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L^α -Regularization of the Beckmann Problem

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We investigate the problem of optimal transport in the so-called Beckmann form, i.e. given two Radon measures on a compact set, we seek an optimal flow field which is a vector valued Radon measure on the same set that describes a flow between these two measures and minimizes a certain linear cost function.

We consider L^α regularization of the problem, which guarantees uniqueness and forces the solution to be an integrable function rather than a Radon measure. This regularization naturally gives rise to a semi-smooth Newton scheme that can be used to solve the problem numerically. Besides motivating and developing the numerical scheme, we also include approximation results for vanishing regularization in the continuous setting.

1. Introduction

The Beckmann formulation of optimal transport is the problem of finding a flow field that describes how to move some measure onto another measure of the same mass such that a certain linear cost functional is minimal. It was first introduced in [5] in a more general form. Specifically, for a domain $\Omega \subset \mathbb{R}^d$, two Radon measures μ^+, μ^- on Ω with $\mu^+(\Omega) = \mu^-(\Omega)$ and a continuous cost function $w : \Omega \rightarrow [0, \infty)$ our goal is to solve

$$\inf_{\substack{q \in \mathfrak{M}(\Omega, \mathbb{R}^d), \\ \operatorname{div} q = \mu}} \int_{\Omega} w \, d|q|, \quad (\text{BP})$$

where we abbreviated $\mu := \mu^+ - \mu^-$ and the divergence constraint has to be understood in a suitable weak sense. Existence of solutions is well known [28, 14], but since the objective functional in (BP) is not strictly convex, solutions may not be unique. Moreover, for general Radon measures μ^+, μ^- , a solution may not admit a density w.r.t. the Lebesgue measure. Hence, standard approximation tools from numerical analysis are not applicable. This motivates the use of regularization of the continuous problem to obtain approximate solutions that are functions instead of measures, which in turn can be treated by classical discretization techniques in order to solve the regularized problem. Here, we aim to employ L^α -regularization which, as we will see, also naturally gives rise to a semi-smooth Newton scheme that can be used to solve the problem numerically.

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The Beckmann problem is closely related to other problems of optimal transport theory, namely the so called Monge problem and the Kantorovich problem as well as the Monge-Kantorovich equation [28, 2]. For example, for $w \equiv 1$ (BP) is equivalent to the Kantorovich problem (with Euclidian cost), [28, § 4.2.1].

1.1. Notation and problem statement

Before we formulate our problem, let us fix the notation that will be used in the remainder. The space of Radon measures and the set of probability measures on $\Omega \subset \mathbb{R}^d$ will be denoted by $\mathfrak{M}(\Omega)$ and $\mathcal{P}(\Omega)$, respectively. The space of vector valued Radon measures will be denoted by $\mathfrak{M}(\Omega, \mathbb{R}^d)$ and we will use the same convention for all other classes of measures and functions as well. With $C(\Omega)$ and $C^k(\Omega)$ we denote the spaces of continuous functions and k times continuously differentiable functions, respectively.

For a Banach space X we will denote its topological dual by X^* . The d -dimensional Lebesgue measure will be denoted by \mathcal{L}^d and, where appropriate, integrals w.r.t. the Lebesgue measure are simply denoted by dx with the appropriate integration variable x . For a set $\Omega \subset \mathbb{R}^d$ we will also use the shorthand notation $|\Omega| := \mathcal{L}^d(\Omega)$. For the space of p -integrable functions on Ω with respect to the Lebesgue measure, the symbol $L^p(\Omega)$ will be used. The symbol $W^{k,p}(\Omega)$ denotes the Sobolev space of functions for which the weak derivatives up to order k are functions in $L^p(\Omega)$.

When a measure ν is absolutely continuous with respect to another measure μ , written as $\nu \ll \mu$, the Radon-Nikodym derivative of ν w.r.t. to μ , i.e. the density of ν w.r.t. μ , will be denoted by $\frac{d\nu}{d\mu}$. Conversely, by $\mathcal{I} : L^1(\Omega, \mathbb{R}^d) \rightarrow \mathfrak{M}(\Omega, \mathbb{R}^d)$ we denote the embedding, which identifies an integrable function with a Radon measure on Ω via

$$(\mathcal{I}(f))(A) := \int_A f \, d\mathcal{L}^d \quad \forall A \subset \Omega.$$

Hence, $\mathcal{I}\left(\frac{d\nu}{d\mathcal{L}^d}\right) = \nu$.

With slight abuse of notation, we will denote the Nemytskii-operator $q \mapsto (\Omega \ni x \mapsto F(x, q(x)))$ associated with a function $F : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^l$ by the same symbol. The characteristic function of a set A will be denoted by $\mathbb{1}_A$. In contrast, ι_A denotes the indicator functional of A . We denote the Euclidian norm on \mathbb{R}^d with $|\cdot|$ and the positive part of a scalar c as $c_+ := \max\{c, 0\}$. The inner product of $x, y \in \mathbb{R}^d$ will be denoted by $x \cdot y$.

In the following we will consider a compact domain Ω . For $f : \Omega \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$, we will use the shorthand notation

$$\{f > c\} := \{x \in \Omega \mid f(x) > c\}$$

and analogously for $\{f \geq c\}$, $\{f < c\}$, $\{f \leq c\}$ and $\{f \neq c\}$.

The regularized Beckmann problem of optimal transport considered in this work now reads as

$$\inf_{\substack{q \in L^\alpha(\Omega, \mathbb{R}^d), \\ \operatorname{div} q = \mu}} \int_{\Omega} w|q| \, d\mathcal{L}^d + \frac{\varepsilon}{\alpha} \|q\|_{L^\alpha(\Omega, \mathbb{R}^d)}^\alpha. \quad (\text{BP}_\varepsilon)$$

Let us summarize our standing assumptions:

Assumption 1.1. We assume that $d \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$ is a compact set, whose interior is a bounded Lipschitz domain in the sense of [18, Chapter 1.2]. The cost function $w : \Omega \rightarrow \mathbb{R}$ is continuous. Assume $1 < \alpha < \frac{d}{d-1}$. Finally, we assume $\mu_i \in \mathcal{P}(\Omega)$, for $i = 1, 2$.

- Remark 1.2.**
1. In contrast to standard notation in PDE literature, we use the symbol Ω for a closed set. Nevertheless, for convenience, we simply write $W^{k,p}(\Omega)$ instead of $W^{k,p}(\text{int}(\Omega))$ for Sobolev spaces.
 2. Note that by standard Sobolev embeddings (e.g. [1, Theorem 4.12]), it holds that $W^{1,\alpha'}(\Omega) \hookrightarrow C(\Omega)$, since $\alpha < \frac{d}{d-1}$. Hence, $\mathfrak{M}(\Omega) \hookrightarrow (W^{1,\alpha'}(\Omega))^*$. This allows us to use arbitrary measures $\mu^+, \mu^- \in \mathcal{P}(\Omega)$ as marginals in (BP_ϵ) .
 3. Note that for the integral $\int_\Omega w|q| \, d\mathcal{L}^d$ to exist, the cost function w does not need to be continuous and the problem may be formulated for more general cost functions. However, some of the results in this work require this assumption and for simplicity it shall be assumed throughout the paper.

1.2. Related Work

Due to its relation with other optimal transport problems, the Beckmann problem has been considered in a number of different settings.

The authors of [29] tackle the Beckmann problem with uniform cost function w from a geometry processing point of view to compute the distances between points on discrete surfaces. The Helmholtz-Hodge decomposition and the spectral decomposition of the Laplacian are used to reformulate the Beckmann problem into an unconstrained problem, where the coefficients of the spectral decomposition are the optimization variables. The authors then pass to a discrete setting and truncate the spectral decomposition, which reduces the problem size and gives an approximation of the original problem.

Several publications employ first order schemes to solve the Beckmann problem. In [22], the authors discretize the problem via a finite differences scheme and employ the Chambolle-Pock algorithm. They, too, only consider uniform cost w , which allows to derive closed form expressions for the involved proximal operators. To ensure uniqueness, they add a regularization term similar to the one in (BP_ϵ) , but only consider the case $\alpha = 2$. The methods of [22] are extended to unbalanced transport (i.e. $\mu^+(\Omega) \neq \mu^-(\Omega)$) in [27] and [23] proposes a multilevel initialization approach to speed up the computation time for fine grids. Another first order scheme is covered in [21], where a variant of the Chambolle-Pock algorithm is analyzed, which involves the computation of optimal step sizes. The results are applied to an ROF formulation of the Beckmann problem in two dimensions with uniform weight and without regularization. Moreover, an estimate for the error in the objective value is derived. In [6] multiple different problems are covered, including the Beckmann problem with general cost or L^p -regularization (in the context of so-called congested transport), but not both at the same time. The problems are solved numerically by solving the dual formulation by the ADMM algorithm. This requires to solve a Laplace equation with Neumann boundary conditions in each iteration step.

The authors of [8, 7] consider the closely related problem of traffic congestion [13]. This problem generalizes the Beckmann problem by allowing the cost function w to depend on q in the sense $w = w(x, |q(x)|)$ and the so-called traffic intensity is computed instead of q , which allows to model a congestion effect. A fast marching algorithm is proposed to treat the problem numerically. In [12] the authors consider regularity results for this line of work and model the congestion by a term $\frac{1}{p}|\cdot|^p$. This

corresponds to our regularization term, however they only consider uniform cost. [19, 11] consider a even more general, anisotropic setting and [19] includes numerical examples, which rely on [6].

A different type of regularization is employed in [4], where the authors use the Monge-Kantorovich equation as starting point and consider the functional $\int_{\Omega} w d|q|^r$ with $r > 1$ after smoothing the marginals μ^+ and μ^- accordingly. After providing a convergence result for $r \rightarrow 0$, the authors switch to a discrete setting and give another approximation result for increasing discretization fineness. The numerical scheme then relies on a fixed-point iteration of the form $|x_i|^{r-2}(x_{i+1} - x_i) + |x_i|^{r-2}x_i$, where an additional regularization is required due to the non-smoothness of $|\cdot|$. We point out that in contrast to (BP_{ε}) this choice of regularization does not preserve the non-smooth structure of (BP) . The setting of [4] is extended to a setting of unbalanced transport in [3].

The authors of [15] propose a dynamic formulation of the Monge-Kantorovich equations (for uniform cost) and conjecture that the solution approximates the solution of the static equations for $t \rightarrow \infty$. However, the conjecture is still open. The authors argue that the dynamic formulation naturally adds a regularization to the problem and derive an Euler scheme for solving the problem numerically.

1.3. Organization

The remainder of this work is organized as follows. We start in Section 2 by rigorously defining the divergence constraint in (BP_{ε}) and proving existence and uniqueness of solutions. Afterwards we derive a semi-smooth Newton iteration in Section 3, which will also involve a second regularization. We detail how to choose appropriate step sizes via an auxiliary minimization problem and make a connection between that problem and (BP_{ε}) . Section 4 is concerned with approximation results. More precisely, we prove weak convergence of minimizers of the regularized problems towards minimizers of (BP_{ε}) and (BP) under suitable assumptions. After discussing numerical examples in Section 5, we finally conclude in Section 6.

2. Existence of solutions

Let us rigorously define the divergence constraint in problem (BP_{ε}) . Motivated by the zero-flux boundary condition, the divergence constraint in (BP) is to be understood as

$$- \int_{\Omega} \text{grad } \varphi \cdot dq = \int_{\Omega} \varphi d\mu \quad \forall \varphi \in C^1(\Omega). \quad (2.1)$$

Therefore, the equality constraint in the regularized problem (BP_{ε}) reads

$$- \int_{\Omega} q \cdot \text{grad } \varphi d\mathcal{L}^d = \int_{\Omega} \varphi d\mu. \quad \forall \varphi \in W^{1,\alpha'}(\Omega) \quad (2.2)$$

Lemma 2.1. *Let $q \in L^{\alpha}(\Omega, \mathbb{R}^d)$ and let Assumption 1.1 hold. Then q solves (2.2) if and only if it solves*

$$- \int_{\Omega} q \cdot \text{grad } \varphi d\mathcal{L}^d = \int_{\Omega} \varphi d\mu \quad \forall \varphi \in W_0^{1,\alpha'}(\Omega), \quad (2.3)$$

where

$$W_0^{1,\alpha'}(\Omega) := \left\{ v \in W^{1,\alpha'}(\Omega) \mid \int_{\Omega} v(x) dx = 0 \right\}.$$

Proof. If q solves (2.2), then it trivially also solves (2.3). On the other hand, if q solves (2.3), then for every $\varphi \in W_0^{1,\alpha'}(\Omega)$ and every $c \in \mathbb{R}$, the assumptions on the marginals imply

$$\begin{aligned} - \int_{\Omega} q(x) \cdot \text{grad}(\varphi(x) + c) \, dx &= \int_{\Omega} \varphi \, d\mu = \int_{\Omega} \varphi \, d\mu + c(\mu^+(\Omega) - \mu^-(\Omega)) \\ &= \int_{\Omega} (\varphi(x) + c) \, d\mu(x). \end{aligned}$$

Since $W^{1,\alpha'}(\Omega) = W_0^{1,\alpha'}(\Omega) + \mathbb{R}$, this gives the assertion. \square

Using the previous result, we can now define the divergence on $L^\alpha(\Omega, \mathbb{R}^d)$ as follows.

Definition 2.2. *Define*

$$\begin{aligned} \text{div}_{L^\alpha} : L^\alpha(\Omega, \mathbb{R}^d) &\rightarrow W_{\perp}^{-1,\alpha}(\Omega) := \left\{ v \in (W^{1,\alpha'}(\Omega))^* \mid \langle v, 1 \rangle = 0 \right\}, \\ \langle \text{div}_{L^\alpha} q, \varphi \rangle &:= - \int_{\Omega} q \cdot \text{grad} \varphi \, d\mathcal{L}^d \quad \forall \varphi \in W_0^{1,\alpha'}(\Omega), \end{aligned}$$

where grad denotes the usual weak gradient.

Remark 2.3. Recalling Remark 1.2, we observe that $\mu \in W_{\perp}^{-1,\alpha}(\Omega)$, since $\langle \mu, 1 \rangle = \mu^+(\Omega) - \mu^-(\Omega) = 0$. Thus, (2.3) (and (2.2), respectively) is equivalent to

$$\text{div}_{L^\alpha} q = \mu \quad \text{in } W_{\perp}^{-1,\alpha}(\Omega).$$

Next, we give a characterization of $(W_0^{1,\alpha'}(\Omega))^*$.

Lemma 2.4. *The space $W_{\perp}^{-1,\alpha}(\Omega)$ is isomorphic to $(W_0^{1,\alpha'}(\Omega))^*$.*

Proof. On the one hand, it is clear that a functional in $W_{\perp}^{-1,\alpha}(\Omega)$ defines a functional on $W_0^{1,\alpha'}(\Omega)$ so that $W_{\perp}^{-1,\alpha}(\Omega) \subset (W_0^{1,\alpha'}(\Omega))^*$.

