^{3/4}}{\sqrt{2\pi}}\frac{2}{\alpha+\beta}+\frac{4\Delta}{\pi}\frac{4}{(\alpha+\beta)^2}\sum_{k=2}^n\frac{1}{\sqrt{k-1}}\right]\\
&\quad\le\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)\left[\frac{4\Delta^{3/4}}{\sqrt{2\pi}}\frac{2}{\alpha+\beta}+\frac{4\Delta}{\pi}\frac{4}{(\alpha+\beta)^2}\int_0^n\frac{1}{\sqrt{x}}dx\right]\\
&\quad=\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)\left[\frac{4\Delta^{3/4}}{\sqrt{2\pi}}\frac{2}{\alpha+\beta}+\frac{4\Delta}{\pi}\frac{4}{(\alpha+\beta)^2}2\sqrt{n}\right]\\
&\quad=\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)\left[\frac{4\Delta^{3/4}}{\sqrt{2\pi}}\frac{2}{\alpha+\beta}+\frac{8\sqrt{\Delta}}{\pi}\frac{4}{(\alpha+\beta)^2}\right],
\end{aligned}
$$

which is bounded.

2.2. *Proof of* $\sum_k \mathbb{E}L_k^{(2)}<\infty$. In the same way, we arrive at

$$
\begin{aligned}
&\mathbb{E}_{\rho,\alpha,\beta}\left[L_k^{(2)}\right]\\
&\quad\le \mathbb{E}_{\rho,\alpha,\beta}\left\{\frac{1}{2\Delta}\left(\frac{X_k-r}{\beta}-\frac{X_{k-1}-r}{\alpha}\right)^2-\frac{(X_k-X_{k-1})^2}{2\Delta\alpha^2}\right\}\mathbf{1}_{\{X_{k-1}\le r<X_k\le\rho\}}
\end{aligned}
$$

$$\leq \frac{(\rho-r)^2}{2\Delta}\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)\mathbb{P}_{\rho,\alpha,\beta}\left(X_{k-1}\leq r<X_k\leq\rho\right)$$
$$+\frac{\rho-r}{\Delta}\left(\frac{1}{\alpha\beta}-\frac{1}{\alpha^2}\right)\mathbb{E}_{\rho,\alpha,\beta}\left[(r-X_{k-1})\mathbf{1}_{\{X_{k-1}\leq r<X_k\leq\rho\}}\right],$$

where

$$\begin{aligned}
&\mathbb{P}_{\rho,\alpha,\beta}\left(X_{k-1}\leq r<X_k\leq\rho\right)\\
&\quad=\int_{-\infty}^{r}\int_{r}^{\rho}p_{\Delta}^{\rho}(x,y)p_{(k-1)\Delta}^{\rho}(\rho,x)dydx\\
&\quad\leq\frac{1}{\sqrt{2\pi\Delta}}\frac{2}{\alpha+\beta}\int_{-\infty}^{r}\int_{r}^{\rho}\exp\left(-\frac{(y-x)^2}{2\Delta\alpha^2}\right)dyp_{(k-1)\Delta}^{\rho}(\rho,x)dx\\
&\quad\leq\frac{\rho-r}{\sqrt{2\pi\Delta}}\frac{2}{\alpha+\beta}\int_{-\infty}^{r}\exp\left(-\frac{(x-r)^2}{2\Delta\alpha^2}\right)p_{(k-1)\Delta}^{\rho}(\rho,x)dx\\
&\quad=\frac{\rho-r}{\sqrt{2\pi\Delta}}\frac{2}{\alpha+\beta}\int_{-\infty}^{r}\frac{1}{\sqrt{2\pi(k-1)\Delta}}\frac{2}{\alpha+\beta}\exp\left(-\frac{(x-r)^2}{2\Delta\alpha^2}\right)\exp\left(-\frac{(x-\rho)^2}{2(k-1)\Delta\alpha^2}\right)dx\\
&\quad\leq\frac{\rho-r}{\sqrt{2\pi\Delta}}\frac{4\alpha}{(\alpha+\beta)^2}\frac{1}{\sqrt{k-1}}\int_{-\infty}^{r}\frac{1}{\sqrt{2\pi\Delta}\alpha}\exp\left(-\frac{(x-r)^2}{2\Delta\alpha^2}\right)dx\\
&\quad=\frac{\rho-r}{\sqrt{2\pi\Delta}}\frac{2\alpha}{(\alpha+\beta)^2}\frac{1}{\sqrt{k-1}}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_{\rho,\alpha,\beta}\left[(r-X_{k-1})\mathbf{1}_{\{X_{k-1}\leq r<X_k\leq\rho\}}\right]\\
&\quad=\int_{-\infty}^{r}\int_{r}^{\rho}(r-x)p_{\Delta}^{\rho}(x,y)p_{(k-1)\Delta}^{\rho}(\rho,x)dydx\\
&\quad\leq\frac{1}{\sqrt{2\pi\Delta}}\frac{2}{\alpha+\beta}\int_{-\infty}^{r}\int_{r}^{\rho}\exp\left(-\frac{(y-x)^2}{2\Delta\alpha^2}\right)dy(r-x)p_{(k-1)\Delta}^{\rho}(\rho,x)dx\\
&\quad\leq\frac{\rho-r}{\sqrt{2\pi\Delta}}\frac{2}{\alpha+\beta}\int_{-\infty}^{r}(r-x)\exp\left(-\frac{(r-x)^2}{2\Delta\alpha^2}\right)p_{(k-1)\Delta}^{\rho}(\rho,x)dx\\
&\quad=\frac{\rho-r}{\sqrt{2\pi\Delta}}\frac{2}{\alpha+\beta}\int_{-\infty}^{r}\frac{r-x}{\sqrt{2\pi(k-1)\Delta}}\frac{2}{\alpha+\beta}\exp\left(-\frac{(x-r)^2}{2\Delta\alpha^2}\right)\exp\left(-\frac{(x-\rho)^2}{2(k-1)\Delta\alpha^2}\right)dx\\
&\quad\leq\frac{\rho-r}{2\pi}\frac{4\alpha^2}{(\alpha+\beta)^2}\frac{1}{\sqrt{k-1}}\int_{-\infty}^{r}\frac{r-x}{\Delta\alpha^2}\exp\left(-\frac{(x-r)^2}{2\Delta\alpha^2}\right)dx\\
&\quad=\frac{\rho-r}{2\pi}\frac{4\alpha^2}{(\alpha+\beta)^2}\frac{1}{\sqrt{k-1}}.
\end{aligned}$$

