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Weak convergence of sample covariance matrices and testing for seasonal unit roots

Discussion Paper

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Weak Convergence of Sample Covariance Matrices and Testing for Seasonal Unit Roots

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The paper has two main contributions. First, weak convergence results are derived from sampling moments of processes that contains a unit root at an arbitrary frequency, where, in contrast to the previous literature, the proofs are mainly based on algebraic manipulations and well known weak convergence results for martingale difference sequences. These convergence results are used to derive the limiting distribution of the ordinary least squares estimator for unit root autoregressions. As as second contribution, a Phillips-Perron type test for a unit root at an arbitrary frequency is introduced and its limiting distributions are derived. This test is further extended to a joint test for multiple unit roots and seasonal integration. The limiting distributions of these test statistics are asymptotically equivalent to various statistics presented earlier in the seasonal unit root literature.

Keywords: Invariance Principle, Weak Convergence, Seasonal Unit Root, Unit Root Test

1. Introduction

Consider the n-dimensional stochastic process $\{x_t\}_{t\in\mathbb{N}}$ in discrete time generated according to the difference equation

$$\begin{aligned} x_t &= Ax_{t-1} + \eta_t, \quad t \in \mathbb{N}, \\ A &= e^{-i\omega} I_n, \end{aligned} \tag{1}$$

for some frequency $\omega \in (-\pi, \pi]$, where we assume that the the starting value x_0 is $\mathcal{O}_{\mathbb{P}}(1)$ and where $\{\eta_t\}_{t\in\mathbb{Z}}$ is a weakly stationary with mean zero. The process $\{x_t\}_{t\in\mathbb{N}_0}$ is called *integrated* at frequency ω or, since $e^{i\omega}$ is the root of the equation $1 - e^{-i\omega}z = 0$, it is also called *unit root process.*

The limiting distributions of the sample covariance matrices $\frac{1}{T^2}\sum_{t=1}^{T} x_t x_t^*$ and $\frac{1}{T}\sum_{t=1}^{T} x_{t-1}\eta_t^*$ are important building blocks in the derivation of an asymptotic theory for unit root test statistics as well as for inference in cointegrating systems. If the process $\{\eta_t\}_{t\in\mathbb{Z}}$ fulfills a functional central limit theorem the limiting distribution of the former can be easily derived by an application of the continuous mapping theorem. The limiting distribution of the latter is more complicated. In case of $\omega = 0$ Phillips (1988b) showed under very general conditions on the process $\{\eta_t\}_{t\in\mathbb{Z}}$ that

$$\frac{1}{T}\sum_{t=1}^{T} x_{t-1}\eta_t \Rightarrow \int_0^1 B(r)\,dB(r) + \Lambda_0,$$

as $T \to \infty$, where $B(r)$ is a vector Brownian motion with covariance matrix given by the long-run variance matrix of $\{\eta_t\}_{t\in\mathbb{Z}}$. The additive bias term Λ_0 defined as the sum of all $\mathbb{E}(\eta_0\eta_h')$ over $h \in \mathbb{N}$ and is therefore also called one sided long-run covariance matrix.

Phillips' proof, however, is quite long and one needs a very deep understanding of certain concepts from probability theory to be able to follow it. Therefore, Phillips (1988a) presented a much simpler proof under marginally more restrictive assumptions. In particular, he requires $\{\eta_t\}_{t\in\mathbb{Z}}$ to be a linear process of the form

$$\eta_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j},$$

with $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ being an i.i.d. sequence with zero mean and finite variance and where the coefficient matrices satisfy

$$\sum_{j=1}^{\infty}\left\{\left\|\sum_{k=j}^{\infty}\psi_k\right\| + \left\|\sum_{k=j}^{\infty}\psi_{-k}\right\|\right\} < \infty. \tag{2}$$

Gregoir (2010) relaxed the i.i.d. assumption on $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ and extended Phillips' approach by deriving the limiting distribution for arbitrary values of ω.

The drawback of the proofs of Phillips (1988a) and Gregoir (2010) is that they are based on the martingale approximation theory of Hall and Heyde (1980), with which many researchers are not familiar with. Thus, one of the aims of this paper is to derive the same results, but without making use of this theory. Instead, we use a decomposition of $\{x_t\}_{t\in\mathbb{N}_0}$ which is based on the so-called Beveridge-Nelson decompisition, and derive a functional central limit theorem following the approach of Phillips and Solo (1992). Furthermore, this decomposition allows us to decompose the sample covariance matrix in such a way that we can derive its asymptotic distribution with simple algebraic transformations and apply well known convergence results for martingale difference sequences. As the only additional assumption we demand that the process $\{\eta_t\}_{t\in\mathbb{Z}}$ is a causal with respect to $\{\varepsilon_t\}_{t\in\mathbb{Z}}$.

We derive the asymptotic distribution of the OLS estimator for A in the regression model (1) as a direct application. With this result in place we generalize the approach of Phillips (1987) and Phillips and Perron (1988) and modify the OLS estimator so that the limiting distribution is free of nuisance parameters. We then use this modified estimator to construct a test for unit roots at any given frequency ω. As an extension of this test we present a joint test for multiple unit roots and for seasonal integration, similar to the tests of Hylleberg et al. (1990) and Ghysels et al. (1994).

The remainder of this paper is organized as follows: In Section 2 we state the precise assumptions and present the decomposition mentioned above. In Section 3 we derive the functional central limit theorem and the limiting distributions of the sample covariance matrices. Section 4 contains the tests for unit roots and seasonal integration. Section 5 concludes. Appendix A contains some auxiliary algebraic results. The proofs of the main mathematical results are relegated to Appendix B.

Throughout the paper we use the following notation: Weak convergence is denoted by $\Rightarrow$ and convergence in probability is signified by $\xrightarrow{\mathbb{P}}$. For convergence in probability to zero we use the small O notation $o_{\mathbb{P}}(1)$ whereas we use $\mathcal{O}_{\mathbb{P}}(1)$ to indicate stochastic boundedness. The integer part of a real number x is given by $[x]$ and the modulus of a complex number $x = \mathrm{Re}(x) + i\,\mathrm{Im}(x)$ is denoted by $|x|$. We use the notation $\|x\|$ to signify the Frobenius norm. For a (possibly complex valued) matrix A we denote its transpose, complex conjugate and Hermitian transpose by A', $\overline{A}$ and A^*, respectively. With L and Δ_ω we denote the lag operator and the seasonal first difference operator, respectively and we use the somewhat sloppy notations $Lx_t = x_{t-1}$ and $\Delta_\omega x_t = x_t - e^{-i\omega}x_{t-1}$.

2. Setup, Assumptions and Decomposition of Unit Root Processes

As mentioned in the introduction, we consider processes generated according to (1) with x_0 being $\mathcal{O}_{\mathbb{P}}(1)$ and $\{\eta_t\}_{t\in\mathbb{Z}}$ satisfying the following assumption.

Assumption 1. The process $\{\eta_t\}_{t\in\mathbb{Z}}$ is a linear process of the form

$$\eta_t = \Psi(L)\varepsilon_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \tag{3}$$

where $\det(\Psi(e^{i\omega})) \neq 0$ and where the coefficient matrices $\psi_j \in \mathbb{C}^{n\times n}$ satisfy the summability condition

$$\sum_{j=0}^{\infty} j\|\psi_j\| < \infty. \tag{4}$$

The innovation process $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a martingale difference sequence with respect to its canonical filtration $\mathcal{F}_t = \sigma\{\varepsilon_{t-j}, j \in \mathbb{N}_0\}$ satisfying $\mathbb{E}(\varepsilon_t\varepsilon_t'|\mathcal{F}_{t-1}) = I_n$ and $\sup_t \mathbb{E}(\|\varepsilon_t\|^{2+\delta}|\mathcal{F}_{t-1}) < \infty$ with probability one for some $\delta > 0$.

Remark 1. The summability condition (4) is common in the unit root literature, as it is, for instance, fulfilled by all causal, stationary and invertible ARMA processes. In particular, since

$$\sum_{j=1}^{\infty}\left\{\left\|\sum_{k=j}^{\infty}\psi_k\right\| + \left\|\sum_{k=j}^{\infty}\psi_{-k}\right\|\right\} = \sum_{j=1}^{\infty}\left\|\sum_{k=j}^{\infty}\psi_k\right\| \leq \sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\|\psi_k\| \leq \sum_{j=0}^{\infty} j\|\psi_j\|,$$

it implies the previously mentioned summability condition (2).

Remark 2. The assumptions stated on the sequence $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ are quite general and are widely applied in the literature. However, the restriction on the (conditional) covariance matrix is imposed only for notational simplicity and can of course be relaxed by assuming that $\mathbb{E}(\varepsilon_t\varepsilon_t'|\mathcal{F}_{t-1}) = \Sigma_\varepsilon$ where Σ_ε is positive definite.

Under Assumption 1 the process $\{\eta_t\}_{t\in\mathbb{Z}}$ has a continuous spectral density, $f(\omega)$ say, and we define

$$\Omega_\omega = 2\pi f(\omega) = \sum_{h=-\infty}^{\infty} e^{-i\omega h}\mathbb{E}(\eta_0\eta_h^*) = \sum_{h=-\infty}^{\infty} e^{-i\omega h}\sum_{j=0}^{\infty}\psi_j\psi_{j+h}^*. \tag{5}$$

Note that $\Omega_\omega = \Psi(e^{i\omega})\Psi(e^{i\omega})^*$. Furthermore, it holds that $\Omega_\omega = \Sigma + \Lambda_\omega + \Lambda_\omega^*$, where

$$\Sigma = \mathbb{E}(\eta_0\eta_0^*) = \sum_{j=0}^{\infty}\psi_j\psi_j^* \tag{6}$$

and

$$\Lambda_\omega = \sum_{h=1}^{\infty} e^{-i\omega h}\mathbb{E}(\eta_0\eta_h^*) = \sum_{h=1}^{\infty} e^{-i\omega h}\sum_{j=0}^{\infty}\psi_j\psi_{j+h}^*. \tag{7}$$

If $\omega = 0$ it is well known that the process $\{x_t\}_{t\in\mathbb{N}_0}$ can be decomposed into a pure random walk, a stationary component and an initial value component. The following result generalizes this decomposition to the arbitrary frequency case.

Proposition 1. *Let $\{x_t\}_{t\in\mathbb{N}_0}$ be a stochastic process in discrete time generated according to the difference equation* (1) *with Assumption 1 in place. Then, it holds that*

$$x_t = e^{-i\omega t}(x_0 + \tilde{\eta}_0) + \Psi(e^{i\omega})e^{-i\omega t}\sum_{j=1}^{t} e^{i\omega j}\varepsilon_j - \tilde{\eta}_t, \quad t = 1, 2\ldots,$$

where $\{\tilde{\eta}_t\}_{t\in\mathbb{Z}}$ is a weakly stationary process with moving average representation

$$\tilde{\eta}_t = \tilde{\Psi}(L)\varepsilon_t = \sum_{j=0}^{\infty} \tilde{\psi}_j \varepsilon_{t-j}, \quad \tilde{\psi}_j = e^{-i\omega j} \sum_{k=j+1}^{\infty} e^{i\omega k}\psi_k.$$

Remark 3. The proof of Proposition 1 is essentially an application of the so-called Beveridge-Nelson decomposition at frequency ω. It states that a matrix polynomial $A(z)$ with matrix coefficients A_j satisfying $\sum_{j=0}^{\infty} j\|A_j\| < \infty$ can be decomposed into

$$A(z) = A(e^{i\omega}) - (1 - e^{-i\omega}z)B(z),$$

where $B(z)$ is a matrix polynomial with absolutely summable matrix coefficients (cf. Phillips and Solo, 1992). We present a simple algebraic proof of this decomposition in Appendix A.

3. Convergence of Sample Covariance Matrices

In this section we present a functional central limit theorem as well as several results on the limiting distributions of sample covariance matrices of integrated processes at some arbitrary frequency. As our main contribution we extend the result of Phillips (1988a) for processes that are integrated at some arbitrary frequency. The following lemma is the central building block for the subsequent results.

Lemma 1. *Let $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ be a martingale difference sequence that satisfies Assumption 1. Then, as $T \to \infty$, it holds that*

$$\left(\frac{1}{\sqrt{T}} \sum_{t=1}^{[rT]} e^{i\omega t}\varepsilon_t, \frac{1}{T} \sum_{t=1}^{[rT]} e^{-i\omega t} \sum_{j=1}^{t-1} e^{i\omega j}\varepsilon_j \varepsilon_t' \right) \Rightarrow \left(\tau_\omega W(r), \tau_\omega^2 \int_0^r W(s)\, dW(s)^* \right).$$

where

$$\tau_\omega = \begin{cases} 1 & \text{if } \omega \in \{0, \pi\}, \\ \frac{1}{\sqrt{2}}, & \text{if } \omega \in (-\pi, 0) \cup (0, \pi) \end{cases} \tag{8}$$

and $W(r)$ is an n-dimensional standard Brownian motion if $\omega \in \{0, \pi\}$ and an n-dimensional standard complex Brownian motion if $\omega \in (-\pi, 0) \cup (0, \pi)$, i.e. $W(r) = W_1(r) + iW_2(r)$ with independent n-dimensional (real valued) standard Brownian motions $W_1(r)$ and $W_2(r)$.