On the other hand, the Hahn-Banach theorem implies that every $\ell \in (W_0^{1,\alpha'}(\Omega))^*$ can be extended to a functional L on $W^{1,\alpha'}(\Omega)$. If we define $\bar{v} := |\Omega|^{-1} \int_{\Omega} v(x) \, dx$, then we observe for the functional L that

$$\langle \ell, v - \bar{v} \rangle = \langle L, v \rangle - \bar{v} \langle L, 1 \rangle \quad \forall v \in W^{1,\alpha'}(\Omega).$$

If we now define $\hat{L} \in (W^{1,\alpha'}(\Omega))^*$ by $\hat{L}(v) := L(v) - \bar{v}L(1)$, then

$\hat{L}(1) = 0$, i.e., $\hat{L} \in W_{\perp}^{-1,\alpha}(\Omega)$, and $\ell(v) = \hat{L}(v)$ for all $v \in W_0^{1,\alpha'}(\Omega)$. \square

Remark 2.5. *In complete analogy to the above argumentation, see that $q \in \mathfrak{M}(\Omega, \mathbb{R}^d)$ solves (2.1) if and only if q solves*

$$- \int_{\Omega} \text{grad} \varphi \cdot dq = \int_{\Omega} \varphi \, d\mu \quad \forall \varphi \in C_0^1(\Omega) := \left\{ v \in C^1(\Omega) \mid \int_{\Omega} v(x) \, dx = 0 \right\}.$$

We can then define

$$\begin{aligned} \operatorname{div}_{\mathfrak{M}} : \mathfrak{M}(\Omega, \mathbb{R}^d) &\rightarrow \mathcal{M}_\perp(\Omega) := \{v \in (C^1(\Omega))^* \mid \langle v, 1 \rangle = 0\}, \\ \langle \operatorname{div}_{\mathfrak{M}} q, \varphi \rangle &:= - \int_{\Omega} \operatorname{grad} \varphi \cdot dq \quad \forall \varphi \in C_0^1(\Omega) \end{aligned}$$

and obtain that $\mathcal{M}_\perp(\Omega)$ is isomorphic to $(C_0^1(\Omega))^*$. Hence, the divergence constraint in (BP) can be understood as

$$\operatorname{div}_{\mathfrak{M}} q = \mu \quad \text{in } \mathcal{M}_\perp(\Omega).$$

Note that clearly $C_0^1(\Omega) \hookrightarrow W_0^{1,\alpha'}(\Omega)$. Hence, for $q \in \mathfrak{M}(\Omega, \mathbb{R}^d)$ with $q \ll \mathcal{L}^d$ and $\frac{dq}{d\mathcal{L}^d} \in L^\alpha(\Omega, \mathbb{R}^d)$, $\operatorname{div}_{\mathfrak{M}} q = \mu$ in $\mathcal{M}_\perp(\Omega)$ immediately implies $\operatorname{div}_{L^\alpha} \frac{dq}{d\mathcal{L}^d} = \mu$ in $W_\perp^{-1,\alpha}(\Omega)$.

The following two corollaries follow directly from the above definitions.

Corollary 2.6. *The adjoint operator $\operatorname{div}_{L^\alpha}^*$ of $\operatorname{div}_{L^\alpha} : L^\alpha(\Omega, \mathbb{R}^d) \rightarrow W_\perp^{-1,\alpha}(\Omega)$ is given by $-\operatorname{grad} : W_0^{1,\alpha'}(\Omega) \rightarrow L^{\alpha'}(\Omega, \mathbb{R}^d)$.*

Corollary 2.7. 1. *The divergence operator $\operatorname{div}_{\mathfrak{M}}$ is continuous w.r.t. weak-* convergence in $\mathfrak{M}(\Omega, \mathbb{R}^d)$.*

2. *The divergence operator $\operatorname{div}_{L^\alpha}$ is continuous w.r.t. weak convergence in $L^\alpha(\Omega, \mathbb{R}^d)$.*

Before proving existence and uniqueness of solutions for (BP $_\epsilon$), we cover surjectivity of the divergence operator under suitable assumptions.

Assumption 2.8. *Assume that Ω is such that the equation*

$$\operatorname{div}_{L^{\alpha'}} \operatorname{grad} y = v \quad \text{in } W_\perp^{-1,\alpha'}(\Omega) \tag{2.4}$$

has a unique solution $y \in W_0^{1,\alpha'}(\Omega)$ for every $v \in W_\perp^{-1,\alpha'}(\Omega)$. Note that the associated solution operator, denoted by $\Delta_{\alpha'}^{-1} : W_\perp^{-1,\alpha'}(\Omega) \rightarrow W_0^{1,\alpha'}(\Omega)$ is continuous by the open mapping theorem.

Remark 2.9. *Note that Assumption 2.8 holds in two and three dimensions provided that the interior of Ω is a bounded Lipschitz domain in the spirit of [18, Chapter 1.2]. See e.g. [17, Theorem 3] for $d = 2$ and [31, Theorem 1.6] for $d = 3$. We will assume Assumption 2.8 to hold for the remainder of this work.*

Lemma 2.10. *Let Assumption 2.8 hold. Then, the divergence operator $\operatorname{div}_{L^\alpha}$ is surjective.*

Proof. We denote the solution operator of (2.4) as $\Delta_{\alpha'}^{-1}$. By identifying $W_0^{1,\alpha'}(\Omega)$ with its bi-dual space, we note that the adjoint operator $(\Delta_{\alpha'}^{-1})^* : W_\perp^{-1,\alpha}(\Omega) \rightarrow W_0^{1,\alpha}(\Omega)$ is continuous as well with $\|(\Delta_{\alpha'}^{-1})^*\| \leq \|\Delta_{\alpha'}^{-1}\|$. Moreover, we observe, that $(\Delta_{\alpha'}^{-1})^* = (\Delta_{\alpha'}^*)^{-1}$ and

$$\Delta_{\alpha'}^* = \operatorname{div}_{L^\alpha} \operatorname{grad} : W_0^{1,\alpha}(\Omega) \rightarrow W_\perp^{-1,\alpha}(\Omega).$$

Hence, the elliptic equation

$$\int_{\Omega} \operatorname{grad} y \cdot \operatorname{grad} \psi \, d\mathcal{L}^d = \langle \psi, v \rangle \quad \forall \psi \in W_0^{1,\alpha'}(\Omega)$$

has a unique solution $y \in W_0^{1,\alpha}(\Omega)$ for all $v \in W_\perp^{-1,\alpha}(\Omega)$. By setting $q = -\operatorname{grad} y$, we find $\operatorname{div}_{L^\alpha} q = v$ in $W_\perp^{-1,\alpha}(\Omega)$, which shows the surjectivity of $\operatorname{div}_{L^\alpha}$ from $L^\alpha(\Omega, \mathbb{R}^d)$ to $W_\perp^{-1,\alpha}(\Omega)$. \square

Remark 2.11. Due to [Remark 1.2 2](#), [Lemma 2.10](#) also implies the surjectivity of $\operatorname{div}_{\mathfrak{M}}$.

Finally, we obtain an existence result.

Corollary 2.12. Let [Assumptions 1.1](#) and [2.8](#) hold. For ever $\varepsilon > 0$ there is a unique solution for problem [\(BP \$_{\varepsilon}\$ \)](#).

Proof. First note that due to [Remark 2.3](#) it holds $\mu \in W_{\perp}^{-1,\alpha}(\Omega)$, so that by [Lemma 2.10](#) the feasible set is non-empty.

Let now $(q_n) \subset \mathfrak{M}(\Omega)$ be a minimizing sequence. Without loss of generality we assume that each q_n is feasible and due to the regularization term $(\frac{dq_n}{d\mathcal{L}^d})$ is bounded in $L^{\alpha}(\Omega)$. We can thus extract a weakly convergent subsequence (denoted by the same symbol) with weak limit $\bar{q} \in L^{\alpha}(\Omega, \mathbb{R}^d)$. As $w \in C(\Omega) \hookrightarrow L^{\alpha'}(\Omega)$, the objective functional is clearly lower semi continuous in $L^{\alpha}(\Omega)$ and thus, \bar{q} is a solution to [\(BP \$_{\varepsilon}\$ \)](#).

Uniqueness of the solution follows trivially from the strict convexity of $\|\cdot\|_{L^{\alpha}(\Omega)}$. \square

3. Semi-Smooth Newton

We first derive the first order optimality system for [\(BP \$_{\varepsilon}\$ \)](#).

Proposition 3.1. There exists a Lagrange multiplier $y \in W_0^{1,\alpha'}(\Omega)$ such that the solution q of [\(BP \$_{\varepsilon}\$ \)](#) fulfills

$$\varepsilon|q|^{\alpha-2}q + \partial|q|_{1,w} + \operatorname{grad} y \ni 0 \quad \text{in } L^{\alpha'}(\Omega, \mathbb{R}^d) \quad (3.1)$$

$$\operatorname{div}_{L^{\alpha}} q = \mu \quad \text{in } W_{\perp}^{-1,\alpha}(\Omega), \quad (3.2)$$

where $|q|_{1,w}(x) := w(x)|q(x)|$.

Proof. Let us denote $C := \{q \in L^{\alpha}(\Omega, \mathbb{R}^d) : \operatorname{div}_{L^{\alpha}} q = \mu\}$ such that [\(BP \$_{\varepsilon}\$ \)](#) is equivalent to

$$\inf_{q \in L^{\alpha}(\Omega, \mathbb{R}^d)} \int_{\Omega} w|q| \, d\mathcal{L}^d + \frac{\varepsilon}{\alpha} \|q\|_{L^{\alpha}(\Omega, \mathbb{R}^d)}^{\alpha} + \iota_C(q).$$

Since the first two addends of the objective are continuous on the whole space $L^{\alpha}(\Omega, \mathbb{R}^d)$ and C is nonempty due to [Lemma 2.10](#), the sum rule for convex subdifferentials is applicable, which gives that the solution q of [\(BP \$_{\varepsilon}\$ \)](#) satisfies

$$\begin{aligned} 0 &\in \varepsilon|q|^{\alpha-2}q + \partial|q|_{1,w} + \partial\iota_C(q) \\ \Leftrightarrow \exists \xi \in \partial|q|_{1,w}, \quad &\int_{\Omega} (\varepsilon|q|^{\alpha-2}q + \xi)(p - q) \, d\mathcal{L}^d \geq 0 \quad \forall p \in C \\ \Leftrightarrow \exists \xi \in \partial|q|_{1,w}, \quad &\varepsilon|q|^{\alpha-2}q + \xi \in \ker(\operatorname{div}_{L^{\alpha}})^{\perp} = \operatorname{ran}(\operatorname{div}_{L^{\alpha}}^*), \end{aligned}$$

where we employed [[24](#), §6.6, Theorem 2], which holds due to the surjectivity of $\operatorname{div}_{L^{\alpha}}$ by [Lemma 2.10](#). Since $\operatorname{div}_{L^{\alpha}}^* = -\operatorname{grad}$, this gives the assertion. \square

We observe that the multi-valued map

$$(x, q) \mapsto \varepsilon|q|^{\alpha-2}q + w(x)\partial|q|$$

has a single-valued inverse, which we denote by

$$F_\varepsilon : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}, \quad F_\varepsilon(x, p) = \left(\frac{1}{\varepsilon} (|p| - w(x)) \right)_+^{\alpha'-1} \frac{p}{|p|}. \quad (3.3)$$

Since (3.1) is a pointwise equation (as identity in $L^{\alpha'}(\Omega, \mathbb{R}^d)$), this yields that (3.1)–(3.2) are equivalent to

$$\operatorname{div}_{L^\alpha} F_\varepsilon(-\operatorname{grad} y) = \mu \quad \text{in } W_\perp^{-1, \alpha}(\Omega). \quad (3.4)$$

where the Nemytskii-operator F_ε maps $L^{\alpha'}(\Omega, \mathbb{R}^d)$ to $L^\alpha(\Omega, \mathbb{R}^d)$. By Definition 2.2, the weak form of (3.4) is given by

$$-\int_\Omega F_\varepsilon(-\operatorname{grad} y) \cdot \operatorname{grad} \varphi \, d\mathcal{L}^d = \int_\Omega \varphi \, d\mu \quad \forall \varphi \in W_0^{1, \alpha'}(\Omega). \quad (3.5)$$

We can now formally write down a semi-smooth Newton iteration for solving (3.5) as follows.

Algorithm 1. Semi-Smooth Newton Iteration for solving (3.5)

Require: $y \in W_0^{1, \alpha'}(\Omega)$

for $k = 1, \dots$ **do**

 Choose a step size $\sigma_k > 0$

 find $\eta \in W_0^{1, \alpha'}(\Omega)$ such that $\forall \varphi \in W_0^{1, \alpha'}(\Omega)$

$$\int_\Omega (D_p F_\varepsilon(-\operatorname{grad} y) \operatorname{grad} \eta) \cdot \operatorname{grad} \varphi \, d\mathcal{L}^d = \int_\Omega F_\varepsilon(-\operatorname{grad} y) \cdot \operatorname{grad} \varphi \, d\mathcal{L}^d + \int_\Omega \varphi \, d\mu \quad (3.6)$$

 Update $y \leftarrow y + \sigma_k \eta$

end for

Remark 3.2. We emphasize that (3.6) is purely formal. For Algorithm 1 to converge, we would need for F_ε to be Newton-differentiable from $L^{\alpha'}(\Omega, \mathbb{R}^d)$ to $L^\alpha(\Omega, \mathbb{R}^d)$ and existence of solutions to (3.6) in the appropriate spaces. While the latter issue will be resolved by an additional Huber-regularization, see (3.7) below, the Newton-differentiability probably requires an additional smoothing step, as applied for instance in [30, Section 6.1]. This is subject to future research.

Due to the positive part in (3.3), $F_\varepsilon(x, p)$ has vanishing slope for $|p| \leq w$, which will clearly lead to illposedness of the Newton step (3.6). As mentioned above, to overcome this issue, we introduce a Huber type regularization term [20] R_δ of the form

$$R_\delta : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad R_\delta(x, p) = \frac{\delta p}{\max(|p|, w)},$$

where $\delta > 0$ is a regularization parameter. Denoting $G_{\varepsilon, \delta} := F_\varepsilon + R_\delta$ we thus replace (3.5) by

$$-\int_\Omega G_{\varepsilon, \delta}(-\operatorname{grad} y) \cdot \operatorname{grad} \varphi \, d\mathcal{L}^d = \int_\Omega \varphi \, d\mu \quad \forall \varphi \in W_0^{1, \alpha'}(\Omega). \quad (3.7)$$

and (3.6) by

$$\begin{aligned} \int_{\Omega} D_q G_{\varepsilon, \delta}(-\text{grad } y) \text{grad } \eta \cdot \text{grad } \varphi \, d\mathcal{L}^d = \\ \int_{\Omega} G_{\varepsilon, \delta}(-\text{grad } y) \cdot \text{grad } \varphi \, d\mathcal{L}^d + \int_{\Omega} \varphi \, d\mu \quad \forall \varphi \in W_0^{1, \alpha'}(\Omega). \end{aligned} \quad (3.8)$$

3.1. Step Size Rule

In order to apply Armijo bracktracking, we lift (3.4) to a minimization problem. To that end, we observe that both F_ε and R_δ admit an antiderivative (w.r.t. p), namely

$$\begin{aligned} \mathcal{F}_\varepsilon : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad \mathcal{F}_\varepsilon(x, p) &:= \frac{\varepsilon}{\alpha'} \left(\frac{1}{\varepsilon} \max\{|p| - w, 0\} \right)^{\alpha'} \text{ and} \\ \mathcal{R}_\delta : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad \mathcal{R}_\delta(x, p) &:= \delta \max\{|p|, w\} + \frac{\delta}{2} \min\left\{ \frac{|p|^2}{w}, w \right\}. \end{aligned}$$

More precisely, we obtain the following result.