Summing over k, we obtain for $\rho - r \leq 2n^{-3/4} = 2\Delta^{3/4}$

$$\begin{aligned}
&\sum_{k=2}^{n} \mathbb{E}_{\rho,\alpha,\beta}\left[L_k^{(2)}\right] \\
&\quad \leq \sum_{k=2}^{n}\left[\frac{(\rho-r)^2}{2\Delta}\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)\frac{\rho-r}{\sqrt{2\pi\Delta}}\frac{2\alpha}{(\alpha+\beta)^2}\frac{1}{\sqrt{k-1}}\right. \\
&\qquad\qquad \left. + \frac{\rho-r}{\Delta}\left(\frac{1}{\alpha\beta}-\frac{1}{\alpha^2}\right)\frac{\rho-r}{2\pi}\frac{4\alpha^2}{(\alpha+\beta)^2}\frac{1}{\sqrt{k-1}}\right] \\
&\quad \leq \frac{2^3\Delta^{\frac{9}{4}}}{2\Delta\sqrt{2\pi\Delta}}\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)\frac{2\alpha}{(\alpha+\beta)^2}2\sqrt{n} \\
&\qquad\qquad + \frac{4\Delta^{\frac{3}{2}}}{2\pi\Delta}\left(\frac{1}{\alpha\beta}-\frac{1}{\alpha^2}\right)\frac{4\alpha^2}{(\alpha+\beta)^2}2\sqrt{n} \\
&\quad = \frac{2^3\Delta^{\frac{1}{4}}}{\sqrt{2\pi}}\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)\frac{2\alpha}{(\alpha+\beta)^2} + \frac{4}{\pi}\left(\frac{1}{\alpha\beta}-\frac{1}{\alpha^2}\right)\frac{4\alpha^2}{(\alpha+\beta)^2},
\end{aligned}$$

which is bounded.

2.3. *Proof of* $\sum_k \mathbb{E}L_k^{(3)} < \infty$. We obtain

$$\begin{aligned}
&\mathbb{E}_{\rho,\alpha,\beta}\left[L_1^{(3)}\right] \\
&\quad \leq \mathbb{E}_{\rho,\alpha,\beta}\left[\left\{\frac{1}{2\Delta}\left(\frac{X_1-r}{\alpha}-\frac{\rho-r}{\beta}\right)^2 - \frac{(X_1-\rho)^2}{2\Delta\alpha^2}\right\}\mathbf{1}_{\{X_1\leq r\}}\right] \\
&\quad \leq \frac{(\rho-r)^2}{2\Delta}\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)\mathbb{P}_{\rho,\alpha,\beta}\left(X_1\leq r\right) \\
&\qquad\qquad + \frac{\rho-r}{\Delta}\left(\frac{1}{\alpha\beta}-\frac{1}{\alpha^2}\right)\mathbb{E}_{\rho,\alpha,\beta}\left[(r-X_1)\mathbf{1}_{\{X_1\leq r\}}\right]
\end{aligned}$$

where

$$\begin{aligned}
&\mathbb{P}_{\rho,\alpha,\beta}\left(X_1\leq r\right) \\
&\quad = \int_{-\infty}^{r} p_\Delta^\rho(\rho,x)dx \\
&\quad = \frac{1}{\sqrt{2\pi\Delta}}\frac{2}{\alpha+\beta}\int_{-\infty}^{r}\exp\left(-\frac{(x-\rho)^2}{2\Delta\alpha^2}\right)dx \\
&\quad = \frac{2\alpha}{\alpha+\beta}\Phi\left(\frac{r-\rho}{\sqrt{\Delta}\alpha}\right) \\
&\quad \leq \frac{\alpha}{\alpha+\beta}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_{\rho,\alpha,\beta}\left[(r - X_1)\mathbf{1}_{\{X_1 \leq r\}}\right] \\
&\quad= \int_{-\infty}^{r} (r-x) p_{\Delta}^{\rho}(\rho, x) dx \\
&\quad= \frac{1}{\sqrt{2\pi\Delta}} \frac{2}{\alpha+\beta} \int_{-\infty}^{r} (r-x) \exp\left(-\frac{(x-\rho)^2}{2\Delta\alpha^2}\right) dx \\
&\quad\leq \frac{1}{\sqrt{2\pi\Delta}} \frac{2}{\alpha+\beta} \int_{-\infty}^{r} (r-x) \exp\left(-\frac{(x-r)^2}{2\Delta\alpha^2}\right) dx \\
&\quad= \frac{\sqrt{\Delta}}{\sqrt{2\pi}} \frac{2\alpha^2}{\alpha+\beta}.
\end{aligned}$$

Thus, for $\rho - r \leq 2n^{-3/4} = 2\Delta^{3/4}$

$$\begin{aligned}
&\mathbb{E}_{\rho,\alpha,\beta}\left[L_1^{(3)}\right] \\
&\quad\leq \frac{(\rho-r)^2}{2\Delta}\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right) \frac{\alpha}{\alpha+\beta} + \frac{\rho - r}{\Delta}\left(\frac{1}{\alpha\beta} - \frac{1}{\alpha^2}\right) \frac{\sqrt{\Delta}}{\sqrt{2\pi}} \frac{2\alpha^2}{\alpha+\beta} \\
&\quad= \frac{2n^{-3/2}}{\Delta}\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right) \frac{\alpha}{\alpha+\beta} + \frac{2n^{-3/4}}{\sqrt{\Delta}}\left(\frac{1}{\alpha\beta} - \frac{1}{\alpha^2}\right) \frac{1}{\sqrt{2\pi}} \frac{2\alpha^2}{\alpha+\beta} \\
&\quad= 2\sqrt{\Delta}\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right) \frac{\alpha}{\alpha+\beta} + 2\Delta^{1/4}\left(\frac{1}{\alpha\beta} - \frac{1}{\alpha^2}\right) \frac{1}{\sqrt{2\pi}} \frac{2\alpha^2}{\alpha+\beta}
\end{aligned}$$