Our first main result is a functional central limit theorem for processes that are integrated at an arbitrary frequency.

Theorem 1. *Let $\{x_t\}_{t\in\mathbb{N}_0}$ be a stochastic process in discrete time generated according to the difference equation* (1) *with Assumption 1 in place. Then, as $T \to \infty$, it holds that*

$$\frac{e^{i\omega[rT]}}{\sqrt{T}} x_{[rT]} \Rightarrow \tau_\omega B(r), \quad r \in (0,1],$$

where $B(r) = \Psi(e^{i\omega})W(r)$ with τ_ω and $W(r)$ defined in Lemma 1.

Theorem 1 can be extended to the following joint convergence result without any additional effort. Let $\{x_{t,k}\}_{t\in\mathbb{N}_0}$, $k = 1,\dots,K$, be n-dimensional processes generated according to $x_{t,k} = e^{-i\omega_k}x_{t-1,k} + \eta_t$ with $\omega_k \neq \omega_j$ for all $k \neq j$. Then, as $T \to \infty$,

$$\left[\frac{e^{i\omega_1[rT]}}{\sqrt{T}} x_{[rT],1}, \dots, \frac{e^{i\omega_K[rT]}}{\sqrt{T}} x_{[rT],K}\right] \Rightarrow \left[\tau_{\omega_1}B_1(r), \dots, \tau_{\omega_K}B_K(r)\right],$$

where $B_k(r) = \Psi(e^{i\omega_k})W_k(r)$ for $k = 1,\dots,K$ and $W_1(r),\dots,W_K(r)$ are independent Brownian motions, complex valued if the corresponding frequency ω_k is different from zero or π. Furthermore, Theorem 1 can be generalized for the weak convergence of the cumulative sum of $e^{i\omega t}x_t$. In particular, it holds that

$$\frac{1}{T^{3/2}} \sum_{t=1}^{[rT]} e^{i\omega t} x_t \Rightarrow \tau_\omega \int_0^r B(s)\,ds,$$

as $T \to \infty$, which is a direct consequence of the continuous mapping theorem. This result can be extended to multiple cumulative summation.

Corollary 1. *Let $\{x_t\}_{t\in\mathbb{N}_0}$ be a stochastic process in discrete time generated according to the difference equation* (1) *with Assumption 1 in place. Then, as $T \to \infty$, it holds that*

$$\frac{1}{T^{(2m+1)/2}} \sum_{t_1=1}^{[rT]} \sum_{t_2=1}^{t_1} \cdots \sum_{t_m=1}^{t_{m-1}} e^{i\omega t_m} x_{t_m} \Rightarrow \tau_\omega \int_0^r \int_0^{s_1} \cdots \int_0^{s_{m-1}} B(s_m)\,ds_m\,ds_{m-1}\dots ds_1,$$

for any $m \in \mathbb{N}$, where the process limiting process $B(r)$ is defined in Theorem 1.

The subsequent proposition states the limiting distribution of the sample covariance matrix between two processes that are integrated at the same frequency as well as the asymptotic orthogonality of two processes that are integrated at different frequencies. The former statement follows again from Theorem 1 and the continuous mapping theorem whereas the latter is an algebraic consequence of the fact that $\sum_{t=1}^{T} e^{i\theta t}$ is bounded if and only if θ is different from zero (cf. Lemma A.1 in the appendix).

Proposition 2. *Let* $\{x_{t,1}\}_{t\in\mathbb{N}_0}$ *and* $\{x_{t,2}\}_{t\in\mathbb{N}_0}$ *be two* n*-dimensional stochastic process, generated according to the difference equations*

$$x_{t,1} = e^{-i\omega_1}x_{t-1,1} + \eta_t,$$
$$x_{t,2} = e^{-i\omega_2}x_{t-1,2} + \eta_t$$

for $t \in \mathbb{N}$*, where* $\{\eta_t\}_{t\in\mathbb{Z}}$ *is a stationary process that satisfies Assumption 1 and the starting values* $x_{0,1}$ *and* $x_{0,2}$ *are* $\mathcal{O}_{\mathbb{P}}(1)$*.*

If $\omega_1 = \omega_2$ *then, as* $T \to \infty$ *it holds that*

$$\frac{1}{T^2}\sum_{t=1}^{T} x_{t,1}x_{t,2}^* \Rightarrow \tau_{\omega_1}^2 \int_0^1 B(r)B(r)^*\,dr,$$

with $B(r) = \Psi(e^{i\omega_1})W(r)$ *being the limiting process from Theorem 1.*

If $\omega_1 \neq \omega_2$ *then, as* $T \to \infty$*, it holds that*

$$\frac{1}{T^2}\sum_{t=1}^{T} x_{t,1}x_{t,2}^* \xrightarrow{\mathbb{P}} 0. \tag{9}$$

Remark 4. Proposition 2 can easily be generalized to covariance matrices of more than two integrated processes as follows. For $k = 1, \ldots, K$ let $\{x_{t,k}\}_{t\in\mathbb{N}_0}$ be n-dimensional processes where for every k the process $\{x_{t,k}\}_{t\in\mathbb{N}_0}$ is generated according to $x_{t,k} = e^{-i\omega_k}x_{t-1,k} + \eta_t$ with $x_{0,k}$ being $\mathcal{O}_{\mathbb{P}}(1)$ and where $\omega_k \neq \omega_j$ for all $k \neq j$. Define

$$X = \begin{bmatrix} x_{1,1} & x_{1,2} & \ldots & x_{1,K} \\ \vdots & \vdots & & \vdots \\ x_{T,1} & x_{T,2} & \ldots & x_{T,K} \end{bmatrix}.$$

Then, as $T \to \infty$, it holds that

$$\frac{1}{T^2}(X^*X) \Rightarrow \begin{bmatrix} \tau_{\omega_1}^2 \int_0^1 B_1(r)B_1(r)^*\,dr & 0 & \ldots & 0 \\ 0 & \tau_{\omega_2}^2 \int_0^1 B_2(r)B_2(r)^*\,dr & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \tau_{\omega_K}^2 \int_0^1 B_K(r)B_K(r)^*\,dr \end{bmatrix},$$

where $B_k(r) = \Psi(e^{i\omega_k})W_k(r)$ for $k = 1, \ldots, K$ and $W_1(r), \ldots, W_K(r)$ are independent Brownian motions, complex valued if the corresponding frequency ω_k is different from zero or π.

Remark 5. The statement of Proposition 2 holds also for processes $\{x_{t,1}\}_{t\in\mathbb{N}_0}$ and $\{x_{t,2}\}_{t\in\mathbb{N}_0}$ that are generated according to the difference equation (1) but with distinct processes $\{\eta_{t,1}\}_{t\in\mathbb{Z}}$ and $\{\eta_{t,2}\}_{t\in\mathbb{Z}}$, i.e.

$$\begin{aligned} x_{t,1} &= e^{-i\omega_1} x_{t-1,1} + \eta_{t,1} \\ x_{t,2} &= e^{-i\omega_2} x_{t-1,2} + \eta_{t,2}, \end{aligned}$$

for $t \in \mathbb{N}$ with starting values $x_{0,1}$ and $x_{0,2}$ being $\mathcal{O}_{\mathbb{P}}(1)$. If the stacked process $\{[\eta_{t,1}', \eta_{t,2}']'\}_{t\in\mathbb{Z}}$ is stationary and fulfills Assumption 1 then it holds that

$$\frac{1}{\sqrt{T}} \begin{bmatrix} e^{i\omega_1 [rT]} x_{[rT],1} \\ e^{i\omega_2 [rT]} x_{[rT],2} \end{bmatrix} \Rightarrow \begin{bmatrix} B_1(r) \\ B_2(r) \end{bmatrix}, \quad r \in (0,1],$$

and, consequently, if $\omega_1 = \omega_2$ we obtain as $T \to \infty$,

$$\frac{1}{T^2} \sum_{t=1}^{T} x_{t,1} x_{t,2}^* \Rightarrow \int_0^1 B_1(r) B_2(r)^* \, dr.$$

whereas if $\omega_1 \neq \omega_2$ it holds that

$$\frac{1}{T^2} \sum_{t=1}^{T} x_{t,1} x_{t,2}^* \xrightarrow{\mathbb{P}} 0.$$

The statement in Remark 4 can be extended in a similar way.

By the same arguments as in the proof of Proposition 2 we can derive the limiting distribution of the sample covariance matrix between a process integrated at some frequency ω and a deterministic sequence.

Corollary 2. *Let $\{x_t\}_{t\in\mathbb{N}_0}$ be generated as in Theorem 1 and let $\{d_t\}_{t\in\mathbb{N}_0}$ be a p-dimensional deterministic sequence with such that $G_D^{-1} e^{i\theta[rT]} D_{[rT]} \Rightarrow D(r)$, as $T \to \infty$, for some $\theta \in (-\pi, \pi]$, where $G_D \in \mathbb{R}^{p\times p}$ is a scaling matrix and $D(r)$ is a càdlàg function.*

If $\theta = \omega$ then, as $T \to \infty$, it holds that

$$\frac{1}{T^{3/2}} G_D^{-1} \sum_{t=1}^{T} d_t x_t^* \Rightarrow \tau_\omega \int_0^1 D(r) B(r)^* \, dr.$$

If $\theta \neq \omega$ then, as $T \to \infty$, it holds that

$$\frac{1}{T^{3/2}} G_D^{-1} \sum_{t=1}^{T} d_t x_t^* \xrightarrow{\mathbb{P}} 0.$$

An important example for a deterministic sequence that satisfy the Assumptions in the Corollary is $\{d_t\}_{t\in\mathbb{N}_0}$, where $d_t = e^{-i\theta t} f_t$ with

$$f_t = [1, t, t^2, \ldots, t^q]'.$$

Then, with $G_D = \text{diag}(1, T, T^2, \ldots, T^q)$ it holds that

$$G_D^{-1} e^{i\theta[rT]} d_{[rT]} = \left[1, \frac{[rT]}{T}, \left(\frac{[rT]}{T}\right)^2, \ldots, \left(\frac{[rT]}{T}\right)^q\right]' \Rightarrow [1, r, r^2, \ldots, r^q]'.$$

Hence, by setting $\theta = 0$, it follows that the sequence of monomials $d_t = [1, t, t^2, \ldots, t^q]'$ is asymptotically orthogonal to any process $\{x_t\}_{t\in\mathbb{N}_0}$ that is integrated at some frequency $\omega \neq 0$.

Next, we discuss the limiting distribution of the sample covariance between x_{t-1} and η_t in model (1), which is the main contribution of this section. If $\{x_t\}_{t\in\mathbb{N}_0}$ is scalar Phillips (1987) showed that the limiting distribution can be easily calculated using the identity

$$x_t^2 = (x_{t-1} + \eta_t)^2 = x_{t-1}^2 + \eta_t^2 + 2x_{t-1}\eta_t.$$

In particular, it holds that

$$\frac{1}{T}\sum_{t=1}^{T} x_{t-1}\eta_t = \frac{1}{2T}\sum_{t=1}^{T}(x_t^2 - x_{t-1}^2) - \frac{1}{2T}\sum_{t=1}^{T}\eta_t^2 = \frac{1}{2T}(x_T^2 - x_0^2) - \frac{1}{2T}\sum_{t=1}^{T}\eta_t^2.$$

The weak law of large numbers implies that the latter term converges to $\Sigma/2$ and it holds that x_0^2/T^2 converges to zero in probability as the starting value x_0 is $\mathcal{O}_{\mathbb{P}}(1)$. Theorem 1, the continuous mapping theorem and Itô's Lemma yield

$$\frac{1}{2T}x_T^2 \Rightarrow \frac{1}{2}B(1)^2 = \frac{\Omega_0}{2}W(1)^2 = \frac{\Omega_0}{2}(W(1)^2 - 1) + \frac{\Omega_0}{2} = \Omega_0\int_0^1 W(r)\,dW(r) + \frac{\Omega_0}{2}.$$

From $\Omega_0 = \Sigma + 2\Lambda_0$ we conclude that

$$\frac{1}{T}\sum_{t=1}^{T} x_{t-1}\eta_t \Rightarrow \Omega_0\int_0^1 W(r)\,dW(r) + \frac{\Omega_0}{2} - \Sigma = \int_0^1 B(r)\,dB(r) + \Lambda_0. \tag{10}$$

Similarly, we can derive the limiting distribution for $\{x_t\}_{t\in\mathbb{N}_0}$ being scalar and generated according to (1) with $\omega = \pi$. In this case it holds that

$$x_t^2 = x_{t-1}^2 - 2x_{t-1}\eta_t + \eta_t^2$$

and, using exactly the same arguments as above, we deduce that

$$\frac{1}{T}\sum_{t=1}^{T} x_{t-1}\eta_t = -\left(\frac{1}{2T}\left(x_T^2 - x_0^2\right) - \frac{1}{2T}\sum_{t=1}^{T}\eta_t^2\right) \Rightarrow -\int_0^1 B(r)\,dB(r) - \Lambda_\pi. \tag{11}$$

