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The Texas Shoot-Out under Knightian Uncertainty

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Abstract

The allocation of a co-owned company to a single owner using the Texas Shoot-Out mechanism with private valuations is investigated. We identify Knightian Uncertainty about the peer's distribution as the reason for its deterrent effect of an immature dissolving. Modeling uncertainty by a compact environment around a reference distribution F in the Prohorov metric, we derive the optimal price announcement for an ambiguity averse divider. The divider hedges against uncertainty for valuations close to the median of F, while extracting expected surplus for high and low valuations. The outcome of the mechanism is efficient for valuations around the median. A risk neutral co-owner prefers to be the chooser, even strictly so for any valuation under low levels of uncertainty and for extreme valuations under high levels of uncertainty.

Key words and phrases: Knightian Uncertainty in Games, Texas Shout-Out, Partnership Dissolution

JEL subject classification: C72, D74, D81, D82

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1 Introduction

Even the friendliest relationships can eventually turn sour. Co-owned companies are no exception, being at risk of feuding partners. In the event that co-owners are not able to cooperatively lead the company effectively anymore, it is reasonable to no longer sustain the partnership. There are two main kinds of solution concepts available to govern the dissolution of a partnership. On the one hand, *external solutions* involving a third party or stock market to serve as an outside option, e.g. a sale of the company to a prospective buyer or a liquidation in order to distribute asset shares can terminate a relationship and pay off the co-owners. However, those options are often not desirable since a liquidation goes along with a loss of jobs, pay-off of the business's debts and far reaching tax consequences while a new owner of the company may not be available on short notice. Furthermore, any external exit solution suffers from possible loss in market value of the company due to the publicly observed hostility within. Even more, private valuations of the co-owners are not taken into account who still might want to lead the company even without a partner. On the other hand, *internal solutions* can mitigate the aforementioned drawbacks, while also taking into account the co-owners valuation for the company. Writing any such *exit mechanism* into the buy-sell contract when founding a business allows an internal solution to be immediately available independent of whether or not an external outside option is at hand.

For a two party co-owned company, a frequently used exit mechanism goes by the name of *Texas Shoot-Out*. Many consulting platforms [1] recommend it for its simplicity and effect as a deterrent to an immature dissolving. The Texas Shoot-Out gives any partner the right to initiate the exit mechanism at any time she desires, thus becoming the so-called "divider". The divider commits to a price p for the sole ownership of the company. Her partner, called "chooser" has exactly two options. Either, he buys the ownership and interest of divider at price p or sells his to her at price p. In the end, exactly one partner remains as the sole owner of the business, having compensated the former co-owner. Although this mechanism is independent of actual shares of the co-owners, it is typically recommended for equally sized shareholders.

While simple, the Texas Shoot-Out is notorious for its deterrent effect on feuding partners: Neither partner knows in which role they will eventually find themselves, what price they might face in case the co-owner initiated

[1] See e.g., ClayconCapitalPartners or ExitPlanningSolutions at `https://claytoncapitalpartners.com/navigator/issue66-biz_continuity_part3.html` and `https://www.exitplanning.com/blog/texas-shootout`.

the exit mechanism or what choice to expect when being they trigger the mechanism themselves. The so created uncertainty leads to a very purposeful execution of the mechanism and thus encourages conciliation in times of minor conflicts while at the same time offering a promptly tool in order to solve a dispute if the partnership reached a dead end.

The focus of this article is to precisely characterize the deterrent component of the Texas Shoot-Out. Since there is no objective information about a peer's valuation, it is natural to assume that co-owners face imprecise probabilistic information about the valuation of their peers. We model this *Knightian Uncertainty* by means of a set of priors about the distribution function of the co-owner's valuation. This prior set allows for more flexibility than uncertainty about the valuation alone as it can be used to model bounds for main characteristics of the distribution, such as its expectation or variance. More precisely, a co-owner is willing to entertain the belief that the partner's valuation is close to a reference distribution F, but prefers a robust approach to the uncertainty about the exact distribution. The proposed framework thus models the Texas Shoot-Out in between the Bayesian setting and the the maxmin case, explaining what happens for intermediate levels of Knightian Uncertainty about the co-owner's distribution. The main finding is the astonishing link that connects those two. The optimal price announcement of divider describes a strictly increasing function in her own valuation with two kinks left and right from the median of F. In between, she announces half her valuation, thus hedging herself against uncertainty. For low (resp. high) valuations she plays as if facing the Bayesian setting induced by the distribution function being stochastically dominated by (resp. stochastically dominating) all other distributions in her prior set. Increasing the level of uncertainty about the co-owner's distribution also increases the interval around the median that corresponds to a full hedging strategy. This completely pins down the continuous transformation from the Bayesian case to the one of full uncertainty: Hedging behavior starts at the median of F and spreads to more and more extreme valuations continuously with an increase in uncertainty. As long as uncertainty is not too high, dividers with very low and high valuations make strategic price announcements. E.g., for low valuations, divider states a price exceeding half her valuation, expecting chooser to still accept the offer and thus generating a revenue.

The shape of the optimal price announcement implies an efficient allocation of the company for valuations of divider in a set close to the median of F. This set is larger the higher the level of uncertainty about the chooser's valuation. Fixing a valuation for the company, the interim expected utility for chooser is always higher than for divider. This preference is strict if the valuation is not too extreme and uncertainty not too high. In combination, our findings explain why the Texas Shoot-Out is preventing an immature selfish end of the partnership: Only co-owners with an extreme valuation expect a revenue from initiating the exit mechanism. However, they thus become divider and suffer from a strictly lower interim expected payoff than the other co-owner. Hence, the higher the uncertainty, the fewer types will execute the Texas Shoot-Out without good reason and expect to get a utility close to half their valuation. At the same time, the Texas Shoot-Out does offer a fair way out of a dead end. Every co-owner emerges with an expected payoff no lower than half her valuation.

Our contribution contrasts two standpoints on the Texas Shoot-Out in the literature by connecting them. In the well-known case of a single distribution ([McA92]) the optimal price announcement describes a strictly increasing function above the line $\frac{x}{2}$ that touches it exactly once - at the median of F. Though chooser can perfectly conclude divider's type, this strategic price announcement can lead to an inefficient outcome in which the co-owner with lower expectation obtains the company. Under full uncertainty ([VEW20]), a co-owner prepares for the worst case the co-owner can inflict on herself by always offering exactly half her valuation. The allocation will thus always be efficient at the cost of extreme behavioral assumptions.

The Texas Shoot-Out can be interpreted as variant of a *cake-cutting mechanism*. Typically stated for divisible objects these mechanisms describe discrete or continuous procedures for proportional or envy-free allocations ([BT95], [BTZ97]). Principally, any cake-cutting mechanism can be extended to settings with indivisible objects by introducing transfer payments, as done in the Texas Shoot-Out, or having a selling third party with sufficient information ([GM89]). The question which agent is cutting and choosing is significant with regard to an efficient allocation as pointed out by [dFK08]. It is not surprising that economic literature, especially game theory and mechanism design have investigated fair division problems, dissolving procedures and their impact on a partnership.

For publicly known shares and private valuations of an indivisible good, [CGK87] derive the class of incentive compatible and individual rational mechanisms that allocate the sole ownership of an object to a shareholder which reconciles the negative result of [MS83] in which for a sole owner the object cannot be efficiently allocated.

The remainder of the article is organized as follows. Section 2 gives a formal introduction to the Texas Shoot-Out and briefly summarizes the results of the stochastic setting and the one with full uncertainty, known from the literature which will be the corner stones of our analysis. Section 3 adds Knightian Uncertainty to the setup. The main result - the optimal price announcement under uncertainty - is stated and unravels the link between the models explained previously. In addition to the efficiency of the allocation, interim expected utility is derived and compared between the co-owners, explaining the deterrent effect of an immature selfish dissolution of the Texas Shoot-Out. Finally, Section 4 wraps up our findings.

2 Bayes-Nash and Maxmin Equilibrium

We consider two equal owners of a company who have come to the point where they want to dissolve their partnership. Both owners have a private valuation $x_i \in [x_l, x_u]$ for being the sole owner of the company. In the Texas Shoot-Out, the first player ("divider") announces a price $p \geq 0$ that he is willing to offer for the company. The second player ("chooser") either pays divider p and becomes the sole owner of the company or sells it at price p and divider and obtains the company.

The chooser clearly buys the company if the announced price exceeds the private value of half of the company, i.e. $p > x_C/2$ and sells the company if $p < x_C/2$.

Let us assume that divider has a belief over chooser's valuation given by the distribution function F on the non-degenerated interval $X := [x_l, x_u]$ with a strictly positive and continuous density function f. Denote by x_m its median. Furthermore, we allow for divider to entertain a utility function which is twice continuously differentiable, concave and strictly increasing ($u' > 0$). Given the chooser's response, divider's interim payoff for valuation x_D is given by

$$\pi_F(p \mid x_D) = u(x_D - p)F(2p) + u(p)(1 - F(2p)).$$

Maxmin and Hedging Let us first consider the maxmin strategy that was recently discussed by [VEW20]. Note that the divider can remove any uncertainty about the payoff by simply bidding half of his own valuation, i.e. $\bar{p} = \frac{x_D}{2}$, guaranteeing a payoff of

$$\pi_F(\bar{p} \mid x_D) = u\left(\frac{x_D}{2}\right) F(x_D) + u\left(\frac{x_D}{2}\right)(1 - F(x_D)) = u\left(\frac{x_D}{2}\right).$$

The choice $\frac{x_D}{2}$ thus hedges the divider's uncertainty completely and we call this strategy *full hedging.*

Proposition 1 ([VEW20])**.** *In maxmin equilibrium, divider bids $\bar{p} = \frac{x_D}{2}$ and chooser accepts if and only if $\bar{p} > x_C/2$.*

Note that the outcome of the maxmin equilibrium is efficient in the sense that the player with the highest valuation obtains the company.

Bayes-Nash Equilibrium In the following, the notion of quasiconcavity will play a crucial role. For completeness, we therefore state some of its definitions and properties in the appendix.

Assumption 1. *For a considered valuation x_D, let divider's payoff function $\pi_F(p \mid x_D)$ be strictly quasiconcave in $2p \in [x_l, x_u]$.*

In the literature, it is common to impose monotone hazard rate conditions on the prior. We show that these common conditions ensure quasiconcavity of the resulting payoff function. We give a sufficient condition for the assumption to hold for all valuations in the next lemma.