Which is bounded. Further

$$\begin{aligned}
&\mathbb{E}_{\rho,\alpha,\beta}\left[L_k^{(3)}\right] \\
&\leq \mathbb{E}_{\rho,\alpha,\beta}\left[\left\{\frac{1}{2\Delta}\left(\frac{X_k - r}{\alpha} - \frac{X_{k-1} - r}{\beta}\right)^2 - \frac{(X_k - X_{k-1})^2}{2\Delta\alpha^2}\right\}\mathbf{1}_{\{X_k \leq r < X_{k-1} \leq \rho\}}\right] \\
&\leq \frac{(\rho-r)^2}{2\Delta}\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)\mathbb{P}_{\rho,\alpha,\beta}\left(X_k \leq r < X_{k-1} \leq \rho\right) \\
&\qquad + \frac{\rho - r}{\Delta}\left(\frac{1}{\alpha\beta} - \frac{1}{\alpha^2}\right)\mathbb{E}_{\rho,\alpha,\beta}\left[(r - X_k)\mathbf{1}_{\{X_k \leq r < X_{k-1} \leq \rho\}}\right]
\end{aligned}$$

where

$$\begin{aligned}
&\mathbb{P}_{\rho,\alpha,\beta}\left(X_k \leq r < X_{k-1} \leq \rho\right) \\
&\quad= \int_{r}^{\rho} \int_{-\infty}^{r} p_{\Delta}^{\rho}(x, y) p_{(k-1)\Delta}^{\rho}(\rho, x) dy dx \\
&\quad\leq \frac{1}{\sqrt{2\pi\Delta}} \frac{2}{\alpha+\beta} \int_{r}^{\rho} \int_{-\infty}^{r} \exp\left(-\frac{(y-x)^2}{2\Delta\alpha^2}\right) dy p_{(k-1)\Delta}^{\rho}(\rho, x) dx
\end{aligned}$$

$$\leq \frac{2\alpha}{\alpha+\beta}\int_r^\rho \int_{-\infty}^x \frac{1}{\sqrt{2\pi\Delta}\alpha} \exp\left(-\frac{(y-x)^2}{2\Delta\alpha^2}\right) dy p^\rho_{(k-1)\Delta}(\rho,x)dx$$
$$\leq \frac{\rho-r}{\sqrt{2\pi\Delta}} \frac{2\alpha}{(\alpha+\beta)^2}\frac{1}{\sqrt{k-1}}$$

and

$$\mathbb{E}_{\rho,\alpha,\beta}\left[(r-X_k)\mathbf{1}_{\{X_k\leq r<X_{k-1}\leq\rho\}}\right]$$
$$= \int_r^\rho \int_{-\infty}^r (r-y)p^\rho_\Delta(x,y)p^\rho_{(k-1)\Delta}(\rho,x)dydx$$
$$\leq \frac{1}{\sqrt{2\pi\Delta}}\frac{2}{\alpha+\beta}\int_r^\rho \int_{-\infty}^r (r-y)\exp\left(-\frac{(y-x)^2}{2\Delta\alpha^2}\right) dy p^\rho_{(k-1)\Delta}(\rho,x)dx$$
$$\leq \frac{1}{\sqrt{2\pi\Delta}}\frac{2}{\alpha+\beta}\int_r^\rho \int_{-\infty}^r (r-y)\exp\left(-\frac{(y-r)^2}{2\Delta\alpha^2}\right) dy p^\rho_{(k-1)\Delta}(\rho,x)dx$$
$$\leq \frac{\rho-r}{2\pi}\frac{4\alpha^2}{(\alpha+\beta)^2}\frac{1}{\sqrt{k-1}}.$$

For $\rho - r \leq 2n^{-3/4} = 2\Delta^{3/4}$ and summing over k, we get the same bound as the for the second term

$$\sum_{k=2}^n \mathbb{E}_{\rho,\alpha,\beta}\left[L_k^{(3)}\right]$$
$$\leq \sum_{k=2}^n \left[\frac{(\rho-r)^2}{2\Delta}\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)\frac{\rho-r}{\sqrt{2\pi\Delta}}\frac{2\alpha}{(\alpha+\beta)^2}\frac{1}{\sqrt{k-1}} + \frac{\rho-r}{\Delta}\left(\frac{1}{\alpha\beta}-\frac{1}{\alpha^2}\right)\frac{\rho-r}{2\pi}\frac{4\alpha^2}{(\alpha+\beta)^2}\frac{1}{\sqrt{k-1}}\right]$$
$$\leq \frac{2^3\Delta^{\frac{1}{4}}}{\sqrt{2\pi}}\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)\frac{2\alpha}{(\alpha+\beta)^2} + \frac{4}{\pi}\left(\frac{1}{\alpha\beta}-\frac{1}{\alpha^2}\right)\frac{4\alpha^2}{(\alpha+\beta)^2}.$$

2.4. *Proof of* $\sum_k \mathbb{E}L_k^{(4)} < \infty$ *and* $\sum_k \mathbb{E}L_k^{(5)} < \infty$. The terms $L_k^{(4)}$ and $L_k^{(5)}$ can be taken care of as $L_k^{(2)}$ and $L_k^{(3)}$.

2.5. *Proof of* $\sum_k \mathbb{E}(L_k^{(6)} + L_k^{(9)}) < \infty$. In case $X_{k-1}, X_k \leq r$ we have

$$\log\left[\frac{\exp\left(-\frac{(X_k-X_{k-1})^2}{2\Delta\alpha^2}\right) + \frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(X_k-2\rho+X_{k-1})^2}{2\Delta\alpha^2}\right)}{\exp\left(-\frac{(X_k-X_{k-1})^2}{2\Delta\alpha^2}\right) + \frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(X_k-2r+X_{k-1})^2}{2\Delta\alpha^2}\right)}\right]$$
$$\leq \frac{\exp\left(-\frac{(X_k-X_{k-1})^2}{2\Delta\alpha^2}\right) + \frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(X_k-2\rho+X_{k-1})^2}{2\Delta\alpha^2}\right)}{\exp\left(-\frac{(X_k-X_{k-1})^2}{2\Delta\alpha^2}\right) + \frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(X_k-2r+X_{k-1})^2}{2\Delta\alpha^2}\right)} - 1$$

$$= -\frac{\alpha-\beta}{\alpha+\beta}\frac{\exp\left(-\frac{(X_k-2r+X_{k-1})^2}{2\Delta\alpha^2}\right)-\exp\left(-\frac{(X_k-2\rho+X_{k-1})^2}{2\Delta\alpha^2}\right)}{\exp\left(-\frac{(X_k-X_{k-1})^2}{2\Delta\alpha^2}\right)+\frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(X_k-2r+X_{k-1})^2}{2\Delta\alpha^2}\right)}$$

and thus

$$\begin{aligned}
&\mathbb{E}_{\rho,\alpha,\beta}\left[L_k^{(6)}\right]\\
&\leq -\frac{\alpha-\beta}{\alpha+\beta}\int_{-\infty}^{r}\int_{-\infty}^{r}\frac{\exp\left(-\frac{(y-2r+x)^2}{2\Delta\alpha^2}\right)-\exp\left(-\frac{(y-2\rho+x)^2}{2\Delta\alpha^2}\right)}{\exp\left(-\frac{(y-x)^2}{2\Delta\alpha^2}\right)+\frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(y-2r+x)^2}{2\Delta\alpha^2}\right)}p_\Delta^\rho(x,y)p_{(k-1)\Delta}^\rho(\rho,x)dydx\\
&= -\frac{\alpha-\beta}{\alpha+\beta}\int_{-\infty}^{r}\int_{-\infty}^{r}\left\{\exp\left(-\frac{(y-2r+x)^2}{2\Delta\alpha^2}\right)-\exp\left(-\frac{(y-2\rho+x)^2}{2\Delta\alpha^2}\right)\right\}\\
&\qquad\times\frac{1}{\sqrt{2\pi\Delta}\alpha}\frac{\exp\left(-\frac{(y-x)^2}{2\Delta\alpha^2}\right)+\frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(y-2\rho+x)^2}{2\Delta\alpha^2}\right)}{\exp\left(-\frac{(y-x)^2}{2\Delta\alpha^2}\right)+\frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(y-2r+x)^2}{2\Delta\alpha^2}\right)}p_{(k-1)\Delta}^\rho(\rho,x)dydx
\end{aligned}$$