Lemma 3.3. *Both $\mathcal{F}_\varepsilon, \mathcal{R}_\delta : L^{\alpha'}(\Omega, \mathbb{R}^d) \rightarrow L^1(\Omega, \mathbb{R}^d)$ are Gateaux-differentiable with Gateaux-derivatives given by*

$$\begin{aligned} d\mathcal{F}_\varepsilon(p; \psi) &= F_\varepsilon(p) \cdot \psi, \\ d\mathcal{R}_\delta(p; \psi) &= R_\delta(p) \cdot \psi \end{aligned}$$

respectively, where $F_\varepsilon, R_\delta : L^{\alpha'}(\Omega, \mathbb{R}^d) \rightarrow L^\alpha(\Omega, \mathbb{R}^d)$.

Proof. Let now $p, \psi \in L^{\alpha'}(\Omega, \mathbb{R}^d)$. Elementary calculations show

$$\lim_{t \rightarrow 0} \frac{\mathcal{F}_\varepsilon(x, (p + t\psi)(x)) - \mathcal{F}_\varepsilon(x, p(x))}{t} \rightarrow F_\varepsilon(x, p(x)) \cdot \psi(x)$$

for a.e. $x \in \Omega$ and similarly for \mathcal{R}_δ . By Lebesgue's dominated convergence theorem, it suffices to show that the right hand side is a function in $L^1(\Omega, \mathbb{R}^d)$ and the mapping $\psi \mapsto F_\varepsilon(p) \cdot \psi$ is continuous.

To that end, note that by Hölder's inequality

$$\begin{aligned} \int_{\Omega} |F_\varepsilon(x, p(x)) \cdot \psi(x)| \, dx &\leq \varepsilon^{1-\alpha'} \|\psi\|_{L^{\alpha'}(\Omega)} \|(|p| - w)_+^{\alpha'-1}\|_{L^\alpha(\Omega)} \\ &\leq \varepsilon^{1-\alpha'} \|\psi\|_{L^{\alpha'}(\Omega, \mathbb{R}^d)} \|p\|_{L^{\alpha'}(\Omega, \mathbb{R}^d)} < \infty. \end{aligned}$$

For \mathcal{R}_δ , we obtain the result by

$$\begin{aligned} \int_{\Omega} |R_\delta(x, p(x)) \cdot \psi(x)| \, dx &\leq \delta \int_{\Omega} \frac{|q||\psi|}{\max(|q|, w)} \, d\mathcal{L}^d \leq \delta \int_{\{q(x) \neq 0\}} \frac{|q|}{|q|} |\psi| \, d\mathcal{L}^d \\ &\leq \delta \|\psi\|_{L^1(\Omega, \mathbb{R}^d)} \leq \delta |\Omega|^{\frac{1}{r}} \|\psi\|_{L^{\alpha'}(\Omega, \mathbb{R}^d)} < \infty, \end{aligned}$$

where $1 = \frac{1}{\alpha'} + \frac{1}{r}$. □

Analogously to $G_{\varepsilon,\delta}$, we will denote $\mathcal{G}_{\varepsilon,\delta} := \mathcal{F}_\varepsilon + \mathcal{R}_\delta$.

In light of the above differentiability results, we observe that (3.7) is nothing else than the necessary optimality conditions of

$$\min_{y \in W_0^{1,\alpha'}(\Omega)} \mathcal{J}(y) := \int_{\Omega} \mathcal{G}_{\varepsilon,\delta}(-\text{grad } y) \, d\mathcal{L}^d - \int_{\Omega} y \, d\mu. \quad (\text{BP}_\dagger)$$

As \mathcal{F}_ε and \mathcal{R}_δ are convex, (3.7) is indeed sufficient for optimality so that (BP_\dagger) is equivalent to (3.7). More precisely, $\mathcal{G}_{\varepsilon,\delta}$ is uniformly convex for $\delta > 0$, as $\frac{\alpha}{\alpha-1} \geq 2$. Now, we can perform a classical Armijo backtracking for \mathcal{J} as detailed in Algorithm 2. Note that

$$D_y \mathcal{J}(y)\eta = - \int_{\Omega} G_{\varepsilon,\delta}(-\text{grad } y) \cdot \text{grad } \eta \, d\mathcal{L}^d - \int_{\Omega} \eta \, d\mu$$

so that the Armijo condition in Algorithm 2 can be written as

$$\begin{aligned} \int_{\Omega} \mathcal{G}_{\varepsilon,\delta}(-\text{grad } y - \sigma_k \text{grad } \eta_k) \, d\mathcal{L}^d &> \int_{\Omega} \mathcal{G}_{\varepsilon,\delta}(-\text{grad } y) \, d\mathcal{L}^d \\ &- \gamma \sigma_k \int_{\Omega} G_{\varepsilon,\delta}(-\text{grad } y) \cdot \text{grad } \eta \, d\mathcal{L}^d + \sigma_k(1-\gamma) \int_{\Omega} \eta \, d\mu. \end{aligned}$$

Algorithm 2. Armijo line search for (BP_\dagger)

Require: $y, \eta \in W_0^{1,\alpha'}(\Omega)$, $\sigma_0 > 0$, $\beta, \gamma \in (0, 1)$

$k \leftarrow 0$

while do $\mathcal{J}(y + \sigma_k \eta) > \mathcal{J}(y) + \gamma \sigma_k D_y \mathcal{J}(y)\eta$

$\sigma_{k+1} \leftarrow \beta \sigma_k$

$k \leftarrow k + 1$

end while

3.2. Connection to Primal Problem

We want to analyze the connection between problem (BP_ε) and problem (BP_\dagger) .

Lemma 3.4. *The Fenchel conjugate $\mathcal{G}_{\varepsilon,\delta}^*$ w.r.t. the second variable is given by*

$$\mathcal{G}_{\varepsilon,\delta}^*(x, q) = \begin{cases} |q|^2 \frac{w}{2\delta} - \delta w, & |q| \leq \delta, \\ \frac{\varepsilon}{\alpha} (|q| - \delta)^\alpha - \frac{3}{2} \delta w + |q|w, & \text{else.} \end{cases}$$

Moreover, $\mathcal{G}_{\varepsilon,\delta}$ is a normal integrand in the sense of [26, Definition 14.27].

Proof. We begin by deriving the conjugate of $\mathcal{G}_{\varepsilon,\delta}$. First note that

$$\text{grad}_p (s \cdot p - \mathcal{G}_{\varepsilon,\delta}(x, p)) = s - G_{\varepsilon,\delta}(x, p).$$

We then observe

$$G_{\varepsilon,\delta}(x, \cdot)^{-1}(z) = \begin{cases} z \frac{w}{\delta}, & |z| \leq \delta, \\ \frac{z}{|z|} \left(\varepsilon(|z| - \delta)^{\alpha-1} + w \right), & \text{else,} \end{cases}$$

so that we can insert $p := G_{\varepsilon,\delta}(x, \cdot)^{-1}(s)$ into $s \cdot p - \mathcal{G}_{\varepsilon,\delta}(x, p)$. By straightforward manipulations, the first claim follows.

For the second claim, we first note that $L^\alpha(\Omega, \mathbb{R}^d)$ is decomposable relative to the Lebesgue measure in the sense of [26, Definition 14.59]. The assertion then follows by [26, Example 4.29] as $\mathcal{G}_{\varepsilon,\delta}(x, \cdot)$ is continuous for all $x \in \Omega$ and $\mathcal{G}_{\varepsilon,\delta}(\cdot, p)$ is measurable for all $p \in \mathbb{R}^d$. \square

Using the above result, we can characterize the connection as follows.

Theorem 3.5. *The predual problem to*

$$-\inf \left\{ \int_{\Omega} \mathcal{G}_{\varepsilon,\delta}^*(q) \, d\mathcal{L}^d \mid q \in \mathfrak{M}(\Omega), q \ll \mathcal{L}^d, \frac{dq}{d\mathcal{L}^d} \in L^\alpha(\Omega), \operatorname{div}_{L^\alpha} \frac{dq}{d\mathcal{L}^d} = \mu \text{ in } W_{\perp}^{-1,\alpha}(\Omega) \right\} \quad (\text{BP}_{\varepsilon,\delta})$$

is given by (BP₊) and strong duality holds.

Proof. Let

$$f : C(\Omega) \rightarrow \mathbb{R} \cup \{\pm\infty\}, \quad f(\xi) = \begin{cases} -\langle \xi, \mu \rangle, & \xi \in W_{\emptyset}^{1,\alpha'}(\Omega), \\ \infty, & \text{else} \end{cases}$$

and $g : C(\Omega, \mathbb{R}^d) \rightarrow \mathbb{R} \cup \{\pm\infty\}$, $g(\xi) = \int_{\Omega} \mathcal{G}_{\varepsilon,\delta}(\xi) \, d\mathcal{L}^d$. It is easy to see that

$$f^*(v) = \iota_{\{-\mu\}}(v) = \begin{cases} 0, & v = -\mu \text{ in } W_{\perp}^{-1,\alpha}(\Omega), \\ \infty, & \text{else.} \end{cases}$$

Finally, by Lemma 3.4, $g^*(v) = \int_{\Omega} \mathcal{G}_{\varepsilon,\delta}^*(q) \, d\mathcal{L}^d$. The assertion then follows by standard arguments, see e.g. [9, Theorem 4.4.3]. \square

4. Approximation Results

Next we turn to results on approximation properties. More precisely, we show that minimizers of the regularized problems converge to minimizers of (BP) under suitable assumptions.

Recall from, e.g., [10], that a sequence (F_n) of functionals $F_n : X \rightarrow \mathbb{R} \cup \{\infty\}$ on a metric space X is said to Γ -converge to a functional $F : X \rightarrow \mathbb{R} \cup \{\infty\}$, written $F = \Gamma\text{-}\lim_{n \rightarrow \infty} F_n$, if

1. for every sequence $\{x_n\} \subset X$ with $x_n \rightarrow x$, it holds $F(x) \leq \liminf_{n \rightarrow \infty} F_n(x_n)$ and
2. for every $x \in X$, there is a sequence $\{x_n\} \subset X$ with $x_n \rightarrow x$ and $F(x) \geq \limsup_{n \rightarrow \infty} F_n(x_n)$.
This sequence is also called a *recovery sequence*.

It is a straightforward consequence of this definition that if F_n Γ -converges to F and x_n is a minimizer of F_n for every $n \in \mathbb{N}$, then every cluster point of the sequence (x_n) is a minimizer to F . Furthermore, Γ -convergence is stable under perturbations by continuous functionals.

To prove the desired approximation results, we will rely on smoothing of measures in order to construct the necessary recovery sequences. Moreover, we need the following technical assumption.

Assumption 4.1. *Assume that Ω is strictly star shaped w.r.t. 0, i.e. for all $x \in \Omega$ and $0 \leq \lambda < 1$, it holds $\lambda x \in \Omega^\circ$.*

Remark 4.2. *We leverage Assumption 4.1 in Lemma 4.3 below. However, while we only use a linear transformation of the domain in the following, the techniques we use in the proof of Lemma 4.3 could be applied in more general settings of nonlinear bi-Lipschitz deformations which would allow us to relax this assumption. We still focus on star shaped domains for the sake of brevity. Additionally, Assumption 4.1 is not overly restrictive as one can always formulate (BP) on a strictly star shaped domain $K \supset \Omega$ and approximate the original problem by choosing w to be large on $K \setminus \Omega$.*

Throughout the rest of this section, for a given sequence $0 < \tau \rightarrow 0$, let $0 \leq \varphi_\tau \in C_c^\infty(\mathbb{R}^d)$ be a sequence of mollifiers. To avoid boundary effects, we will need to slightly extend the domain Ω . More precisely, for every $\tau > 0$, we choose $s > 1$ such that $\Omega_\tau := (1+s)\Omega \supset \Omega + \text{spt } \varphi_\tau$. W.l.o.g. we may assume $\Omega_\tau \supset \Omega_\vartheta$ whenever $\tau > \vartheta$. Note that this is possible thanks to Assumption 4.1. Moreover, we denote $\tilde{\Omega} = \cup_\tau \Omega_\tau$. Given a function (or measure) f , we will denote by \tilde{f} the extension of f onto $\tilde{\Omega}$ by zero. With \hat{w} we will denote a continuous extension of $w \in C(\Omega)$ onto $\tilde{\Omega}$ which also satisfies $\min_{\tilde{\Omega}} \hat{w} = \min_\Omega w$.

For $\nu \in \mathcal{M}_\perp(\Omega)$ and $\nu_\varepsilon \in \mathcal{M}_\perp(\Omega_\varepsilon)$ let now $H_{\varepsilon,\delta}^{\nu_\varepsilon} : \mathfrak{M}(\Omega_\varepsilon) \rightarrow \mathbb{R} \cup \{\pm\infty\}$ and $H^\nu : \mathfrak{M}(\Omega) \rightarrow \mathbb{R} \cup \{\pm\infty\}$ be defined by

$$H_{\varepsilon,\delta}^{\nu_\varepsilon}(q) = \begin{cases} \int_{\Omega_\varepsilon} \mathcal{G}_{\varepsilon,\delta}^* \left(\frac{dq}{d\mathcal{L}^d} \right) d\mathcal{L}^d, & q \ll \mathcal{L}^d, \frac{dq}{d\mathcal{L}^d} \in L^\alpha(\Omega_\varepsilon, \mathbb{R}^d), \text{div}_{L^\alpha} \frac{dq}{d\mathcal{L}^d} = \nu_\varepsilon \text{ in } W_\perp^{-1,\alpha}(\Omega_\varepsilon), \\ \infty, & \text{else,} \end{cases}$$

where $\mathcal{G}_{\varepsilon,\delta}^*(\cdot, q)$ is extended onto Ω_ε by extending w with \hat{w} , and

$$H^\nu(q) = \begin{cases} \int_\Omega w d|q|, & \text{div}_{\mathfrak{M}} q = \nu \text{ in } \mathcal{M}_\perp(\Omega), \\ \infty, & \text{else,} \end{cases}$$

respectively. Note that

$$H_{\varepsilon,0}^{\nu_\varepsilon}(q) = \begin{cases} \int_{\Omega_\varepsilon} \hat{w} d|q| + \frac{\varepsilon}{\alpha} \left\| \frac{dq}{d\mathcal{L}^d} \right\|_{L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)}^\alpha, & q \ll \mathcal{L}^d, \frac{dq}{d\mathcal{L}^d} \in L^\alpha(\Omega_\varepsilon, \mathbb{R}^d), \text{div}_{L^\alpha} \frac{dq}{d\mathcal{L}^d} = \nu_\varepsilon \text{ in } W_\perp^{-1,\alpha}(\Omega_\varepsilon), \\ \infty, & \text{else.} \end{cases}$$

Note that we can extend $H_{\varepsilon,\delta}^\nu$ and H^ν to be defined on measures on $\tilde{\Omega}$ by extending the argument onto $\tilde{\Omega}$ by zero as described above. Strictly speaking, the approximating problems that we consider are given as problems on Ω_ε , i.e.

$$\min_{q \in \mathfrak{M}(\Omega_\varepsilon, \mathbb{R}^d)} H_{\varepsilon,0}^{\hat{\mu}}(q) \quad \text{and} \quad \min_{q \in \mathfrak{M}(\Omega_\varepsilon, \mathbb{R}^d)} H_{\varepsilon,\delta}^{\hat{\mu}}(q),$$

respectively. For convenience, we will refer to these problems by (BP_ε) and $(BP_{\varepsilon,\delta})$, too.