We cannot apply this approach when $\omega \in (-\pi, 0) \cup (0, \pi)$ since in this case it holds that

$$x_t\overline{x}_t = (e^{-i\omega}x_{t-1} + \eta_t)(e^{i\omega}\overline{x}_{t-1} + \overline{\eta}_t) = x_{t-1}\overline{x}_{t-1} + \eta_t\overline{\eta}_t + e^{-i\omega}x_{t-1}\overline{\eta}_t + e^{i\omega}\eta_t\overline{x}_{t-1}.$$

Hence, as $T \to \infty$,

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}(e^{-i\omega}x_{t-1}\overline{\eta}_t + e^{i\omega}\eta_t\overline{x}_{t-1}) &= \frac{1}{T}\sum_{t=1}^{T}(x_t\overline{x}_t - x_{t-1}\overline{x}_{t-1}) - \frac{1}{T}\sum_{t=1}^{T}\eta_t\overline{\eta}_t \\ &= \frac{1}{T}x_T\overline{x}_T - \frac{1}{T}x_0\overline{x}_0 - \frac{1}{T}\sum_{t=1}^{T}\eta_t\overline{\eta}_t \\ &\Rightarrow B(1)\overline{B}(1) - \Sigma.\end{aligned}$$

Without any effort, for multivariate $\{x_t\}_{t\in\mathbb{N}_0}$ we obtain analogously

$$\frac{1}{T}\sum_{t=1}^{T}(e^{-i\omega}x_{t-1}\eta_t^* + e^{i\omega}\eta_t x_{t-1}^*) \Rightarrow B(1)B(1)^* - \Sigma. \tag{12}$$

By an application of the multivariate integration-by-parts formula for Brownian motions[1] and noting that $\Psi(e^{i\omega})\Psi(e^{i\omega})^* - \Sigma = \Lambda_\omega + \Lambda_\omega^*$ we can rewrite (12) as

$$B(1)B(1)^* - \Sigma = \int_0^1 B(r)\,dB(r)^* + \int_0^1 dB(r)\,B(r)^* + \Lambda_\omega + \Lambda_\omega^*.$$

Whilst the above considerations lead one to expect that

$$\frac{1}{T}\sum_{t=1}^{T} e^{-i\omega}x_{t-1}\eta_t^* \Rightarrow \int_0^1 B(r)\,dB(r)^* + \Lambda_\omega, \tag{13}$$

[1]The integration-by-parts formula also applies for complex Brownian motions. Let $V(r) = V_1(r) + iV_2(r)$ and $Z(r) = Z_1(r) + iZ_2(r)$ be two complex Brownian motions. Then, by the definition of the complex Itô-Integral it holds that

$$\int_0^1 V(r)\,dZ(r)^* = \int_0^1 V_1(r)\,dZ_1(r) + \int_0^1 V_2(r)\,dZ_2(r) + i\int_0^1 V_2(r)\,dZ_1(r) - i\int_0^1 V_1(r)\,dZ_2(r).$$

The complex integration-by-parts formula follows from an application of the multivariate real integration-by-parts formula for each of the integrals and rearranging the resulting terms.

as $T \to \infty$, this claim cannot be deduced from (12). This is similar to the case where $\{x_t\}_{t\in\mathbb{N}_0}$ is multivariate with $\omega \in \{0, \pi\}$. In particular, if $\omega = 0$ it holds that

$$x_t x_t' = (x_{t-1} + \eta_t)(x_{t-1} + \eta_t)' = x_{t-1}x_{t-1}' + \eta_t\eta_t' + x_{t-1}\eta_t' + \eta_t x_{t-1}'$$

and, instead of a multivariate version of (10), we now obtain

$$\frac{1}{T}\sum_{t=1}^{T}(x_{t-1}\eta_t' + \eta_t x_{t-1}') \Rightarrow B(1)B(1)' - \Sigma. \tag{14}$$

If $\omega = \pi$ it holds that $x_t x_t' = x_{t-1}x_{t-1}' - x_{t-1}\eta_t - \eta_t x_{t-1} + \eta_t'$ and, hence,

$$-\frac{1}{T}\sum_{t=1}^{T}(x_{t-1}\eta_t' + \eta_t x_{t-1}') \Rightarrow B(1)B(1)' - \Sigma. \tag{15}$$

Phillips (1988a,b) has proven (13) for $\omega = 0$. The general result for arbitrary frequencies is the main result of this section.

Theorem 2. *Let $\{x_t\}_{t\in\mathbb{N}_0}$ be a stochastic process in discrete time generated according to the difference equation* (1) *with Assumption 1 in place. Then, as $T \to \infty$, it holds that*

$$\frac{1}{T}\sum_{t=1}^{T} x_{t-1}\eta_t^* \Rightarrow e^{i\omega}\left(\tau_\omega^2 \int_0^1 B(r)\,dB(r)^* + \Lambda_\omega\right), \tag{16}$$

where τ_ω and Λ_ω are introduced in (8) *and* (7)*, respectively and $B(r)$ is defined in Theorem 1.*

Remark 6. As mentioned in the introduction this result has also been established by Gregoir (2010). However, his proof is a generalization of the proof of Phillips (1988a) and therefore it crucially relies on the martingale approximation of Hall and Heyde (1980). Our proof of Theorem 2 is much simpler as we only require weak convergence results for martingale difference sequences, presented in Lemma 1, and the decomposition stated in Proposition 1.

By the same arguments as in the proof of Theorem 2 we can also derive the limiting distribution of the sample covariance matrix between $\{\eta_t\}_{t\in\mathbb{Z}}$ and a deterministic sequence.

Corollary 3. *Let $\{\eta_t\}_{t\in\mathbb{Z}}$ be a stochastic process that fulfills Assumption 1 and let $\{d_t\}_{t\in\mathbb{N}_0}$ be a deterministic sequence that satisfies the assumptions stated in Corollary 2. Then, as $T \to \infty$, it holds that*

$$\frac{1}{T^{1/2}} G_F^{-1} \sum_{t=1}^{T} d_t\eta_t^* \Rightarrow \tau_\omega \int_0^1 D(r)\,dB(r)^*.$$

Note that there is no additive bias appearing in the limiting distribution which is due to the obvious independence between deterministic sequences and stochastic processes.

At the end of this section we present the limiting distribution of the ordinary least squares estimator (OLS) for A in (1), given by

$$\hat{A} = \left(\sum_{t=1}^{T} x_t x_{t-1}^* \right) \left(\sum_{t=1}^{T} x_{t-1} x_{t-1}^* \right)^{-1}, \tag{17}$$

which is an important building block for the asymptotic theory of seasonal unit root tests discussed in the next section.

Theorem 3. *Let $\{x_t\}_{t\in\mathbb{N}_0}$ be a stochastic process in discrete time generated according to the difference equation* (1) *with Assumption 1 in place. Then, as $T \to \infty$, it holds that*

$$T(\hat{A} - A) \Rightarrow e^{-i\omega} \left(\tau_\omega^2 \int_0^1 dB(r)\, B(r)^* + \Lambda_\omega^* \right) \left(\tau_\omega^2 \int_0^1 B(r) B(r)^* \, dr \right)^{-1}, \tag{18}$$

where Λ_ω and τ_ω are introduced in (7) *and* (8)*, respectively and $B(r)$ is defined in Theorem 1.*

We can extend this result for unit root processes that contain a deterministic component. In particular, consider the n-dimensional stochastic process $\{y_t\}_{t\in\mathbb{N}_0}$ generated according to

$$y_t = B_d d_t + x_t, \quad t \in \mathbb{N}, \tag{19}$$

where the process $\{x_t\}_{t\in\mathbb{N}_0}$ is generated according to (1) and $\{d_t\}_{t\in\mathbb{N}_0}$ is a deterministic sequence satisfying the assumptions stated in Corollary 2 and Corollary 3. Clearly, (19) is equivalent to

$$y_t = B_d d_t + B_x x_{t-1} + \eta_t, \quad t \in \mathbb{N}, \tag{20}$$

where $B_x = e^{-i\omega} I_n$. Setting $z_t = [d_t', x_{t-1}']'$ the OLS estimator for $B = [B_d, B_x]$ is given by

$$\hat{B} = \left(\sum_{t=1}^{T} y_t z_t^* \right) \left(\sum_{t=1}^{T} z_t z_t^* \right)^{-1}.$$

The limiting distribution of the scaled and centered OLS estimator follows now from several results presented previously in this section. Note that from the different convergence rates required in Proposition 2 and Corollary 2 as well as in Theorem 2 and Corollary 3 we deduce that the coefficient estimates must also converge at different rates. We therefore define the scaling matrix

$$G = \begin{bmatrix} G_d & 0 \\ 0 & T^{1/2} I_n \end{bmatrix},$$

where G_d is defined in Corollary 2.

Corollary 4. *Let* $\{y_t\}_{t\in\mathbb{N}_0}$ *be defined as in* (20) *where* $\{x_t\}_{t\in\mathbb{N}_0}$ *is generated according to* (1) *and* $\{d_t\}_{t\in\mathbb{N}_0}$ *is a deterministic sequence satisfying the assumptions stated in Corollary 2. Then, as* $T \to \infty$*, it holds that*

$$\sqrt{T}(\hat{B} - B)G \Rightarrow e^{-i\omega} \left(\tau_\omega^2 \int_0^1 dB(r)\, J(r)^* + [0_{n\times p}, \Lambda_\omega^*] \right) \left(\tau_\omega^2 \int_0^1 J(r)J(r)^*\, dr \right)^{-1},$$

where $J(r) = [\tau_\omega^{-1} e^{i\omega} D(r)', B(r)']'$ *with* $D(r)$ *and* $B(r)$ *defined in Theorem 1 and Corollary 2, respectively.*

4. Testing for Unit Roots

As an application of the results in Section 3 we generalize the tests of Phillips (1987) and Phillips and Perron (1988) to test for unit roots at an arbitrary frequency. Additionally, we extend our approach to test for seasonal integration by jointly testing for unit roots at multiple seasonal frequencies in the spirit of Hylleberg et al. (1990) and Ghysels et al. (1994).

4.1. A Phillips-Perron Type Test for a Unit Root

Let $\{x_t\}_{t\in\mathbb{N}_0}$ be a scalar process be generated according to (1) for some $\omega \in (-\pi, \pi]$ and $\{\eta_t\}_{t\in\mathbb{Z}}$ being a stationary process that satisfies Assumption 1. Subtracting $e^{-i\omega}x_{t-1}$ on both sides of (1) yields to the equivalent representation

$$\Delta_\omega x_t = (A - e^{-i\omega})x_{t-1} + \eta_t \tag{21}$$

with $A - e^{-i\omega} = 0$. Hence, a test for a unit root at frequency ω can be carried out by testing the null hypothesis $H_0 : \gamma = 0$ in the linear regression model

$$\Delta_\omega x_t = \gamma x_{t-1} + \eta_t. \tag{22}$$

The OLS estimator for γ is given by

$$\hat{\gamma} = \frac{\sum_{t=1}^T \overline{x}_{t-1} \Delta_\omega x_t}{\sum_{t=1}^T |x_{t-1}|^2}.$$

Note that $\Delta_\omega x_t = \eta_t$ under the null hypothesis. Hence, using the identity $\overline{\Lambda}_\omega = \Lambda_{-\omega}$ we immediately deduce from Theorem 3 that, as $T \to \infty$,

$$T\hat{\gamma} \Rightarrow \frac{e^{-i\omega} \left(\tau_\omega^2 \int_0^1 \overline{B}(r)\, dB(r) + \Lambda_{-\omega} \right)}{\tau_\omega^2 \int_0^1 |B(r)|^2\, dr},$$

where $B(r) = \Psi(e^{i\omega})W(r)$ with $W(r) = W_1(r) + iW_2(r)$ as in Lemma 1.

The Phillips-Perron type modification of the OLS estimator hinges upon the consistent estimation of the additive bias term $\Lambda_{-\omega}$. Consider the kernel estimator for Λ_ω of the form

$$\hat{\Lambda}_\omega = \frac{1}{T}\sum_{h=1}^{T-1} k\left(\frac{h}{M_T}\right) e^{-i\omega h}\frac{1}{T}\sum_{t=1}^{T-h} \hat{\eta}_t \overline{\hat{\eta}}_{t+h}, \tag{23}$$

where $\hat{\eta}_t$ are the OLS residuals from (22), M_T is the so-called bandwidth parameter and k is the kernel weighting function. We impose the following assumption, which is due to Jansson (2002), on the kernel function and the bandwidth parameter:

Assumption 2. The kernel function $k : \mathbb{R} \to \mathbb{R}$ is even, bounded and continuous. Furthermore, the function $\tilde{k} : \mathbb{R} \to \mathbb{R}$, defined by $\tilde{k}(x) = \sup_{y \geq |x|} |k(y)|$ is integrable on $\mathbb{R}$. The bandwidth parameter M_T satisfies $M_T^{-1} \to 0$ and $M_T/T^{1/2} \to 0$ as $T \to \infty$.