With

$$f(r)=\exp\left(-\frac{(y-x)^2}{2\Delta\alpha^2}\right)+\frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(y-2r+x)^2}{2\Delta\alpha^2}\right)$$

we use the following expansion

$$f(\rho)=f(r)+(\rho-r)f'(\widetilde{r})$$

for some $r\leq\widetilde{r}\leq\rho$, where

$$f'(\widetilde{r})=2\frac{\alpha-\beta}{\alpha+\beta}\frac{y-2\widetilde{r}+x}{\Delta\alpha^2}\exp\left(-\frac{(y-2\widetilde{r}+x)^2}{2\Delta\alpha^2}\right).$$

Therfore,

$$\begin{aligned}
&\mathbb{E}_{\rho,\alpha,\beta}\left[L_k^{(6)}\right]\\
&\leq -\frac{\alpha-\beta}{\alpha+\beta}\int_{-\infty}^{r}\int_{-\infty}^{r}\left\{\exp\left(-\frac{(y-2r+x)^2}{2\Delta\alpha^2}\right)-\exp\left(-\frac{(y-2\rho+x)^2}{2\Delta\alpha^2}\right)\right\}\\
&\qquad\times\frac{1}{\sqrt{2\pi\Delta}\alpha}\frac{f(r)+(\rho-r)f'(\widetilde{r})}{f(r)}p_{(k-1)\Delta}^\rho(\rho,x)dydx\\
&= -\frac{\alpha-\beta}{\alpha+\beta}\int_{-\infty}^{r}\int_{-\infty}^{r}\left\{\exp\left(-\frac{(y-2r+x)^2}{2\Delta\alpha^2}\right)-\exp\left(-\frac{(y-2\rho+x)^2}{2\Delta\alpha^2}\right)\right\}\\
&\qquad\times\frac{1}{\sqrt{2\pi\Delta}\alpha}p_{(k-1)\Delta}^\rho(\rho,x)dydx
\end{aligned}$$

$$+\int_{-\infty}^{r}\int_{-\infty}^{r}\frac{1}{\sqrt{2\pi\Delta}\alpha}\frac{f(\rho)-f(r)}{f(r)}(\rho-r)f'(\widetilde{r})p^{\rho}_{(k-1)\Delta}(\rho,x)dydx$$
$$=: L_k^{(6.1)}+L_k^{(6.2)}.$$

We bound

$$\begin{aligned}
L_k^{(6.1)} &= -\frac{\alpha-\beta}{\alpha+\beta}\int_{-\infty}^{r}\frac{1}{\sqrt{2\pi}}\int_{-\frac{2\rho-r-x}{\sqrt{\Delta}\alpha}}^{-\frac{r-x}{\sqrt{\Delta}\alpha}}\exp\left(-\frac{z^2}{2}\right)dz\,p^{\rho}_{(k-1)\Delta}(\rho,x)dx\\
&\le -\frac{\alpha-\beta}{\alpha+\beta}\frac{2}{\sqrt{2\pi}}\frac{\rho-r}{\sqrt{\Delta}\alpha}\int_{-\infty}^{r}\exp\left(-\frac{(2\rho-r-x)^2}{2\Delta\alpha^2}\right)p^{\rho}_{(k-1)\Delta}(\rho,x)dx\\
&= -2\frac{\alpha-\beta}{\alpha+\beta}\frac{\rho-r}{\sqrt{2\pi\Delta}\alpha}\int_{-\infty}^{r}\exp\left(-\frac{(2\rho-r-x)^2}{2\Delta\alpha^2}\right)p^{\rho}_{(k-1)\Delta}(\rho,x)dx. \qquad (2.2)
\end{aligned}$$

Observe that this term is negative, it will be used to compensate another term. As concerns $L_k^{(6.2)}$, note first that because of $x, y \le r$ we have

$$\begin{aligned}
&0 \le (r-y)(r-x)\\
\iff &\exp\left(-\frac{(y-x)^2}{2\Delta\alpha^2}\right) \ge \exp\left(-\frac{(y-2r+x)^2}{2\Delta\alpha^2}\right).
\end{aligned}$$

Therefore,

$$f(r) \ge \frac{2\alpha}{\alpha+\beta}\exp\left(-\frac{(y-2r+x)^2}{2\Delta\alpha^2}\right)$$

and

$$\begin{aligned}
L_k^{(6.2)} &= \int_{-\infty}^{r}\int_{-\infty}^{r}\frac{1}{\sqrt{2\pi\Delta}\alpha}\frac{(\rho-r)^2f'(\widetilde{r})^2}{f(r)}p^{\rho}_{(k-1)\Delta}(\rho,x)dydx\\
&\le \frac{(\rho-r)^2}{\sqrt{2\pi\Delta}\alpha}\frac{\alpha+\beta}{2\alpha}\int_{-\infty}^{r}\int_{-\infty}^{r}f'(\widetilde{r})^2\exp\left(\frac{(y-2r+x)^2}{2\Delta\alpha^2}\right)p^{\rho}_{(k-1)\Delta}(\rho,x)dydx\\
&= 2\frac{(\rho-r)^2}{\sqrt{2\pi\Delta}\alpha^2}\frac{(\alpha-\beta)^2}{\alpha+\beta}\int_{-\infty}^{r}\int_{-\infty}^{r}\left(\frac{y-2\widetilde{r}+x}{\Delta\alpha^2}\right)^2\exp\left(-\frac{2(y-2\widetilde{r}+x)^2}{2\Delta\alpha^2}\right)\\
&\qquad\qquad\times\exp\left(\frac{(y-2r+x)^2}{2\Delta\alpha^2}\right)p^{\rho}_{(k-1)\Delta}(\rho,x)dydx.
\end{aligned}$$

Furthermore, since

$$(y-2r+x)^2-2(y-2\widetilde{r}+x)^2 = -(y+x-4\widetilde{r}+2r)^2+8(\widetilde{r}-r)^2$$

we have, using $(\widetilde{r}-r)^2 \le (\rho-r)^2$,

$$L_k^{(6.2)} \le 2\frac{(\rho-r)^2}{\sqrt{2\pi\Delta}\alpha^2}\frac{(\alpha-\beta)^2}{\alpha+\beta}\exp\left(4\frac{(\rho-r)^2}{\Delta\alpha^2}\right)$$

$$\times \int_{-\infty}^{r}\int_{-\infty}^{r}\left(\frac{y-2\widetilde{r}+x}{\Delta\alpha^2}\right)^2 \exp\left(-\frac{(y+x-4\widetilde{r}+2r)^2}{2\Delta\alpha^2}\right) p^{\rho}_{(k-1)\Delta}(\rho,x)dydx.$$

Now we consider the inner integral.