Before we present the first approximation result, we state two auxiliary results.

Lemma 4.3. Let [Assumptions 1.1, 2.8 and 4.1](#) hold and let $\tau > 0$. Let $v_\tau \in W_\perp^{-1,\alpha}(\Omega_\tau)$. Then the elliptic equation

$$\int_{\Omega_\tau} \text{grad } y_\tau \cdot \text{grad } \psi \, d\mathcal{L}^d = -\langle \psi, v_\tau \rangle \quad \forall \psi \in W_0^{1,\alpha'}(\Omega_\tau).$$

has a unique solution $y_\tau \in W_0^{1,\alpha}(\Omega_\tau)$. Moreover, the solution operator $\Delta_\tau^{-1} : W_\perp^{-1,\alpha}(\Omega_\tau) \rightarrow W_0^{1,\alpha}(\Omega_\tau)$ is uniformly bounded for $\tau \rightarrow 0$.

A proof of [Lemma 4.3](#) is given in [Appendix B](#).

Lemma 4.4. Let [Assumptions 1.1, 2.8 and 4.1](#) hold and let $q \in \mathfrak{M}(\Omega, \mathbb{R}^d)$ and $\mu \in \mathcal{M}_\perp(\Omega)$. Let $q_\varepsilon \in \mathfrak{M}(\Omega_\varepsilon, \mathbb{R}^d)$ such that $\tilde{q}_\varepsilon \xrightarrow{*} \tilde{q}$ in $\mathfrak{M}(\tilde{\Omega}, \mathbb{R}^d)$ and $\text{div}_{\mathfrak{M}} q_\varepsilon = \tilde{\mu}$ in $\mathcal{M}_\perp(\Omega_\varepsilon)$. Then also $\text{div}_{\mathfrak{M}} q = \mu$ in $\mathcal{M}_\perp(\Omega)$.

Proof. Let $\psi \in C^1(\Omega)$, $\eta > 0$ and w.l.o.g. assume $\varepsilon < \eta$. Then with $\psi_\eta := \psi(\cdot \cdot (1 + \eta)^{-1})$ it holds $\psi_\eta|_{\Omega_\varepsilon} \in C^1(\Omega_\varepsilon)$ and $\psi_\eta \rightarrow \psi$ in $C^1(\Omega)$. Let now $\xi_\eta \in C(\tilde{\Omega}, \mathbb{R}^d)$ be a continuous extension of $\text{grad } \psi_\eta$ onto $\tilde{\Omega}$. Then

$$-\int_{\Omega} \text{grad } \psi_\eta \, d\mu = \int_{\tilde{\Omega}} \xi_\eta \cdot d\tilde{q}_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \int_{\tilde{\Omega}} \xi_\eta \cdot d\tilde{q} = \int_{\Omega} \text{grad } \psi_\eta \cdot dq$$

and passing to the limit $\eta \rightarrow 0$ concludes the proof. \square

We are now in the position to state our first approximation result, which covers convergence of the minimizers of [\(BP\) \$_\varepsilon\$](#) .

Theorem 4.5. Let [Assumptions 1.1, 2.8 and 4.1](#) hold. It holds $\Gamma\text{-lim}_{\varepsilon \rightarrow 0} H_{\varepsilon,0}^{\tilde{\mu}} = H^\mu$ w.r.t. weak-* convergence in $\mathfrak{M}(\tilde{\Omega}, \mathbb{R}^d)$.

Proof. 1. *lim inf-condition:* Let $q_\varepsilon \in \mathfrak{M}(\Omega_\varepsilon, \mathbb{R}^d)$ be such that $\tilde{q}_\varepsilon \xrightarrow{*} q \in \mathfrak{M}(\tilde{\Omega})$ in $\mathfrak{M}(\tilde{\Omega})$. Due to the weak-* convergence, (\tilde{q}_ε) is bounded in $\mathfrak{M}(\tilde{\Omega})$, so that $\int_{\tilde{\Omega}} \hat{w} \, d|q| < \infty$. Resort now to a subsequence such that $H_{\varepsilon,0}^{\tilde{\mu}}(q_\varepsilon) < \infty$. Then by [Lemma A.2](#), $\text{spt } q \subset \Omega$. Together with [Lemma 4.4](#), q is feasible for [\(BP\)](#) and the assertion then follows directly from $\int_{\tilde{\Omega}} \hat{w} \, d|\cdot|$ being l.s.c. w.r.t. weak-* convergence in $\mathfrak{M}(\tilde{\Omega})$ and $\|\cdot\|_{L^\alpha(\tilde{\Omega}, \mathbb{R}^d)} \geq 0$.

2. *lim sup-condition:* Let $q \in \mathfrak{M}(\Omega, \mathbb{R}^d)$ be arbitrary. In the case $H^\mu(q) = \infty$, the assertion holds trivially. Hence, assume $H^\mu(q) < \infty$.

Let now φ_ε be as above and w.l.o.g. assume $\varepsilon \|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^d)}^\alpha \rightarrow 0$ for $\varepsilon \rightarrow 0$. Set $q_\varepsilon := \varphi_\varepsilon * \tilde{q}$ and $\mu_\varepsilon := \varphi_\varepsilon * \tilde{\mu}$. Then $q_\varepsilon \in L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)$ and $\mathcal{I}(\tilde{q}_\varepsilon) \rightarrow \tilde{q}$ in $\mathfrak{M}(\tilde{\Omega}, \mathbb{R}^d)$ by [Lemma A.1](#). Define now $e_\varepsilon := \tilde{\mu}_\varepsilon - \tilde{\mu}$. It is straightforward to see that $\langle e_\varepsilon, 1 \rangle = 0$, i.e. $e_\varepsilon \in W_\perp^{-1,\alpha}(\Omega_\varepsilon)$. Then by [Lemma 4.3](#) there is $y_\varepsilon \in W_0^{1,\alpha}(\Omega_\varepsilon)$ solving

$$\int_{\Omega_\varepsilon} \text{grad } y_\varepsilon \cdot \text{grad } \psi \, d\mathcal{L}^d = - \int_{\Omega_\varepsilon} \psi \, de_\varepsilon \quad \forall \psi \in W_0^{1,\alpha'}(\Omega_\varepsilon). \quad (4.1)$$

Moreover, [Remark 1.2](#) and [Lemma A.1](#) yield $e_\varepsilon \rightarrow 0$ in $W_\perp^{-1,\alpha}(\Omega_\varepsilon)$ and hence, $\tilde{y}_\varepsilon \rightarrow 0$ in $W_\emptyset^{1,\alpha'}(\tilde{\Omega})$ by [Lemma 4.3](#). Thus, $\widetilde{\text{grad } y_\varepsilon} \rightarrow 0$ in $L^\alpha(\tilde{\Omega})$ and by defining $\check{q}_\varepsilon \in \mathfrak{M}(\Omega_\varepsilon)$ as

$$\check{q}_\varepsilon := q_\varepsilon + \text{grad } y_\varepsilon,$$

we obtain $\widetilde{\mathcal{I}(\check{q}_\varepsilon)} \rightarrow \tilde{q}$ in $\mathfrak{M}(\tilde{\Omega}, \mathbb{R}^d)$. For $\psi \in W_\emptyset^{1,\alpha'}(\Omega_\varepsilon)$, leveraging [Lemma A.1 2](#) now yields

$$\begin{aligned} - \int_{\Omega_\varepsilon} \text{grad } \psi \cdot d\check{q}_\varepsilon &= - \int_{\Omega_\varepsilon} \text{grad } \psi \cdot q_\varepsilon \, d\mathcal{L}^d - \int_{\Omega_\varepsilon} \text{grad } \psi \cdot \text{grad } y_\varepsilon \, d\mathcal{L}^d \\ &= \int_{\Omega_\varepsilon} \psi \, d\mu_\varepsilon + \int_{\Omega} \psi \, d\mu - \int_{\Omega_\varepsilon} \psi \, d\mu_\varepsilon = \int_{\Omega_\varepsilon} \psi \, d\tilde{\mu}, \end{aligned}$$

so that $\text{div}_{L^\alpha} \check{q}_\varepsilon = \tilde{\mu}$ in $W_\perp^{-1,\alpha}(\Omega_\varepsilon)$. Thus, \check{q}_ε is feasible for [\(BP \$_\varepsilon\$ \)](#). Going on, we note that $\int_{\Omega} w|\check{q}_\varepsilon| \, d\mathcal{L}^d \rightarrow \int_{\Omega} w \, d|q|$ due to $\widetilde{\mathcal{I}(\check{q}_\varepsilon)} \rightarrow \tilde{q} \in \mathfrak{M}(\tilde{\Omega})$. Moreover,

$$|q_\varepsilon(x)| \leq \sup_{h \in \mathbb{R}^d, |h| \leq 1} \left| \int_{\tilde{\Omega}} \varphi_\varepsilon(x-y) \, d(\tilde{q}(y) \cdot h) \right| \leq \int_{\tilde{\Omega}} |\varphi_\varepsilon(x-y)| \, d|\tilde{q}(y)|, \quad (4.2)$$

which gives

$$\|q_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq \|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^d)} |q|(\Omega). \quad (4.3)$$

Hence,

$$\left(\frac{\varepsilon}{\alpha}\right)^{\frac{1}{\alpha}} \|\check{q}_\varepsilon\|_{L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)} \leq \left(\frac{\varepsilon}{\alpha}\right)^{\frac{1}{\alpha}} \left(|\tilde{\Omega}| \|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^d)} (|q|(\Omega)) + \|\text{grad } y_\varepsilon\|_{L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)} \right),$$

which, due to the assumption on φ_ε , vanishes for $\varepsilon \rightarrow 0$. This yields the desired assertion and concludes the proof. \square

Corollary 4.6. *In the setting of [Theorem 4.5](#), let $w \geq w_\diamond > 0$. Let $\varepsilon_n > 0$ be a vanishing sequence and $(q_n) \subset L^\alpha(\Omega_{\varepsilon_n}, \mathbb{R}^d)$ be the sequence of corresponding solutions of [\(BP \$_\varepsilon\$ \)](#). Then (q_n) admits a subsequence that converges to a solution of [\(BP\)](#) w.r.t. weak-* convergence in $\mathfrak{M}(\tilde{\Omega})$.*

Proof. Let $q_0 \in L^\alpha(\Omega, \mathbb{R}^d)$ be fixed such that $\text{div}_{L^\alpha} q_0 = \mu$ in $W_\perp^{-1,\alpha}(\Omega)$, which exists due to [Lemma 2.10](#). Then q_n satisfies

$$\int_{\tilde{\Omega}} \hat{w} |\tilde{q}_n| \, d\mathcal{L}^d + \frac{\varepsilon_n}{\alpha} \|\tilde{q}_n\|_{L^\alpha(\tilde{\Omega}, \mathbb{R}^d)}^\alpha \leq \int_{\tilde{\Omega}} \hat{w} |\tilde{q}_0| \, d\mathcal{L}^d + \frac{\varepsilon_n}{\alpha} \|\tilde{q}_0\|_{L^\alpha(\tilde{\Omega}, \mathbb{R}^d)}^\alpha.$$

Thus, due to $\hat{w} \geq w_\diamond > 0$ and $\frac{\varepsilon_n}{\alpha} \|\tilde{q}_0\|_{L^\alpha(\tilde{\Omega}, \mathbb{R}^d)}^\alpha \rightarrow 0$ for $n \rightarrow \infty$, it holds

$$w_\diamond \|\tilde{q}_n\|_{L^1(\tilde{\Omega})} \leq \int_{\tilde{\Omega}} \hat{w} |\tilde{q}_0| \, d\mathcal{L}^d < \infty.$$

Hence, $(\mathcal{I}(\tilde{q}_n))$ is bounded in $\mathfrak{M}(\tilde{\Omega})$ and by the Banach-Alaoglu theorem there exists a subsequence (denoted by the same symbol), which converges to $q \in \mathfrak{M}(\tilde{\Omega}, \mathbb{R}^d)$ w.r.t. weak-* convergence in $\mathfrak{M}(\tilde{\Omega})$. The assertion then follows directly from [Theorem 4.5](#) and the properties of Γ -convergence. \square

4.1. Convergence for vanishing Huber regularization

Going on, we turn to problem $(BP_{\varepsilon,\delta})$. As a first step, we only consider the convergence for $\delta \rightarrow 0$. We start by proving an auxiliary result.

Lemma 4.7. *Let Assumption 1.1 hold. For $\delta \rightarrow 0$, the functional $q \mapsto \int_{\Omega_\varepsilon} \mathcal{G}_{\varepsilon,\delta}^*(q) \, d\mathcal{L}^d$ converges locally uniformly to $q \mapsto \int_{\Omega_\varepsilon} \mathcal{G}_{\varepsilon,0}^*(q) \, d\mathcal{L}^d$ on $L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)$.*

Proof. Let $K \subset L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)$ be a bounded set. We want to show

$$\limsup_{\delta \rightarrow 0} \sup_{q \in K} \left| \int_{\Omega_\varepsilon} \mathcal{G}_{\varepsilon,\delta}^*(q) - \mathcal{G}_{\varepsilon,0}^*(q) \, d\mathcal{L}^d \right| = 0.$$

Clearly,

$$\begin{aligned} \left| \int_{\Omega_\varepsilon} \mathcal{G}_{\varepsilon,\delta}^*(q) - \mathcal{G}_{\varepsilon,0}^*(q) \, d\mathcal{L}^d \right| &\leq \left| \int_{\{|q(x)| \leq \delta\}} \mathcal{G}_{\varepsilon,\delta}^*(x, q(x)) - \hat{w}|q(x)| - \frac{\varepsilon}{\alpha}|q(x)|^\alpha \, dx \right| \\ &\quad + \left| \int_{\{|q(x)| > \delta\}} \mathcal{G}_{\varepsilon,\delta}^*(x, q(x)) - \hat{w}|q(x)| - \frac{\varepsilon}{\alpha}|q(x)|^\alpha \, dx \right|. \end{aligned}$$

We first consider the first term:

$$\begin{aligned} \left| \int_{\{|q| \leq \delta\}} \mathcal{G}_{\varepsilon,\delta}^*(x, q(x)) - \hat{w}|q(x)| - \frac{\varepsilon}{\alpha}|q(x)|^\alpha \, dx \right| &= \left| \int_{\{|q| \leq \delta\}} |q|^2 \frac{\hat{w}}{\delta} - \delta \hat{w} - \hat{w}|q| - \frac{\varepsilon}{\alpha}|q|^\alpha \, d\mathcal{L}^d \right| \\ &\leq 3\delta \|\hat{w}\|_{L^1(\Omega_\varepsilon)} + \frac{\varepsilon}{\alpha} \delta^\alpha |\Omega_\varepsilon| \xrightarrow{\delta \rightarrow 0} 0, \end{aligned}$$

independent of q . For the second term it holds

$$\begin{aligned} \left| \int_{\{|q| > \delta\}} \mathcal{G}_{\varepsilon,\delta}^*(x, q(x)) - \hat{w}|q(x)| - \frac{\varepsilon}{\alpha}|q(x)|^\alpha \, dx \right| &= \left| \int_{\{|q| > \delta\}} \frac{\varepsilon}{\alpha} ((|q| - \delta)^\alpha - |q|^\alpha) - \frac{3}{2} \delta \hat{w} \, d\mathcal{L}^d \right| \\ &\leq \left| \int_{\{|q| > \delta\}} \frac{\varepsilon}{\alpha} ((|q| - \delta)^\alpha - |q|^\alpha) \, d\mathcal{L}^d \right| + \frac{3}{2} \delta \|\hat{w}\|_{L^1(\Omega_\varepsilon)}. \end{aligned}$$