Lemma 1. *Assumption 1 is satisfied if the* standard hazard rate conditions *(SHRC, [McA92][2]) are fulfilled, i.e.*

$$\frac{\partial}{\partial x}\left(x + \frac{F(x)}{f(x)}\right) \geq 0 \quad \textit{and} \quad \frac{\partial}{\partial x}\left(x - \frac{1-F(x)}{f(x)}\right) \geq 0. \tag{1}$$

Proof. All of the proofs are delegated to the appendix. □

Proposition 2 ([McA92])**.** *The optimal price announcement $B(x_D)$ for divider is strictly increasing in x_D. If $x \lesseqgtr x_m$ then $x \lesseqgtr 2B(x) \lesseqgtr x_m$. For any valuation x, interim expected utility is strictly larger for the chooser.*

Example 1. *Let us consider the particularly transparent case of the uniform distribution $f(x) = 1$ for $x \in [0,1]$ and $u = \mathrm{id}$. By Lemma 1, Assumption 1 is satisfied. For $0 < p < 1/2$ we have*

$$\pi_F(p \mid x_D) = (x_D - p)2p + p(1-2p) = 2x_D p + p - 4p^2.$$

The optimal bid is thus given by the first order condition

$$2x_D + 1 - 8p = 0$$

or

$$p = x_D/4 + 1/8.$$

Note that the outcome of the Bayes-Nash equilibrium is not efficient, in contrast to the maxmin equilibrium outcome. There is an incentive for low co-owners with low valuation to bid a positive price because the chooser is going to accept with a certain probability. It can thus happen that the divider obtains the company although he has a lower valuation than the chooser in the case of low valuations and vice versa in case of both players having a high valuations.

[2]Note that there's a typo at the second condition in the reference which is clear from their proof. See also our proof.

3 Equilibria under Knightian Uncertainty

We now turn towards the case of uncertainty about the distribution of the chooser's valuation. We consider the case of a divider who is willing to entertain the belief F about the chooser's valuation, yet prefers a robust approach to account for the uncertainty about the exact distribution. More explicitly, he considers a whole family of distributions $G \in \mathcal{G}$ that are close to F in the Prohorov metric[3]., i.e. for some $\varepsilon > 0$ we have

$$\inf\{\eta > 0 : F(x - \eta) - \eta \leq G(x) \leq F(x + \eta) + \eta \text{ for all } x \in [0, 1]\} \leq \varepsilon.$$

The Prohorov metric is a standard way to model uncertainty about the exact distribution; it allows for small changes in the probability weights of each type as well as small shifts of the entire distribution to higher or lower types.

The agent is uncertainty-averse in the sense of Gilboa and Schmeidler [GS89]. He thus considers his worst expected utility in the class $\mathcal{G}$, given by

$$\pi(p \mid x_D) = \min_{G \in \mathcal{G}} \pi_G(p \mid x_D).$$

There are two extreme distributions G_0, G_1 in the class $\mathcal{G}$ that are particularly relevant for the analysis because they are the extreme points in the sense of stochastic dominance[4]. For the Prohorov-ball they are explicitly given by

$$G_0(x) = \begin{cases} 0 & , x_l \leq x < \epsilon + F^{-1}(\epsilon), \\ F(x - \epsilon) - \epsilon & , \epsilon + F^{-1}(\epsilon) \leq x < x_u, \\ 1 & , x = x_u \end{cases}$$

$$G_1(x) = \begin{cases} F(x + \epsilon) + \epsilon & , x_l \leq x \leq F^{-1}(1 - \epsilon) - \epsilon, \\ 1 & , F^{-1}(1 - \epsilon) - \epsilon < x \leq x_u, \end{cases}.$$

G_0 is stochastically dominating every distribution in $\mathcal{G}$. It is obtained from the prior distribution F by reducing masses on valuations by ϵ and shifting the remaining mass to the highest valuation. Similarly, G_1 shifts masses to the low valuations and reduces the mass on other valuations accordingly, making it stochastically dominated by all distributions in $\mathcal{G}$. An illustration of the these distributions. is depicted in Figure 1.

[3]Also called Lévy-Prohorov metric, c.f. [Bil13]

[4]The notion of stochastic dominance is as usual: G stochastically dominates G' if $G(x) \leq G'(x)$ for all x. Equivalently, $\int u(x)\, G(\mathrm{d}x) \geq \int u(x)\, G'(\mathrm{d}x)$ for all functions u with $u' \geq 0$, see [Lev92] or [RS70]

Note that for high values of ϵ some of the above intervals involve a higher left boundary than the right one. Throughout the paper we implicitly make the appropriate adjustment to treat those intervals as empty and thus as not contributing to a functions definition. Our results remain valid in those cases as well.

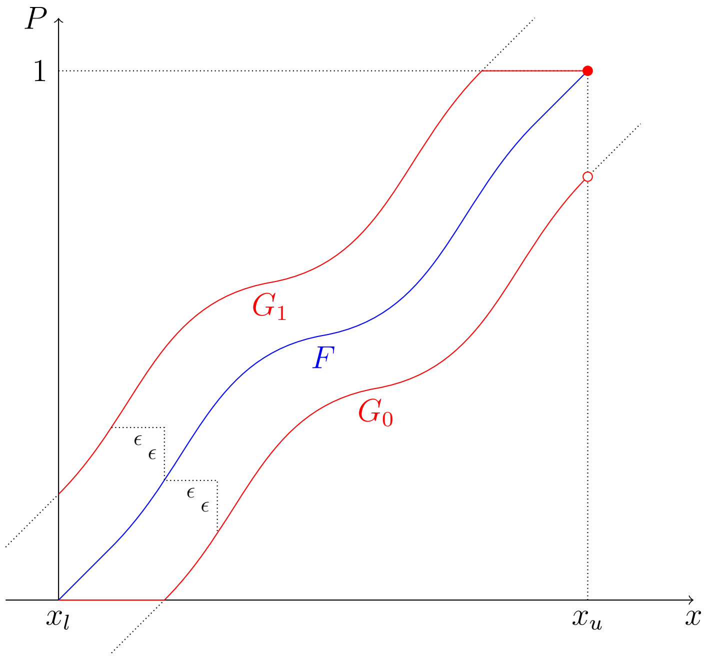

Figure 1: Prohorov ϵ-ball around a distribution function F.

Optimal Price Announcement Recall the payoff function

$$\pi_G(p \mid x_D) = u(x_D - p)G(2p) + u(p)(1 - G(2p)).$$

By the strict monotonicity of u, $u(x_D - p) > u(p)$ if and only if $x_D > 2p$. Thus, if the divider announces a price fulfilling $x_D > 2p$, her worst case belief is $G = G_0$. Vice versa, her worst case belief for $x_D < 2p$ is $G = G_1$. Consequently, only G_0 and G_1 are relevant for evaluating the worst-cases for divider, i.e.

$$\begin{aligned}\pi(p \mid x_D) &= \min_{G \in \{G_0, G_1\}} u(x_D - p)G(2p) + u(p)(1 - G(2p)) \\ &= \begin{cases} u(x_D - p)G_0(2p) + u(p)(1 - G_0(2p)) & , 2p < x_D, \\ u(x_D - p)G_1(2p) + u(p)(1 - G_1(2p)) & , x_D \leq 2p \end{cases}.\end{aligned}$$

Note that π is continuous, not only in x_D but also in p: The discontinuity of $\pi_{G_0}(. \mid x_D)$ in $2p = x_u$ only occurs if $x_D < x_u$. But in that case, $\pi(. \mid x_D)$ switches to the continuous function $\pi_{G_1}(. \mid x_D)$ before hitting $2p = x_u$.

Thus, the optimal price announcement (correspondence) can be defined

$$B(x_D) := \arg\max_p \pi(p \mid x_D). \tag{2}$$

Under Assumption 2, the driver of our main theorem, it will turn out, that the maximizer is unique, so the optimal price announcement under uncertainty is indeed a function rather than a correspondence.

An immediate observation is that optimal price announcements cannot be extreme:

Lemma 2. *An optimal price announcement $p^* \in \mathbb{R}$ fulfills $p^* \in [x_l, x_u]$.*

In order to give a precise and complete game theoretic description of the Texas Shoot-Out under multiple priors, we also allow chooser to face uncertainty about the distribution of divider's valuation. However, as chooser's decision in her action phase does not depend on her belief about divider's valuation or the valuation itself, her perception of this uncertainty will only play a role when talking about interim efficiency later on. For the following equilibrium concept, assume that chooser faces uncertainty about divider's valuation distribution given by an ϵ'-Prohorov ball around some distribution function F' on X.

Definition 1. *A strategy profile (b, c) consisting of a divider strategy $b\colon X \to \mathbb{R}$ and a chooser strategy $c\colon X \times \mathbb{R} \to \{buy, sell\}$ is called* interim (subgame-perfect) Knight-Nash equilibrium*, if the strategy profile defines a (subgame-perfect) Nash-equilibrium of the corresponding extensive form game where utility functions are given by the worst-case expected utility w.r.t. the resp. agent's multiple set for fixed valuations x_D, x_C.*

The definition is a straightforward extension of a Nash equilibrium for agents who resolve uncertainty by maximizing their worst case expected utility. Our focus on interim equilibria is based on the assumption that both co-owners already perfectly know their private valuations before engaging in the exit mechanism.[5] A direct consequence of the chooser's decision rule and the continuity of divider' payoff function is the existence of an equilibrium.

[5] If one is to model an ex-ante setting, additional assumptions on the expected/uncertain own future valuation need to be made in order to describe an agent's own belief about her future valuation.

Proposition 3. *In the Texas Shoot-Out, the following interim subgame-perfect Knight-Nash equilibrium always exist. For an announced price p, a chooser with valuation x_C will buy the company if and only if $x_C \geq 2p$. A divider with valuation x_D will make an optimal price announcement p belonging to $B(x_D)$.*

We will now provide a leading example for the uniform distribution that sheds light on the general results. It turns out that, on the one hand, a divider with a rather average valuation will play cautiously and fully-hedge himself against any losses by playing half his valuation. On the other hand, a divider with very low (resp. high) valuation will still try to strategically extract some revenue by stating a slightly higher (resp. lower price), thinking that chooser will still take the offer (resp. refuse) it.

Example 2. *Let F be the uniform distribution on the unit interval $[0,1]$ and let $u = \mathrm{id}$. For $\epsilon \leq 1/4$ the crucial distributions are given by*

$$G_0(x) = \begin{cases} 0 & , 0 \leq x \leq 2\epsilon, \\ x - 2\epsilon & , 2\epsilon < x < 1, \\ 1 & , x = 1. \end{cases}$$

and

$$G_1(x) = \begin{cases} x + 2\epsilon & , 0 \leq x \leq 1 - 2\epsilon \\ 1 & , 1 - 2\epsilon < x \leq 1. \end{cases}.$$

The optimal price announcement for the uncertainty-averse divider is given by the function

$$B(x_D) = \begin{cases} \frac{x_D}{4} - \frac{\epsilon}{2} + \frac{1}{8} & , 0 \leq x_D < \frac{1}{2} - 2\epsilon, \\ \frac{x_D}{2} & , \frac{1}{2} - 2\epsilon \leq x_D \leq \frac{1}{2} + 2\epsilon, \\ \frac{x_D}{4} + \frac{\epsilon}{2} + \frac{1}{8} & , \frac{1}{2} + 2\epsilon < x_D \leq 1. \end{cases}$$

which is depicted in Figure 2.

The assumption $\epsilon \leq 1/4$ is made to exclude degenerated cases. The results and formulae stay correct for any value of $\epsilon \geq 0$ provided the obvious adjustment to neglect empty intervals in the piecewise definition. In the special case $\epsilon \geq \frac{1}{2}$ full uncertainty is faced and thus, full hedging will be played.