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi\Delta}}\int_{-\infty}^{r}\left(\frac{y-2\widetilde{r}+x}{\Delta\alpha^2}\right)^2 \exp\left(-\frac{(y+x-4\widetilde{r}+2r)^2}{2\Delta\alpha^2}\right)dy \\
&\quad\leq \frac{1}{\sqrt{2\pi\Delta}}\int_{-\infty}^{r}\left(\frac{y-2\rho+x}{\Delta\alpha^2}\right)^2 \exp\left(-\frac{(y+x-2r)^2}{2\Delta\alpha^2}\right)dy \\
&\quad= \frac{1}{\sqrt{2\pi\Delta}}\int_{-\infty}^{r}\left(\frac{y+x-2r}{\Delta\alpha^2}-2\frac{\rho-r}{\Delta\alpha^2}\right)^2 \exp\left(-\frac{(y+x-2r)^2}{2\Delta\alpha^2}\right)dy \\
&\quad= \frac{1}{\Delta\alpha^2}\left\{\left(4\frac{(\rho-r)^2}{\Delta\alpha}+\alpha\right)\Phi\left(-\frac{r-x}{\sqrt{\Delta}\alpha}\right)+\frac{4\rho-3r-x}{\sqrt{2\pi\Delta}}\exp\left(-\frac{(x-r)^2}{2\Delta\alpha^2}\right)\right\}.
\end{aligned}$$

Finally,

$$\begin{aligned}
L_k^{(6.2)} &\leq 2\frac{(\rho-r)^2}{\Delta\alpha^4}\frac{(\alpha-\beta)^2}{\alpha+\beta}\exp\left(4\frac{(\rho-r)^2}{\Delta\alpha^2}\right)\int_{-\infty}^{r}\left\{\left(4\frac{(\rho-r)^2}{\Delta\alpha}+\alpha\right)\Phi\left(-\frac{r-x}{\sqrt{\Delta}\alpha}\right)\right. \\
&\qquad\left.+\frac{4\rho-3r-x}{\sqrt{2\pi\Delta}}\exp\left(-\frac{(x-r)^2}{2\Delta\alpha^2}\right)\right\} p^{\rho}_{(k-1)\Delta}(\rho,x)dx \\
&\leq 2\frac{(\rho-r)^2}{\Delta\alpha^4}\frac{(\alpha-\beta)^2}{\alpha+\beta}\exp\left(4\frac{(\rho-r)^2}{\Delta\alpha^2}\right)\int_{-\infty}^{r}\left\{\left(4\frac{(\rho-r)^2}{\Delta\alpha}+\alpha\right)\exp\left(-\frac{(x-r)^2}{2\Delta\alpha^2}\right)\right. \\
&\qquad\left.+\frac{4\rho-3r-x}{\sqrt{2\pi\Delta}}\exp\left(-\frac{(x-r)^2}{2\Delta\alpha^2}\right)\right\}\frac{1}{\sqrt{2\pi(k-1)\Delta}}\frac{2}{\alpha+\beta}dx \\
&= 4\frac{(\rho-r)^2}{\Delta\alpha^4}\frac{(\alpha-\beta)^2}{(\alpha+\beta)^2}\exp\left(4\frac{(\rho-r)^2}{\Delta\alpha^2}\right)\frac{1}{\sqrt{k-1}}\left\{2\frac{(\rho-r)^2}{\Delta}+\frac{\alpha^2}{2}+2\alpha\frac{\rho-r}{\sqrt{2\pi\Delta}}+\frac{\alpha^2}{2\pi}\right\}.
\end{aligned}$$

For $\rho-r\leq 2n^{-3/4}=2\Delta^{3/4}$ and summing up we get

$$\begin{aligned}
\sum_{k=2}^{n} L_k^{(6.2)} &\leq \sum_{k=2}^{n} 4\frac{(\rho-r)^2}{\Delta\alpha^4}\frac{(\alpha-\beta)^2}{(\alpha+\beta)^2}\exp\left(4\frac{(\rho-r)^2}{\Delta\alpha^2}\right)\frac{1}{\sqrt{k-1}}\left\{2\frac{(\rho-r)^2}{\Delta}+\frac{\alpha^2}{2}+2\alpha\frac{\rho-r}{\sqrt{2\pi\Delta}}+\frac{\alpha^2}{2\pi}\right\} \\
&\leq 4\frac{4\Delta^{\frac{3}{2}}}{\Delta\alpha^4}\frac{(\alpha-\beta)^2}{(\alpha+\beta)^2}\exp\left(4\frac{4\Delta^{\frac{3}{2}}}{\Delta\alpha^2}\right)\left\{2\frac{4\Delta^{\frac{3}{2}}}{\Delta}+\frac{\alpha^2}{2}+2\alpha\frac{2\Delta^{\frac{3}{4}}}{\sqrt{2\pi\Delta}}+\frac{\alpha^2}{2\pi}\right\}\sum_{k=2}^{n}\frac{1}{\sqrt{k-1}} \\
&\leq 2^4\frac{\Delta^{\frac{1}{2}}}{\alpha^4}\frac{(\alpha-\beta)^2}{(\alpha+\beta)^2}\exp\left(2^4\frac{\Delta^{\frac{1}{2}}}{\alpha^2}\right)\left\{2^3\Delta^{\frac{1}{2}}+\frac{\alpha^2}{2}+4\alpha\frac{\Delta^{\frac{1}{4}}}{\sqrt{2\pi}}+\frac{\alpha^2}{2\pi}\right\}2\sqrt{n} \\
&= \frac{2^5}{\alpha^4}\frac{(\alpha-\beta)^2}{(\alpha+\beta)^2}\exp\left(2^4\frac{\Delta^{\frac{1}{2}}}{\alpha^2}\right)\left\{2^3\Delta^{\frac{1}{2}}+\frac{\alpha^2}{2}+4\alpha\frac{\Delta^{\frac{1}{4}}}{\sqrt{2\pi}}+\frac{\alpha^2}{2\pi}\right\}
\end{aligned}$$