Denoting $Q_\delta : \mathbb{R} \rightarrow \mathbb{R}$, $Q_\delta(x) = \max\{x, \delta\}$, we see that

$$\begin{aligned} 0 &> \int_{\{|q| > \delta\}} (|q| - \delta)^\alpha - |q|^\alpha \, d\mathcal{L}^d = \int_{\{|q| > \delta\}} (Q_\delta(|q|) - \delta)^\alpha - Q_\delta(|q|)^\alpha \, d\mathcal{L}^d \\ &> \int_{\Omega_\varepsilon} (Q_\delta(|q|) - \delta)^\alpha - Q_\delta(|q|)^\alpha \, d\mathcal{L}^d. \end{aligned}$$

By noting that $Q_\delta(|q|) - \delta$ and $Q_\delta(|q|)$ are non-negative, we thus obtain

$$\sup_{q \in K} \frac{\varepsilon}{\alpha} \left| \int_{\{|q| > \delta\}} (|q| - \delta)^\alpha - |q|^\alpha \, d\mathcal{L}^d \right| \leq \sup_{q \in K} \frac{\varepsilon}{\alpha} \left| \|Q_\delta(|q|) - \delta\|_{L^\alpha(\Omega_\varepsilon)}^\alpha - \|Q_\delta(|q|)\|_{L^\alpha(\Omega_\varepsilon)}^\alpha \right|.$$

Because $(\cdot)^\alpha$ is locally Lipschitz on \mathbb{R} , there is a constant $C_K \geq 0$ such that $|\|v\|_{L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)}^\alpha - \|w\|_{L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)}^\alpha| \leq C_K |\|v\|_{L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)} - \|w\|_{L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)}|$ for all $v, w \in K$. Together with the reverse triangle inequality, this yields

$$\begin{aligned} \sup_{q \in K} \frac{\varepsilon}{\alpha} \left| \|Q_\delta(|q|) - \delta\|_{L^\alpha(\Omega_\varepsilon)}^\alpha - \|Q_\delta(|q|)\|_{L^\alpha(\Omega_\varepsilon)}^\alpha \right| &\leq \sup_{q \in K} \frac{\varepsilon}{\alpha} C_K \left| \|Q_\delta(|q|) - \delta\|_{L^\alpha(\Omega_\varepsilon)} - \|Q_\delta(|q|)\|_{L^\alpha(\Omega_\varepsilon)} \right| \\ &\leq \sup_{q \in K} \frac{\varepsilon}{\alpha} C_K \|\delta\|_{L^\alpha(\Omega_\varepsilon)} = \frac{\varepsilon}{\alpha} C_K |\Omega_\varepsilon| \delta \xrightarrow{\delta \rightarrow 0} 0 \end{aligned}$$

and concludes the proof. \square

Now we're in a position to prove the desired result on Γ -convergence.

Theorem 4.8. *Let Assumptions 1.1, 2.8 and 4.1 hold. Then for $v \in \mathcal{M}_\perp(\Omega_\varepsilon)$ it holds $\Gamma\text{-}\lim_{\delta \rightarrow 0} H_{\varepsilon, \delta}^v = H_{\varepsilon, 0}^v$ w.r.t. weak-* convergence in $\mathfrak{M}(\Omega_\varepsilon, \mathbb{R}^d)$.*

Proof. 1. *lim sup-condition:* Let $q \in \mathfrak{M}(\Omega_\varepsilon, \mathbb{R}^d)$ be arbitrary. As recovery sequence, we use the constant sequence, i.e. $q_\delta \equiv q$. In the case $H_{\varepsilon, 0}^v(q) = \infty$, the assertion holds trivially.

Hence, we assume $H_{\varepsilon, 0}^v(q) < \infty$. In this case, $\frac{dq}{d\mathcal{L}^d} - \delta \in L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)$ with $\text{div}_{L^\alpha} \frac{dq}{d\mathcal{L}^d} = v$ in $W_\perp^{-1, \alpha}(\Omega_\varepsilon)$ and thus, $H_{\varepsilon, \delta}^v(q) < \infty$. Then, by Lemma 4.7, $H_{\varepsilon, \delta}^v(q) \rightarrow H_{\varepsilon, 0}^v(q)$ for $\delta \rightarrow 0$.

2. *lim inf-condition:* Let $q \in \mathfrak{M}(\Omega_\varepsilon, \mathbb{R}^d)$ be arbitrary and let $\mathfrak{M}(\Omega_\varepsilon, \mathbb{R}^d) \ni q_\delta \xrightarrow{*} q$ in $\mathfrak{M}(\Omega_\varepsilon, \mathbb{R}^d)$. Moreover, we have $\int_{\Omega_\varepsilon} \hat{w} d|q| < \infty$ analogously to the proof of Theorem 4.5

First, assume $H_{\varepsilon, 0}^v(q) < \infty$ and w.l.o.g. resort to a subsequence of q_δ (denoted by the same symbol) such that $\lim_{\delta \rightarrow 0} H_{\varepsilon, \delta}^v(q_\delta) = \liminf_{\delta \rightarrow 0} H_{\varepsilon, \delta}^v(q_\delta) < \infty$. We may w.l.o.g. assume $\delta \leq 1$ so that

$$\mathcal{G}_{\varepsilon, \delta}^*(x, p) \geq \hat{w} \left(|q| - \frac{3}{2} \right) + \frac{\varepsilon}{\alpha} ((|p| - 1)_+)^{\alpha}. \quad (4.4)$$

Thanks to Lemma 2.10 we may choose $q_0 \in L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)$ fixed with $\text{div}_{L^\alpha} q_0 = v$ in $W_\perp^{-1, \alpha}(\Omega_\varepsilon)$ and obtain

$$\int_{\Omega_\varepsilon} \frac{\varepsilon}{\alpha} ((|\frac{dq_\delta}{d\mathcal{L}^d}| - 1)_+)^{\alpha} \leq \int_{\Omega_\varepsilon} \mathcal{G}_{\varepsilon, \delta}^*(\frac{dq_\delta}{d\mathcal{L}^d}) d\mathcal{L}^d \leq \int_{\Omega_\varepsilon} \mathcal{G}_{\varepsilon, \delta}^*(q_0) d\mathcal{L}^d < \infty \quad (4.5)$$

and hence $\frac{dq_\delta}{d\mathcal{L}^d}$ is bounded in $L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)$, i.e. there is some $K \subset L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)$ bounded such that $(\frac{dq_\delta}{d\mathcal{L}^d}) \subset K$. This also yields weak convergence of a subsequence of $\frac{dq_\delta}{d\mathcal{L}^d}$ in $L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)$ and together with $q_\delta \xrightarrow{*} q$ in $\mathfrak{M}(\Omega_\varepsilon, \mathbb{R}^d)$ we have $\frac{dq_\delta}{d\mathcal{L}^d} \rightharpoonup \frac{dq}{d\mathcal{L}^d}$ in $L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)$. Hence, Lemma 4.7 yields

$$\begin{aligned} \liminf_{\delta \rightarrow 0} H_{\varepsilon, \delta}^v(q_\delta) &\geq \liminf_{\delta \rightarrow 0} (H_{\varepsilon, \delta}^v(q_\delta) - H_{\varepsilon, 0}^v(q_\delta)) + \liminf_{\delta \rightarrow 0} H_{\varepsilon, 0}^v(q_\delta) \\ &\geq - \sup_{p \in K} |H_{\varepsilon, \delta}^v(\mathcal{I}(p)) H_{\varepsilon, 0}^v(\mathcal{I}(p))| + \liminf_{\delta \rightarrow 0} H_{\varepsilon, 0}^v(q_\delta) \\ &= \liminf_{\delta \rightarrow 0} H_{\varepsilon, 0}^v(q_\delta) \geq H_{\varepsilon, 0}^v(q), \end{aligned}$$

where the last inequality holds due to $H_{\varepsilon,0}^v$ being l.s.c. w.r.t. weak convergence in $L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)$.

Assume now that $H_{\varepsilon,0}^v(q) = \infty$ with either $\frac{dq}{d\mathcal{L}^d} \notin L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)$ or $q \not\ll \mathcal{L}^d$. For a contradiction, assume $\liminf_{\delta \rightarrow 0} H_{\varepsilon,\delta}^v(q_\delta) < \infty$. As seen in (4.5), this implies boundedness of $\frac{dq_\delta}{d\mathcal{L}^d}$ in $L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)$ and as above, we obtain $\frac{dq_\delta}{d\mathcal{L}^d} \rightharpoonup \frac{dq}{d\mathcal{L}^d}$ in $L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)$, which is the desired contradiction.

Finally, we are left with the case $H_{\varepsilon,0}^v(q) = \infty$ with $q \ll \mathcal{L}^d$ and $\frac{dq}{d\mathcal{L}^d} \in L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)$ but $\operatorname{div}_{L^\alpha} \frac{dq}{d\mathcal{L}^d} \neq v$ in $W_{\perp}^{-1,\alpha}(\Omega_\varepsilon)$. For a contradiction, we assume $\liminf_{\delta \rightarrow 0} H_{\varepsilon,\delta}^v(q_\delta) < \infty$ and pass to a subsequence (denoted by the same symbol) such that $H_{\varepsilon,\delta}^v(q_\delta)$ converges. As above, it follows that $\frac{dq_\delta}{d\mathcal{L}^d} \rightharpoonup \frac{dq}{d\mathcal{L}^d}$ in $L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)$ and therefore

$$-\langle \varphi, \tilde{v} \rangle = \int_{\Omega_\varepsilon} \frac{dq_\delta}{d\mathcal{L}^d} \cdot \operatorname{grad} \varphi \, d\mathcal{L}^d \rightarrow \int_{\Omega_\varepsilon} \frac{dq}{d\mathcal{L}^d} \cdot \operatorname{grad} \varphi \, d\mathcal{L}^d \quad \forall \varphi \in W_0^{1,\alpha'}(\Omega_\varepsilon).$$

This implies $\operatorname{div}_{L^\alpha} \frac{dq}{d\mathcal{L}^d} = v$ in $W_{\perp}^{-1,\alpha}(\Omega_\varepsilon)$, thus yielding the desired contradiction and concluding the proof. \square

Corollary 4.9. *In the setting of Theorem 4.8, let $\delta_n > 0$ be a vanishing sequence and $(q_n) \subset L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)$ be the sequence of corresponding solutions of $(\text{BP}_{\varepsilon,\delta})$. Then (q_n) admits a subsequence that converges to a solution of (BP_ε) w.r.t. weak-* convergence in $\mathfrak{M}(\Omega_\varepsilon)$.*

Proof. Analogously to the argument involving (4.5) in the proof of Theorem 4.8, we obtain a subsequence of q_n (denoted by the same symbol) with $q_n \rightharpoonup q \in L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)$ in $L^\alpha(\Omega_\varepsilon, \mathbb{R}^d)$. This also implies $I(q_n) \xrightarrow{*} I(q)$ in $\mathfrak{M}(\Omega_\varepsilon, \mathbb{R}^d)$ and the assertion follows directly from Theorem 4.8 and the properties of Γ -convergence. \square

4.2. Simultaneous Convergence of ε and δ

Lastly, we aim to show Γ -convergence for $\delta \rightarrow 0$ and $\varepsilon \rightarrow 0$ *simultaneously*.

Theorem 4.10. *Let Assumptions 1.1, 2.8 and 4.1 hold and let $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $h(\delta) \rightarrow \infty$ for $\delta \rightarrow 0$. Denote $\tau := (\varepsilon, \delta)$ and let $\tau \rightarrow 0$ such that*

$$\varepsilon \cdot h(\delta)^\alpha \rightarrow 0. \tag{4.6}$$

Then $\Gamma\text{-}\lim_{\tau \rightarrow 0} H_{\varepsilon,\delta}^{\tilde{\mu}} = H^\mu$ w.r.t. weak- convergence in $\mathfrak{M}(\tilde{\Omega}, \mathbb{R}^d)$.*

Proof. In the following, we will abbreviate $H_\tau := H_{\varepsilon,\delta}^{\tilde{\mu}}$.

1. *lim inf-condition:* Let $q \in \mathfrak{M}(\tilde{\Omega}, \mathbb{R}^d)$ and $(q_\tau) \subset \mathfrak{M}(\Omega_\tau, \mathbb{R}^d)$ such that $\tilde{q}_\tau \xrightarrow{*} q$ in $\mathfrak{M}(\tilde{\Omega}, \mathbb{R}^d)$. Analogously to the lim inf case in the proof of Theorem 4.5, we obtain that q is feasible for (BP) with $\int_{\tilde{\Omega}} w \, d|q| < \infty$.

Without renaming, we resort to a subsequence of (q_τ) such that $\lim_{\tau \rightarrow 0} H_\tau(q_\tau) = \liminf_{\tau \rightarrow 0} H_\tau(q_\tau) < \infty$. Then we obtain

$$\begin{aligned} \liminf_{\tau \rightarrow 0} \int_{\left\{ \left| \frac{d\tilde{q}_\tau}{d\mathcal{L}^d} \right| > \delta \right\}} \hat{w} \, d|\tilde{q}_\tau(x)| &= \liminf_{\tau \rightarrow 0} \int_{\tilde{\Omega}} \hat{w} \, d|\tilde{q}_\tau(x)| - \int_{\left\{ \left| \frac{d\tilde{q}_\tau}{d\mathcal{L}^d} \right| \leq \delta \right\}} \hat{w} \, d|\tilde{q}_\tau(x)| \\ &\geq \liminf_{\tau \rightarrow 0} \int_{\tilde{\Omega}} \hat{w} \, d|\tilde{q}_\tau(x)| - \delta \|\hat{w}\|_{L^1(\tilde{\Omega})} = \int_{\Omega} w \, d|q| \end{aligned}$$

thanks to $\int_{\tilde{\Omega}} \hat{w} \, d|\cdot|$ being l.s.c. w.r.t. weak-* convergence in $\mathfrak{M}(\tilde{\Omega})$. Going on, we see that

$$\int_{\left\{ \left| \frac{d\tilde{q}_\tau}{d\mathcal{L}^d} \right| > \delta \right\}} \frac{\varepsilon}{\alpha} \left(\left| \frac{d\tilde{q}_\tau}{d\mathcal{L}^d} \right| - \delta \right)^\alpha - \frac{3}{2} \delta \hat{w} \, d\mathcal{L}^d \geq -\frac{3}{2} \delta \|\hat{w}\|_{L^1(\tilde{\Omega})} \xrightarrow{\tau \rightarrow 0} 0$$

as well as

$$\int_{\left\{ \left| \frac{d\tilde{q}_\tau}{d\mathcal{L}^d} \right| \leq \delta \right\}} \left| \frac{d\tilde{q}_\tau}{d\mathcal{L}^d} \right|^2 \frac{\hat{w}}{2\delta} - \delta \hat{w} \, d\mathcal{L}^d \geq \delta \|\hat{w}\|_{L^1(\tilde{\Omega})} \xrightarrow{\tau \rightarrow 0} 0.$$

In summary, this yields $\liminf_{\tau \rightarrow 0} H_\tau(q_\tau) \geq H^\mu(q)$.