Under Assumption 2 the convergence results of Jansson (2002) for the frequency $\omega = 0$ can be easily extended to the arbitrary frequency case, i.e.

$$\hat{\Lambda}_\omega \xrightarrow{\mathbb{P}} \Lambda_\omega, \tag{24}$$

as $T \to \infty$. With this consistency result we can modify the estimator $\hat{\gamma}$ in order to obtain an asymptotically nuisance parameter free limiting distribution. In particular, a Phillips-Perron type modification of $\hat{\gamma}$ is given by

$$\hat{\gamma}_+ = \frac{\left(\sum_{t=1}^T \overline{x}_{t-1}\Delta_\omega x_t - Te^{-i\omega}\hat{\Lambda}_{-\omega}\right)}{\sum_{t=1}^T |x_{t-1}|^2}. \tag{25}$$

The following proposition states the limiting distribution of $\hat{\gamma}_+$.

Proposition 3. *Let $\{x_t\}_{t\in\mathbb{N}_0}$ be a scalar stochastic process generated according to the difference equation* (21) *with Assumption 1 in place. Let $\hat{\Lambda}_\omega$ be defined as in* (23) *with bandwidth parameter M_T and kernel function $k(r)$ satisfying Assumption 2. Then, as $T \to \infty$, it holds that*

$$T\hat{\gamma}_+ \Rightarrow \frac{e^{-i\omega}\int_0^1 \overline{W}(r)\, dW(r)}{\int_0^1 |W(r)|^2\, dr},$$

with Brownian motion $W(r)$, defined in Lemma 1.

Remark 7. If $W(r)$ is a complex Brownian motion, i.e. $W(r) = W_1(r) + iW_2(r)$, it holds for the real and imaginary parts of the stochastic integral and for the denominator that appears in the limiting distribution that

$$\mathrm{Re}\left\{\int_0^1 \overline{W}(r)\,dW(r)\right\} = \int_0^1 W_1(r)\,dW_1(r) + \int_0^1 W_2(r)\,dW_2(r),$$
$$\mathrm{Im}\left\{\int_0^1 \overline{W}(r)\,dW(r)\right\} = \int_0^1 W_1(r)\,dW_2(r) - \int_0^1 W_2(r)\,dW_1(r)$$

and

$$\int_0^1 |W(r)|^2\,dr = \int_0^1 W_1^2(r)\,dr + \int_0^1 W_2^2(r)\,dr.$$

The limiting distribution in Proposition 3 is a complex valued distribution if $\omega \in (-\pi, 0)\cup(0, \pi)$ and, hence, the null hypothesis can be equivalently stated as

$$H_0 : \mathrm{Re}\{\gamma\} = 0 \text{ and } \mathrm{Im}\{\gamma\} = 0.$$

To test this hypothesis we consider the following F-type statistic:

$$F_\omega = \frac{|\hat{\gamma}_+|^2}{\hat{\Omega}_\omega}\sum_{t=1}^{T}|x_{t-1}|^2, \tag{26}$$

where $\hat{\Omega}_\omega = \hat{\Sigma} + \hat{\Lambda}_\omega + \hat{\Lambda}_{-\omega}$ with $\hat{\Lambda}_\omega$ defined in (23) and $\hat{\Sigma}$ denoting the consistent sample covariance estimator, i.e.

$$\hat{\Sigma} = \frac{1}{T}\sum_{t=1}^{T}\hat{\eta}_t\overline{\hat{\eta}}_t.$$

Since $\hat{\Sigma} \xrightarrow{\mathbb{P}} \Sigma$ and $\hat{\Lambda}_\omega \xrightarrow{\mathbb{P}} \Lambda_\omega$ it holds that $\hat{\Omega}_\omega$ is a consistent estimator for $\Omega_\omega = |\Psi(e^{i\omega})|^2$. Proposition 3 and the continuous mapping theorem immediately imply that

$$T^2|\hat{\gamma}_+|^2 \Rightarrow \frac{\mathrm{Re}\left\{\int_0^1\overline{W}(r)\,dW(r)\right\}^2 + \mathrm{Im}\left\{\int_0^1\overline{W}(r)\,dW(r)\right\}^2}{\left(\int_0^1|W(r)|^2\,dr\right)^2}.$$

Furthermore, from Proposition 2 we deduce that

$$\frac{1}{T^2\hat{\Omega}_\omega}\sum_{t=1}^{T}|x_{t-1}|^2 \Rightarrow \frac{\tau_\omega^2}{\Omega_\omega}\int_0^1|B(r)|^2\,dr = \tau_\omega^2\int_0^1|W(r)|^2\,dr.$$

The continuous mapping theorem and the identities stated in Remark 7 yield the limiting distribution of the test statistic F_ω under the null hypothesis.

Theorem 4. *Under the assumptions of Proposition 3 it holds for $T \to \infty$ that*

$$F_\omega \Rightarrow \frac{\left(\int_0^1 W(r)\, dW(r)\right)^2}{\int_0^1 W^2(r)\, dr},$$

with standard Brownian motion $W(r)$ if $\omega \in \{0, \pi\}$ and

$$F_\omega \Rightarrow \frac{1}{2}\frac{\left(\int_0^1 W_1(r)\, dW_1(r) + \int_0^1 W_2(r)\, dW_2(r)\right)^2}{\int_0^1 W_1^2(r)\, dr + \int_0^1 W_2^2(r)\, dr} + \frac{1}{2}\frac{\left(\int_0^1 W_1(r)\, dW_2(r) - \int_0^1 W_2(r)\, dW_1(r)\right)^2}{\int_0^1 W_1^2(r)\, dr + \int_0^1 W_2^2(r)\, dr},$$

where $W_1(r)$ and $W_2(r)$ are independent standard Brownian motions if $\omega \in (-\pi, 0) \cup (0, \pi)$.

In practice time series models of the form (1) or (21) with $\omega \in (-\pi, 0) \cup (0, \pi)$ are found rather rarely, as these models are complex valued by design. Instead, one often assumes that the observed time series is a realization of a process, $\{y_t\}_{t \in \mathbb{N}_0}$ say, of the form

$$y_t = 2\cos(\omega) y_{t-1} + y_{t-2} + \eta_t, \tag{27}$$

with $\{\eta_t\}_{t \in \mathbb{Z}}$ satisfying assumptions such as Assumption 1. The process $\{y_t\}_{t \in \mathbb{N}_0}$ is integrated at both frequencies ω and $-\omega$ since (27) can be rewritten equivalently as

$$\Delta_\omega \Delta_{-\omega} y_t = (1 - e^{-i\omega} L)(1 - e^{i\omega} L) y_t = \eta_t.$$

Define $\{y_{t,1}\}_{t \in \mathbb{N}_0}$ and $\{y_{t,2}\}_{t \in \mathbb{N}_0}$ via $y_{t,1} = \Delta_{-\omega} y_t$ and $y_{t,2} = \Delta_\omega y_t$, respectively. Then, it holds that $y_{t,1} = \overline{y}_{t,2}$ and $y_t = \mu_1 y_{t,1} + \mu_2 y_{t,2}$ with

$$\mu_1 = \frac{e^{-i\omega}}{e^{-i\omega} - e^{i\omega}}$$

and $\mu_2 = \overline{\mu}_1$. Note that $\mu = 0$ if and only if $\overline{\mu} = 0$ and, consequently, testing for a unit root at frequency ω is equivalent to testing for a unit root at frequency $-\omega$. In particular, we can either test $H_0 : \gamma_1 = 0$ in the regression model

$$\Delta_\omega y_{t,1} = \gamma_1 y_{t-1,1} + \eta_t$$

using the test statistic F_ω defined in (26) or we can test $H_0 : \gamma_2 = 0$ in the regression model

$$\Delta_{-\omega} y_{t,2} = \gamma_2 y_{t-1,2} + \eta_t$$

using the test statistic $F_{-\omega}$. Clearly, since $\Delta_\omega y_{t,1} = \Delta_{-\omega} y_{t,2}$ and $y_{t-1,2} = \overline{y}_{t-1,1}$ it holds that $\gamma_2 = \overline{\gamma}_1$ and $F_\omega = F_{-\omega}$, which once again implies the equivalence of both test approaches.

4.2. Testing Multiple Unit Roots and Seasonal Integration

Let now $\{x_t\}_{t\in\mathbb{N}_0}$ be a real valued scalar process, observed in S equidistant periods per season, e.g a series which is observed quarterly ($S = 4$) or monthly ($S = 12$) within a year or daily ($S = 7$) within a week. If the process is non-stationary and generated according to the difference equation

$$x_t = \alpha x_{t-S} + \eta_t, \quad t \in \mathbb{N}, \tag{28}$$

with $\alpha = 1$, where $\{\eta_t\}_{t\in\mathbb{Z}}$ is a stationary process satisfying Assumption 1 and the initial values $x_0, \ldots, x_{-S+1}$ are assumed to be $\mathcal{O}_{\mathbb{P}}(1)$ then we say that $\{x_t\}_{t\in\mathbb{N}_0}$ is *seasonally integrated*. If $\{\eta_t\}_{t\in\mathbb{Z}}$ is a martingale difference sequence, i.e. $\eta_t = \sigma\varepsilon_t$ for $\sigma \in \mathbb{R}$ and $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ satisfying Assumption 1 then $\{x_t\}_{t\in\mathbb{N}_0}$ is said to be a *saisonal random walk*. For the remainder of this section we focus on quarterly seasonally integrated process, i.e. $S = 4$.

Define the processes $\{x_{t,j}\}_{t\in\mathbb{N}_0}$, $j = 1, \ldots, 4$ via

$$\begin{aligned}
x_{t,1} &= x_t + x_{t-1} + x_{t-2} + x_{t-3}, \\
x_{t,2} &= x_t - x_{t-1} + x_{t-2} - x_{t-3}, \\
x_{t,3} &= x_t + ix_{t-1} - x_{t-2} - ix_{t-3}, \\
x_{t,4} &= x_t - ix_{t-1} - x_{t-2} + ix_{t-3}.
\end{aligned}$$

By simple algebra we deduce that $x_{t,1} = x_{t-1,1} + \eta_t$, $x_{t,2} = -x_{t-1,2} + \eta_t$, $x_{t,3} = -ix_{t-1,3} + \eta_t$ and $x_{t,4} = ix_{t-1,4} + \eta_t$ or, equivalently,

$$\Delta_0 x_{t,1} = \Delta_\pi x_{t,2} = \Delta_{\pi/2} x_{t,3} = \Delta_{-\pi/2} x_{t,4} = \eta_t.$$

Hence, $\{x_{t,j}\}_{t\in\mathbb{N}_0}$, $j = 1, \ldots, 4$, satisfy (21) with $\omega_1 = 0$, $\omega_2 = \pi$, $\omega_3 = \frac{\pi}{2}$ and $\omega_4 = -\frac{\pi}{2}$, respectively. Obviously, it holds that $x_{t,3} = \overline{x}_{t,4}$ and one easily verifies that

$$x_t = \frac{1}{4}\left(x_{t,1} + x_{t,2} + x_{t,3} + x_{t,4}\right). \tag{29}$$

We can therefore test the null hypothesis of seasonal integration by testing separately, using the test statistic F_ω presented in (26), whether $\{x_{t,1}\}_{t\in\mathbb{N}_0}$ is integrated at frequency $\omega = 0$, $\{x_{t,2}\}_{t\in\mathbb{N}_0}$ is integrated at frequency $\omega = \pi$ and $\{x_{t,3}\}_{t\in\mathbb{N}_0}$ is integrated at frequency $\omega = \frac{\pi}{2}$ or, equivalently, whether $\{x_{t,4}\}_{t\in\mathbb{N}_0}$ is integrated at frequency $\omega = -\frac{\pi}{2}$.

We can also test the seasonal integration hypothesis in the spirit of Hylleberg et al. (1990) Ghysels et al. (1994) by carrying out multiple tests for different unit roots jointly. In particular, consider the test regression

$$y_t = x_{t-1,1}\gamma_1 + x_{t-1,2}\gamma_2 + x_{t-1,3}\gamma_3 + x_{t-1,4}\gamma_4 + \eta_t, \quad t = 1, \ldots, T, \tag{30}$$

where $y_t = x_t - x_{t-4}$. Setting $\Gamma = [\gamma_1, \gamma_2, \gamma_3, \gamma_4]'$ we can rewrite this equation in matrix notation as

$$y = X\Gamma + \eta,$$

where

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_T \end{bmatrix}, \quad X = \begin{bmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ \vdots & \vdots & \vdots & \vdots \\ x_{T-1,1} & x_{T-1,2} & x_{T-1,3} & x_{T-1,4} \end{bmatrix}, \quad \eta = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_T \end{bmatrix}.$$

A multivariate extension of the estimator $\hat{\gamma}_+$ is given by

$$\hat{\Gamma}_+ = (X^*X)^{-1}(X^*y - T\hat{\Theta}), \tag{31}$$

where

$$\hat{\Theta} = \left[\hat{\Lambda}_0, -\hat{\Lambda}_\pi, -i\hat{\Lambda}_{-\pi/2}, i\hat{\Lambda}_{\pi/2}\right]'.$$

Using the asymptotic orthogonality of the processes $\{x_{t,j}\}_{t\in\mathbb{N}_0}$, $j = 1, \ldots, 4$, stated in Proposition 2 and Remark 4, we can easily derive the asymptotic distribution of $\hat{\Gamma}_+$.