which is bounded. And for $X_{k-1} \le r < \rho < X_k$ we have

$$\log\left[\frac{\exp\left(-\frac{1}{2\Delta}\left(\frac{X_k-\rho}{\beta}-\frac{X_{k-1}-\rho}{\alpha}\right)^2\right)}{\exp\left(-\frac{1}{2\Delta}\left(\frac{X_k-r}{\beta}-\frac{X_{k-1}-r}{\alpha}\right)^2\right)}\right]$$
$$= -\frac{(\rho-r)}{2\Delta\alpha\beta}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\{\alpha(2X_k-\rho-r)+\beta(\rho+r-2X_{k-1})\}$$

and thus

$$\begin{aligned}
&\mathbb{E}_{\rho,\alpha,\beta}\left[L_k^{(9)}\right]\\
&\quad= 2\frac{\rho-r}{\sqrt{2\pi\Delta}\alpha}\frac{\alpha-\beta}{\alpha+\beta}\int_{-\infty}^{r}\exp\left(-\frac{(x-\rho)^2}{2\Delta\alpha^2}\right)p^{\rho}_{(k-1)\Delta}(\rho,x)dx\\
&\qquad+\frac{(\rho-r)^2}{2\Delta}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)^2\frac{2\beta}{\alpha+\beta}\int_{-\infty}^{r}\left(1-\Phi\left(\frac{\rho-x}{\sqrt{\Delta}\alpha}\right)\right)p^{\rho}_{(k-1)\Delta}(\rho,x)dx\\
&\quad= L_k^{(9.1)}+L_k^{(9.2)}.
\end{aligned}$$

Observe that $L_k^{(9.2)}$ can be treated as $L_k^{(6.2)}$. Moreover, for some $\rho \le \widetilde{r} \le 2\rho - r$ we have

$$\begin{aligned}
&L_k^{(9.1)}+L_k^{(6.1)}\\
&\quad\le 2\frac{\rho-r}{\sqrt{2\pi\Delta}\alpha}\frac{\alpha-\beta}{\alpha+\beta}\int_{-\infty}^{r}\exp\left(-\frac{(x-\rho)^2}{2\Delta\alpha^2}\right)p^{\rho}_{(k-1)\Delta}(\rho,x)dx\\
&\qquad\qquad-2\frac{\alpha-\beta}{\alpha+\beta}\frac{\rho-r}{\sqrt{2\pi\Delta}\alpha}\int_{-\infty}^{r}\exp\left(-\frac{(2\rho-r-x)^2}{2\Delta\alpha^2}\right)p^{\rho}_{(k-1)\Delta}(\rho,x)dx\\
&\quad\le 2\frac{(\rho-r)^2}{\sqrt{2\pi\Delta}\Delta\alpha^3}\frac{\alpha-\beta}{\alpha+\beta}\int_{-\infty}^{r}(2\rho-r-x)\exp\left(-\frac{(x-\rho)^2}{2\Delta\alpha^2}\right)p^{\rho}_{(k-1)\Delta}(\rho,x)dx\\
&\quad\le 2\frac{(\rho-r)^2}{\sqrt{2\pi\Delta}\Delta\alpha^3}\frac{\alpha-\beta}{\alpha+\beta}\frac{2}{\alpha+\beta}\int_{-\infty}^{r}(2\rho-r-x)\frac{1}{\sqrt{2\pi(k-1)\Delta}}\exp\left(-\frac{(x-\rho)^2}{2\Delta\alpha^2}\right)dx\\
&\quad\le 2\frac{(\rho-r)^3}{\sqrt{2\pi\Delta}\Delta\alpha^2}\frac{\alpha-\beta}{\alpha+\beta}\frac{2}{\alpha+\beta}\frac{1}{\sqrt{k-1}}\int_{-\infty}^{\rho}\frac{1}{\sqrt{2\pi\Delta}\alpha}\exp\left(-\frac{(x-\rho)^2}{2\Delta\alpha^2}\right)dx\\
&\qquad\qquad+2\frac{(\rho-r)^2}{2\pi\Delta\alpha}\frac{\alpha-\beta}{\alpha+\beta}\frac{2}{\alpha+\beta}\frac{1}{\sqrt{k-1}}\int_{-\infty}^{\rho}\frac{\rho-x}{\Delta\alpha^2}\exp\left(-\frac{(x-\rho)^2}{2\Delta\alpha^2}\right)dx\\
&\quad=\left\{\frac{(\rho-r)^3}{\sqrt{2\pi\Delta}\Delta\alpha^2}+2\frac{(\rho-r)^2}{2\pi\Delta\alpha}\right\}\frac{\alpha-\beta}{\alpha+\beta}\frac{2}{\alpha+\beta}\frac{1}{\sqrt{k-1}}.
\end{aligned}$$

For $\rho - r \leq 2n^{-3/4} = 2\Delta^{3/4}$ and summing up we get

$$\begin{aligned}
&\sum_{k=2}^{n} L_k^{(9.1)} + L_k^{(6.1)} \\
&\quad \leq \sum_{k=2}^{n} \left\{ \frac{(\rho - r)^3}{\sqrt{2\pi\Delta}\Delta\alpha^2} + 2\frac{(\rho - r)^2}{2\pi\Delta\alpha} \right\} \frac{\alpha - \beta}{\alpha + \beta} \frac{2}{\alpha + \beta} \frac{1}{\sqrt{k-1}} \\
&\quad \leq \left\{ \frac{2^3\Delta^{\frac{9}{4}}}{\sqrt{2\pi\Delta}\Delta\alpha^2} + 2\frac{4\Delta^{\frac{3}{2}}}{2\pi\Delta\alpha} \right\} \frac{\alpha - \beta}{\alpha + \beta} \frac{2}{\alpha + \beta} \sum_{k=2}^{n} \frac{1}{\sqrt{k-1}} \\
&\quad \leq \left\{ \frac{2^3\Delta^{\frac{9}{4}}}{\sqrt{2\pi\Delta}\Delta\alpha^2} + 2\frac{4\Delta^{\frac{3}{2}}}{2\pi\Delta\alpha} \right\} \frac{\alpha - \beta}{\alpha + \beta} \frac{2}{\alpha + \beta} 2\sqrt{n} \\
&\quad = \left\{ \frac{2^3\Delta^{\frac{1}{4}}}{\sqrt{2\pi}\alpha^2} + 2\frac{4}{2\pi\alpha} \right\} \frac{\alpha - \beta}{\alpha + \beta} \frac{4}{\alpha + \beta}
\end{aligned}$$

which is bounded.