It is worth mentioning that for $\epsilon \in [\frac{1}{4}, \frac{1}{2})$ the optimal price announcement is already full hedging for every valuation, while divider does not face full uncertainty, i.e. $\mathcal{G}$ is not the full set of probability distributions on $[0,1]$. Complementary to [VEW20], this indicates that full uncertainty is sufficient, but not necessary for full hedging to be optimal for every valuation.

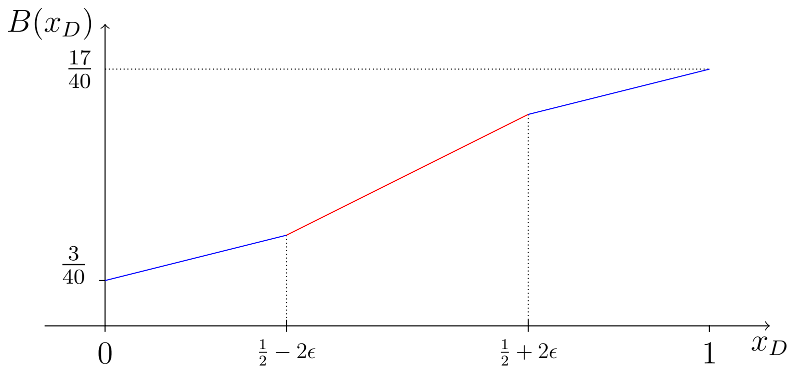

Figure 2: Maxmin price announcement for $F \sim \mathcal{U}([0,1]), u = \text{id}$ and $\epsilon = \frac{1}{10}$.

We will now turn towards our intuition, that hedging around the median valuation is not a coincidence, but holds in fairly general cases. We key assumption will be strict quasiconcavity of the induced payoff functions π_{G_0} and π_{G_1}.

Assumption 2. *For a considered valuation $x_D \in X$ and ϵ, let both the divider's payoff functions π_{G_0} and π_{G_1} be strictly quasiconcave in $2p \in [x_l, x_u)$.*

A sufficient condition for this is given in the following lemma.

Lemma 3. *Assumption 2 is satisfied for all valuations if the following* Prohorov-Knight hazard rate conditions *are fulfilled, i.e.*

$$\frac{\partial}{\partial x}\left(x + \frac{F(x)}{f(x)}\right) - \epsilon \cdot \left|\frac{\partial}{\partial x}\frac{1}{f(x)}\right| \geq 0$$

and

$$\frac{\partial}{\partial x}\left(x - \frac{1 - F(x)}{f(x)}\right) - \epsilon \cdot \left|\frac{\partial}{\partial x}\frac{1}{f(x)}\right| \geq 0$$

One can view this as a sharper version of the SHRC, where F is not only required to fulfill the SHRC, but in addition the slope of $1/f$ is sufficiently bounded.

We will find out later in our main theorem, that we only need to consider values of ϵ up to $\frac{1}{2}$, as for $\epsilon > \frac{1}{2}$ the optimal price announcement will necessary invoke full hedging behavior for all valuations. Thus, $\epsilon = \frac{1}{2}$ can serve as a bound to include all valuations and levels of uncertainty.

Example 3. *Examples that fulfill Lemma 3 include the uniform distribution or (shifted) truncated standard normal distributions, on $[0,1]$. However, the*

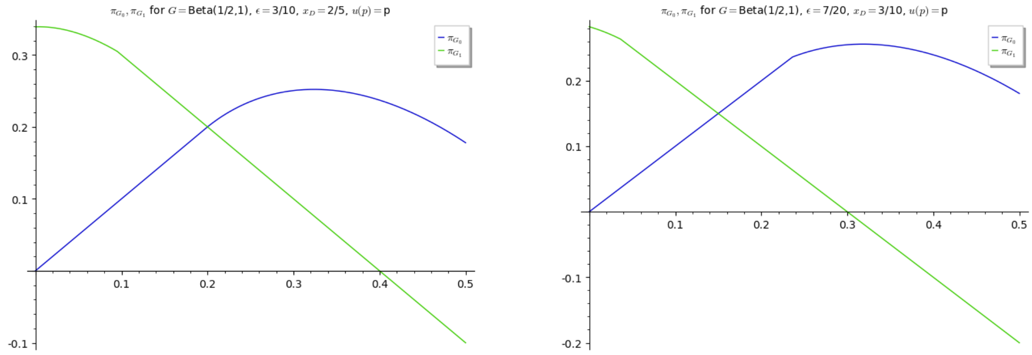

Figure 3: Quasiconcavity of payoff functions under a beta distribution for different values of x_D and ϵ.

sufficient condition is far from sharp as in the case of risk neutrality, $u = \mathrm{id}$. *For example, beta distributions induce strictly quasiconcave utility functions although the Prohorov-Knight hazard rate conditions are not satisfied, see Figure 3.*

Strict quasiconcavity is driving our main result - a full characterization of the optimal price announcement of divider.

Theorem 1. *In interim Knight-Nash-equilibrium, divider will play full hedging for valuations close to the median* x_m*, whereas making strategic announcements for extreme valuations.*

More precisely, we have

$$B(x_D) = \begin{cases} p_1(x_D) & , x_l \leq x_D < F^{-1}(\frac{1}{2} - \epsilon) - \epsilon, \\ \frac{x_D}{2} & , F^{-1}(\frac{1}{2} - \epsilon) - \epsilon \leq x_D \leq F^{-1}(\frac{1}{2} + \epsilon) + \epsilon, \\ p_0(x_D) & , F^{-1}(\frac{1}{2} + \epsilon) + \epsilon < x_D \leq x_u, \end{cases}$$

where the outer intervals may be empty. Here, $p_0(x_D)$ *resp.* $p_1(x_D)$ *denote the unique and local maxima of* $\pi_{G_0}(p \mid x_D)$ *resp.* $\pi_{G_1}(p \mid x_D)$ *w.r.t.* $2p$ *on* $[x_l, x_u]$ *in the considered cases for* x_D*.*

Theorem 1 is the link between [McA92] and [VEW20] for intermediate levels of uncertainty. It explicitly describes the deformation process of the stochastic case ($\epsilon = 0$) to the setting of full uncertainty ($\epsilon \gg 0$) by two observations: Firstly, the divider will hedge himself in an expanding environment around the median x_m of the reference distribution for increasing levels of uncertainty. Secondly, for low resp. high valuations, divider will make the optimal price announcement corresponding to facing the distribution G_1 resp. G_0.

Corollary 1. *The optimal price announcement $B(x_D)$ is strictly increasing in x_D and thus measurable and almost everywhere continuous.*

Corollary 2. *We have $x_D \leq 2B(x_D) < x_m$ if $x_D < x_m$ (resp. $x_m < 2B(x_D) \leq x_D$ if $x_m < x_D$) with a strict inequality if and only if $x_D < F^{-1}(1/2 - \epsilon) - \epsilon$ (resp. $F^{-1}(1/2 + \epsilon) + \epsilon < x_D$).*

The Corollaries do not only state that the optimal price announcement is well-behaved, but also encodes that co-owners with low valuations will announce a price above half their valuation (but below the median), while for high ones will announce a price less than half their valuation (but above the median). For valuations around the median on the other hand, agents expect to end up in in the wrong place when gambling and thus fully hedge themselves by reporting half their valuation.

Is is also worth noting that the kinks of the optimal price announcement happen at the medians of G_0 and G_1, giving rise to a generalization of the found result.

Efficiency We now turn towards an analysis of efficiency with two points of interest. Firstly, allocation efficiency, i.e., whether the mechanism allocates the object to the agent with the highest valuation. Secondly, interim efficiency, i.e. whether an agent with a fixed valuation prefers to be the divider or the divider in equilibrium.

The results on allocation efficiency are an immediate consequence of Theorem 1 and Corollary 2: If divider's valuation x_D is within $[F^{-1}(1/2 - \epsilon) - \epsilon, F^{-1}(1/2 + \epsilon) + \epsilon]$ the allocation induced by the Texas Shoot-Out under Knightian Uncertainty is efficient. For valuations outside of this interval, the mechanism might not lead to an efficient allocation.

In the case of interim efficiency, we will see that a risk neutral agent with valuation x prefers to be the chooser. This preference is strict for all valuations if the uncertainty is not too high. If uncertainty is high, it still is strict for extreme valuations, unless full uncertainty is faced, leading to indifference everywhere.

We begin the analysis of interim efficiency, by deriving the interim worst case EU of a divider with valuation x_D, which is given by

$$\begin{aligned}\Phi_D(x_D) :=& \pi(B(x_D) \mid x_D) \\ =& \min_{G \in \{G_0, G_1\}} u(x_D - B(x_D)) \cdot G(2B(x_D)) + u(B(x_D)) \cdot (1 - G(2B(x_D))).\end{aligned}$$

By Theorem 1, we can write this as

$$\Phi_D(x_D) = \begin{cases} \pi_{G_1}(p_1(x_D) \mid x_D) & , x_l \leq x_D < F^{-1}(\frac{1}{2} - \epsilon) - \epsilon, \\ \frac{x_D}{2} & , F^{-1}(\frac{1}{2} - \epsilon) - \epsilon \leq x_D \leq F^{-1}(\frac{1}{2} + \epsilon)\epsilon, , \\ \pi_{G_0}(p_0(x_D) \mid x_D) & , F^{-1}(\frac{1}{2} + \epsilon) + \epsilon < x_D \leq x_u. \end{cases}$$

disregarding empty intervals if ϵ is large.

So far, we have not investigated chooser's interim expected utility, but will do so now. To this end, assume chooser faces the same uncertainty as divider, i.e. she beliefs divider's valuation to be drawn from the set $\mathcal{G}$, the Prohorov ϵ-ball around the same distribution function F and having valuation x_C. Her interim worst case expected payoff in equilibrium is given by

$$\Phi_C(x_C) := \min_{G \in \mathcal{G}} \mathbb{E}_G \left[\max\{u(x_C - B(z)), u(B(z))\} \right],$$

where the expectation is taken w.r.t. the cdf G over divider's valuations, denoted by the variable z, and B is divider's optimal price announcement.