2. *lim sup-condition*: Let $q \in \mathfrak{M}(\Omega, \mathbb{R}^d)$ be arbitrary. In the case $H^\mu(q) = \infty$ the assertion again holds trivially.

Hence, let $H^\mu(q) < \infty$. Let (φ_τ) be as above and w.l.o.g. assume $\|\varphi_\tau\|_{L^\infty(\mathbb{R}^d)} \leq h(\delta)$. Denote $q_\tau := \varphi_\tau * q \in L^\alpha(\Omega_\tau, \mathbb{R}^d)$ and similarly for μ_τ . By [Lemma A.1](#), it holds $\mathcal{I}(\tilde{q}_\tau) \rightarrow \tilde{q}$ in $\mathfrak{M}(\tilde{\Omega}, \mathbb{R}^d)$ and $\operatorname{div}_{\mathfrak{M}} \mathcal{I}(q_\tau) = \mu_\tau$ in $\mathcal{M}_\perp(\tilde{\Omega})$. Moreover, we obtain $\|\tilde{q}_\tau\|_{L^\infty(\tilde{\Omega})} \leq Ch(\delta)$ for some constant $C > 0$ similarly to [\(4.2\)](#) and [\(4.3\)](#). Analogously to the proof of [Theorem 4.5](#), we set $\check{q}_\tau := \tilde{q}_\tau + \operatorname{grad} y_\tau$, where $y_\tau \in W_0^{1,\alpha'}(\Omega_\tau)$ solves the analogue to [\(4.1\)](#). Then we have $\widetilde{\mathcal{I}(\check{q}_\tau)} \rightarrow \tilde{q}$ in $\mathfrak{M}(\tilde{\Omega}, \mathbb{R}^d)$, $\operatorname{div}_{L^\alpha} \check{q}_\tau = \tilde{\mu}$ in $W_\perp^{-1,\alpha}(\tilde{\Omega})$ and $\widetilde{\operatorname{grad} y_\tau} \rightarrow 0$ in $L^\alpha(\tilde{\Omega}, \mathbb{R}^d)$. Going on, it holds

$$\int_{\left\{ |\check{q}_\tau| < \delta \right\}} |\check{q}_\tau|^2 \frac{\hat{w}}{2\delta} - \delta \hat{w} \, d\mathcal{L}^d \leq \int_{\tilde{\Omega}} \delta \hat{w} - \delta \hat{w} \, d\mathcal{L}^d = 0$$

and

$$\int_{\left\{ |\check{q}_\tau| \geq \delta \right\}} \mathcal{G}_{\varepsilon,\delta}^*(\check{q}_\tau) \, d\mathcal{L}^d \leq \frac{\varepsilon}{\alpha} \left(\|\check{q}_\tau + \widetilde{\operatorname{grad} y_\tau}\|_{L^\alpha(\tilde{\Omega}, \mathbb{R}^d)} \right)^\alpha + \int_{\tilde{\Omega}} |\check{q}_\tau| \hat{w} \, d\mathcal{L}^d. \quad (4.7)$$

Due to [\(4.6\)](#),

$$\left(\frac{\varepsilon}{\alpha} \right)^{\frac{1}{\alpha}} \|\check{q}_\tau\|_{L^\alpha(\tilde{\Omega}, \mathbb{R}^d)} \leq C \left(\frac{\varepsilon}{\alpha} \right)^{\frac{1}{\alpha}} h(\delta) \xrightarrow{\tau \rightarrow 0} 0.$$

Together with the strong convergence of $\mathcal{I}(\check{q}_\tau)$ for $\tau \rightarrow 0$ and $\widetilde{\operatorname{grad} y_\tau} \rightarrow 0$ in $L^\alpha(\tilde{\Omega}, \mathbb{R}^d)$, the right hand side of [\(4.7\)](#) converges to $H^\mu(q)$, thus concluding the proof. \square

Corollary 4.11. *In the setting of [Theorem 4.10](#), let $w \geq w_\diamond > 0$ and let $\varepsilon_n, \delta_n > 0$ be a vanishing sequences such that [\(4.6\)](#) holds. Let $(q_n) \subset L^\alpha(\Omega_{\varepsilon_n}, \mathbb{R}^d)$ be the sequence of corresponding solutions of [\(BP\) \$_{\varepsilon,\delta}\$](#) . Then (q_n) admits a subsequence that converges to a solution of [\(BP\)](#) w.r.t. weak-* convergence in $\mathfrak{M}(\Omega_\varepsilon)$.*

Proof. Let $q_0 \in L^\alpha(\Omega, \mathbb{R}^d)$ be fixed such that $\operatorname{div}_{L^\alpha} q_0 = \mu$ in $W_\perp^{-1,\alpha}(\Omega)$, which exists due to [Lemma 2.10](#). Then thanks to [\(4.4\)](#) q_n satisfies

$$\int_{\tilde{\Omega}} \hat{w} \left(|\tilde{q}_n| - \frac{3}{2} \right) + \frac{\varepsilon_n}{\alpha} \left((|\tilde{q}_n| - 1)_+ \right)^\alpha \, d\mathcal{L}^d \leq \int_{\tilde{\Omega}} \mathcal{G}_{\varepsilon_n, \delta_n}^*(\tilde{q}_n) \, d\mathcal{L}^d \leq \int_{\tilde{\Omega}} \mathcal{G}_{\varepsilon_n, \delta_n}^*(\tilde{q}) \, d\mathcal{L}^d,$$

where for the right hand side it holds

$$\limsup_{n \rightarrow \infty} \int_{\tilde{\Omega}} \mathcal{G}_{\varepsilon_n, \delta_n}^*(\tilde{q}_0) \, d\mathcal{L}^d \leq \int_{\tilde{\Omega}} \hat{w} \tilde{q}_0 \, d\mathcal{L}^d < \infty$$

by similar argumentation as in lim sup case of the preceding proof. Thus, due to $w \geq w_\diamond > 0$, the sequence (\tilde{q}_n) is bounded in $L^1(\tilde{\Omega}, \mathbb{R}^d)$ and the assertion then follows by argumentation analogous to the proof [Corollary 4.6](#). \square

5. Numerical Examples

In this section, we report on the conducted numerical experiments. We start by briefly explaining our discretization scheme.

5.1. Discretization via Finite Elements

To discretize the Newton equation in [\(3.8\)](#), we employ standard piecewise linear and continuous finite elements. The nodal basis associated with nodes x_1, \dots, x_n , $n \in \mathbb{N}$, of a given triangular grid is denoted by $\varphi_1, \dots, \varphi_n$ such that the discretized ansatz and trial space is $V_n = \text{span}(\varphi_1, \dots, \varphi_n)$. Now, given an iterate $y_h \in V_n$, the discrete counterpart of [\(3.8\)](#) reads

$$\begin{aligned} \eta_h \in V_n, \quad \int_{\Omega} \eta_h \, d\mathcal{L}^d &= 0, \\ \int_{\Omega} G'_{\varepsilon, \delta}(-\text{grad } y_h) \text{grad } \eta_h \cdot \text{grad } \psi \, d\mathcal{L}^d &= \int_{\Omega} G_{\varepsilon, \delta}(-\text{grad } y_h) \cdot \text{grad } \psi \, d\mathcal{L}^d + \int_{\Omega} \psi \, d\mu \quad (5.1) \\ \forall \psi \in V_n : \int_{\Omega} \psi \, d\mathcal{L}^d &= 0. \end{aligned}$$

We introduce the matrices

$$A(y_h)_{ij} := \int_{\Omega} G'_{\varepsilon, \delta}(-\text{grad } y_h) \text{grad } \varphi_i \cdot \text{grad } \varphi_j \, d\mathcal{L}^d, \quad M_{ij} := \int_{\Omega} \varphi_i \varphi_j \, d\mathcal{L}^d$$

and the vectors

$$b(y_h)_i := \int_{\Omega} G_{\varepsilon, \delta}(-\text{grad } y_h) \cdot \text{grad } \varphi_i \, d\mathcal{L}^d, \quad d_i := \int_{\Omega} \varphi_i \, d\mu. \quad (5.2)$$

Then [\(5.1\)](#) is equivalent to

$$\begin{aligned} \mathbf{1}^\top M \eta &= 0, \\ \eta^\top A(y_h) v &= (b(y_h) + d)^\top v \quad \forall v \in \mathbb{R}^n : \mathbf{1}^\top M v = 0 \end{aligned} \quad (5.3)$$

where $\eta \in \mathbb{R}^n$ denotes the coefficient vector of η_h for the basis $\varphi_1, \dots, \varphi_n$ and $\mathbf{1} = [1, \dots, 1]^\top$. If we introduce a scalar Lagrange multiplier r associated with [\(5.3\)](#), then the system is equivalent to the saddle point problem

$$\begin{pmatrix} A(y_h) & M\mathbf{1} \\ \mathbf{1}^\top M & 0 \end{pmatrix} \begin{pmatrix} \eta \\ r \end{pmatrix} = \begin{pmatrix} b(y_h) + d \\ 0 \end{pmatrix}$$

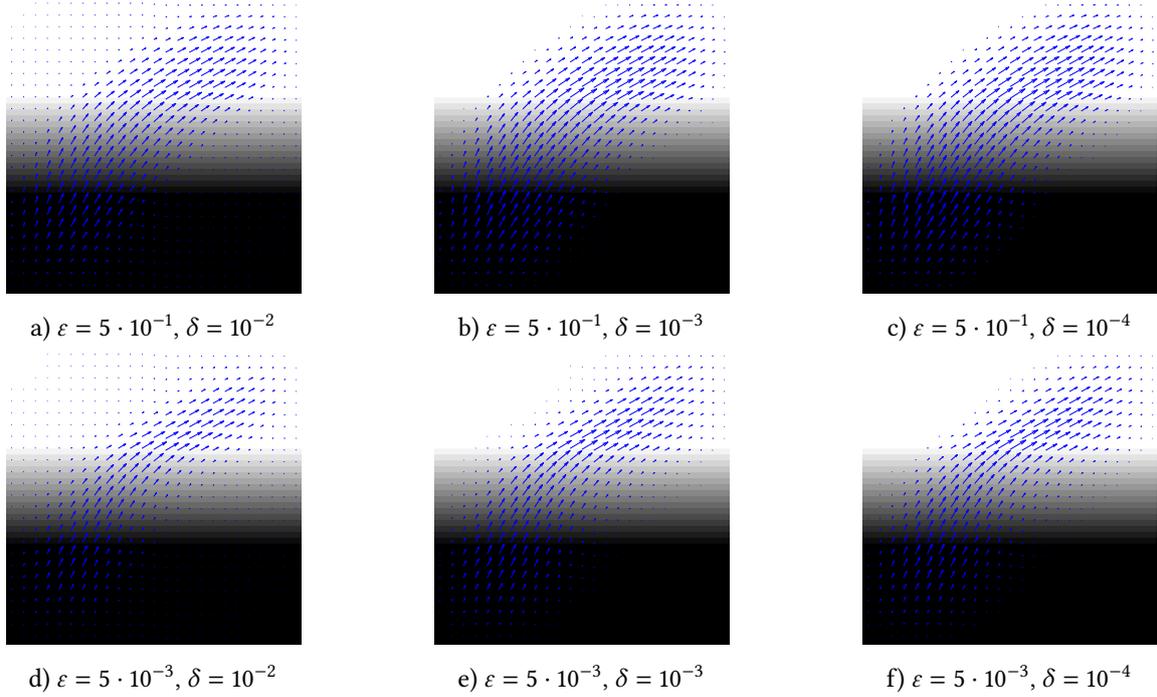


Figure 1: Visualization of the flow field for an example with piecewise linear/constant cost function. Both μ^+ and μ^- are Gaussians centered in the bottom left and top right quadrant, respectively.

Remark 5.1. *If w is chosen piecewise constant on the triangular grid, the entries of $A(y_h)$ and $b(y_h)$ can be evaluated exactly, since $\text{grad } y_h$ is constant on each element. The same holds for the objective $\mathcal{J}(y_h)$ in the Armijo line search, as the second integral only involves linear combinations of piecewise linear functions on the elements, which can be integrated exactly.*

5.2. Influence of ε and δ

We first illustrate effect of the regularization parameters ε and δ on the solutions of $(\text{BP}_{\varepsilon,\delta})$. We choose a simple Friedrich-Keller grid, i.e. we divide the domain $\Omega = [0, 1]^2$ into a regular partition of equally sized squares and divide each square into two congruent triangles.

Both the marginals μ^+ , μ^- and the cost function w are non-negative functions which are constant on the squares, i.e. they are constant across two adjacent triangles. For the exponent α , we choose $\alpha = 2$. Note that we required $\alpha < \frac{d}{d-1}$ in [Assumption 1.1](#), so that $\alpha = 2$ is actually a limit case. We start the iteration with $y \equiv 0$ and use the parameters $\sigma_0 = 1$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{10}$ for the Armijo line search ([Algorithm 2](#)). As stopping criterion, we use the relative error of the optimality condition [\(3.7\)](#). More precisely, for each $i = 1, \dots, n$ we calculated d_i and $b(y)_i$ as in [\(5.2\)](#) and use the relative error

$$\frac{|d - b(y)|}{|b(y)|} \tag{5.4}$$

and stopped the iteration once this error dropped below 10^{-8} or after 1000 iterations.

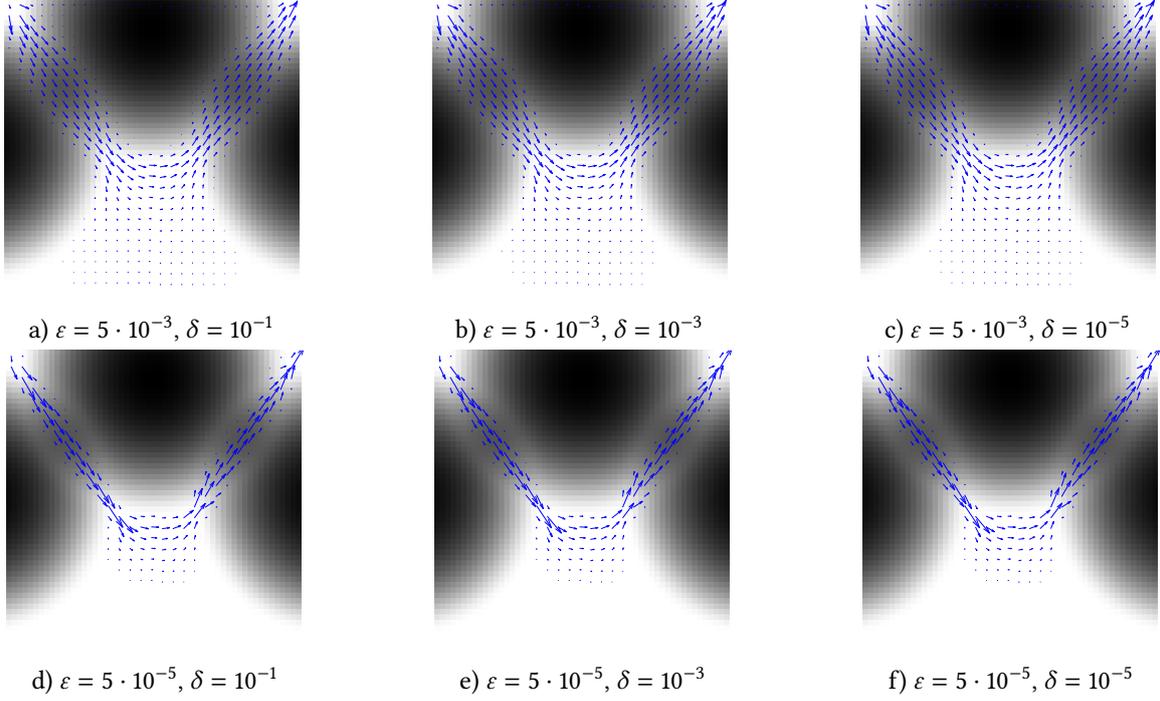


Figure 2.: Visualization of the flow field for an example where the cost function is a mixture of three Gaussians. Both μ^+ and μ^- are concentrated on a single square in the top left and top right corner, respectively.