2.6. *Proof of* $\sum_k \mathbb{E}(L_k^{(7)} + L_k^{(8)}) < \infty$. First note that for $X_{k-1}, X_k > \rho$, we have

$$\begin{aligned}
&\log \left[\frac{\exp\left(-\frac{(X_k - X_{k-1})^2}{2\Delta\beta^2}\right) - \frac{\alpha-\beta}{\alpha+\beta} \exp\left(-\frac{(X_k - 2\rho + X_{k-1})^2}{2\Delta\beta^2}\right)}{\exp\left(-\frac{(X_k - X_{k-1})^2}{2\Delta\beta^2}\right) - \frac{\alpha-\beta}{\alpha+\beta} \exp\left(-\frac{(X_k - 2r + X_{k-1})^2}{2\Delta\beta^2}\right)} \right] \\
&\quad \leq \frac{\exp\left(-\frac{(X_k - X_{k-1})^2}{2\Delta\beta^2}\right) - \frac{\alpha-\beta}{\alpha+\beta} \exp\left(-\frac{(X_k - 2\rho + X_{k-1})^2}{2\Delta\beta^2}\right)}{\exp\left(-\frac{(X_k - X_{k-1})^2}{2\Delta\beta^2}\right) - \frac{\alpha-\beta}{\alpha+\beta} \exp\left(-\frac{(X_k - 2r + X_{k-1})^2}{2\Delta\beta^2}\right)} - 1 \\
&\quad = -\frac{\alpha - \beta}{\alpha + \beta} \frac{\exp\left(-\frac{(X_k - 2\rho + X_{k-1})^2}{2\Delta\beta^2}\right) - \exp\left(-\frac{(X_k - 2r + X_{k-1})^2}{2\Delta\beta^2}\right)}{\exp\left(-\frac{(X_k - X_{k-1})^2}{2\Delta\beta^2}\right) - \frac{\alpha-\beta}{\alpha+\beta} \exp\left(-\frac{(X_k - 2r + X_{k-1})^2}{2\Delta\beta^2}\right)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbb{E}_{\rho,\alpha,\beta}\left[L_k^{(7)}\right] \\
&\quad \leq -\frac{\alpha - \beta}{\alpha + \beta} \int_\rho^\infty \int_\rho^\infty \frac{\exp\left(-\frac{(y-2\rho+x)^2}{2\Delta\beta^2}\right) - \exp\left(-\frac{(y-2r+x)^2}{2\Delta\beta^2}\right)}{\exp\left(-\frac{(y-x)^2}{2\Delta\beta^2}\right) - \frac{\alpha-\beta}{\alpha+\beta} \exp\left(-\frac{(y-2r+x)^2}{2\Delta\beta^2}\right)} \\
&\qquad\qquad \times p_\Delta^\rho(x,y) p_{(k-1)\Delta}^\rho(\rho, x) dy dx \\
&\quad = -\frac{\alpha - \beta}{\alpha + \beta} \int_\rho^\infty \int_\rho^\infty \left\{ \exp\left(-\frac{(y - 2\rho + x)^2}{2\Delta\beta^2}\right) - \exp\left(-\frac{(y - 2r + x)^2}{2\Delta\beta^2}\right) \right\}
\end{aligned}$$

$$\times \frac{1}{\sqrt{2\pi\Delta}\beta} \frac{\exp\left(-\frac{(y-x)^2}{2\Delta\beta^2}\right) - \frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(y-2\rho+x)^2}{2\Delta\beta^2}\right)}{\exp\left(-\frac{(y-x)^2}{2\Delta\beta^2}\right) + -\frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(y-2r+x)^2}{2\Delta\beta^2}\right)} p^{\rho}_{(k-1)\Delta}(\rho, x)dydx.$$

Now, with

$$g(r) = \exp\left(-\frac{(y-x)^2}{2\Delta\beta^2}\right) - \frac{\alpha-\beta}{\alpha+\beta}\exp\left(-\frac{(y-2r+x)^2}{2\Delta\beta^2}\right)$$

we use the following expansion

$$g(\rho) = g(r) + (\rho - r)g'(\widetilde{r})$$

for some $r \leq \widetilde{r} \leq \rho$, where

$$g'(\widetilde{r}) = -2\frac{\alpha-\beta}{\alpha+\beta}\frac{y-2\widetilde{r}+x}{\Delta\beta^2}\exp\left(-\frac{(y-2\widetilde{r}+x)^2}{2\Delta\beta^2}\right).$$

Therefore, we have

$$\begin{aligned}
&\mathbb{E}_{\rho,\alpha,\beta}\left[L_k^{(7)}\right] \\
&\quad \leq -\frac{\alpha-\beta}{\alpha+\beta}\int_\rho^\infty\int_\rho^\infty\left\{\exp\left(-\frac{(y-2\rho+x)^2}{2\Delta\beta^2}\right) - \exp\left(-\frac{(y-2r+x)^2}{2\Delta\beta^2}\right)\right\} \\
&\qquad\qquad \times \frac{1}{\sqrt{2\pi\Delta}\beta}\frac{g(r)+(\rho-r)g'(\widetilde{r})}{g(r)}p^{\rho}_{(k-1)\Delta}(\rho,x)dydx \\
&\quad = -\frac{\alpha-\beta}{\alpha+\beta}\int_\rho^\infty\int_\rho^\infty\left\{\exp\left(-\frac{(y-2\rho+x)^2}{2\Delta\beta^2}\right) - \exp\left(-\frac{(y-2r+x)^2}{2\Delta\beta^2}\right)\right\} \\
&\qquad\qquad \times \frac{1}{\sqrt{2\pi\Delta}\beta}p^{\rho}_{(k-1)\Delta}(\rho,x)dydx \\
&\qquad + \int_\rho^\infty\int_\rho^\infty \frac{1}{\sqrt{2\pi\Delta}\beta}\frac{g(\rho)-g(r)}{g(r)}(\rho-r)g'(\widetilde{r})p^{\rho}_{(k-1)\Delta}(\rho,x)dydx \\
&\quad =: L_k^{(7.1)} + L_k^{(7.2)}
\end{aligned}$$

We bound

$$\begin{aligned}
L_k^{(7.1)} &= -\frac{\alpha-\beta}{\alpha+\beta}\int_\rho^\infty\frac{1}{\sqrt{2\pi}}\int_{\frac{x-\rho}{\sqrt{\Delta}\beta}}^{\frac{x+\rho-2r}{\sqrt{\Delta}\beta}}\exp\left(-\frac{z^2}{2}\right)dyp^{\rho}_{(k-1)\Delta}(\rho,x)dx \\
&\leq -2\frac{\alpha-\beta}{\alpha+\beta}\int_\rho^\infty\frac{1}{\sqrt{2\pi}}\frac{\rho-r}{\sqrt{\Delta}\beta}\exp\left(-\frac{(x+\rho-2r)^2}{2\Delta\beta^2}\right)p^{\rho}_{(k-1)\Delta}(\rho,x)dx \\
&= -2\frac{\alpha-\beta}{\alpha+\beta}\frac{\rho-r}{\sqrt{2\pi\Delta}\beta}\int_\rho^\infty\exp\left(-\frac{(x+\rho-2r)^2}{2\Delta\beta^2}\right)p^{\rho}_{(k-1)\Delta}(\rho,x)dx.
\end{aligned}$$