Since $2B$ is strictly increasing with range $[2B(x_l), 2B(x_u)]$ we can draw the following conclusions:

1. If $x_C < 2B(x_l)$, the max-function will always choose $u(B(z))$. Thus, as $z \mapsto u(B(z))$ is strictly increasing, the worst case is $G = G_1$ which is stochastically dominated by every other distribution in $\mathcal{G}$ and puts most weight on the lowest price announcements. Interim worst case EU is thus
$$\mathbb{E}_{G_1}[u(B(z))].$$

2. If $x_C > 2B(x_u)$ the max-function will always choose $u(x_C - B(z))$. Thus, as $z \mapsto u(x_C - B(z))$ is strictly decreasing, the worst case is $G = G_0$, i.e. the distribution stochastically dominating all other distributions in $\mathcal{G}$ and puts most weight on the highest price announcements. Interim worst case EU is thus
$$\mathbb{E}_{G_0}[u(x_C - B(z))].$$

3. If $x_C \in [2B(x_l), 2B(x_u)]$, the worst case is putting as much weight at (and around) the valuation z^* that induces a price announcement $2B(z^*) = x_C$ as possible. More precisely, consider the partition $X = [x_l, z^*) \cup [z^*, x_u]$. Note that for any $x \in [x_l, z^*)$, the max-function will choose $u(x_C - B(z))$, thus the expectation is minimized by G_0. For

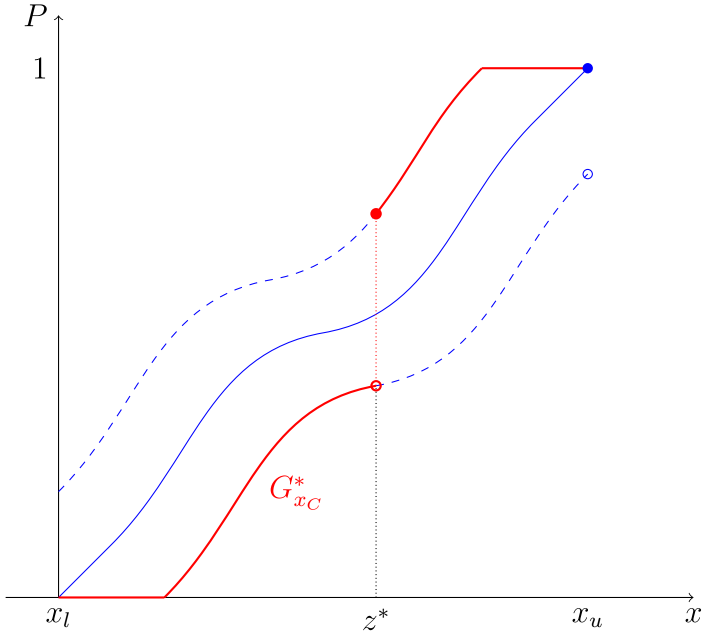

Figure 4: Worst case distribution of chooser in case 3 with $2B(z^*) = x_C$.

$x \in [z^*, x_u]$ the max-function selects $u(B(z))$, thus the expectation is minimized by G_1. As we can piece both cases together into a distribution function $G^*_{x_C} \in \mathcal{G}$ (note that it must be right-continuous, thus $G^*_{x_C}(z^*) = G_1(z^*)$), this is the worst case distribution. It is explicitly given by

$$G^*_{x_C}(z) = \begin{cases} G_0(z) & , x_l \leq z < z^*, \\ G_1(z) & , z^* \leq z \leq x_u. \end{cases}$$

Figure 4 depicts $G^*_{x_C}$. Interim worst case EU is thus

$$\begin{aligned} &\mathbb{E}_{G^*_{x_C}}[\max\{u(x_C - B(z)), u(B(z))\}] \\ &= \int_{[x_l, z^*)} u(x_C - B(z))\ G_0(\mathrm{d}z) + (G_1(z^*) - G_0(z^*)) \cdot u(x_C/2) \\ &\quad + \int_{(z^*, x_u]} u(B(z))\ G_1(\mathrm{d}z). \end{aligned}$$

In the following, we will restrict to the case of risk neutral agents only, i.e. $u = \mathrm{id}$. A useful tool in the upcoming analysis is the characterization of the derivatives of Φ_D and Φ_C.

Lemma 4. *Let* $u = \mathrm{id}$*. The functions* Φ_D *and* Φ_C *are increasing and - except for* Φ_C *in the pasting points - differentiable with derivatives*

$$\Phi_D'(x) = \begin{cases} G_1(2p_1(x)) & , x_l \leq x < F^{-1}(\frac{1}{2} - \epsilon) - \epsilon, \\ \frac{1}{2} & , F^{-1}(\frac{1}{2} - \epsilon) - \epsilon \leq x \leq F^{-1}(\frac{1}{2} + \epsilon) + \epsilon, \\ G_0(2p_0(x)) & , F^{-1}(\frac{1}{2} + \epsilon) + \epsilon < x \leq x_u. \end{cases}$$

and

$$\Phi_C'(x) = \begin{cases} 0 & , x_l \leq x < 2p_1(x_l), \\ \frac{1}{2} \cdot \left(G_1(B^{-1}(\frac{x}{2})) + G_0(B^{-1}(\frac{x}{2}))\right) & , 2p_1(x_l) \leq x \leq 2p_0(x_u) \\ 1 & , 2p_0(x_u) < x \leq x_u. \end{cases}$$

Thus, the higher the valuation one has, the higher the interim worst case expected utility. In fact, from the functional form of the derivatives, the utility strictly increases with increasing valuation except for a chooser facing very low valuations.

By means of the above lemma we can prove that knowing ones valuation, one always strictly prefers to be chooser if ϵ is sufficiently small. If ϵ is large, for intermediate valuations, one is indifferent between being chooser or divider while for extreme valuations the preference is still strict to be chooser, unless full uncertainty is faced.

Theorem 2. *If* $\epsilon > 0$ *fulfills* $F^{-1}(\epsilon) + \epsilon < F^{-1}(1 - \epsilon) - \epsilon$*, we have* $\Phi_D(x) < \Phi_C(x)$ *for all* $x \in X$*.*

Otherwise, $\Phi_D(x) = \Phi_C(x)$ *for* $x \in [F^{-1}(1 - \epsilon) - \epsilon, F^{-1}(\epsilon) + \epsilon]$ *and* $\Phi_D(x) < \Phi_C(x)$ *for all other valuations* x*.*

Especially, under full uncertainty both, divider and chooser have an interim worst case expected utility equal to half their valuation.

We conclude this section by extending our uniform example to incorporate the utility comparison.

Example 4. *Consider again* $F \sim \mathcal{U}([0, 1])$ *for risk-neutral agents, i.e.* $u = \mathrm{id}$*.*

Since we have already calculated $B(x)$*, we can calculate* $\pi(B(x) \mid x)$ *for a divider with valuation* x*, obtaining her interim worst-case expected utility*

$$\Phi_D(x) = \begin{cases} \frac{1}{4}x^2 + (\frac{1}{4} + \epsilon) \cdot x + \epsilon^2 - \frac{\epsilon}{2} + \frac{1}{16} & , 0 \leq x < \frac{1}{2} - 2\epsilon \\ \frac{x}{2} & , \frac{1}{2} - 2\epsilon \leq x \leq \frac{1}{2} + 2\epsilon, \\ \frac{1}{4}x^2 + (\frac{1}{4} - \epsilon) \cdot x + \epsilon^2 - \frac{\epsilon}{2} + \frac{1}{16} & , \frac{1}{2} + 2\epsilon < x \leq 1, \end{cases}$$

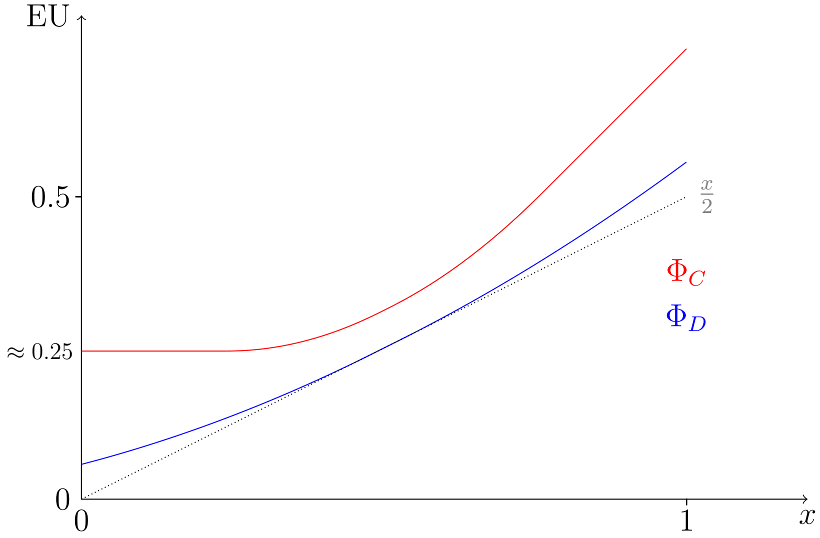

Figure 5: Interim worst case EU for $\epsilon = 0.01$

where for $\epsilon > \frac{1}{4}$ the function should be read as being equal to the function $\frac{x}{2}$ everywhere.

In order to calculate Φ_C we need to distinguish three cases $0 \leq \epsilon \leq \frac{1}{8}$, $\frac{1}{8} < \epsilon \leq \frac{1}{4}$ and $\frac{1}{4} < \epsilon \leq \frac{1}{2}$ since not only the pasting points for B (or B^{-1}) play a role, but also the kinks of G_0, G_1.[6] *Also note that for $\epsilon \geq \frac{1}{2}$ we have full uncertainty and the situation won't change anymore. All the explicit formulae are delegated to the appendix.*

Figures 5, 6 and 7 graphically illustrate Φ_D and Φ_C for $\epsilon = 0.01, 0.2, 0.3$.

For $\epsilon < \frac{1}{4}$ we see that $\Phi_C(x) > \Phi_D(x)$ everywhere. In the case $\epsilon = 0.3 > \frac{1}{4}$ we have $\Phi_C(x) = \Phi_D(x)$ precisely for $x \in [1 - 2 \cdot 0.3, 2 \cdot 0.3] = [0.4, 0.6]$ as is clear from Theorem 2. We also note that for increasing ϵ both functions are deforming into $x \mapsto \frac{x}{2}$, and the curvature of Φ_D is only visible for very small values of ϵ.

The example combines and stresses two points of our argument for the Texas Shoot-Out under Knightian Uncertainty to being a deterrent exit mechanism. For small amounts of uncertainty, i.e. small ϵ, the divider's interim worst case expected payoff is close to the full hedging payoff given by $\frac{x}{2}$. Thus, the Texas Shoot-Out lowers an agent's material incentive to initiate the mechanism and become a divider. We conclude that only co-owners with extremely high or low valuations might consider the exit profitable. Furthermore, the higher the uncertainty the more likely the mechanism is to achieve

[6] If $\epsilon \leq \frac{1}{8}$ we have $2\epsilon \leq \frac{1}{2} - 2\epsilon \leq \frac{1}{2} + 2\epsilon \leq 1 - 2\epsilon$ while for $\frac{1}{8} < \epsilon \leq \frac{1}{4}$ we have $\frac{1}{2} - 2\epsilon < 2\epsilon \leq 1 - 2\epsilon < \frac{1}{2} + 2\epsilon$ and $\frac{1}{4} < \epsilon$ implies $\frac{1}{2} - 2\epsilon < 1 - 2\epsilon < 2\epsilon < \frac{1}{2} + 2\epsilon$.