Figures 1 to 3 show solutions of $(BP_{\epsilon,\delta})$ for different choices of μ^+ , μ^- and w . The cost function w is encoded by the gray scale background, where darker shades denote higher costs. In all cases, w is bounded away from zero. The vector field q is encoded by the blue arrows. For purposes of visualization, we display a downsampled version of q , which was achieved by taking the average over the value of q across 4 squares (i.e. 8 triangles) each. Moreover, we only plot arrows who's Euclidian norm is larger than 1% of the largest Euclidian norm of an entry in the averaged q . Note that the arrows are scaled for each subfigure independently. The mesh consists of 5000 triangles for Fig. 1, 6050 triangles for Fig. 2 and 8450 triangles for Fig. 3.

We can observe that for $\epsilon \rightarrow 0$, the solutions q become more singular, while for large ϵ the regularization terms dominates the transportation cost so that the mass is transported more evenly through the domain. As for the parameter δ controlling the Huber regularization term, we can observe that while having only small influence on the regularity of q , the overall objective value is reduced for large δ . This can be seen best in Fig. 3, where the maze has multiple solutions. While for small δ the shortest path is preferred, we see that other paths are used as well for larger δ . This observation is in accordance to Theorem 3.5 due to the terms $-\delta w$ in $\mathcal{G}_{\epsilon,\delta}^*$.

5.3. Speed of Convergence

Figure 4 shows the observed relative errors in the optimality condition (as described above) in dependence on the number of iterations for selected instances of the examples from Figs. 2 and 3. We

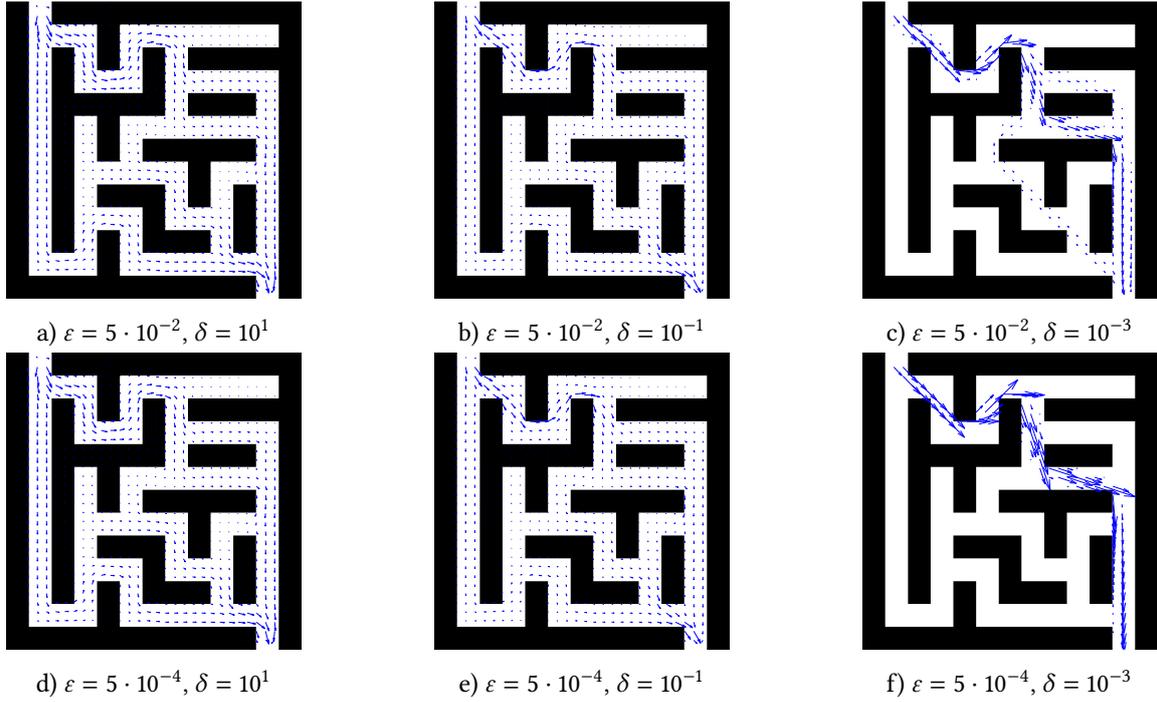


Figure 3.: Visualization of the flow field for an example where the cost function encodes a maze. Both μ^+ and μ^- are concentrated on a single square in the top left and bottom right corner, respectively.

observe that larger regularization parameters, both for δ and ε significantly speed up convergence. In fact, for some combinations of δ and ε the iteration failed to terminate for the given stopping criterion within the given maximum number of iterations. These cases mostly correspond to very small regularization parameters. However, for most test cases, we see quadratic convergence once we're close to the solution.

Note that larger values for ε are interesting in the context of traffic congestion [12]. The effect studied here can be observed by comparing Figs. 3c) and 3f). Here we can see, that the larger value of ε promotes the shortest path, while the larger value promotes to spread the flow of mass across the different possible paths even if they are longer.

We also point out that our stopping criterion (5.4) is rather strict. Among the literature reviewed in Section 1.2, a similar criterion is used only in [6, 19]. In these publications first order methods are employed, which naturally need a much higher number of iterations to achieve the same accuracy. In [27] a fixed-point residual of the Chambolle-Pock iteration is used as stopping criterion, which is not as easy to interpret. For the ROF-Model in [21], the authors are mainly interested in the objective value. Hence, they only consider experiments where the objective value is known and use the error in the objective value as stopping criterion. Finally, [4, 15] use the relative change in the iterates \underline{Q}^j and $\mu_h(t)$ (roughly corresponding to q and $|q|$ in our notation) as stopping criterion.

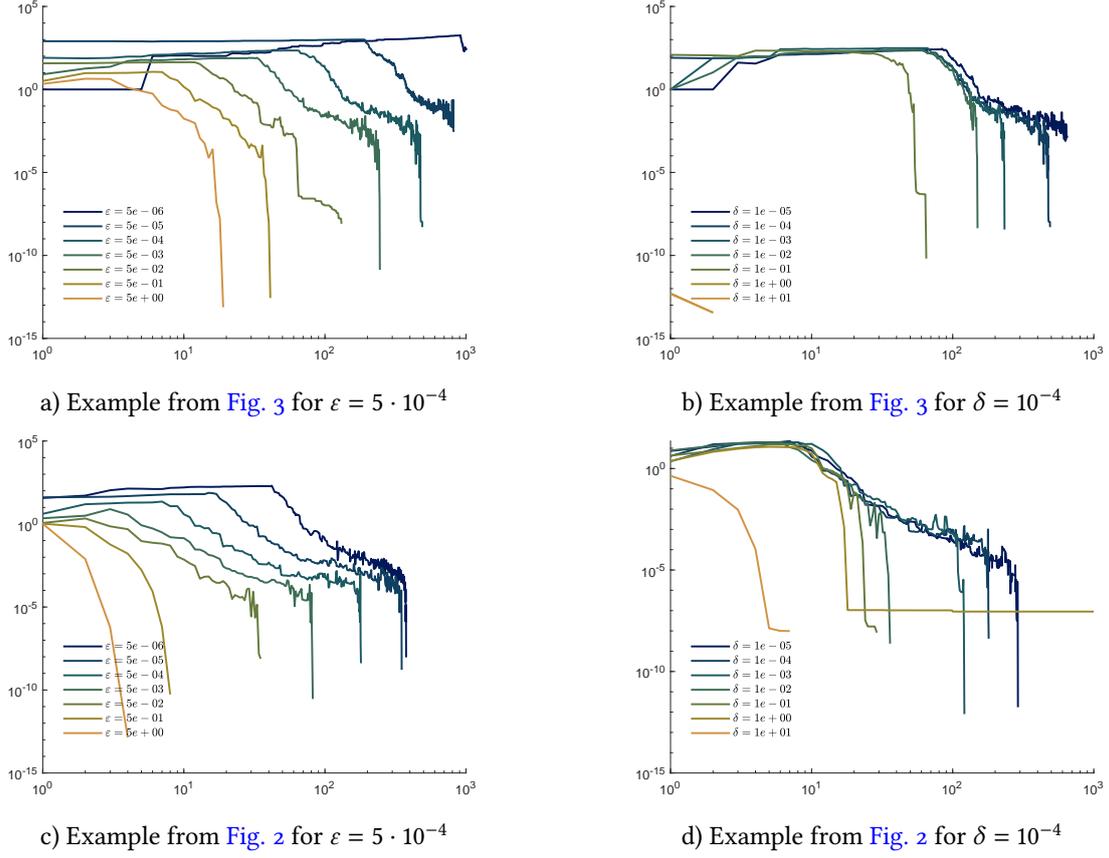


Figure 4.: Relative error as in (5.4) for each iteration for selected instances of the examples shown in Figs. 2 and 3.

6. Conclusion & Outlook

In contrast to the original Beckmann problem, the L^α -regularized counterpart has unique solutions even for $\mu^+, \mu^- \in \mathfrak{M}(\Omega)$. Moreover, this regularization naturally gives rise to a semi-smooth Newton scheme that can be used to solve the problem numerically. For the iteration step to be well posed, we add a second regularization term of Huber type. Convergence towards the original problem for vanishing regularization parameters can be proven, if the regularization parameters are coupled in an appropriate way.

This work can be extended both on the theoretic part and the numerical part. On the theoretical part, a rigorous convergence theory for the proposed semi-smooth Newton iteration Algorithm 1 is still missing. Regarding numerics, we have only worked with simple, fixed grids and similar to [4] one could explore whether mesh adaption techniques are beneficial for the speed of convergence and accuracy of the solution. Moreover one could employ path following schemes to try and improve the convergence speed.

Appendix

For the following result, $\tilde{\Omega}$ and $\tilde{\cdot}$ are defined as in [Section 4](#).

Lemma A.1. *Let $0 < \tau \rightarrow 0$ and let the notation of [Section 4](#) hold. Let $q \in \mathfrak{M}(\Omega, \mathbb{R}^d)$ such that $\operatorname{div}_{\mathfrak{M}} q = \mu$ in $\mathcal{M}_{\perp}(\Omega)$ and denote $q^{\tau} := \varphi_{\tau} * \tilde{q}$. Then*

1. $q^{\tau} \in L^{\alpha}(\tilde{\Omega}, \mathbb{R}^d)$
2. $\operatorname{div}_{L^{\alpha}} q^{\tau} = \varphi_{\tau} * \tilde{\mu}$ in $W_{\perp}^{-1, \alpha}(\tilde{\Omega})$
3. $q^{\tau} \rightarrow \tilde{q}$ strongly in $\mathfrak{M}(\tilde{\Omega}, \mathbb{R}^d)$.
4. $|q^{\tau}| \rightarrow |\tilde{q}|$ strongly in $\mathfrak{M}(\tilde{\Omega})$ and hence also w.r.t. weak-* convergence in $\mathfrak{M}(\tilde{\Omega})$.

Proof. 1. See [[16](#), Proposition 1.16].

2. Let $\psi \in W_{\emptyset}^{1, \alpha'}(\tilde{\Omega})$. By [Definition 2.2](#) and Fubini's theorem, it holds

$$\begin{aligned} -\langle \operatorname{div}_{L^{\alpha}} q^{\tau}, \psi \rangle &= \int_{\tilde{\Omega}} q^{\tau} \cdot \operatorname{grad} \psi \, d\mathcal{L}^d = \int_{\tilde{\Omega}} \int_{\tilde{\Omega}} \varphi_{\tau}(y-x) \, d(\tilde{q}(x) \cdot \operatorname{grad} \psi(y)) \, dy \\ &= \int_{\tilde{\Omega}} \left(\int_{\tilde{\Omega}} \varphi_{\tau}(y-x) \operatorname{grad} \psi(y) \, dy \right) \cdot d\tilde{q}(x). \end{aligned} \quad (\text{A.1})$$

Denote $\psi_{\tau}(x) := \int_{\tilde{\Omega}} \varphi_{\tau}(y-x) \psi(y) \, dy$. Then for every $x \in \Omega$, integration by parts yields

$$\int_{\tilde{\Omega}} \varphi_{\tau}(y-x) \operatorname{grad} \psi(y) \, dy = \operatorname{grad} \psi_{\tau}(x)$$

due to $\varphi_{\tau}(\cdot - x) = 0$ on $\partial\tilde{\Omega}$. Moreover, we note that $\psi_{\tau}|_{\Omega} \in W^{1, \alpha'}(\Omega)$. Hence, since \tilde{q} is the extension by zero of q onto $\tilde{\Omega}$

$$-\langle \operatorname{div}_{L^{\alpha}} q^{\tau}, \psi \rangle = \int_{\tilde{\Omega}} \operatorname{grad} \psi_{\tau} \cdot d\tilde{q} = -\langle \tilde{\mu}, \psi_{\tau} \rangle = -\langle \mu * \psi_{\tau}, \psi \rangle,$$

where the last equation follows analogously to [\(A.1\)](#).

3. Let $A \subset \tilde{\Omega}$. Then

$$\begin{aligned} |q^{\tau} - \tilde{q}|(A) &= \left| \int_A \int_{\tilde{\Omega}} \varphi_{\tau}(y-x) \, d\tilde{q}(x) \, dy - \tilde{q}(A) \right| \\ &= \left| \int_{\tilde{\Omega}} \int_A \varphi_{\tau}(y-x) \, dy \, d\tilde{q}(x) - \tilde{q}(A) \right|. \end{aligned}$$

Clearly, the mapping $x \mapsto \int_A \varphi_{\tau}(y-x) \, dy$ is bounded by 1 for all $\tau > 0$ and converges to $\mathbf{1}_A$ pointwise thanks to $\operatorname{spt} \tilde{q} \subset \tilde{\Omega}$. Hence, by dominated convergence, $|q^{\tau} - \tilde{q}|(A) \rightarrow 0$.

4. The last assertion is an immediate consequence of the reverse triangle inequality. \square

Lemma A.2. *Let $0 < \tau \rightarrow 0$ and let the notation of [Section 4](#) hold. Let $q_{\tau} \in \mathfrak{M}(\Omega_{\tau}, \mathbb{R}^d)$ and $q \in \mathfrak{M}(\tilde{\Omega}, \mathbb{R}^d)$ such that $\tilde{q}_{\tau} \xrightarrow{*} q$ in $\mathfrak{M}(\tilde{\Omega}, \mathbb{R}^d)$. Then $\operatorname{spt} q \subset \Omega$.*

Proof. Assume the contrary such that there is a Borel set $A \subset \tilde{\Omega} \setminus \Omega$ with $|q|(A) = M > 0$. Then there is a compact set $K \subset A$ such that $|q|(K) > \frac{M}{2}$ and $\text{dist}(K, \Omega) = r > 0$. Set $N := K + B_{\frac{r}{2}}(0)$. From the weak-* lower semicontinuity of $\mathfrak{M}(\tilde{\Omega}, \mathbb{R}^d) \ni p \mapsto |p|(A) \in \mathbb{R}$ for an arbitrary (relatively) open set A (c.f. e.g. [25]), we deduce

$$0 < \frac{M}{2} < |\tilde{q}|(K) \leq |\tilde{q}|(N) \leq \liminf_{\tau \rightarrow 0} |\tilde{q}_\tau|(N) = 0,$$

where we used that $\text{spt } \tilde{q}_\tau \subset \Omega_\tau$ and $\Omega_\tau \cap N = \emptyset$ for $\tau > 0$ sufficiently small. \square

B. Proof of Lemma 4.3

In order to derive a proof for Lemma 4.3, we aim to express the transformation of the domain through a transformation of the differential operator. First, we present a special case of [17, Theorem 1] that is adapted to our setting.