Observe that this term is negative, it will be used to compensate another term. As concerns $L_k^{(7.2)}$, because of $x, y > \rho > r$

$$\begin{aligned} &0 \le (y-r)(x-r) \\ \iff &\exp\left(-\frac{(y-x)^2}{2\Delta\beta^2}\right) \ge \exp\left(-\frac{(y-2r+x)^2}{2\Delta\beta^2}\right). \end{aligned}$$

it holds that

$$g(r) \ge \frac{2\beta}{\alpha+\beta}\exp\left(-\frac{(y-2r+x)^2}{2\Delta\beta^2}\right),$$

we obtain

$$\begin{aligned} L_k^{(7.2)} &= \int_\rho^\infty \int_\rho^\infty \frac{1}{\sqrt{2\pi\Delta}\beta} \frac{(\rho-r)^2 g'(\widetilde{r})^2}{g(r)} p^\rho_{(k-1)\Delta}(\rho,x)dydx \\ &\le \frac{(\rho-r)^2}{\sqrt{2\pi\Delta}\beta}\frac{\alpha+\beta}{2\beta}\int_\rho^\infty\int_\rho^\infty g'(\widetilde{r})^2 \exp\left(\frac{(y-2r+x)^2}{2\Delta\beta^2}\right) p^\rho_{(k-1)\Delta}(\rho,x)dydx \\ &= 2\frac{(\rho-r)^2}{\sqrt{2\pi\Delta}\beta^2}\frac{(\beta-\alpha)^2}{\alpha+\beta}\int_\rho^\infty\int_\rho^\infty \left(\frac{y-2\widetilde{r}+x}{\Delta\beta^2}\right)^2 \exp\left(-\frac{2(y-2\widetilde{r}+x)^2}{2\Delta\beta^2}\right) \\ &\qquad \exp\left(\frac{(y-2r+x)^2}{2\Delta\beta^2}\right) p^\rho_{(k-1)\Delta}(\rho,x)dydx. \end{aligned}$$

Finally, as above,

$$\begin{aligned} L_k^{(7.2)} &\le 2\frac{(\rho-r)^2}{\sqrt{2\pi\Delta}\beta^2}\frac{(\beta-\alpha)^2}{\alpha+\beta}\exp\left(4\frac{(\rho-r)^2}{\Delta\beta^2}\right) \\ &\quad \times \int_\rho^\infty\int_\rho^\infty \left(\frac{y-2r+x}{\Delta\beta^2}\right)^2 \exp\left(-\frac{(y+x-4\rho+2r)^2}{2\Delta\beta^2}\right) p^\rho_{(k-1)\Delta}(\rho,x)dydx. \end{aligned}$$

As concerns the inner integral,

$$\begin{aligned} &\frac{1}{\sqrt{2\pi\Delta}}\int_\rho^\infty \left(\frac{y-2r+x}{\Delta\beta^2}\right)^2 \exp\left(-\frac{(y+x-4\rho+2r)^2}{2\Delta\beta^2}\right)dy \\ &\quad = \frac{1}{\Delta\beta^2}\left\{\left(16\frac{(\rho-r)^2}{\Delta\beta}+\beta\right)\left(1-\Phi\left(\frac{x-3\rho+2r}{\sqrt{\Delta}\beta}\right)\right) + \frac{x+5\rho-6r}{\sqrt{2\pi\Delta}}\exp\left(-\frac{(x-3\rho+2r)^2}{2\Delta\beta^2}\right)\right\}, \end{aligned}$$

so that

$$\begin{aligned} L_k^{(7.2)} \le 2\frac{(\rho-r)^2}{\Delta\beta^4}\frac{(\beta-\alpha)^2}{\alpha+\beta}\exp\left(4\frac{(\rho-r)^2}{\Delta\beta^2}\right)\int_\rho^\infty &\left\{\left(16\frac{(\rho-r)^2}{\Delta\beta}+\beta\right)\left(1-\Phi\left(\frac{x-3\rho+2r}{\sqrt{\Delta}\beta}\right)\right)\right. \\ &\left. +\frac{x+5\rho-6r}{\sqrt{2\pi\Delta}}\exp\left(-\frac{(x-3\rho+2r)^2}{2\Delta\beta^2}\right)\right\} p^\rho_{(k-1)\Delta}(\rho,x)dx. \end{aligned}$$

This term can be treated as $L_k^{(6.2)}$. For $X_k \leq r < \rho < X_{k-1}$ we have

$$\log\left[\frac{\exp\left(-\frac{1}{2\Delta}\left(\frac{X_k-\rho}{\alpha}-\frac{X_{k-1}-\rho}{\beta}\right)^2\right)}{\exp\left(-\frac{1}{2\Delta}\left(\frac{X_k-r}{\alpha}-\frac{X_{k-1}-r}{\beta}\right)^2\right)}\right]$$
$$= -\frac{(\rho-r)}{2\Delta\alpha\beta}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\{\beta(\rho+r-2X_k)+\alpha(2X_{k-1}-\rho-r)\}$$

and therefore,

$$\mathbb{E}_{\rho,\alpha,\beta}\left[L_k^{(8)}\right]$$
$$= 2\frac{\rho-r}{\sqrt{2\pi\Delta}\beta}\frac{\alpha-\beta}{\alpha+\beta}\int_\rho^\infty \exp\left(-\frac{1}{2\Delta}\left(\frac{r-\rho}{\alpha}-\frac{x-\rho}{\beta}\right)^2\right)p^\rho_{(k-1)\Delta}(\rho,x)dx$$
$$+\frac{(\rho-r)^2}{2\Delta}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)^2\frac{2\alpha}{\alpha+\beta}\int_\rho^\infty \Phi\left(-\frac{\rho-r}{\sqrt{\Delta}\alpha}-\frac{x-\rho}{\sqrt{\Delta}\beta}\right)p^\rho_{(k-1)\Delta}(\rho,x)dx$$
$$= L_k^{(8.1)}+L_k^{(8.2)}.$$

$L_k^{(8.2)}$ can be bounded with similar arguments as $L_k^{(6.2)}$ and $L_k^{(7.1)}+L_k^{(8.1)}$ as $L_k^{(6.1)}+L_k^{(9.1)}$. □

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