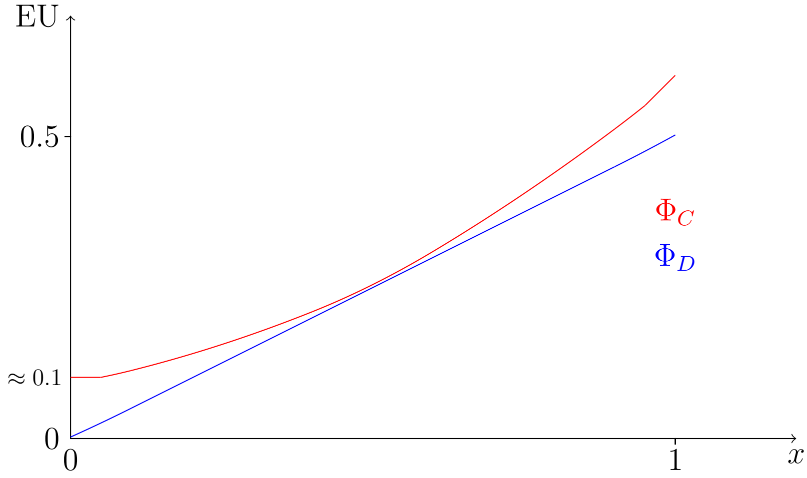

Figure 6: Interim worst case EU for $\epsilon = 0.2$

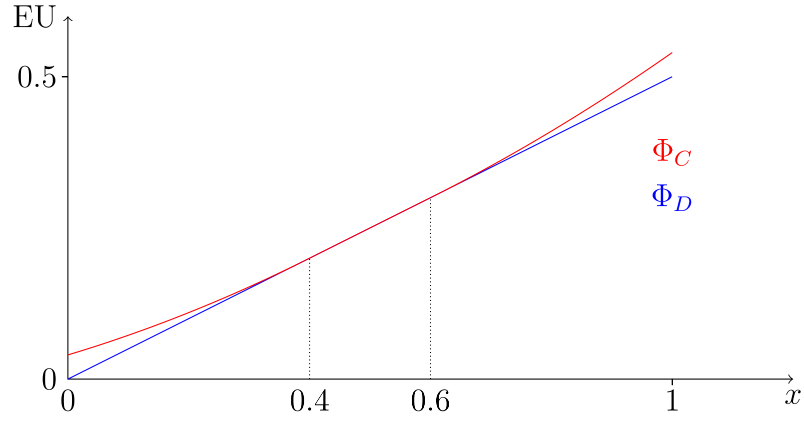

Figure 7: Interim worst case EU for $\epsilon = 0.3$

an efficient allocation. Secondly, even if a co-owner has an extreme valuation and is thus materially interested in ending the partnership under the terms of the Texas Shoot-Out, she would rather be in the position of chooser and thus not triggering the mechanism in the first place.

4 Conclusion

When founding a co-owned company it is preventative and advantageous to agree in advance on a dissolution mechanism in case things get sour. Any such dissolution mechanism should be readily available, simple, and it should discourage a selfish premature termination of the partnership. The so-called Texas Shoot-Out is a well-known example of a exit mechanism for two co-owners. The co-owner initiating the mechanism announces a price for the sole ownership of the company while the other can choose to sell or buy the company at that price. While simple and independent of external outside options, it is notorious for its deterrent effect on a premature dissolving.

This article invokes Knightian Uncertainty as an explanatory source for this discouragement. Having in mind that co-owners have some idea about the distribution of their partner, we allow for any degree of confidence in a reference distribution F by modeling the uncertainty as a compact neighborhood of F in the Prohorov metric that can range anywhere in between the Bayesian setting towards one of full uncertainty. For quasiconcave induces worst case expected utility functions, we derive the co-owners' optimal actions and interim expected payoffs. Our main result is the explicit characterization of the divider's price announcement which is a surprising mixture of the optimal strategies under no and full uncertainty: She will play cautiously for valuations close the the median valuation of the reference distribution while still trying to generate a revenue that exceeds half her valuation for low or high valuations. Hence, only co-owners with extreme valuations have a material incentive to initiate a Texas Shoot-Out. However, it turns out that it are the co-owners with high (resp. low) valuations themselves who prefer the other co-owner to trigger the exit mechanism the most as they fear to find themselves in an unfavorable position (being forced to leave the company resp. take over the sole ownership). All these consequences are already visible for small levels of uncertainty and can thus explain why consultancies recommend to include the Texas Shoot-Out in buy-sell agreements.

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A Additional Material & Proofs

Definition 1 (Quasiconcavity). *Let $S \subseteq \mathbb{R}^L$ be a convex set. A function $f\colon S \to \mathbb{R}$ is called* quasiconcave *if one of the following equivalent statements holds true:*

(i) Every super level set is convex, i.e. for any $\xi \in \mathbb{R}$ the set $\{x \mid f(x) \geq \xi\}$ is convex.

(ii) For any $x, x' \in S$ and any $\lambda \in (0,1)$ we have $\lambda f(x) + (1-\lambda) f(x') \geq \min\{f(x), f(x')\}$.

Furthermore, a function is called strictly quasiconcave*, if one can replace the above inqequalities by strict ones.*
The special case $S \subseteq \mathbb{R}$ yields another characterization suitable for our analysis. A function $f\colon S \to \mathbb{R}$ is quasiconcave if and only if one of the following three conditions hold.

(a) f is non-decreasing,

(b) f is non-increasing,

(c) there is a point $c \in S$ such that f is non-decreasing on $\{x \mid x \leq c\}$ and non-increasing on $\{x \mid c \leq x\}$.

In this case, f is strictly quasiconcave if its graph has no horizontal sections. Especially, it is either strictly increasing, strictly decreasing or has a unique point c such that f is strictly increasing on $\{x \mid x \leq c\}$ and strictly decreasing on $\{x \mid c \leq x\}$. The latter condition is often referred to as unimodality *in statistics.*

Note that concave functions are quasiconcave, but quasiconcave functions are neither necessarily concave nor even continuous.

Proof of Lemma 1. The proof essentially follows the lines of Lemma 7 in [McA92].

We start by calculating

$$\begin{aligned}
&\frac{\partial}{\partial p}\pi_F(p \mid x_D) \\
=&f(2p) \cdot \left\{ 2u(x_D - p) - 2u(p) + \frac{F(2p)}{f(2p)} \cdot (-u'(x_D - p) - u'(p)) + \frac{u'(p)}{f(2p)} \right\}.
\end{aligned}$$

Now, the second derivative of π_F with respect to p and evaluated in a point $q \in [x_l, x_u]$ (if it exists) with $\frac{\partial}{\partial p}\pi_F(p \mid x_D) \mid_{p=q} = 0$ is

$$\begin{aligned}\frac{\partial^2}{\partial p^2}\pi_F(p \mid x_D) \mid_{p=q} = &- f(2q)\frac{\partial}{\partial p}\left(2p + \frac{F(2p)}{f(2p)}\right) \mid_{p=q} \cdot u'(x_D - q)\\ &- f(2q)\frac{\partial}{\partial p}\left(2p - \frac{1-F(2p)}{f(2p)}\right) \mid_{p=q} \cdot u'(q)\\ &+ u''(x_D - q)F(2q) + u''(q)(1 - F(2q)) < 0.\end{aligned}$$

Thus, any q with $\frac{\partial}{\partial p}\pi_F(p \mid x_D) \mid_{p=q} = 0$ is an isolated local maximum. There can't be more than one such q as otherwise one would also have a local minimum in between the two due to π_F being differentiable.

Now, note that if there's a unique such q, then $\frac{\partial}{\partial p}\pi_F(p \mid x_D)$ must be positive before q and negative afterwards since it is a local isolated maximum (i.e. negative second derivative at q). Thus, it's strictly quasiconcave. Finally, if there's no such q, $\frac{\partial}{\partial p}\pi_F$ must either be positive or negative and thus strict quasiconcavity of $\pi_F(p \mid x_D)$ holds in both cases. □

A direct consequence of the proof of Lemma 1 is the following.

Corollary A1. *If F is a continuous function on $[x_l, x_u)$ fulfilling the SHRC on $[a, b) \subseteq [x_l, x_u)$ while being 0 on $[x_l, a]$ and 1 on $[b, x_u)$, then the associated function π_F is strictly quasiconcave on $2p \in [x_l, x_u)$.*

Proof of Corollary A1. Note that by the proof of the above lemma, π_F is strictly quasiconcave in $2p \in [a, b)$. Since $\pi_F = u(p)$ is strictly increasing on $2p \in [x_l, a)$ and $\pi_F = u(x_D - p)$ is strictly decreasing on $2p \in [b, x_u)$ and π_F is continuous on $[x_l, x_u)$, it still is strictly quasiconcave. □

Proof of Lemma 2. Fix any $x_D \in [x_l, x_u]$ and consider a possible price announcement $p \in \mathbb{R}$. If $2p < x_l$, we have $\pi(p \mid x_D) = u(p) < u(\frac{x_D}{2})$ and if $x_u < 2p$, we find $\pi(p \mid x_D) = u(x_D - p) < u(\frac{x_D}{2})$. As the full hedging price announcement of $\frac{x_D}{2}$ guarantees a payoff of $u(\frac{x_D}{2})$, price announcements with $2p \notin [x_l, x_u]$ are not optimal. □

Explicit calculations to Example 2. It is straightforward to show that the induced respective utility functions are given by

$$\pi_{G_0}(x_D, p) = \begin{cases} p & , 0 \leq 2p \leq 2\epsilon, \\ -4p^2 + 4p\epsilon + x_D(2p - 2\epsilon) + p & , 2\epsilon \leq x < 1, \\ x_D - p & , 2p = 1. \end{cases}$$

$$\pi_{G_1}(x_D, p) = \begin{cases} -4p^2 - 4p\epsilon + x_D(2p + 2\epsilon) + p & , 0 \leq 2p \leq 1 - 2\epsilon \\ x_D - p & , 1 - 2\epsilon < 2p \leq 1. \end{cases}.$$

Note that π is the function that stays π_{G_0} until $2p = x_D$ and is π_{G_1} afterwards. Moreover, on $2p \in [x_l, x_u)$ both, π_{G_0} and π_{G_1} are strictly quasiconcave.

The unique local maxima of the parabolas are located at $2p_0(x_D) = \frac{x_D}{2} + \epsilon + \frac{1}{4}$ and $2p_1(x_D) = \frac{x_D}{2} - \epsilon + \frac{1}{4}$, respectively. Note that $p_1(x_D) < p_0(x_D)$ and moreover $2p_1(x_D) \leq x_D \iff \frac{1}{2} - 2\epsilon \leq x_D$ as well as $x_D \leq 2p_0(x_D) \iff x_D \leq \frac{1}{2} + 2\epsilon$. The following conclusions arise:

Firstly, for $x_D \in [\frac{1}{2} - 2\epsilon, \frac{1}{2} + 2\epsilon]$ the highest value of the function π is attained at $2p = x_D$ since it is strictly increasing for values below it and strictly decreasing for higher values of $2p$.

Secondly, for $x_D < \frac{1}{2} - 2\epsilon$ we find $x_D < 2p_1(x_D) \leq 1$ and the maximum of π is attained at $p = p_1(x_D)$ (π_{G_0} is increasing, switching into π_{G_1} an still strictly increasing until $p = p_1(x_D)$ and strictly decreasing afterwards).