Theorem B.1. *Let Assumptions 1.1 and 2.8 hold with $\alpha' = r \geq 2$ and by r' denote the conjugate exponent, i.e. $\frac{1}{r} + \frac{1}{r'} = 1$. Let $D : \Omega \rightarrow \mathbb{R}^{d \times d}$ be a measurable map satisfying $mI \leq D(x) \leq MI$ for all $x \in \Omega$ with $0 < m \leq M$. Define*

$$A : W_0^{1,r}(\Omega) \rightarrow W_{\perp}^{-1,r}(\Omega), \quad \langle Ay, \varphi \rangle := \int_{\Omega} (D(x) \text{grad } y(x)) \cdot \text{grad } \varphi(x) \, dx \quad \forall \varphi \in W_0^{1,r'}(\Omega) \quad (\text{B.1})$$

and $\Delta_r := \text{div}_{L^r} \text{grad} : W_0^{1,r}(\Omega) \rightarrow W_{\perp}^{-1,r}(\Omega)$. Note that Δ_r is continuously invertible by Assumption 2.8. Let m and M denote the infimum and supremum over $x \in \Omega$ of the smallest and largest eigenvalue of $D(x)$, respectively. Finally, set $k := (1 - \frac{m^2}{M^2})$.

If $k \|\Delta_r^{-1}\| < 1$, then A is bijective. Moreover, A^{-1} is continuous with

$$\|A^{-1}\|_{W_{\perp}^{-1,r}(\Omega) \rightarrow W_0^{1,r}(\Omega)} \leq \frac{m \|\Delta_r^{-1}\|}{M^2(1 - k \|\Delta_r^{-1}\|)}.$$

Proof. This proof follows the outline of the proof given in [17, Theorem 1].

We first note that A is well defined and bounded as mapping from $W_0^{1,r'}(\Omega)$ to $W_{\perp}^{-1,r}(\Omega)$, which can be seen by applying Hölder's inequality. Moreover, A is injective, which can be seen as follows. Let $y_1, y_2 \in W_0^{1,r'}(\Omega)$ with $Ay_1 = Ay_2$. Due to $r \geq 2$, we may choose $\varphi = y_1 - y_2 \in W_0^{1,r}(\Omega) \hookrightarrow W_0^{1,r'}(\Omega)$ in (B.1), which yields $0 = \int_{\Omega} (D(\text{grad } y_1 - \text{grad } y_2)) \cdot (\text{grad } y_1 - \text{grad } y_2)$. Because D has positive definite values, this implies $\|y_1 - y_2\| = 0$, as conjectured.

Let now $t := mM^{-2}$ and let $B : L^r(\Omega, \mathbb{R}^d) \rightarrow L^r(\Omega, \mathbb{R}^d)$, $(By)(x) := y(x) - tD(x)y(x)$. Clearly, B is linear. Moreover, B is bounded with $\|B\| \leq k$.

Going on, let $v \in W_{\perp}^{-1,r}(\Omega)$ and set

$$\begin{aligned} Q_v : W_0^{1,r}(\Omega) &\rightarrow W_0^{1,r}(\Omega), \\ \langle Q_v y, \varphi \rangle &= \langle \Delta_r^{-1}(-\text{div}_{L^r} B \text{grad } y + tv), \varphi \rangle \quad \forall \varphi \in W_{\perp}^{-1,r'}(\Omega). \end{aligned} \quad (\text{B.2})$$

Note that

$$\begin{aligned} \langle Q_\nu y, \varphi \rangle &= \langle y, \varphi \rangle - t \langle D \operatorname{grad} y, \operatorname{grad} (\Delta_r^{-1})^* \varphi \rangle_{L^r(\Omega) \times L^{r'}(\Omega)} + t \langle \nu, (\Delta_r^{-1})^* \varphi \rangle \\ &= \langle y - t \Delta_r^{-1} (A y - \nu), \varphi \rangle. \end{aligned} \quad (\text{B.3})$$

From (B.2), it is straight forward to derive

$$\begin{aligned} \|Q_\nu \zeta - Q_\nu \xi\|_{W_0^{1,r}(\Omega)} &\leq \|\Delta_r^{-1} (-\operatorname{div}_{L^r}) B \operatorname{grad}\| \|\zeta - \xi\|_{W_0^{1,r}(\Omega)} \\ &\leq \|\Delta_r^{-1}\| \|(-\operatorname{div}_{L^r}) B \operatorname{grad}\| \|\zeta - \xi\|_{W_0^{1,r}(\Omega)} \leq k \|\Delta_r^{-1}\| \|\zeta - \xi\|_{W_0^{1,r}(\Omega)} \end{aligned}$$

for all $\zeta, \xi \in W_0^{1,r}(\Omega)$. Hence, Q_ν is Lipschitz continuous with Lipschitz constant $k \|\Delta_r^{-1}\|$. Due to the assumption $k \|\Delta_r^{-1}\| < 1$, Q_ν is also strictly contractive and by (B.3) the fixed point $y \in W^{1,r}(\Omega)$ of Q_ν is a solution of $A y = \nu$. Hence, A is surjective and it remains to prove the conjectured continuity constant.

To that end, let $\nu, \rho \in W^{-1,r}(\Omega)$ and let $\xi, \zeta \in W^{1,r}(\Omega)$ be the corresponding fixed points of Q_ν and Q_ρ . Then,

$$\begin{aligned} \|\zeta - \xi\|_{W^{1,r}(\Omega)} &= \|Q_\rho \zeta - Q_\nu \xi\|_{W^{1,r}(\Omega)} \leq \|Q_\rho \zeta - Q_\rho \xi\|_{W^{1,r}(\Omega)} + \|Q_\rho \xi - Q_\nu \xi\|_{W^{1,r}(\Omega)} \\ &\leq k \|\Delta_r^{-1}\| \|\zeta - \xi\|_{W^{1,r}(\Omega)} + \|t \Delta_r^{-1} (\rho - \nu)\|_{W^{1,r}(\Omega)} \\ &\leq k \|\Delta_r^{-1}\| \|\zeta - \xi\|_{W^{1,r}(\Omega)} + t \|\Delta_r^{-1}\| \|\rho - \nu\|_{W^{-1,r}(\Omega)}. \end{aligned}$$

Therefore, we obtain

$$\|A^{-1} \rho - A^{-1} \nu\|_{W^{1,r}(\Omega)} (1 - k \|\Delta_r^{-1}\|) = \|\zeta - \xi\|_{W^{1,r}(\Omega)} (1 - k \|\Delta_r^{-1}\|) \leq t \|\Delta_r^{-1}\| \|\rho - \nu\|_{W^{-1,r}(\Omega)},$$

which concludes the proof. \square

Theorem B.1 now allows us to solve the Poisson equation on Ω_τ , which is covered by the following Lemma.

Lemma B.2. *In the setting of Theorem B.1, let Assumption 4.1 hold in addition and let $\tau > 0$. Then, the equation*

$$\int_{\Omega_\tau} \operatorname{grad} y_\tau \cdot \operatorname{grad} \varphi \, d\mathcal{L}^d = \langle \varphi, \nu_\tau \rangle \quad \forall \varphi \in W_0^{1,r'}(\Omega_\tau). \quad (\text{B.4})$$

has a unique solution $y_\tau \in W_0^{1,r}(\Omega_\tau)$ for every $\nu_\tau \in W_\perp^{-1,r}(\Omega)$. Moreover, the solution operator $\Delta_{r,\tau}^{-1} : W_\perp^{-1,r}(\Omega_\tau) \rightarrow W_0^{1,r}(\Omega_\tau)$ of (B.4) is continuous with

$$\|\Delta_{r,\tau}^{-1}\|_{W_\perp^{-1,r}(\Omega_\tau) \rightarrow W_0^{1,r}(\Omega_\tau)} \leq \|\Delta_r^{-1}\| (1 + \tau)^2.$$

Proof. W.l.o.g. we assume $\Omega_\tau = (1 + \tau)\Omega$. Let $\Phi_\tau \varphi(x) = \varphi\left(\frac{x}{1+\tau}\right)$ and note that Φ_τ is a homeomorphism from $W_0^{1,s}(\Omega)$ to $W_0^{1,s}(\Omega_\tau)$ for every $s > 1$. Moreover, let $\omega_\tau := (1 + \tau)^{d-2} I \in \mathbb{R}^{d \times d}$. Note that the only eigenvalue of ω_τ is $(1 + \tau)^{d-2}$. Let now $g \in W_\perp^{-1,r}(\Omega)$ and consider the equation

$$\int_{\Omega} \omega_\tau \operatorname{grad} y \cdot \operatorname{grad} v \, d\mathcal{L}^d = \langle g, v \rangle \quad \forall v \in W_0^{1,r'}(\Omega). \quad (\text{B.5})$$

By [Theorem B.1](#), [\(B.5\)](#) has a unique solution $y \in W_0^{1,r}(\Omega)$. By defining $g \in W_{\perp}^{-1,r}(\Omega)$ via $g := (1+\tau)^d \Phi_{\tau}^{-1} v_{\tau}$ as well as $y_{\tau} := \Phi_{\tau} y \in W_0^{1,r}(\Omega_{\tau})$ and inserting both into [\(B.5\)](#), we obtain

$$\int_{\Omega} (\omega_{\tau} \operatorname{grad} \Phi_{\tau}^{-1} y_{\tau}) \cdot \operatorname{grad} \Phi_{\tau}^{-1} \psi = \langle \psi, v_{\tau} \rangle \quad \forall \psi \in W_0^{1,r'}(\Omega_{\tau}), \quad (\text{B.6})$$

where we have used that Φ_{τ} is a bijection. Using the transformation formula, [\(B.6\)](#) can be seen to be equivalent to [\(B.4\)](#) and hence, y_{τ} is a solution of [\(B.4\)](#). Note that y_{τ} is the unique solution, since y is the unique solution of [\(B.5\)](#) and Φ_{τ} is a bijection.

To show continuity of the solution operator, we first note that thanks to [\(B.5\)](#) for y_{τ} it holds

$$\|y_{\tau}\|_{W_0^{1,r}(\Omega_{\tau})} = \omega_{\tau}^{-1} \|\Phi_{\tau} \Delta_r^{-1} g\|_{W_0^{1,r}(\Omega_{\tau})} \leq \frac{\|\Phi_{\tau}\|_{W_0^{1,r}(\Omega) \rightarrow W_0^{1,r}(\Omega_{\tau})} \|\Delta_r^{-1}\| \|g\|_{W_{\perp}^{-1,r}(\Omega)}}{(1+\tau)^{d-2}},$$

where

$$\|g\|_{W_{\perp}^{-1,r}(\Omega)} \leq \sup_{1=\|v\|_{W_0^{1,r'}(\Omega)}} \|v_{\tau}\|_{W_{\perp}^{-1,r}(\Omega_{\tau})} \|\Phi_{\tau} v\|_{W_0^{1,r'}(\Omega_{\tau})} \leq \|v_{\tau}\|_{W_{\perp}^{-1,r}(\Omega_{\tau})} \|\Phi_{\tau}\|_{W_0^{1,r'}(\Omega) \rightarrow W_0^{1,r'}(\Omega_{\tau})}$$

and it remains to compute the operator norm $\|\Phi_{\tau}\|_{W_0^{1,s}(\Omega) \rightarrow W_0^{1,s}(\Omega_{\tau})}$. To this end, let $\varphi \in W_0^{1,s}(\Omega)$. Using the transformation formula, it is straightforward to compute

$$\begin{aligned} \|\Phi_{\tau} \varphi\|_{L^s(\Omega_{\tau})} &= (1+\tau)^{\frac{d}{s}} \|\varphi\|_{L^s(\Omega)}, \\ \left\| \frac{1}{1+\tau} \operatorname{grad} \Phi_{\tau} \varphi \right\|_{L^s(\Omega_{\tau})} &= (1+\tau)^{\frac{d}{s}-1} \|\operatorname{grad} \varphi\|_{L^s(\Omega, \mathbb{R}^d)}, \end{aligned}$$

such that

$$\begin{aligned} \|\Phi_{\tau}\|_{W_0^{1,s}(\Omega) \rightarrow W_0^{1,s}(\Omega_{\tau})} &= \sup_{1=\|\varphi\|_{W_0^{1,s}(\Omega)}} (1+\tau)^{\frac{d}{s}} \|\varphi\|_{L^s(\Omega)} + (1+\tau)^{\frac{d}{s}-1} \|\operatorname{grad} \varphi\|_{L^s(\Omega, \mathbb{R}^d)} \\ &\leq \sup_{\substack{\zeta, \xi \in [0,1] \\ \zeta + \xi = 1}} \zeta (1+\tau)^{\frac{d}{s}} + \xi (1+\tau)^{\frac{d}{s}-1} = (1+\tau)^{\frac{d}{s}}. \end{aligned}$$

Hence,

$$\frac{\|\Delta_{r,\tau}^{-1} v_{\tau}\|_{W_0^{1,r}(\Omega_{\tau})}}{\|v_{\tau}\|_{W_{\perp}^{-1,r}(\Omega_{\tau})}} = \frac{\|y_{\tau}\|_{W_0^{1,r}(\Omega_{\tau})}}{\|v_{\tau}\|_{W_{\perp}^{-1,r}(\Omega_{\tau})}} \leq \frac{\|\Delta_r^{-1}\|}{(1+\tau)^{d-2}} (1+\tau)^{\frac{d}{r}} (1+\tau)^{\frac{d}{r'}} = \|\Delta_r^{-1}\| (1+\tau)^2,$$

which yields $\|\Delta_{r,\tau}^{-1}\|_{W_{\perp}^{-1,r}(\Omega_{\tau}) \rightarrow W_0^{1,r}(\Omega_{\tau})} \leq \|\Delta_r^{-1}\| (1+\tau)^2$ and concludes the proof. \square

Finally, we're in the position to prove [Lemma 4.3](#).

Proof. Choosing $r = \alpha'$ in [Lemma B.2](#), we obtain that

$$- \int_{\Omega_{\tau}} \operatorname{grad} \zeta \cdot \operatorname{grad} \varphi = \int_{\Omega_{\tau}} \varphi \, d\xi \quad \forall \varphi \in W_0^{1,\alpha'}(\Omega_{\tau}),$$

has a unique solution $\zeta \in W_0^{1,\alpha'}(\Omega_{\tau})$ for all $\xi \in W_{\perp}^{-1,\alpha'}(\Omega_{\tau})$, which corresponds to [\(2.4\)](#) on Ω_{τ} . Moreover, the corresponding solution operator is uniformly bounded for $\tau \rightarrow 0$. The assertion now follows analogously to the proof of [Lemma 2.10](#). \square

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