Proposition 4. *Let $\{x_t\}_{t\in\mathbb{N}_0}$ be a scalar stochastic process generated according to the difference equation*

$$x_t = x_{t-4} + \eta_t, \quad t \in \mathbb{N},$$

with $x_0, x_{-1}, x_{-2}, x_{-3}$ being $\mathcal{O}_\mathbb{P}(1)$ and $\{\eta_t\}_{t\in\mathbb{Z}}$ satisfying Assumption 1. Let $\hat{\Lambda}_\omega$ be defined as in (23) with bandwidth parameter M_T and kernel function $k(r)$ satisfying Assumption 2. Then, as $T \to \infty$, it holds that

$$T\hat{\Gamma}_+ \Rightarrow \left[\frac{\int_0^1 W_1(r)\, dW_1(r)}{\int_0^1 W_1^2(r)\, dr}, -\frac{\int_0^1 W_2(r)\, dW_2(r)}{\int_0^1 W_2^2(r)\, dr}, -i\frac{\int_0^1 \overline{W}_c(r)\, dW_c(r)}{\int_0^1 |W_c(r)|^2\, dr}, i\frac{\int_0^1 W_c(r)\, d\overline{W}_c(r)}{\int_0^1 |W_c(r)|^2\, dr}\right]',$$

with $W_c(r) = W_3(r) + iW_4(r)$ and where $W_1(r), \ldots, W_4(r)$ are independent standard Brownian motions.

Remark 8. Note that since $x_{t,1}$ and $x_{t,2}$ are real valued and $x_{t,4} = \overline{x}_{t,3}$ it holds that $\hat{\gamma}_4 = \overline{\hat{\gamma}}_3$. Hence, one might wonder whether it is necessary to keep both $x_{t-1,3}$ and $x_{t-1,4}$ as independent variables in the regression model (30), especially since both regressors are asymptotically orthogonal due to Proposition 2. However, omitting one of these regressors causes $\hat{\gamma}_1$ and $\hat{\gamma}_2$ to be complex valued. Since the asymptotic distributions are real valued, which implies that the imaginary parts are $o_\mathbb{P}(1)$, one should work with real valued estimators.

We can now perform a joint test of the hypothesis

$$H_0 : \Gamma = 0$$

for investigating whether $\{x_t\}_{t\in\mathbb{N}_0}$ contains unit roots at all of the four considered frequencies. To this end we define $\tilde{X} = X\hat{\Phi}^{-1}$ with

$$\hat{\Psi} = \begin{bmatrix} \hat{\Omega}_0^{1/2} & 0 & 0 & 0 \\ 0 & \hat{\Omega}_\pi^{1/2} & 0 & 0 \\ 0 & 0 & \hat{\Omega}_{\pi/2}^{1/2} & 0 \\ 0 & 0 & 0 & \hat{\Omega}_{\pi/2}^{1/2} \end{bmatrix}.$$

Then, an extended version of the statistic F_ω is given by

$$F = \hat{\Gamma}_+^*(\tilde{X}^*\tilde{X})\hat{\Gamma}_+.$$

Using the asymptotic orthogonality from Remartk 4 we obtain the following limiting distribution of the test statistic F under the null hypothesis.

Theorem 5. *Under the assumptions of Proposition 4 it holds for $T \to \infty$ that*

$$\begin{aligned} F \Rightarrow & \frac{\left(\int_0^1 W_1(r)\, dW_1(r)\right)^2}{\int_0^1 W_1^2(r)\, dr} + \frac{\left(\int_0^1 W_2(r)\, dW_2(r)\right)^2}{\int_0^1 W_2^2(r)\, dr} \\ & + \frac{\left(\int_0^1 W_3(r)\, dW_3(r) + \int_0^1 W_4(r)\, dW_4(r)\right)^2}{\int_0^1 W_3^2(r)\, dr + \int_0^1 W_3^2(r)\, dr} + \frac{\left(\int_0^1 W_3(r)\, dW_4(r) - \int_0^1 W_4(r)\, dW_3(r)\right)^2}{\int_0^1 W_3^2(r)\, dr + \int_0^1 W_4^2(r)\, dr}, \end{aligned}$$

where $W_1(r), \ldots, W_4(r)$ are independent standard Brownian motions.

Remark 9. Consider the following test statistic which results from adding the F_ω statistics calculated at the four frequencies separately, i.e.

$$\tilde{F} = F_0 + F_\pi + F_{\pi/2} + F_{-\pi/2},$$

where F_0 is calculated from $x_{t,1}$, F_π is calculated from $x_{t,2}$, and so on. According to Proposition 2 the off-diagonal elements of $\tilde{X}^*\tilde{X}$ in the definition of the test statistic F vanish asymptotically implying that $F = \tilde{F} + o_{\mathbb{P}}(1)$. Hence, both test statistics F and $\tilde{F}$ have the same asymptotic distribution.

We can easily modify the testing approach in order to test for unit roots at $\omega = \pm\pi/2$ only. The corresponding test statistic is given by

$$F_{\pm\pi/2} = (R_{\pm\pi/2}\hat{\Gamma}_+)^*(R_{\pm\pi/2}\tilde{X}^*\tilde{X}^*R'_{\pm\pi/2})(R_{\pm\pi/2}\hat{\Gamma}_+),$$

with

$$R_{\pm\pi/2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It holds that $F_{\pm\pi/2} = F_{\pi/2} + F_{-\pi/2} + o_{\mathbb{P}}(1)$ and it holds that

$$F_{\pm\pi/2} \Rightarrow \frac{\left(\int_0^1 W_3(r)\,dW_3(r) + \int_0^1 W_4(r)\,dW_4(r)\right)^2}{\int_0^1 W_3^2(r)\,dr + \int_0^1 W_3^2(r)\,dr} + \frac{\left(\int_0^1 W_3(r)\,dW_4(r) - \int_0^1 W_4(r)\,dW_3(r)\right)^2}{\int_0^1 W_3^2(r)\,dr + \int_0^1 W_4^2(r)\,dr}.$$

Similarly, we can easily modify the testing approach in order to test for seasonal unit roots only, i.e. to test whether $\{x_t\}_{t\in\mathbb{N}_0}$ is integrated at $\omega = \pm\pi/2$ and $\omega = \pi$. The corresponding test statistic is given by

$$F_S = (R_S\hat{\Gamma}_+)^*(R_S\tilde{X}^*\tilde{X}^*R_S')(R_S\hat{\Gamma}_+),$$

with

$$R_S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Clearly, it holds that $F_S = \tilde{F}_S + o_{\mathbb{P}}(1)$ where $\tilde{F}_S = F_\pi + F_{\pi/2} + F_{-\pi/2}$ and the limiting distribution of F_S follows immediately.

Remark 10. The previously discussed tests can be extended by adding deterministic regressors in (30) to capture deterministic cyclical patterns or time trends under the alternative hypothesis. In particular, consider the regression model

$$y_t = d_t'\delta + x_{t-1,1}\gamma_1 + x_{t-1,2}\gamma_2 + x_{t-1,3}\gamma_3 + x_{t-1,4}\gamma_4 + \eta_t, \quad t = 1, \dots, T.$$

The deterministic term d_t might contain an intercept, a linear trend, seasonal dummies or a combination of those. Note that since $\{x_{t,j}\}_{t\in\mathbb{N}_0}$, $j \in \{2,3,4\}$, are integrated at frequencies other than zero they are asymptotically orthogonal to constants and linear trends (cf. Corollary 3). Hence, including these deterministic terms only changes the limiting distribution of F_0. In particular, if $d_t = 1$ then the Brownian motion that appears in the limiting distribution of F_0 is replaced by a demeaned Brownian motion and if $d_t = [1, t]'$ then the Brownian motion is replaced by a detrended Brownian motion. On the other hand, if d_t contains a full set of seasonal dummies then all Brownian motions in the limiting distributions F_0, F_π, $F_{\pi/2}$ and $F_{-\pi/2}$ have to be replaced by demeand ones.

Remark 11. The testing approach of Hylleberg et al. (1990) and Ghysels et al. (1994) is similar to our test procedures. However, they consider the test real valued test regression

$$y_t = x_{t-1,1}\pi_1 + x_{t-1,2}\pi_2 + x_{t-2,5}\pi_3 + x_{t-1,5}\pi_4 + \eta_t, \quad t = 1, \dots, T, \tag{32}$$

where $\{x_{t,1}\}_{t\in\mathbb{N}_0}$ and $\{x_{t,2}\}_{t\in\mathbb{N}_0}$ are defined as above and $\{x_{t,5}\}_{t\in\mathbb{N}_0}$ is defined via

$$x_{t,5} = \Delta_0\Delta_\pi x_t = x_t - x_{t-2}.$$

Clearly, it holds that $\Delta_0\Delta_\pi x_{t,5} = \eta_t$ so that $\{x_{t,5}\}_{t\in\mathbb{N}_0}$ has unit roots at both annual frequencies $\omega = \pi/2$ and $\omega = -\pi/2$. From the discussion at the end of Subsection 4.1 it becomes apparent that

$$\begin{bmatrix} y_{t,1} \\ y_{t,2} \end{bmatrix} = \begin{bmatrix} 1 & -e^{i\omega} \\ 1 & -e^{-i\omega} \end{bmatrix} \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}.$$

Hence, testing the null hypothesis $H_0 : \pi_3 = 0, \pi_4 = 0$ in the regression model (32) is equivalent to test the null hypothesis $H_0 : \gamma_3 = 0, \gamma_4 = 0$ in the regression model (30).

Remark 12. If $\{\eta_t\}_{t\in\mathbb{Z}}$ is white noise then the OLS estimators for $[\gamma_1, \ldots, \gamma_4]'$ in the regression model (30) is pivotal. In this case there is no need for a $\hat{\Gamma}_+$ type correction and textbook OLS-based F-statistics are sufficient to test the hypotheses discussed above. Furthermore, from Remark 11 we deduce that such OLS based F-statistics are identical to the corresponding OLS-based F-statistics in the regression model (32). If, however, $\{\eta_t\}_{t\in\mathbb{Z}}$ is not white noise Hylleberg et al. (1990) suggest to augment the test regression (32) by adding additional lagged values of y_t to whiten the regression errors. This procedure is identical to the well known augmentation of the test regression in the unit root test of Dickey and Fuller (1979), which is widely known as ADF-test. Therefore, the test of Hylleberg et al. (1990) can be interpreted as a seasonal extension of the ADF-test whereas our test procedure is obviously a seasonal extension of the Phillips-Perron test.

Remark 13. The test statistics presented in this section or straightforward modifications of them have the same asymptotic distributions as the tests statistics presented in Hylleberg et al. (1990) and Ghysels et al. (1994) (see also Engle et al., 1993, Ghysels and Osborn, 2001). In particular, the test statistics F_0 and F_π are asymptotically equivalent to the squared t-statistics for the parameters π_1 and π_2 in Hylleberg et al. (1990), respectively. The test statistic $F_{\pi/2}$ has exactly the same asymptotic distribution as the joint F statistic for the parameters π_3 and π_4 in Hylleberg et al. (1990). The test statistics F and F_S are asymptotically equivalent to four times the test statistic F_{1234} and three times the test statistic F_{234} in Ghysels et al. (1994), respectively.

5. Conclusion

In this paper we have derived several limiting results processes that are integrated at an arbitrary frequency. In particular, we have established a functional central limit theorem and the limiting distribution of sample covariance matrices $\frac{1}{T^2}\sum_{t=1}^{T} x_t x_t^*$ and $\frac{1}{T}\sum_{t=1}^{T} x_{t-1}\eta_t^*$. Where the former follows quite easily from the functional central limit theorem and the continuous mapping theorem, the proof of the latter is somewhat more demanding. In contrast to Phillips (1988a) and Gregoir (2010), our proof is mainly based on algebraic manipulations and we do not rely on the martingale approximation theory of Hall and Heyde (1980). As a direct application of these results, we have presented unit root tests for arbitrary frequencies, which rely on a Phillips-Perron-like modification of the OLS estimator. These estimators are asymptotically equivalent to the estimators of Hylleberg et al. (1990) and Ghysels et al. (1994), so we did not need to simulate critical values.