Finally, for $\frac{1}{2} + 2\epsilon < x_D$ we find $0 \leq 2p_0(x_D) < x_D$ an the maximum of π is attained at $p = p_0(x_D)$ (π_{G_0} strictly increases, reaches its top and starts to strictly decrease and keeps doing to after switching to π_{G_1}). □

Proof of Lemma 3. Note that π_{G_0} might be 0 before taking on the functional form $F(x-\epsilon)-\epsilon$ and π_{G_1} will be constantly 1 after $F(x+\epsilon)+\epsilon$ is hitting 1. We will now derive a condition that will make both the functions $F(x \pm \epsilon) \pm \epsilon$ fulfill Lemma 1. Quasiconcavity on the whole interval $2p \in [x_l, x_u)$ then follows by Corollary A1. Let us start by calculating their first SHRCs, which are given by

$$\frac{\partial}{\partial x}\left(x + \frac{F(x \pm \epsilon) \pm \epsilon}{f(x \pm \epsilon)}\right) = \frac{\partial}{\partial y}\left(y + \frac{F(y)}{f(y)} \pm \frac{\epsilon}{f(y)}\right),$$

where we suppressed (and unified) the domains to which the variables belong and made a change of variables ($y = x \pm \epsilon$). As we need to make sure that both expressions are non-negative, we must take into account the worst case $-\epsilon \left|\frac{\partial}{\partial y}\frac{1}{f(y)}\right|$.

A similar calculation yields the second expression as a sufficient condition for the second SHRC. □

Proposition 3. The proof is obvious by backward induction. □

The proof of Theorem 1 is split into several parts that we now briefly sketch. First, we will label the 'candidate' unique maximum price announcements $p_0(x_D), p_1(x_D)$ for the two payoff functions π_{G_0} and π_{G_1}. They determine the 'go-to' points for the price announcements if offering $2p = x_D$ is worse. We will characterize the situations when exactly $p_0(x_D), p_1(x_D)$ or $\frac{x_D}{2}$ mark the maximum of π . Linking this to the medians of G_0, G_1, optimal price announcements is thus elicited.

Lemma A1. *For π_{G_0} and π_{G_1} there are unique points $p_0(x_D)$ resp. $p_1(x_D)$ such that π_{G_0} resp. π_{G_1} is strictly decreasing on $[x_l, 2p_0(x_D))$ resp. on $[x_l, 2p_1(x_D))$ and strictly decreasing on $(2p_0(x_D), x_u)$ resp. $(2p_1(x_D), x_u]$. Furthermore, $x_l < F^{-1}(\epsilon) + \epsilon \leq 2p_0(x_D)$ and $2p_1(x_D) \leq F^{-1}(1-\epsilon) - \epsilon < x_u$.*

Proof. We start with π_{G_1} and compactly write $x_a := F^{-1}(a)$ for $a \in [0,1]$ in the following. Note that π_{G_1} cannot be everywhere strictly increasing, as it's equal to the strictly decreasing function $u(x_D - p)$ where $G_1(2p) = 1$ which happens exactly for $x_{1-\epsilon} - \epsilon \leq x_D \leq x_u$. Thus $2p_1(x_D) \leq x_{1-\epsilon} - \epsilon < x_u$. If it is everywhere strictly decreasing we obviously have $2p_1(x_D) = x_l$. If it is unimodal, $2p_1(x_D)$ will be its mode. Note that $2p_1(x_D)$ is the unique maximum of π_{G_1} since it is continuous.

Turning towards π_{G_0} we first note that it cannot be everywhere strictly decreasing as it's equal to $u(p)$ where $G_0(2p) = 0$ which happens exactly if $x_l \leq x_D \leq x_\epsilon + \epsilon$. Thus $x_l < x_\epsilon + \epsilon \leq 2p_0(x_D)$. We mention, that we might face a discontinuity in $2p = x_u$ if $x_D \neq x_u$. If $x_D = x_u$, it's continuous and we can argue similarly as before. If it is not continuous, only the cases that π_{G_0} is strictly increasing on $2p \in [x_l, x_u)$ or it's mode is x_u in case of unimodality are critical. However, in both those cases we can set $2p_0(x_D) := x_u$ as π_{G_0} is then strictly increasing on $[x_l, x_u)$, even if we don't attain the maximum.[7] □

The next lemma gives a characterization of the relation between x_D and the values $p_0(x_D)$ and $p_1(x_D)$ that we will use afterwards to pin down the optimal price announcement.

Lemma A2. *We have $2p_1(x_D) \gtreqless x_D$ if and only if $F^{-1}(\frac{1}{2} - \epsilon) - \epsilon \gtreqless x_D$ for all x_D except for the degenerated case where $F^{-1}(1-\epsilon) - \epsilon \leq x_l$ in which $2p_1(x_D) = x_l$ and $F^{-1}(\frac{1}{2} - \epsilon) - \epsilon < x_D$ for all x_D.*

Furthermore we have $x_D \gtreqless 2p_0(x_D)$ if and only if $x_D \gtreqless F^{-1}(\frac{1}{2} + \epsilon) + \epsilon$ except for the degenerated case where $x_u \leq F^{-1}(\epsilon) + \epsilon$ in which $2p_0(x_D) = x_u$ and $F^{-1}(\frac{1}{2} + \epsilon) + \epsilon > x_D$ for all x_D.

Proof. We compactly write $x_a := F^{-1}(a)$ for $a \in [0,1]$. We begin with the first statement by distinguishing the degenerated (where G_1 is constant to 1) and non-degenerated case:

(a) $x_{1-\epsilon} - \epsilon \leq x_l < x_u$:
Note that in this case, $G_1 \equiv 1$. Thus, $\pi_{G_1}(p \mid x_D) = u(x_D - p)$ and hence $2p_1(x_D) = x_l$ for all x_D. Now note that the case implies $x_{1-\epsilon} \leq x_D + \epsilon$. Thus, $\frac{1}{2} - \epsilon < 1 - \epsilon = F(x_{1-\epsilon}) \leq F(x_l + \epsilon) \leq F(x_D + \epsilon)$.

[7] We will see later, that $p_0(x_D)$ will only be played in cases where it indeed attains the maximum of π_{G_0}.

(b) $x_l < x_{1-\epsilon} - \epsilon \leq x_u$:
Note that the continuous function π_{G_1} takes on the following form:

$$\pi_{G_1}(x_D, p) = \begin{cases} \varphi_1 := \pi_{F(x+\epsilon)+\epsilon} & , 2p \in [x_l, x_{1-\epsilon} - \epsilon], \\ u(x_D - p) & , 2p \in (x_{1-\epsilon} - \epsilon, x_u]. \end{cases}$$

Since $u(x_D - p)$ is strictly decreasing in p, it suffices to look at the strictly quasiconcave function φ_1. Note that

$$\begin{aligned} & \frac{\partial}{\partial p}\varphi_1(p \mid x_D) \mid_{2p=x_D} \gtreqqless 0 \\ \iff & \underbrace{u'(\tfrac{x_D}{2})}_{>0} \cdot (1 - 2 \cdot (F(x_D + \epsilon) + \epsilon)) \gtreqqless 0 \\ \iff & \frac{1}{2} - F(x_D + \epsilon) - \epsilon \gtreqqless 0 \\ \iff & F^{-1}(\tfrac{1}{2} - \epsilon) - \epsilon \gtreqqless x_D, \end{aligned}$$

thus it suffices to look at the continuous function $\partial\varphi_1(p \mid x_D) := \frac{\partial}{\partial p}\varphi_1(. \mid x_D)$ and its evaluation in $2p = x_D$. Note that by Lemma A1 $2p_1(x_D) \leq x_{1-\epsilon} - \epsilon$. We first show the following claims.

(bi) $\partial\varphi_1(\frac{x_D}{2} \mid x_D) < 0 \Rightarrow 2p_1(x_D) < x_D$:
Note that the continuous function $\partial\varphi_1$ is thus negative in an open environment of $2p = x_D$ and that $\frac{\partial}{\partial p}u(x_D - p) = -u'(x_D - p) < 0$ as well. Thus, by the defining property of $p_1(x_D)$, we conclude $2p_1(x_D) < x_D$.

(bii) $\partial\varphi_1(\frac{x_D}{2} \mid x_D) > 0 \Rightarrow 2p_1(x_D) \geq x_D$:
The continuous function $\partial\varphi_1$ is thus strictly increasing around $2p = x_D$ and not saturated. That implies that $x_D \leq 2p_1(x_D) \leq x_{1-\epsilon} - \epsilon$.
The first part of the next point shows, that this inequality is indeed a strict one.

(biii) $2p_1(x_D) = x_D \Leftrightarrow \partial\varphi_1(\frac{x_D}{2} \mid x_D) = 0$:
Firstly, let $2p_1(x_D) = x_D$. Then $x_D = 2p_1(x_D) \leq x_{1-\epsilon} - \epsilon$. We claim that $x_D = 2p_1(x_D) = x_{1-\epsilon} - \epsilon$ is impossible. To this end, observe that this would imply $x_D = 2p_1(x_D) = F^{-1}(1 - \epsilon) - \epsilon > F^{-1}(\frac{1}{2} - \epsilon) - \epsilon$. This implies $\partial\varphi_1(\frac{x_D}{2} \mid x_D) < 0$ and thus $2p_1(x_D) < x_D$ as shown above - a contradiction. Hence $x_D = 2p_1(x_D) < x_{1-\epsilon} - \epsilon$. This shows, that we have $\pi_{G_1} = \varphi_1$ in a neighborhood of x_D. By the defining property of $p_1(x_D)$ and the

continuity of $\partial\varphi_1(. \mid x_D)$, $2p = x_D$ defined a local extremum of $\partial\varphi_1(. \mid x_D)$. We especially conclude, that point (bii) must involve a strict inequality.
Secondly, recall that the condition $\partial\varphi_1(\frac{x_D}{2} \mid x_D) = 0$ is equivalent to $F^{-1}(\frac{1}{2} - \epsilon) - \epsilon = x_D$. This equation has a most one solution. Since the inequalities in the former points are all strict ones this leaves $2p_1(x_D) = x_D$ as the only possible option.

By a complete case distinction, the former points together prove the first part of the Lemma.

#

We now turn towards the second part of the Lemma whose proof is similar to the above one and just given for sake of completeness. We again distinguish the degenerated and non-degenerated cases:

(a) $x_l < x_u \leq x_\epsilon + \epsilon$:
Note that in this case $G_0(x) = 1_{x \geq x_u}$ and thus $\pi_{G_0}(p \mid x_D) = u(p)$ for $x_D < x_u$ and $\pi_{G_0}(\frac{x_u}{2} \mid x_D) = u(x_D - \frac{x_u}{2})$. As explained in Lemma A1 this implies $2p_0(x_D) = x_u$ for all x_D. Finally note that the case implies $x_u - \epsilon < x_\epsilon$ and thus $F(x_D - \epsilon) \leq F(x_u - \epsilon) \leq F(x_\epsilon) = \epsilon < \frac{1}{2} + \epsilon$.