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A. Auxiliary Resuls

In this section we provide some algebraic auxiliary results that are required for the proofs of the main results. The first result is states that the cumulative the summation of $e^{i\omega t}$ for $t = 1, \ldots, T$ is bounded for every $T \in \mathbb{N}$ if and only if $\omega \neq 0$. This algebraic property is the reason why processes that are integrated at different frequencies are asymptotically orthogonal.

Lemma A.1. *For $T \in \mathbb{N}$ it holds that*

$$\sup_{t \in \mathbb{N}} \left| \sum_{t=1}^{T} e^{i\omega t} \right| \leq C < \infty$$

if and only if $\omega \neq 0$.

Proof. For $\omega \neq 0$ the sum formula for a geometric progression yields

$$\left| \sum_{t=1}^{T} e^{i\omega t} \right| = \left| e^{i\omega} \sum_{t=0}^{T-1} e^{i\omega t} \right| = \left| \frac{e^{i\omega}}{1 - e^{i\omega}} \left(1 - e^{i\omega T}\right) \right| \leq \left| \frac{e^{i\omega}}{1 - e^{i\omega}} \right| \left|1 - e^{i\omega T}\right| \leq \frac{2}{|1 - e^{i\omega}|},$$

which is bounded and independent of T. The statement for $\omega = 0$ is trivial. $\square$

The next lemma is a generalization of the summation by parts formula (sometimes referred to as Abel's Lemma).

Lemma A.2. *For two sequences $\{a_t\}_{t\in\mathbb{Z}}$ and $\{b_t\}_{t\in\mathbb{Z}}$ with $a_t, b_t \in \mathbb{C}^{n\times n}$ it holds that*

$$\sum_{t=1}^{T} a_t b_t^* = e^{-i\omega T}\sum_{t=1}^{T} e^{i\omega t} a_t b_T^* - e^{-i\omega}\sum_{t=1}^{T} e^{-i\omega(t-1)}\sum_{j=1}^{t-1} e^{i\omega j} a_j \Delta_\omega b_t^*.$$

Proof. Using simple algebra we deduce that

$$\begin{aligned}
\sum_{t=1}^{T} a_t b_t^* &= \sum_{t=1}^{T} \Delta_\omega \left\{ e^{-i\omega t}\sum_{j=1}^{t} e^{i\omega j} a_j \right\} b_t^* \\
&= \sum_{t=1}^{T} \left(e^{-i\omega t}\sum_{j=1}^{t} e^{i\omega j} a_j - e^{-i\omega} e^{-i\omega(t-1)} \sum_{j=1}^{t-1} e^{i\omega j} a_j \right) b_t^* \\
&= \sum_{t=1}^{T} e^{-i\omega t}\sum_{j=1}^{t} e^{i\omega j} a_j b_t^* - \sum_{t=1}^{T} e^{-i\omega} e^{-i\omega(t-1)} \sum_{j=1}^{t-1} e^{i\omega j} a_j b_t^* \\
&= \sum_{t=1}^{T} e^{-i\omega t}\sum_{j=1}^{t} e^{i\omega j} a_j b_t^* - \sum_{t=0}^{T-1} e^{-i\omega} e^{-i\omega t} \sum_{j=1}^{t} e^{i\omega j} a_j b_{t+1}^*. \qquad (33)
\end{aligned}$$

The first sum in (33) can be decomposed into

$$\sum_{t=1}^{T} e^{-i\omega t}\sum_{j=1}^{t} e^{i\omega j} a_j b_t^* = e^{-i\omega T}\sum_{j=1}^{T} e^{i\omega j} a_t b_T^* + \sum_{t=1}^{T-1} e^{-i\omega t}\sum_{j=1}^{t} e^{i\omega j} a_j b_t^*. \qquad (34)$$

The sum of the the second term on the right hand side of (33) and the term sum on the right hand side of (34) is equal to

$$\begin{aligned}
\sum_{t=1}^{T-1} e^{-i\omega t}\sum_{j=1}^{t} e^{i\omega j} a_j (b_t^* - e^{-i\omega} b_{t+1}^*) &= -e^{-i\omega}\sum_{t=1}^{T-1} e^{-i\omega t}\sum_{j=1}^{t} e^{i\omega j} a_j (b_{t+1}^* - e^{i\omega} b_t^*) \\
&= -e^{-i\omega}\sum_{t=1}^{T-1} e^{-i\omega t}\sum_{j=1}^{t} e^{i\omega j} a_j \Delta_\omega b_{t+1}^* \\
&= -e^{-i\omega}\sum_{t=1}^{T} e^{-i\omega(t-1)}\sum_{j=1}^{t-1} e^{i\omega j} a_j \Delta_\omega b_t^*,
\end{aligned}$$

as claimed. $\square$

The last auxiliary result is the Beveridge-Nelson decomposition at some arbitrary frequency. This decomposition has been excessively discussed in the literature for the case $\omega = 0$ where a simple algebraic proof is given in Neusser (2016). However, to our knowledge the general case has yet not been proven using only simple algebra.

Lemma A.3 (Beveridge-Nelson Decomposition)**.** *Let $A(z) = \sum_{j=0}^{\infty} \alpha_j z^j$ be a matrix polynomial with $\alpha_j \in \mathbb{C}^{n\times n}$. Then, it holds that*

$$A(z) = A(e^{i\omega}) - (1 - e^{-i\omega} z) B(z),$$

where $B(z) = \sum_{j=0}^{\infty} \beta_j z^j$ is a matrix polynomial with coefficient matrices $B_j \in \mathbb{C}^{n\times n}$ given by

$$B_j = e^{-i\omega j} \sum_{k=j+1}^{\infty} B_k e^{i\omega k},$$

with $\sum_{j=0}^{\infty} \|B_j\| < \infty$. If the coefficient matrices A_j satisfy $\sum_{j=0}^{\infty} j^l \|A_j\| < \infty$ for some $l \in \mathbb{N}$ then it holds that $\sum_{j=0}^{\infty} j^{l-1} \|B_j\| < \infty$.

Proof of Lemma A.3. It holds that

$$\Psi(L) - \Psi(e^{i\omega}) = \sum_{j=0}^{\infty} \psi_j L^j - \sum_{j=0}^{\infty} \psi_j e^{i\omega j} = \sum_{j=0}^{\infty} \psi_j \left(L^j - e^{i\omega j} I_n \right).$$

By simple algebra we deduce that

$$L^j - e^{i\omega j} I_n = -\left(I_n - e^{-i\omega} L \right) \sum_{k=0}^{j-1} e^{i\omega(j-k)} L^k.$$

Hence,

$$\begin{aligned}
\Psi(L) - \Psi(e^{i\omega}) &= -\sum_{j=0}^{\infty} \psi_j \left(I_n - e^{-i\omega} L \right) \sum_{k=0}^{j-1} e^{i\omega(j-k)} L^k \\
&= -\left(I_n - e^{-i\omega} L \right) \sum_{j=0}^{\infty} \sum_{k=0}^{j-1} e^{i\omega(j-k)} L^k \\
&= -\left(I_n - e^{-i\omega} L \right) \sum_{k=0}^{\infty} e^{-i\omega k} \sum_{j=k+1}^{\infty} \psi_j e^{i\omega j} L^k \\
&= -\left(I_n - e^{-i\omega} L \right) \sum_{k=0}^{\infty} \tilde{\psi}_k L^k,
\end{aligned}$$

where the last equation follow from a change of variables.

Note that for some arbitrary $l \in \mathbb{N}$ it holds that

$$\sum_{j=0}^{\infty} j^{l-1} \|\tilde{\psi}_j\| = \sum_{j=0}^{\infty} j^{l-1} \left\| e^{-i\omega j} \sum_{k=j+1}^{\infty} e^{i\omega k} \psi_k \right\| \leq \sum_{j=0}^{\infty} j^{l-1} \sum_{k=j+1}^{\infty} \|\psi_k\| \leq \sum_{j=0}^{\infty} j^l \|\psi_j\|,$$

where we used the fact that $|e^{ix}| = 1$ for all $x \in \mathbb{R}$. This concludes the proof. □

B. Proofs of the Main Results

Proof of Proposition 1. Solving the difference equation 1 recursively leads to

$$x_t = e^{-i\omega t} x_t + e^{-i\omega t} \sum_{j=1}^{t} e^{i\omega j} \eta_j.$$

Since $\eta_t = \Psi(L)\varepsilon_t$ we can apply the Beveridge-Nelson decomposition (cf. Lemma A.3) on the lag polynomial $\Psi(L)$ and obtain

$$\begin{aligned}
x_t - e^{-i\omega t} x_0 &= e^{-i\omega t} \sum_{j=1}^{t} e^{i\omega j} \eta_j \\
&= e^{-i\omega t} \sum_{j=1}^{t} e^{i\omega j} \Psi(L) \varepsilon_j \\
&= \Psi(e^{i\omega}) e^{-i\omega t} \sum_{j=1}^{t} e^{i\omega j} \varepsilon_j - e^{-i\omega t} \sum_{j=1}^{t} e^{i\omega j} \Delta_\omega \tilde{\Psi}(L) \varepsilon_j \\
&= \Psi(e^{i\omega}) e^{-i\omega t} \sum_{j=1}^{t} e^{i\omega j} \varepsilon_j - \tilde{\Psi}(L) \varepsilon_t + e^{-i\omega t} \tilde{\Psi}(L) \varepsilon_0
\end{aligned}$$

It remains to show that the process $\{\tilde{\eta}_t\}_{t \in \mathbb{Z}}$ defined by $\tilde{\eta}_t = \tilde{\Psi}(L)\varepsilon_t$ is stationary. Therefore, it is sufficient to show that the coefficient matrices $\tilde{\psi}_j$ are absolutely summable. This, however, follows immediately from Lemma A.3 with $l = 1$. □

Proof of Lemma 1. Define the sequence of continuous time processes $\{W_T(r); r \in [0, 1]\}_{T \in \mathbb{N}_0}$ as

$$W_T(r) = \frac{1}{\sqrt{T}} \sum_{t=1}^{[rT]} e^{i\omega t} \varepsilon_t.$$

It is well known from Chan and Wei (1988) that $W_T(r) \Rightarrow \tau_\omega W(r)$. Clearly, the process $\{X_{t,T}\}_{0\le t\le T}$, defined via $X_{t,T} = W_T(t/T)$, is a martingale with respect to its canonical filtration. Hence, with

$$I_T(r) = \sum_{t=1}^{[rT]} X_{t-1,T}(X_{t,T} - X_{t-1,T})^*,$$

it follows from Hansen (1992) and Kurtz and Protter (1991) (with $\mathbb{C}^n \simeq \mathbb{R}^{2n}$) that

$$(W_T(r), I_T(r)) \Rightarrow \left(\tau_\omega W(r), \tau_\omega^2 \int_0^r W(r)\, dW(r)^*\right).$$

Since

$$\begin{aligned}
I(r) &= \sum_{t=1}^{[rT]} W_T\left(\frac{t-1}{T}\right)\left[W_T\left(\frac{t}{T}\right) - W_T\left(\frac{t-1}{T}\right)\right]^* \\
&= \frac{1}{T}\sum_{t=1}^{[rT]}\sum_{j=1}^{t-1} e^{i\omega j}\varepsilon_j \left[\sum_{j=1}^{t} e^{i\omega j}\varepsilon_j - \sum_{j=1}^{t-1} e^{i\omega j}\varepsilon_j\right]^* \\
&= \frac{1}{T}\sum_{t=1}^{[rT]}\sum_{j=1}^{t-1} e^{i\omega j}\varepsilon_j \left(e^{i\omega t}\varepsilon_t\right)^*,
\end{aligned}$$

the stated result follows. □

Proof of Theorem 1. From Proposition 1 we immediately deduce that

$$\frac{e^{i\omega[rT]}}{\sqrt{T}} x_{[rT]} = \frac{1}{\sqrt{T}}(x_0 + \tilde{\eta}_0) + \Psi(e^{i\omega})\frac{1}{\sqrt{T}}\sum_{j=1}^{[rT]} e^{i\omega j}\varepsilon_j - \frac{e^{i\omega[rT]}}{\sqrt{T}}\tilde{\eta}_{[rT]}.$$

It follows from Lemma 1 that

$$\Psi(e^{i\omega})\frac{1}{\sqrt{T}}\sum_{t=1}^{[rT]} e^{i\omega t}\varepsilon_t \Rightarrow \tau_\omega \Psi(e^{i\omega})W(r),$$

where $W(r)$ is a real valued standard Brownian motion if $\omega \in \{0, \pi\}$ and $W(r) = W_1(r) + iW_2(r)$ with independent standard Brownian motions $W_1(r)$ and $W_2(r)$ if $\omega \in (0, \pi)$. It remains to show that (cf. Billingsley, 1968, Theorem 4.1)

$$\sup_{r\in[0,1]}\left\|\frac{e^{i\omega[rT]}}{\sqrt{T}} x_{[rT]} - \Psi(e^{i\omega})\frac{1}{\sqrt{T}}\sum_{t=1}^{[rT]} e^{i\omega t}\varepsilon_t\right\| \xrightarrow{\mathbb{P}} 0. \tag{35}$$