(b) $x_l \leq x_\epsilon + \epsilon < x_u$:
Then, we have

$$\pi_{G_0}(x_D, p) = \begin{cases} u(p) & , 2p \in [x_l, x_\epsilon + \epsilon), \\ \varphi_0 := \pi_{F(x-\epsilon)-\epsilon} & , 2p \in [x_\epsilon + \epsilon, x_u)], \\ u(x_D - x_u/2) & , 2p \in \{x_u\}, \end{cases}$$

with a discontinuity in $2p = x_u$ if and only if $x_D < x_u$. Note that $u(p)$ is strictly increasing in p. It thus suffices to look at the strictly quasiconcave function φ_0 as also $2p = x_u$ is not optimal unless $x_D = x_u$ in which case the following argument also works. We thus restrict to $x_D < x_u$ in the following. First of, note that

$$\begin{aligned} & \frac{\partial}{\partial p}\varphi_0(p \mid x_D) \mid_{2p=x_D} \gtreqless 0 \\ \iff \quad & u'(\tfrac{x_D}{2}) \cdot (1 - 2 \cdot (F(x_D - \epsilon) - \epsilon) \gtreqless 0 \\ \iff \quad & \frac{1}{2} - F(x_D - \epsilon) + \epsilon \gtreqless 0 \\ \iff \quad & F^{-1}(\tfrac{1}{2} + \epsilon) + \epsilon \gtreqless x_D. \end{aligned}$$

Thus, it suffices to look at the continuous function $\partial\varphi_0(p \mid x_D) := \frac{\partial}{\partial p}\varphi_0(p \mid x_D)$ and its evaluation in $2p = x_D$. Recall that $x_\epsilon + \epsilon \le 2p_0(x_D)$ by Lemma A1. We now show the following claims.

(i) $\partial\varphi_0(\frac{x_D}{2} \mid x_D) > 0 \Rightarrow 2p_0(x_D) > x_D$**:** Note that thus the continuous function $\partial\varphi_0(p \mid x_D)$ is positive in a neighborhood of $2p = x_D$ as is $\frac{\partial}{\partial p}u(p) = u'(p)$. Thus, by the defining property of $p_0(x_D)$ and since $x_D < x_u$ we find $x_D < 2p_0(x_D)$.

(ii) $\partial\varphi_0(\frac{1}{2} \mid x_D) < 0$**:**
The continuous function $\partial\varphi_0$ is thus strictly decreasing in a neighborhood and after $2p = x_D$. Thus by the defining property of $p_0(x_D)$ we must have $2p_0(x_D) \le x_D$.
Indeed, the first part of the next point shows, that the found inequality must be a strict one.

(iii) $2p_0(x_D) = x_D \Leftrightarrow \partial\varphi_0(\frac{x_D}{2} \mid x_D) = 0$**:**
Firstly, let $2p_0(x_D) = x_D$. Then $x_\epsilon + \epsilon \le 2p_0(x_D) = x_D$. We claim that the inequality can't hold with equality. Assume so, then $x_D = 2p_0(x_D) = F^{-1}(\epsilon) + \epsilon < F^{-1}(\frac{1}{2} + \epsilon) + \epsilon$. This implies $\partial\varphi_0(\frac{x_D}{2} \mid x_D) > 0$ and thus as seen above $2p_0(x_D) > x_D$ - a contradiction. Hence $x_\epsilon + \epsilon < 2p_0(x_D) = x_D$. This shows that $\pi_{G_0} = \varphi_0$ locally around x_D. By the defining property of $p_0(x_D)$ we conclude that the continuous function $\partial\varphi_0(p \mid x_D)$ has a zero in $2p = x_D$. Especially, the inequality in point (bii) mus be a strict one.
Secondly, recall that the condition $\partial\varphi_0(\frac{x_D}{2} \mid x_D) = 0$ is equivalent to $F^{-1}(\frac{1}{2} + \epsilon) + \epsilon = x_D$. This equation has at most one solution. Since the inequalities in the former points are all strict ones, this implies $2p_0(x_D) = x_D$ by a full case distinction.

The above points complete the proof of the second part of the Lemma.

□

The following corollary characterizes the optimal price announcement.

Corollary A1. *The following conditions summarize the optimal price announcement function $B(x_D)$ for any value of x_D.*

(i) $B(x_D) = \frac{x_D}{2}$, *if* $2p_1(x_D) \le x_D \le 2p_0(x_D)$.

(ii) $B(x_D) = p_1(x_D)$ *if* $x_D < 2p_1(x_D)$.

(iii) $B(x_D) = p_0(x_D)$ *if* $2p_0(x_D) < x_D$.

Proof. Recall that π is the max of π_{G_0} and π_{G_1} and that the switch happens exactly at $2p = x_D$ for increasing p. Also, we know from Lemma A1 where exactly π_{G_0} resp. π_{G_1} are increasing or decreasing. The rest is just a matter of case distinctions.

(i) In this case, π_{G_0} is is strictly increasing until $2p = x_D$ and π_{G_1} is strictly decreasing afterwards. Thus, the optimal action is to play the intersection $2p = x_D$.

(ii) According to Lemma A2, in this case we have $x_D < 2p_0(x_D)$ as well and thus, π is strictly increasing until $2p_1(x_D)$ where it attains its maximum and strictly decreasing afterwards.

(iii) In this case, π is strictly increasing until $2p_0(x_D)$ where it indeed attains its maximum (note that $2p_0(x_D) < x_u$ here) and strictly decreasing afterwards, since also $2p_1(x_D) < x_D$ by Lemma A2.

□

We now identify $p_0(x_D)$ and $p_1(x_D)$ as local optima.

Corollary A1. *The points $p_0(x_D)$ and $p_1(x_D)$ denote the unique local maxima of π_{G_0} resp. π_{G_1} in the domains where they are played.*

Proof. For the notation of $x_a, \varphi_1, \varphi_0$ we refer to the proof of, e.g. Lemma A2.

We start with $x_D < 2p_1(x_D)$. Then $x_l \leq x_D < 2p_1(x_D) \leq x_{1-\epsilon} - \epsilon$. We now show that $x_D < 2p_1(x_D) = x_{1-\epsilon} - \epsilon$ is impossible. Suppose otherwise, then this implies $\pi_{G_1}(p_1(x_D) \mid x_D) = u(x_D - p_1(x_D)) < u(\frac{x_D}{2})$ which can be reached by announcing $p = \frac{x_D}{2}$. This is a contradiction to the definition of $p_1(x_D)$ and strict quasiconcavity of π_{G_1}. Hence we face $x_D < 2p_1(x_D) < x_{1-\epsilon} - \epsilon$. Then, $\pi_{G_1} = \partial\varphi_1$ in $p = p_1(x_D)$. By the defining property of $p_1(x_D)$ and continuity of $\partial\varphi_1$ we conclude that necessarily $\partial\varphi_1(p_1(x_D) \mid x_D) = 0$, i.e. $p_1(x_D)$ is a local extremum of φ_1 and thus π_{G_1} and by strict quasiconcavity a unique maximum.

Vice versa for $2p_0(x_D) < x_D$. Then $x_\epsilon + \epsilon \leq 2p_0(x_D) < x_D$. Suppose for the moment that $x_\epsilon + \epsilon = 2p_0(x_D) < x_D$. Then we find $\pi_{G_0}(p_0(x_D) \mid x_D) = u(p_0(x_D)) < u(\frac{x_D}{2})$ which is incompatible with the definition of $p_0(x_D)$. Thus $x_\epsilon + \epsilon < 2p_0(x_D) < x_D$, meaning that $\pi_{G_0} = \varphi_0$ in an environment of $p = p_0(x_D)$. By continuity of $\partial\varphi_0$ we deduce that $p_0(x_D)$ marks a local extremum of π_{G_0}. By strict quasiconcavity, it is a maximum and unique. □

Together, Lemma A1, Corollary A1, Lemma A2 and Corollary A1 imply Theorem 1.

We also point out, that, depending on X, F and ϵ, one might encounter $F^{-1}(1/2-\epsilon)-\epsilon < x_l$ and/or $F^{-1}(1/2+\epsilon)+\epsilon > x_u$. In either case, the respective interval in the definition of B are to be considered empty, but remain valid with the obvious adjustment.

Proof of Corollary 1. Since $x \mapsto \frac{x}{2}$ is strictly increasing, it suffices to show that $\frac{\partial^2}{\partial p \partial x_D}\pi_G > 0$ for $G = G_1$ on $x_D < F^{-1}(\frac{1}{2}-\epsilon)-\epsilon$ and for $G = G_0$ on $F^{-1}(\frac{1}{2}+\epsilon)+\epsilon < x_D$. Recall that on those domains $p_1(x_D)$ resp. $p_0(x_D)$ denote local maxima by Corollary A1 and thus are a zero of $\frac{\partial}{\partial p}\pi_G$. We calculate and find

$$\frac{\partial^2}{\partial p \partial x_D}\pi_G = 2 \cdot g(2p) \cdot u'(x_D - p) - G(2p) \cdot u''(x-p) > 0,$$

since $g > 0$, $G(2p) \geq 0$ and u is strictly increasing and concave. □

Proof of Corollary 2. The statement is immediate from Corollary A1 and Theorem 1 and remembering that $x_m \in [F^{-1}(\frac{1}{2}-\epsilon)-\epsilon, F^{-1}(\frac{1}{2}+\epsilon)+\epsilon]$. □

Proof of Lemma 4. Before we get into the calculations, note that Φ_D and Φ_C are continuous (recall $2p_1(F^{-1}(1/2-\epsilon)-\epsilon) = F^{-1}(1/2-\epsilon)-\epsilon$ and similarly for $2p_0$), so it suffices to the differentiation on the corresponding open intervals. The derivatives of Φ_D for the center interval and of Φ_C for the outer ones are obvious. We now turn towards the remaining ones.

Firstly, for $G \in \{G_0, G_1\}$ consider $\Phi_D(x) = \pi_G(B(x) \mid x)$ on the respective (open) interval. We already know that $B(x)$ is the unique interior optimum of $\pi_G(p \mid x)$ and and thus, a calculation with $u = \mathrm{id}$ reveals

$$\begin{aligned}\frac{\partial}{\partial x}\Phi_D(x) =& \frac{\partial}{\partial x}\left((x - B(x)) \cdot G(2B(x) + B(x) \cdot (1 - G(2B(x)))\right) \\ =& G(2B(x)) + B'(x) \cdot \underbrace{\frac{\partial}{\partial p}\pi_G(p \mid x) \mid_{p=B(x)}}_{=0} \\ =& G(2B(x)).\end{aligned}$$

Note that indeed $B(x)$ is differentiable on the resp. interval considered, see Corollary 1. This concludes the differentiability of Φ_D and its derivative.

Secondly, consider Φ_C for $x \in (2B(x_l), 2B(x_u))$ and recall $z^* = z^*(x) = B^{-1}(x/2)$. We use the measure theoretic version of the differentiation of parameter integrals for the following calculation. Therefore, $C := x_u + 1$ serves as a constant bounding z^* from above, μ^G describes the probability measure associated to the distribution function $G \in \{G_0, G_1\}$ with densities g_0, g_1, and we again use the integration variable z to avoid confusion.

$$\begin{aligned}
&\frac{\partial}{\partial x}\Phi_C(x) \\
=&\frac{\partial}{\partial x}\left(\int_{[x_l,z^*)} x - B(z)\; G_0(\mathrm{d}z) + (G_1(z^*) - G_0(z^*)) \cdot \frac{x}{2} + \int_{(z^*,x_u]} u(B(z))\; G_1(\mathrm{d}z)\right) \\
= & \int_{[x_l,z^*)} 1\, G_0(\mathrm{d}z) - (x - B(z^*)) \cdot \frac{\partial}{\partial x}\left(\mu^{G_0}((z^*, C))\right) \\
&+ \left(g_1(z^*) \cdot \left(\frac{\partial}{\partial x}(z^*)\right) - g_0(z^*) \cdot \left(\frac{\partial}{\partial x}(z^*)\right)\right) \cdot \frac{x}{2} + (G_1(z^*) - G_0(z^*)) \cdot \frac{1}{2} \\
&+ \int_{(z^*,x_u]} 0\, G_1(\mathrm{d}z) + B(z^*) \cdot \frac{\partial}{\partial x}\left(\mu^{G_1}([z^*, C])\right).
\end{aligned}$$

Realizing $\mu^{G_0}((z^*, C)) = 1 - G_0(z^*)$ we find $\frac{\partial}{\partial x}\mu^{G_0}((z^*, C]) = -g_0(z^*) \cdot \frac{\partial}{\partial x}(z^*)$ and similarly for G_1. Since $B(z^*) = x/2$ we thus obtain

$$\begin{aligned}
&\frac{\partial}{\partial x}\Phi_C(x) \\
=&G_0(z^*) - \frac{x}{2} \cdot (-g_0(z^*)\frac{\partial}{\partial x}(z^*)) + \left(g_1(z^*) \cdot \left(\frac{\partial}{\partial x}(z^*)\right) - g_0(z^*) \cdot \left(\frac{\partial}{\partial x}(z^*)\right)\right) \cdot \frac{x}{2} \\
&+ \frac{1}{2} \cdot (G_1(z^*) - G_0(z^*)) + \frac{x}{2} \cdot (-g_1(z^*)\frac{\partial}{\partial x}(z^*)) \\
=&\frac{1}{2} \cdot (G_1(z^*) + G_0(z^*)).
\end{aligned}$$

Finally, since the derivatives are non-negative, Φ_D and Φ_C are increasing. □

Proof of Theorem 2.

Case $F^{-1}(\epsilon) + \epsilon < F^{-1}(1 - \epsilon) - \epsilon$:
Under this assumption, we have $G_1(z) - G_0(z) < 1$ for all $z \in X$. In other words, for all $z \in X$ there is no $G \in \mathcal{G}$ that can assign point mass 1 to z. Since B is measurable, as it's strictly increasing, we find $\mu^G(\{z \mid B(z) = \frac{x}{2}\}) < 1$ for all $G \in \mathcal{G}$.

We now prove that $\Phi_D(x) < \Phi_C(x)$ for all $x \in [F^{-1}(\frac{1}{2} - \epsilon) - \epsilon, F^{-1}(\frac{1}{2} + \epsilon) + \epsilon]$. On the one hand for any such x we have $B(x) = \frac{x}{2}$ and thus $\Phi_D(x) = \frac{x}{2}$. On the other hand

$$\Phi_C(x) = \min_{G \in \{G_0, G_1, G_x^*\}} \mathbb{E}_G\left[\underbrace{\max\{x - B(z), B(z)\}}_{\geq \frac{x}{2}}\right] > \tfrac{x}{2} = \Phi_D(x),$$

where the strict inequality results from the introductory argument.

In the following, we compare the derivatives on the outer intervals. We start with $x \in [x_l, F^{-1}(\frac{1}{2}-\epsilon)-\epsilon)$. We have the following chain of arguments

$$\begin{aligned}
& x < F^{-1}(\tfrac{1}{2}-\epsilon)-\epsilon \\
\implies & x < 2B(2B(x)) \qquad \text{(Corollary 2)} \\
\implies & B^{-1}(\tfrac{x}{2}) < 2B(x) \\
\implies & G_1(B^{-1}(\tfrac{x}{2})) < G_1(2B(x)) \qquad (\star) \\
\implies & \frac{1}{2}\big(G_1(B^{-1}(\tfrac{x}{2})) + G_0(B^{-1}(\tfrac{x}{2}))\big) < G_1(2B(x)) \qquad (G_0 \leq G_1) \\
\implies & \Phi'_C(x) < \Phi'_D(x),
\end{aligned}$$

where we note that $\Phi'_C(x)$ might be zero, but cannot be 1 since $x < x_m \leq 2p_1(x_u) = 2B(x_u)$ as $2B$ is strictly increasing with fixed point x_m (see Corollary 1). The inequality in $(\star)$ remains a strict one, since (using Corollary 2) $B^{-1}(\frac{x}{2}) < x < F^{-1}(\frac{1}{2}-\epsilon)-\epsilon < F^{-1}(1-\epsilon)-\epsilon$ and thus, by its functional form, G_1 is strictly increasing in a neighborhood of $B^{-1}(\frac{x}{2})$.

We now turn towards the case $x \in (F^{-1}(\frac{1}{2}+\epsilon)+\epsilon, x_u]$. In analogy to the above, the following chain of arguments applies:

$$\begin{aligned}
& F^{-1}(\tfrac{1}{2}+\epsilon)+\epsilon < x \\
\implies & 2B(2B(x)) < x \\
\implies & 2B(x) < B^{-1}(\tfrac{x}{2}) \\
\implies & G_0(2B(x)) < G_0(B^{-1}(\tfrac{x}{2})) \\
\implies & G_0(2B(x)) < \frac{1}{2}\big(G_0(B^{-1}(\tfrac{x}{2}) + G_1(B^{-1}(\tfrac{x}{2})\big) \\
\implies & \Phi'_D(x) < \Phi'_C(x),
\end{aligned}$$

where we note that indeed $\Phi'_C(x)$ could be equal to 1 and that applying G_0 preserves the strict inequality since $F^{-1}(\epsilon)+\epsilon < F^{-1}(\frac{1}{2}+\epsilon)+\epsilon < x < B^{-1}(\frac{x}{2})$.

Together, the above steps imply $\Phi_D(x) < \Phi_C(x)$ for all $x \in X$.

Case $F^{-1}(1-\epsilon)-\epsilon \leq F^{-1}(\epsilon)+\epsilon$:
For $x \in [F^{-1}(1-\epsilon)-\epsilon, F^{-1}(\epsilon)+\epsilon]$ we have by Theorem 1 $z^* := B^{-1}(\frac{x}{2}) = x$. Note that if $x = z^*$ is within the subinterval $[F^{-1}(1-\epsilon)-\epsilon, F^{-1}(\epsilon)+\epsilon]$ there is $G \in \mathcal{G}$, namely $G = G^*_x$, with $\mu^G(\{z^* | B(z^*) = \frac{x}{2}\}) = 1$. Hence, the argument of the first part of the previous case reveals that $\Phi_D(x) = \Phi_C(x)$ if $x = z^* \in [F^{-1}(1-\epsilon)-\epsilon, F^{-1}(\epsilon)+\epsilon]$ and $\Phi_D(x) < \Phi_C(x)$ otherwise.

Since our considerations from the first case remain valid for valuations outside of the interval $[F^{-1}(\frac{1}{2}-\epsilon)-\epsilon, F^{-1}(\frac{1}{2}+\epsilon)+\epsilon]$ we conclude

$$\begin{cases} \Phi_D(x) = \Phi_C(x) & , x \in [F^{-1}(1-\epsilon)-\epsilon, F^{-1}(\epsilon)+\epsilon], \\ \Phi_D(x) < \Phi_C(x) & , \text{otherwise} \end{cases}.$$

□

Explicit formulae for Φ_C in Example 4. For $\frac{1}{4} < \epsilon \leq \frac{1}{2}$ we find

$$\Phi_C(x) = \begin{cases} \frac{1}{x^2} + \epsilon x + \epsilon^2 - \epsilon + \frac{1}{4} & , 0 \leq x < 1-2\epsilon, \\ \frac{x}{2} & , 1-2\epsilon \leq x \leq 2\epsilon, \\ \frac{1}{4}x^2 + (\frac{1}{2}-\epsilon)x + \epsilon^2 & , 2\epsilon < x \leq 1, \end{cases}$$

while for $\frac{1}{8} < \epsilon \leq \frac{1}{4}$ it is given by

$$\Phi_C(x) = \begin{cases} \frac{1}{2}\epsilon^2 - \epsilon + \frac{9}{32} & , 0 \leq x < 2p_1(0) = \frac{1}{4} - \epsilon, \\ \frac{1}{2}x^2 + (2\epsilon - \frac{1}{4})x + 2\epsilon^2 - \frac{3}{2}\epsilon + \frac{5}{16} & , \frac{1}{4} - \epsilon \leq x < \frac{1}{2} - 2\epsilon \\ \frac{1}{4}x^2 + \epsilon x + \epsilon^2 - \epsilon + \frac{1}{4} & , \frac{1}{2} - 2\epsilon \leq x < 2\epsilon, \\ \frac{1}{2}x^2 + 2\epsilon^2 - \epsilon + \frac{1}{4} & , 2\epsilon \leq x \leq 1 - 2\epsilon, \\ \frac{1}{4}x^2 + (\frac{1}{2} - \epsilon)x + \epsilon^2 & , 1 - 2\epsilon < x \leq \frac{1}{2} + 2\epsilon, \\ \frac{1}{2}x^2 + (\frac{1}{4} - 2\epsilon)x + 2\epsilon^2 + \frac{1}{2}\epsilon + \frac{1}{16} & , \frac{1}{2} + 2\epsilon < x \leq \frac{3}{4} - \epsilon, \\ x + \frac{1}{2}\epsilon^2 - \epsilon - \frac{7}{32} & , \frac{3}{4} + \epsilon = 2p_0(1) < x \leq 1, \end{cases}$$

and for $0 \leq \epsilon \leq \frac{1}{8}$ we finally have

$$\Phi_C(x) = \begin{cases} -\frac{3}{2}\epsilon^2 - \frac{1}{2}\epsilon + \frac{1}{4} & , 0 \leq x < 2p_1(0) = \frac{1}{4} - \epsilon, \\ \frac{1}{2}x^2 + (2\epsilon - \frac{1}{4})x - \epsilon + \frac{9}{32} & , \frac{1}{4} - \epsilon \leq x < \frac{1}{4}, \\ x^2 + (2\epsilon - \frac{1}{2})x - \epsilon + \frac{5}{16} & , \frac{1}{4} \leq x < \frac{1}{2} - 2\epsilon, \\ \frac{1}{2}x^2 - 2\epsilon^2 + \frac{3}{16} & , \frac{1}{2} - 2\epsilon \leq x \leq \frac{1}{2} + 2\epsilon, \\ x^2 - (2\epsilon + \frac{1}{2})x + \epsilon + \frac{5}{16} & , \frac{1}{2} + 2\epsilon < x \leq \frac{3}{4} \\ \frac{1}{2}x^2 + (\frac{1}{4} - 2\epsilon)x + \epsilon + \frac{1}{32} & , \frac{3}{4} < x \leq \frac{3}{4} + \epsilon, \\ x - \frac{3}{2}\epsilon^2 - \frac{1}{2}\epsilon - \frac{1}{4} & , \frac{3}{4} + \epsilon = 2p_0(1) < x \leq 1. \end{cases}$$

□