It holds that

$$\left\| \frac{e^{i\omega[rT]}}{\sqrt{T}} x_{[rT]} - \Psi(e^{i\omega}) \frac{1}{\sqrt{T}} \sum_{t=1}^{[rT]} e^{i\omega t} \varepsilon_t \right\| \leq \frac{1}{\sqrt{T}} \|x_0\| + \frac{1}{\sqrt{T}} \|\tilde{\eta}_0\| + \frac{1}{\sqrt{T}} \left\|\tilde{\eta}_{[rT]}\right\|,$$

which vanishes asymptotically if and only if

$$\max_{0\leq t\leq T} \frac{1}{\sqrt{T}} \|\tilde{\eta}_t\| \xrightarrow{\mathbb{P}} 0,$$

since x_0 is $\mathcal{O}_{\mathbb{P}}(1)$. Let $\delta > 0$ be such that $\sup_t \mathbb{E}(\|\varepsilon_t\|^{2+\delta}|\mathcal{F}_{t-1}) < \infty$. Then, it holds that

$$\begin{aligned} \|\tilde{\eta}_t\| &= \left\| \sum_{j=0}^{\infty} \tilde{\psi}_j \eta_{t-j} \right\| \\ &\leq \sum_{j=0}^{\infty} \|\tilde{\psi}_j\|^{\frac{1+\delta}{2+\delta}} \|\tilde{\psi}_j\|^{\frac{1}{2+\delta}} \|\varepsilon_{t-j}\| \\ &\leq \left(\sum_{j=0}^{\infty} \|\tilde{\psi}_j\| \right)^{\frac{1+\delta}{2+\delta}} \left(\sum_{j=0}^{\infty} \|\tilde{\psi}_j\| \|\varepsilon_{t-j}\|^{2+\delta} \right)^{\frac{1}{2+\delta}}, \end{aligned}$$

by Hölder's inequality. This implies

$$\max_{0\leq t\leq T} \mathbb{E}(\|\tilde{\eta}_t\|^{2+\delta}|\mathcal{F}_{t-1}) \leq \left(\sum_{j=0}^{\infty} \|\psi_j\| \right)^{2+\delta} \max_{0\leq t\leq T} \mathbb{E}(\|\varepsilon_t\|^{2+\delta}|\mathcal{F}_{t-1}) < \infty. \tag{36}$$

For some arbitrary $\alpha > 0$ it holds that

$$\mathbb{P}\left(\max_{0\leq t\leq T} \frac{1}{\sqrt{T}} \|\tilde{\eta}_t\| > \alpha \right) \leq \sum_{t=0}^{T} \mathbb{P}\left(\|\tilde{\eta}_t\| > \alpha\sqrt{T} \right) \leq \frac{1}{T^{\delta/2}} \sum_{t=0}^{T} \frac{\mathbb{E}\|\tilde{\eta}_t\|^{2+\delta}}{\alpha^{2+\delta} T}$$

by the Bonferroni and Markov inequalities. This expression converges to zero since (36) implies that the expected value is uniformly bounded. This concludes the proof. □

Proof of Proposition 2. If $\omega_1 = \omega_2$ the statement follows immediately from Theorem 1 and the continuous mapping theorem. To prove the statement for $\omega_1 \neq \omega_2$ generalize the proof of Johansen and Schaumburg (1999). Let

$$\begin{aligned}
\tilde{x}_{t,1} &= e^{i\omega_1 t} x_{t,1}, \\
\tilde{x}_{t,2} &= e^{i\omega_2 t} x_{t,2}, \\
E_t &= \sum_{j=1}^{t} e^{i(\omega_2-\omega_1)t}.
\end{aligned}$$

Using the identity $e^{i(\omega_2-\omega_1)t} = \Delta_0 E_t$ it holds that

$$\frac{1}{T^2}\sum_{t=1}^{T} x_{t,1}x_{t,2}^* = \frac{1}{T^2}\sum_{t=1}^{T} \tilde{x}_{t,1} e^{i(\omega_2-\omega_1)t} \tilde{x}_{t,2}^*$$

and we have to verify that the right hand side is $o_{\mathbb{P}}(1)$. It holds that

$$\begin{aligned}
\frac{1}{T^2}\sum_{t=1}^{T} \tilde{x}_{t,1}\tilde{x}_{t,2}^*\Delta_0 E_t &= \frac{1}{T^2}\sum_{t=1}^{T} \tilde{x}_{t,1}\tilde{x}_{t,2}^* E_t - \frac{1}{T^2}\sum_{t=1}^{T} (\Delta_0 x_{t,1} + \tilde{x}_{t-1,1})(\Delta_0 \tilde{x}_{t,2} + \tilde{x}_{t-1,2})^* E_{t-1} \\
&= \frac{1}{T^2}\sum_{t=1}^{T} \tilde{x}_{t,1}\tilde{x}_{t,2}^* E_t - \frac{1}{T^2}\sum_{t=1}^{T} \tilde{x}_{t-1,1}\tilde{x}_{t-1,2}^* E_{t-1} - \frac{1}{T^2}\sum_{t=1}^{T} \Delta_0\tilde{x}_{t,1}\Delta_0\tilde{x}_{t,2}^* E_{t-1} \\
&\quad - \frac{1}{T^2}\sum_{t=1}^{T} \tilde{x}_{t-1,1}\Delta_0\tilde{x}_{t,2}^* E_{t-1} - \frac{1}{T^2}\sum_{t=1}^{T} \Delta_0\tilde{x}_{t,1}\tilde{x}_{t-1,2}^* E_{t-1} \qquad (37)
\end{aligned}$$

The first two terms are equal to

$$\frac{1}{T^2}\sum_{t=1}^{T} \left(\tilde{x}_{t,1}\tilde{x}_{t,2}^* E_t - \tilde{x}_{t-1,1}\tilde{x}_{t-1,2}^* E_{t-1}\right) = \frac{1}{\sqrt{T}}\tilde{x}_{T,1}\frac{1}{\sqrt{T}}\tilde{x}_{T,2}^*\frac{1}{T}E_T,$$

which is $o_{\mathbb{P}}(1)$ since $T^{-1/2}\tilde{x}_{t,1}$ and $T^{-1/2}\tilde{x}_{t,2}$ are $\mathcal{O}_{\mathbb{P}}(1)$ by Theorem 1 and $E_T/T \to 0$ by Lemma A.1.

For the third term in (37) it holds that

$$\begin{aligned}
\mathbb{E}\left\|\frac{1}{T^2}\sum_{t=1}^{T} \Delta_0\tilde{x}_{t,1}\Delta_0\tilde{x}_{t,2}^* E_{t-1}\right\| &\leq \max_{0\leq t\leq T}|E_t|\frac{1}{T^2}\sum_{t=1}^{T}\mathbb{E}\|\Delta_0\tilde{x}_{t,1}\Delta_0\tilde{x}_{t,2}^*\| \\
&\leq \max_{0\leq t\leq T}|E_t|\frac{1}{T^2}\sum_{t=1}^{T}\left(\mathbb{E}\|\Delta_0\tilde{x}_{t,1}\|^2\right)^{1/2}\left(\mathbb{E}\|\Delta_0\tilde{x}_{t,2}\|^2\right)^{1/2}. \qquad (38)
\end{aligned}$$

From Proposition 1 we deduce that

$$\tilde{x}_{t,1} = \Psi(e^{i\omega_1}) \sum_{j=1}^{t} e^{i\omega_1 j}\varepsilon_j - e^{i\omega_1 t}\tilde{\eta}_t + y_0 + \tilde{\eta}_0,$$

and, hence,

$$\Delta_0 \tilde{x}_{t,1} = \Psi(e^{i\omega_1}) e^{i\omega_1 t}\varepsilon_t - \left(e^{i\omega_1 t}\tilde{\eta}_t\right) - e^{i\omega_1 (t-1)}\tilde{\eta}_{t-1},$$

which immediately implies that $\mathbb{E}\|\Delta_0 \tilde{x}_{t,1}\|$ is bounded. By the same arguments it follows that $\mathbb{E}\|\Delta_0 \tilde{x}_{t,2}\|$ is bounded and, since $|E_t|$ is uniformly bounded, we deduce that the right hand side of (38) vanishes asymptotically.

For the fourth term in (37) we obtain, using similar arguments as above, that

$$\mathbb{E}\left\|\frac{1}{T^2}\sum_{t=1}^{T} \tilde{x}_{t-1,1}\Delta_0 \tilde{x}^*_{t,2} E_{t-1}\right\| \le \max_{0\le t\le T} |E_t| \frac{1}{T^2}\sum_{t=1}^{T} \left(\mathbb{E}\|\tilde{x}_{t-1,1}\|^2\right)^{1/2} \left(\mathbb{E}\|\sqrt{T}\Delta_0 \tilde{x}_{t,2}\|^2\right)^{1/2}. \quad (39)$$

Since $\mathbb{E}\|\tilde{x}_{t,1}\|^2 = \mathcal{O}(t)$ it follows that the right hand side of (39) is $\mathcal{O}(T^{-1/2})$. Using exactly the same arguments it can be shown that the same asymptotic bound holds for the last term in (37). This concludes the proof. □

Proof of Theorem 2. Using the the Beveridge-Nelson decomposition in Lemma A.3 we rewrite the process $\{\eta_t\}_{t\in\mathbb{Z}}$ as

$$\eta_t = \Psi(e^{i\omega})\varepsilon_t - \Delta_\omega \tilde{\eta}_t.$$

Assumption (4) guarantees that the coefficient matrices $\tilde{\psi}_j$ are absolutely sumable und that $\{\tilde{\eta}_t\}_{t\in\mathbb{Z}}$ is stationary and ergodic. With this decomposition and Proposition 1 in place, we immediately obtain

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T} x_{t-1}\eta_t^* = {} & \Psi(e^{i\omega})\frac{1}{T}\sum_{t=1}^{T} e^{-i\omega(t-1)}\sum_{j=1}^{t-1} e^{i\omega j}\varepsilon_j\varepsilon_t'\Psi(e^{i\omega})^* - \frac{1}{T}\sum_{t=1}^{T}\tilde{\eta}_{t-1}\varepsilon_t'\Psi(e^{i\omega})^* \\ & - \Psi(e^{i\omega})\frac{1}{T}\sum_{t=1}^{T} e^{-i\omega(t-1)}\sum_{j=1}^{t-1} e^{i\omega j}\varepsilon_j(\Delta_\omega\tilde{\eta}_t)^* + \frac{1}{T}\sum_{t=1}^{T}\tilde{\eta}_{t-1}(\Delta_\omega\tilde{\eta}_t)^* \quad (40) \\ & + (x_0 + \tilde{\eta}_0)\frac{1}{T}\sum_{t=1}^{T} e^{-i\omega(t-1)}\eta_t^*.\end{aligned}$$

It can easily be verified that the last term is $o_{\mathbb{P}}(1)$.

By Lemma 1 we obtain

$$\frac{1}{T}\sum_{t=1}^{T} e^{-i\omega(t-1)}\sum_{j=1}^{t-1} e^{i\omega j}\varepsilon_j\varepsilon_t' = e^{i\omega}\sum_{t=1}^{T}\left(\frac{1}{\sqrt{T}}\sum_{j=1}^{t-1} e^{i\omega j}\varepsilon_j\right)\left(\frac{e^{i\omega t}}{\sqrt{T}}e_t\right)^*$$
$$\Rightarrow e^{i\omega}\tau_\omega^2\int_0^1 W(r)\,dW(r)^*.$$

Since $\Omega_\omega^{1/2} = \Psi(e^{i\omega})$ it follows for the first term in (40) that

$$\Psi(e^{i\omega})\frac{1}{T}\sum_{t=1}^{T} e^{-i\omega(t-1)}\sum_{j=1}^{t-1} e^{i\omega j}\varepsilon_j\varepsilon_t'\Psi(e^{i\omega})^* \Rightarrow e^{i\omega}\tau_\omega^2\int_0^1 B(r)\,dB(r)^*.$$

Since $\{\tilde{\eta}_t\}_{t\in\mathbb{Z}}$ is stationary and ergodic and the output of a linear filter, we have that $\{(\tilde{\eta}_t, \varepsilon_t)\}_{t\in\mathbb{Z}}$ is also stationary and ergodic and, consequently,

$$\frac{1}{T}\sum_{t=1}^{T}\tilde{\eta}_{t-1}\varepsilon_t' \Rightarrow \mathbb{E}(\tilde{\eta}_{t-1}\varepsilon_t') = \mathbb{E}\left(\sum_{j=0}^{\infty}\tilde{\psi}_j\varepsilon_{t-1-j}\varepsilon_t'\right) = 0.$$

Thus, it remains to show that the two remaining terms in (40) converge to $e^{i\omega}\Lambda_\omega$.

It follows immediately from Lemma A.2 that

$$\frac{1}{T}\sum_{t=1}^{T} e^{-i\omega(t-1)}\sum_{j=1}^{t-1} e^{i\omega j}\varepsilon_j(\Delta_\omega\tilde{\eta}_t)^* = \frac{e^{i\omega}}{T}e^{-i\omega T}\sum_{j=1}^{T} e^{i\omega j}\varepsilon_j\tilde{\eta}_T^* - \frac{e^{i\omega}}{T}\sum_{t=1}^{T}\varepsilon_t\tilde{\eta}_t^*.$$

Clearly, the first term is $o_{\mathbb{P}}(1)$. For the second term, using the same arguments as above, we obtain

$$\frac{e^{i\omega}}{T}\sum_{t=1}^{T}\varepsilon_t\tilde{\eta}_t^* \Rightarrow e^{i\omega}\mathbb{E}(\varepsilon_t\tilde{\eta}_t^*) = e^{i\omega}\sum_{j=0}^{\infty}\mathbb{E}(\varepsilon_t\varepsilon_{t-j}')\tilde{\psi}_j^* = e^{i\omega}\tilde{\psi}_0^* = e^{i\omega}\sum_{k=1}^{\infty} e^{-i\omega k}\psi_k^*.$$

Consequently,

$$\Psi(e^{i\omega})\frac{1}{T}\sum_{t=1}^{T} e^{-i\omega(t-1)}\sum_{k=1}^{t-1} e^{i\omega k}\varepsilon_j(\Delta_\omega\tilde{\eta}_t)^* \Rightarrow \Psi(e^{i\omega})e^{i\omega}\sum_{k=1}^{\infty} e^{-i\omega k}\psi_j^* = e^{i\omega}\sum_{j=0}^{\infty}\sum_{k=1}^{\infty} e^{i\omega j}\psi_j\psi_k^* e^{-i\omega k} \tag{41}$$

At last, it holds that

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}\tilde{\eta}_{t-1}(\Delta_\omega\tilde{\eta}_t)^* &= \frac{1}{T}\sum_{t=1}^{T}\tilde{\eta}_{t-1}\tilde{\eta}_t^* - e^{i\omega}\frac{1}{T}\sum_{t=1}^{T}\tilde{\eta}_{t-1}\tilde{\eta}_{t-1}^* \\
&\Rightarrow \mathbb{E}(\tilde{\eta}_{t-1}\tilde{\eta}_t^*) - e^{i\omega}\mathbb{E}(\tilde{\eta}_t\tilde{\eta}_t^*) \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\tilde{\psi}_j\mathbb{E}(\varepsilon_{t-1-j}\varepsilon_{t-k}')\tilde{\psi}_k^* - e^{i\omega}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\tilde{\psi}_j\mathbb{E}(\varepsilon_{t-j}\varepsilon_{t-k}')\tilde{\psi}_k^* \\
&= \sum_{j=0}^{\infty}\tilde{\psi}_j\tilde{\psi}_{j+1}^* - e^{i\omega}\sum_{j=0}^{\infty}\tilde{\psi}_j\tilde{\psi}_j^* \\
&= \sum_{j=0}^{\infty}\tilde{\psi}_j\left(\tilde{\psi}_{j+1} - e^{-i\omega}\tilde{\psi}_j\right)^*.
\end{aligned}$$

From the definition of $\tilde{\psi}_j$ we obtain the following expression for the terms in parentheses.

$$\tilde{\psi}_{j+1} - e^{-i\omega}\tilde{\psi}_j = e^{-i\omega(j+1)}\sum_{k=j+2}^{\infty}e^{i\omega k}\psi_k - e^{-i\omega}e^{-i\omega j}\sum_{k=j+1}^{\infty}e^{i\omega k}\psi_k = -\psi_{j+1}.$$

Hence,

$$\begin{aligned}
\sum_{j=0}^{\infty}\tilde{\psi}_j\left(\tilde{\psi}_{j+1} - e^{-i\omega}\tilde{\psi}_j\right)^* &= -\sum_{j=0}^{\infty}e^{-i\omega j}\sum_{k=j+1}^{\infty}\psi_k e^{i\omega k}\psi_{j+1}^* \\
&= -\sum_{j=1}^{\infty}e^{-i\omega(j-1)}\sum_{k=j}^{\infty}\psi_k e^{i\omega k}\psi_j^* \\
&= -e^{i\omega}\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}e^{-i\omega j}\psi_k\psi_j^* e^{i\omega k} \\
&= -e^{i\omega}\sum_{k=1}^{\infty}\sum_{j=1}^{k}e^{-i\omega j}\psi_k\psi_j^* e^{i\omega k}. \qquad (42)
\end{aligned}$$

Combining (41) and (42) yields

$$\begin{aligned}
e^{i\omega}\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}e^{i\omega j}\psi_j\psi_k^* e^{-i\omega k} - e^{i\omega}\sum_{j=1}^{\infty}\sum_{k=1}^{j}e^{i\omega j}\psi_j\psi_k^* e^{-i\omega k} &= e^{i\omega}\sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty}e^{i\omega j}\psi_j\psi_k^* e^{-i\omega k} \\
&= e^{i\omega}\sum_{j=0}^{\infty}e^{-i\omega(k-j)}\sum_{k=j+1}^{\infty}\psi_j\psi_k^* \\
&= e^{i\omega}\sum_{l=1}^{\infty}e^{-i\omega l}\sum_{j=0}^{\infty}\psi_j\psi_{j+l}^*,
\end{aligned}$$

which is equal to $e^{i\omega}\Lambda_\omega$. Putting everything together, we deduce that

$$\frac{1}{T}\sum_{t=1}^{T} x_{t-1}\eta_t^* \Rightarrow e^{i\omega}\left(\tau_\omega^2 \int_0^1 B(r)\, dB(r)^* + \Lambda_\omega\right),$$

as claimed. □

Proof of Theorem 3. The limiting distribution of the OLS estimator follows from Proposition 2 and Theorem 2 since both sums converge jointly. To be more precise, it holds that

$$\frac{1}{T}\sum_{t=1}^{T} \eta_t x_{t-1}^* = e^{i\omega}\frac{1}{T}\sum_{t=1}^{T} \Psi(e^{i\omega})e^{i\omega t}\varepsilon_t e^{-i\omega(t-1)}\sum_{j=1}^{t-1} e^{-i\omega j}\varepsilon_j'\Psi(e^{i\omega})^* + o_{\mathbb{P}}(1) \tag{43}$$

and

$$\frac{1}{T^2}\sum_{t=1}^{T} x_{t-1}x_{t-1}^* = \frac{1}{T^2}\sum_{t=1}^{T} \Psi(e^{i\omega})e^{i\omega(t-1)}\sum_{j=1}^{t-1} e^{i\omega j}\varepsilon_j e^{-i\omega(t-1)}\sum_{k=1}^{t-1} e^{-i\omega k}\varepsilon_k'\Psi(e^{i\omega})^* + o_{\mathbb{P}}(1), \tag{44}$$

and both (43) and (44) converge jointly according to Lemma 1. The claim follows now immediately from the continuous mapping theorem.

□

Proof of Proposition 3. Under the null hypothesis it holds that $\Delta_\omega x_t = \eta_t$. Hence,

$$T\hat{\gamma}_+ = \frac{\frac{1}{T}\sum_{t=1}^{T}\overline{x}_{t-1}\eta_t}{\frac{1}{T^2}\sum_{t=1}^{T}|x_{t-1}|^2} - \frac{e^{-i\omega}\hat{\Lambda}_{-\omega}}{\frac{1}{T^2}\sum_{t=1}^{T}|x_{t-1}|^2}.$$

It follows from Theorem 3 that

$$\frac{\frac{1}{T}\sum_{t=1}^{T}\overline{x}_{t-1}\eta_t}{\frac{1}{T^2}\sum_{t=1}^{T}|x_{t-1}|^2} \Rightarrow \frac{e^{-i\omega}\int_0^1 \overline{B}(r)\, dB(r)}{\int_0^1 |B(r)|^2\, dr} + \frac{e^{-i\omega}\Lambda_{-\omega}}{\tau_\omega^2\int_0^1 |B(r)|^2\, dr}$$

and from Proposition 2 and (24) that

$$\frac{e^{-i\omega}\hat{\Lambda}_{-\omega}}{\frac{1}{T^2}\sum_{t=1}^{T}|x_{t-1}|^2} \Rightarrow \frac{e^{-i\omega}\Lambda_{-\omega}}{\tau_\omega^2\int_0^1 |B(r)|^2\, dr}.$$

Combining both limiting expressions completes the proof. □

Proof of Theorem 4. Since $\{\eta_t\}_{t\in\mathbb{Z}}$ is stationary it holds that $\hat{\Gamma}_0$ converges to $\mathbb{E}(\eta_t\eta_t^*)$ and from (24) we deduce that $\hat{\Lambda}_\omega + \hat{\Lambda}_{-\omega}$ goes to $\Lambda_\omega + \Lambda_{-\omega}$. Consequently, $\hat{\Omega}$ converges to $\mathbb{E}(\eta_t\eta_t^*) + \Lambda_\omega + \Lambda_\omega^*$

which is equal to Ω_ω. The claim now follows immediately from Proposition 2, Proposition 3 and the continuous mapping theorem. □

Proof of Proposition 4. First, we derive the limiting distribution of the OLS estimator

$$\hat{\Gamma} = (X^*X)^{-1}X^*y.$$

It holds that

$$X^*X = \begin{bmatrix} \sum_{t=1}^T |x_{t-1,1}|^2 & \sum_{t=1}^T \overline{x}_{t-1,1}x_{t-1,2} & \sum_{t=1}^T \overline{x}_{t-1,1}x_{t-1,3} & \sum_{t=1}^T \overline{x}_{t-1,1}x_{t-1,4} \\ \sum_{t=1}^T \overline{x}_{t-1,2}x_{t-1,1} & \sum_{t=1}^T |x_{t-1,2}|^2 & \sum_{t=1}^T \overline{x}_{t-1,2}x_{t-1,3} & \sum_{t=1}^T \overline{x}_{t-1,2}x_{t-1,4} \\ \sum_{t=1}^T \overline{x}_{t-1,3}x_{t-1,1} & \sum_{t=1}^T \overline{x}_{t-1,3}x_{t-1,2} & \sum_{t=1}^T |x_{t-1,3}|^2 & \sum_{t=1}^T \overline{x}_{t-1,3}x_{t-1,4} \\ \sum_{t=1}^T \overline{x}_{t-1,4}x_{t-1,1} & \sum_{t=1}^T \overline{x}_{t-1,4}x_{t-1,2} & \sum_{t=1}^T \overline{x}_{t-1,4}x_{t-1,3} & \sum_{t=1}^T |x_{t-1,4}|^2 \end{bmatrix}$$

and from the asymptotic orthogonality in Proposition 2 and Remark 4 we deduce that

$$\frac{1}{T^2}X^*X \Rightarrow \begin{bmatrix} \int_0^1 B_0^2(r)\,dr & 0 & 0 & 0 \\ 0 & \int_0^1 B_\pi^2(r)\,dr & 0 & 0 \\ 0 & 0 & \frac{1}{2}\int_0^1 |B_{\pi/2}(r)|^2\,dr & 0 \\ 0 & 0 & 0 & \frac{1}{2}\int_0^1 |B_{\pi/2}(r)|^2\,dr \end{bmatrix}.$$

Similarly, using Theorem 2 we obtain

$$\frac{1}{T}X^*y = \begin{bmatrix} \frac{1}{T}\sum_{t=1}^T x_{t-1,1}y_t \\ \frac{1}{T}\sum_{t=1}^T x_{t-1,2}y_t \\ \frac{1}{T}\sum_{t=1}^T \overline{x}_{t-1,3}y_t \\ \frac{1}{T}\sum_{t=1}^T x_{t-1,3}y_t \end{bmatrix} \Rightarrow \begin{bmatrix} \int_0^1 B_0(r)\,dB_0(r) + \Lambda_0 \\ -\int_0^1 B_\pi(r)\,dB_\pi(r) - \Lambda_\pi \\ -\frac{i}{2}\int_0^1 \overline{B}_{\pi/2}(r)\,dB_{\pi/2}(r) - \Lambda_{-\pi/2} \\ \frac{i}{2}\int_0^1 B_{\pi/2}(r)\,d\overline{B}_{\pi/2}(r) + \Lambda_{\pi/2} \end{bmatrix}. \tag{45}$$

The modified OLS estimator is given by $\hat{\Gamma}_+ = (X^*X)^{-1}(X^*y - T\hat{\Theta})$ where the the bias correction term $\hat{\Theta} = [\hat{\Lambda}_0, -\hat{\Lambda}_\pi, -i\hat{\Lambda}_{-\pi/2}, i\hat{\Lambda}_{\pi/2}]'$ removes the additive components that appear in the limiting expression (45) in exactly the same way as discussed in the proof of Proposition 3. Combining these results yields the limiting distribution of $\hat{\Gamma}_+$. □

Proof of Theorem 5. Follows directly from Proposition 4 in conjunction with the asymptotic orthogonality stated in Remark 4 and the continuous mapping theorem. □